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Efficient Regression in Time Series Partial Linear Models[1]

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Abstract

This paper studies efficient estimation of partial linear regression in time series models. In particular, it combines two topics that have attracted a good deal of attention in econometrics, viz. spectral regression and partial linear regression, and proposes an efficient frequency domain estimator for partial linear models with serially correlated residuals. A nonparametric treatment of regression errors is permitted so that it is not necessary to be explicit about the dynamic specification of the errors other than to assume stationarity. A new concept of weak dependence is introduced based on regularity conditions on the joint density. Under these and some other regularity conditions, it is shown that the spectral estimator is root-n-consistent, asymptotically normal, and asymptotically efficient.

JEL Codes: C14 C22 C13

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1 Introduction

The subject of this paper is the partially linear regression model

$$y_t = \beta' x_t + g(z_t) + u_t, t = 1, 2, ..., n, \tag{1}$$

where $\{x_t, z_t, u_t\}_{t=1}^n$ are $\mathcal{R}^p \times \mathcal{R}^l \times \mathcal{R}$ -valued random variables, $g(\cdot)$ is an unknown real function, and β is the vector of unknown parameters that we want to estimate. In this model, the mean response is assumed to be linearly related to one or more variables, and nonparametrically related to some other variables. This specification arises when the primary interest is precise estimation of β, while the building of a full parametric model may be of secondary importance or the relation of the mean response to additional variables is not easily parameterized. This compromising modeling strategy is more flexible than the standard linear model, and affords greater precision than a pure nonparametric one.

Partial linear models have been an important object of study in econometrics and statistics. One approach to estimation in these models is based on the penalized least squares method and has been employed by Wabba (1984), Engle et al. (1986), and Shiau et al. (1986), among others. Estimation is obtained by adding a penalty term to the ordinary nonlinear least squares criterion to penalize for roughness in the fitted function $g(\cdot)$. Heckman (1986) and Chen (1988) proved that this estimator of β can achieve a $\sqrt{n}$ convergence rate if x and z are not related to each other. Rice (1986) obtained the asymptotic bias of a partial smoothing spline estimator of β in the presence of dependence between x and z and showed that it is not generally possible to attain the $\sqrt{n}$ convergence rate for β.

Green et al. (1985) and Speckman (1988) suggested a simultaneous equation method of estimating both β and g. $\sqrt{n}$-consistency and asymptotic normality are established in Speckman (1988). Juhl and Xiao (2000) studied partially linear models with unit roots and show that the autoregressive parameter can be estimated at rate n even though part of the model is estimated nonparametrically. For additional work on partial linear regression models, see Cosslett (1984), Chen (1988), Shiller (1984), Eubank (1986), and Schick (1986), among others.

Most of the above estimators are not efficient. Robinson (1988) addressed efficiency issues when the regression errors are *iid* normal. In particular, he used (higher order) Nadaraya-Watson kernel estimates to eliminate the unknown function and introduced a feasible least squares estimator for β. Under regularity and smoothness conditions, $\sqrt{n}$-consistency and asymptotic normality are obtained by this approach. When the errors are *iid* normal, this estimator achieves the semiparametric information bound. A higher order asymptotic analysis of this estimator is given by Linton (1995). Fan and Li (1999) extended the Robinson estimator to regressions with weakly dependent disturbances and established $\sqrt{n}$-consistency and asymptotic normality for the density weighted version of the Robinson estimator. However, when the unobserved disturbances are autocorrelated, this estimator is no longer efficient and, moreover, when the correlation structure is not parameterized, it is generally not possible to estimate the full covariance matrix and the conventional time domain GLS estimator is infeasible.

We believe that frequency domain regression can address some of the efficiency issues in the presence of serially correlated residuals. Spectral regression estimators were introduced by Hannan (1963), following earlier work by Whittle (1953). Hannan (1963) showed that a frequency domain GLS estimator achieves asymptotically the Gauss-Markov efficiency bound under general smoothness conditions on the residual spectral density. Hannan (1971) and Robinson (1972) extended this method to nonlinear models. Phillips (1991) showed how to apply it to cointegrating regressions in the presence of integrated time series and developed a new asymptotic theory for this case. These frequency domain estimators are semiparametric since they rely upon a nonparametric treatment of the regression errors.

