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THE COURNOT EQUILIBRIUM IN A NONSYMMETRIC OLIGOPOLISTIC MARKET

(A Business Game for Teaching and Research Purposes: Part VI)

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1. Introduction

In several previous papers we have explored the application of different solution concepts to the same market structure.[1] Previously we have considered models in which both price and production are independent variables. These models were solved for: (1) joint maximum, (2) efficient production (3) the noncooperative equilibrium and (4) maximization of profit share.

The noncooperative equilibrium solutions (pure strategy or "Edgeworth cycle" [2]) were based on one set of classical mathematical economic models with price and quantity as the independent variables. In contrast we can consider the Cournot model which has only quantity as the independent variable. To some extent this is somewhat less realistic than the previous model; however for experimental purposes it is useful to test for the predictive value of this solution. Furthermore when

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there is product differentiation the generation of prices in the extended Cournot model poses an interesting problem closely related to general equilibrium and rationing analysis.

We assume that consumer preferences can be represented by a general quadratic utility function. Our somewhat strong special assumption is that to a first approximation there is no income effect between this class of goods and the remainder of the consumer's purchases. In terms of quantity the consumer's utility function is:

$$\frac{u(x)}{\lambda} = V \sum_{i=1}^{n} x_i - \frac{1}{2\beta(1+\gamma)} \left\{ \frac{\sum_{i=1}^{n} x_i^2}{w_i} + \gamma(\Sigma\, x_i)^2 \right\}$$

where λ = marginal worth of money.

In the simple case of quantity duopoly with an undifferentiated product, price is easily determined by the demand relation

$$p = f\left(\sum_{i=1}^{n} q_i \right) .$$

When there are differentiated products we must solve a limited general equilibrium model to determine the set of prices which will just clear all markets. A detailed analysis of this problem and its relationship to rationing and general equilibrium has been given by Levitan.[3/]

2. The Cournot Equilibrium

Let

x = the vector $(x_1, x_2, \ldots, x_n)$ of quantities offered by the firms.

p = the vector $(p_1, p_2, \ldots, p_n)$ of prices.

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & w_n \end{bmatrix}$$ where w_i is the "weight" reflecting the "size" of the i^{th} firm.

$$1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & \cdots & & 0 \\ 0 & 1 & & \\ \vdots & & 1 & \\ & & & \ddots \\ 0 & & & 1 \end{bmatrix} .$$

c = the vector $(c_1, c_2, \ldots, c_n)$ of costs of the firms.

β , V and γ are parameters.

We may express demand in terms of price by equation (1).

(1) $$x = \beta W(V\hat{1} - ((1+\gamma)I - \gamma SW)p) \; ,$$

hence

$$p = [(1+\gamma)I - \gamma SW]^{-1}(V\hat{1} - \frac{1}{\beta} W^{-1}x)$$

$$= \frac{1}{\gamma} W^{-1} (\frac{1+\gamma}{\gamma} W^{-1} - S)^{-1} (V\hat{1} - \frac{1}{\beta} W^{-1}x)$$

$$= \frac{1}{1+\gamma} (I + \gamma SW)(V\hat{1} - \frac{1}{\beta} W^{-1}x)$$

or

(2) $$p = V\hat{1} \quad \frac{1}{\beta(1+\gamma)} (W^{-1} + \gamma S)x \; .$$

The i^{th} payoff may be expressed as

(3) $$\Pi_i = x_i(p_i - c_i)$$

$$= x_i(V - \frac{1}{\beta(1+\gamma)} (\frac{x_i}{w_i} + \gamma \Sigma x_j) - c_i) \; .$$

Differentiating with respect to quantity we obtain:

(4) $$\frac{\partial \Pi_i}{\partial x_i} = V - \frac{1}{\beta(1+\gamma)} (\frac{x_i}{w_i} + \gamma \Sigma x_j) - c_i - \frac{x_i}{\beta(1+\gamma)} (\frac{1}{w_i} + \gamma)$$

Setting (4) equal to zero we have:

(5) $$V - c_i - \frac{1}{\beta(1+\gamma)} ((\frac{2}{w_i} + \gamma)x_i + \gamma \Sigma x_j) = 0$$