Consider the time series regression $y_t = \beta' x_t + u_t$, where u_t is stationary and has absolutely continuous spectral density. Roughly speaking, the discrete Fourier transform (dft) of this regression has residuals that are locally asymptotically independent (Phillips, 2000, showed this property for frequencies in the neighborhood of zero) making the frequency domain a natural setting for efficient estimation. In particular, efficient methods of estimating β in the frequency domain are possible via weighted regression and have been used in a variety of econometric applications (see, inter alia, Engle, 1974, Robinson 1991, Corbae, Ouliaris, and Phillips 2001). Such a technique has the advantage that it is not necessary to be explicit about the generating mechanism for the errors other than to assume stationarity.

In this paper, we show that spectral regression methods can be applied to partial linear models with serially correlated residuals to obtain an asymptotically efficient estimator of β. In particular, we consider a partial linear regression model whose residuals follow a linear process. Such a linear process includes quite general stationary time series and facilitates the asymptotic analysis. We also allow for serial correlation in the regressors x and z, as long as they are independent of the residuals. We introduce a new concept of weak dependence that controls the temporal correlation in z and is formulated directly in terms of the joint probability density. This concept is particularly useful in developing an asymptotic theory of regression in the present context. We construct nonparametric preliminary estimators for the conditional expectations as in Robinson (1988), and then propose a frequency domain efficient regression estimation of β. Under some regularity conditions and assumptions on the kernel function and bandwidth expansion rates, we show that the proposed spectral regression estimator is root-n consistent, asymptotically normal, and asymptotically efficient.

The paper is organized as follows: The next section describes the model and assumptions. Analysis for the estimators are presented in Sections 3 and 4. Section 5 studies the more general model. Section 6 concludes.

2 Assumptions and Econometric Estimation

Our interest is in the efficient estimation of β in the presence of serially correlated residuals in (1). For convenience of exposition, the case of scalar z_t will be examined first, the more general case being considered in Section 4.

Taking expectations in (1) conditional on z_t, we have

$$E(y_t|z_t) = \beta' E(x_t|z_t) + g(z_t), \tag{2}$$

and combining (1) and (2) leads to

$$y_t - E(y_t|z_t) = \beta'(x_t - E(x_t|z_t)) + u_t,$$

which has the following parametric regression form

$$y_t^* = \beta' x_t^* + u_t, \tag{3}$$

where $y_t^* = y_t - E(y_t|z)$, and $x_t^* = x_t - E(x_t|z)$.

If $E(y_t|z_t)$ and $E(x_t|z_t)$ were known, y_t^* and x_t^* could be calculated and then (3) would be amenable to regression. In practice, $E(y_t|z_t)$ and $E(x_t|z_t)$ are unknown and appropriate estimation of these quantities is needed to construct a feasible estimator for β. The conditional means $E(y_t|z_t)$ and $E(x_t|z_t)$ can be estimated nonparametrically by standard Nadaraya-Watson kernel methods giving the estimates

$$\widehat{E(y_t|z_t)} = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) y_j / \widehat{f}(z_t), \tag{4}$$

$$\widehat{E(x_t|z)} = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) x_j / \widehat{f}(z_t), \tag{5}$$

where

$$\widehat{f}(z_t) = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) \tag{6}$$

is a consistent density estimator for $f(z_t)$ under certain bandwidth conditions, $K(\cdot)$ is a kernel function, and h is a bandwidth parameter. Constructing deviations of the data from these nonparametrically fitted conditional means gives

$$\widehat{y}_t^* = y_t - \widehat{E(y_t|z_t)}, \text{ and } \widehat{x}_t^* = x_t - \widehat{E(x_t|z_t)}, \tag{7}$$

and then a feasible estimator for β can be formed by least squares regression giving

$$\widehat{\beta}_{OLS} = \left[\sum \widehat{x}_t^* \widehat{x}_t^{*\prime}\right]^{-1} \left[\sum \widehat{x}_t^* \widehat{y}_t^*\right].$$

Robinson (1988) studied the semiparametric estimator $\widehat{\beta}_{OLS}$ when $(x_t',\ z_t,\ u_t)$ are iid, and showed that, under certain regularity conditions, $\widehat{\beta}_{OLS}$ is $\sqrt{n}$-consistent, asymptotically normal, and asymptotically efficient. In the case where the residuals are weakly dependent, Fan and Li (1999) considered the density weighted version of $\widehat{\beta}_{OLS}$ and showed that under certain regularity conditions, the density weighted regression estimator is still $\sqrt{n}$-consistent and asymptotic normal. However, in the presence of autocorrelated residuals, the OLS estimator $\widehat{\beta}_{OLS}$ is no longer efficient, just as in convention linear time series regression.