Writing this in matrix notation:

(6) $$\frac{\gamma}{\beta(1+\gamma)} (\frac{1}{\gamma} (2W^{-1} + \gamma I) + S)x = V\hat{1} - c \; .$$

We note that

(7) $$(Z^{-1} + S)^{-1} = (Z - qZSZ)$$

where

$$q = \frac{1}{1 + \Sigma z_i} \quad .$$

In this case (equation (6)).

$$z_i = \frac{\gamma w_i}{2 + \gamma w_i} \quad .$$

Specifically

$$q = \frac{1}{1 + \gamma \, \Sigma \frac{w_j}{2 + \gamma w_j}}$$

Solving for x we obtain:

(8) $$x = \frac{\beta(1 + \gamma)}{\gamma} (Z - qZSZ)(V\hat{1} - c)$$

and

(9) $$p = V\hat{1} - \frac{1}{\gamma} (W^{-1} + \gamma S)(Z - qZSZ)(V\hat{1} - c)$$

Multiplying out the factors on the right of (9)

(10) $$p = V\hat{1} - \frac{1}{\gamma} (W^{-1} + \gamma \, S - qW^{-1}ZS - \gamma q \, \hat{1}\hat{1}^T Z\hat{1}\,\hat{1}^T)(Z)(V\hat{1} - c) \, ,$$

however

$$\hat{1}^T Z\hat{1} = \frac{1 - q}{q}$$

hence

(11) $$p = V\hat{1} - \frac{1}{\gamma} (W^{-1} + \gamma S - qW^{-1}ZS - \gamma(1 - q)S)(Z)(V\hat{1} - c)$$

or:

(12) $$p = V\hat{1} - \frac{1}{\gamma} (W^{-1} + q(\gamma I - W^{-1}Z)S)(Z)(V\hat{1} - c)$$

which simplifies to:

(13) $$p = V[\hat{1} - \frac{1}{\gamma}(W^{-1}Z\hat{1} + q(\gamma I - W^{-1}Z)SZ\hat{1}] + \frac{1}{\gamma}(W^{-1} + q(\gamma I - W^{-1}Z)S)ZC$$

writing $S = \hat{1}\,\hat{1}^T$ in (13) gives us $SZ\hat{1} = \hat{1}\,\hat{1}^T Z\hat{1} = \hat{1}\,(\frac{1-q}{q})$ hence

(14) $$p = V[\hat{1}(1 - (1-q)) - \frac{1}{\gamma}W^{-1}Z\hat{1}(\hat{1} - (1-q))] + \frac{1}{\gamma}W^{-1}ZC + \frac{q}{\gamma}(\gamma I - W^{-1}Z)SZC$$

or:

(15) $$p = qV[\hat{1} - \frac{1}{\gamma}W^{-1}Z\hat{1}] + \left\{\frac{c_i}{2 + \gamma w_i}\right\} + q(\gamma I - W^{-1}Z)\,\Sigma\,\frac{w_j c_j}{2 + \gamma w_j}$$

hence

(16) $$p_i = qV\left\{\frac{1 + \gamma w_i}{2 + \gamma w_i}\right\} + \left\{\frac{c_i}{2 + \gamma w_i}\right\} + q\gamma\left\{\frac{1 + \gamma w_i}{2 + \gamma w_i}\right\}\Sigma\,\frac{w_j c_j}{2 + \gamma w_j}$$

or

(17) $$\boxed{p_i = q\left(V + \gamma\,\Sigma\,\frac{w_j c_j}{2 + \gamma w_j}\right)\left\{\frac{1 + \gamma w_i}{2 + \gamma w_i}\right\} + \left\{\frac{c_i}{2 + \gamma w_i}\right\}}$$

We may check some special cases.

(A) Suppose $\gamma = 0$. This implies $q = 1$

$$p = \frac{1}{2}V\hat{1} + \frac{1}{2}c$$

$$= \frac{1}{2}(V\hat{1} + c)$$

which is the monopoly solution.

(B) Suppose $\gamma \to \infty$. This implies $z_i \to 1$ hence $q \to \frac{1}{n+1}$.

$$p \to \frac{1}{n+1} (V + \Sigma c_j) + 0$$

$$= \frac{V}{n+1} + \frac{n}{n+1} \bar{c}$$

where $\bar{c}$ is the average cost.

(C) Suppose $w_i = \frac{1}{n}$ this formula becomes

$$(18) \quad \boxed{p_i = \frac{n+\gamma}{2n+(n+1)\gamma} (V + \frac{\gamma n}{2n+\gamma} \bar{c}) + \frac{nc_i}{2n+\gamma}} .$$

We now compare the noncooperative equilibrium obtained from regarding price as the independent variable or quantity as the independent variable. For $n = 1$ we see below from the two general symmetric market formulae that we obtain the same result. Similarly for $n \to \infty$ we obtain the same limit (when all $c_i = \bar{c}$) .

$$(19) \quad p_i(\text{Edgeworth}) = \frac{n}{2n+(n-1)\gamma} \left(V + \left\{ \frac{n+(n-1)\gamma}{2n+(2n-1)\gamma} \right\} \gamma\bar{c} \right) + \frac{(n+(n-1)\gamma)\bar{c}}{2n+(2n-1)\gamma}$$

$$(20) \quad p_i(\text{Cournot}) = \frac{n+\gamma}{2n+(n+1)\gamma} \left(V + \frac{\gamma n}{2n+\gamma} \bar{c} \right) + \frac{n}{2n+\gamma} \bar{c}$$

For $n = 1$

$$p_i(\text{Edgeworth}) = \frac{1}{2} \left(V + \frac{\gamma}{2+\gamma} \bar{c} \right) + \frac{\bar{c}}{2+\gamma} = \frac{1}{2} (V + \bar{c})$$

$$p_i(\text{Cournot}) = \frac{1+\gamma}{2(1+\gamma)} \left(V + \frac{\gamma}{2+\gamma} \bar{c} \right) + \frac{\bar{c}}{2+\gamma} = \frac{1}{2} (V + \bar{c}) .$$

For $n \to \infty$

$$p_i(\text{Edgeworth}) = \frac{1}{2+\gamma}\left(V + \frac{1}{2}\left(\frac{1+\gamma}{1+\gamma}\right)\gamma\bar{c}\right) + \frac{\bar{c}}{2+\gamma} = \frac{V}{2+\gamma} + \bar{c}\,\frac{(\gamma+1)}{(\gamma+2)}$$

$$p_i(\text{Cournot}) = \frac{1}{2+\gamma}\left(V + \frac{\gamma}{2}\,\bar{c}\right) + \frac{\bar{c}}{2+\gamma} = \frac{V}{2+\gamma} + \bar{c}\,\left(\frac{\gamma+1}{\gamma+2}\right)$$

We note that for $\gamma \to \infty$ these both give $p_i = \bar{c}$.

Returning to (19) and (20) for $c_i = c$ we may observe that for $n > 1$ and $\gamma > 0$:

$$p_c > p_e$$

This follows from observing that

$$\frac{n+\gamma}{2n+(n+1)\gamma} > \frac{n}{2n+(n-1)\gamma} \quad ,$$

then observing that price is a weighted average of costs c_i and V , however $V \geq \bar{c}$ and V appears with a larger weighting in (20).

FOOTNOTES

1/ Levitan, Richard and Martin Shubik, "A Business Game for Teaching and Research Purposes, Part I., General Description of the Game," RC-730, IBM Watson Research Center, Yorktown Heights, July 17, 1962.

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Levitan, Richard and Martin Shubik, "Part IV: Mathematical Structure and Analysis of the Nonsymmetric Game."

Shubik, Martin, "Part V: The Nonsymmetric Game: Joint Maximum, Efficient Solution and Measures of Collusion and Welfare."

2/ __________________, Strategy and Market Structure, John Wiley and Sons, New York, 1959, Chapter 5.

3/ Levitan, Richard, "Demand in an Oligopolistic Market and the Theory of Rationing," RC-1545, IBM Watson Research Center, Yorktown Heights, January 21, 1966.