We propose an efficient estimation procedure for β when u_t is a linear process whose coefficients satisfy certain summability conditions as in Phillips and Solo (1992, hereafter PS). We employ the following condition, which is convenient for our development but which involves some strengthening of the conditions in PS.

Assumption A: *$u_t = C(L)\varepsilon_t$, where ε_t is iid $(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 > 0$, and $C(L) = \sum_{j=0}^{\infty} c_j L^j$, $C(1) \neq 0$, and $|c_j| < \rho^j$, for any j and some $0 < \rho < 1$.*

While stronger than the summability conditions in PS, the dominance requirement $|c_j| < \rho^j$ is general enough to include leading cases like autoregressive moving average (ARMA) models in stationary time series. This dominance condition is useful in our technical development and, in particular, provides a sufficient condition for controlling the order of magnitude of various summations involving c_j (such as sums of the form $\sum_j \sum s \sum_r c_r c_{r+s-j}$). No doubt this dominance condition could be weakened, but we do not attempt to do so or to find minimal conditions under which our results hold.

Notice that regression (3) is a (parametric) time series regression with stationary residuals. Thus, if x_t^* and y_t^* were known, efficient methods of estimating β by spectral methods would be possible and have been developed by Hannan (1963), following the ideas of Whittle (1953). Performing a discrete Fourier transform (dft) of regression (3) and assuming for simplicity that n is an even number, we have

$$w_{y^*}(\lambda_t) = \beta w_{x^*}(\lambda_t) + w_u(\lambda_t),$$

where $\lambda_t = 2\pi t/n$, $(t = -n/2+1, ..., n/2)$ are the fundamental frequencies and $w_a(\lambda)$ is the dft of time series a_t at frequency λ. Under smoothness conditions on the residual spectral density, the error terms in the above frequency domain regression, $w_u(\lambda_t)$, are, roughly speaking, asymptotically independent but heteroskedastic. The frequency domain GLS estimator for β suggested by Hannan (1963), which is based on smoothed periodogram estimates, has the following form:

$$\widehat{\beta}_H = \left[\sum_{j=-M+1}^{M} \widehat{f}_{x^*x^*}(\omega_j)\widehat{f}_{uu}(\omega_j)^{-1}\right]^{-1} \left[\sum_{j=-M+1}^{M} \widehat{f}_{x^*y^*}(\omega_j)\widehat{f}_{uu}(\omega_j)^{-1}\right], \tag{8}$$

where

$$\widehat{f}_{uu}(\omega_j) = m^{-1} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{uu}(\lambda_s),$$

$$\widehat{f}_{x^*x^*}(\omega_j) = m^{-1} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{x^*x^*}(\lambda_s),$$

$$\widehat{f}_{x^*y^*}(\omega_j) = m^{-1} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{x^*y^*}(\lambda_s),$$

$\omega_j = \pi j/M$, $L(\cdot)$ is a kernel function, and $I_{uu}(\lambda_s)$, $I_{x^*x^*}(\lambda_s)$, and $I_{x^*y^*}(\lambda_s)$ are periodograms defined as $w_u(\lambda_t)w_u(\lambda_t)^*$, $w_{x^*}(\lambda_t)w_{x^*}(\lambda_t)^*$, and $w_{x^*}(\lambda_t)w_{y^*}(\lambda_t)^*$, respectively. Under general smoothness conditions on the spectral density, the frequency domain GLS estimator achieves asymptotically the Gauss-Markov efficiency bound.

An alternative frequency domain estimator of β can be constructed based on a weighted average of periodogram estimates at the fundamental frequencies $\omega_j = 2\pi j/n$,

$$\widehat{\beta} = \left[\sum_{j=-n/2+1}^{n/2} I_{x^*x^*}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} \right]^{-1} \left[\sum_{j=-n/2+1}^{n/2} I_{x^*y^*}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} \right]. \tag{9}$$

This spectral estimator of β is first-order equivalent to the estimator $\widehat{\beta}_H$, is widely used in spectral regression applications (e.g., Robinson 1991) and has certain advantages over $\widehat{\beta}_H$. In particular, the estimator (9) has been found to be second-order more efficient than the original Hannan estimator (2) – see Xiao and Phillips, 1998. Our analysis in this paper will focus on $\widehat{\beta}$ and, while similar results can be derived for $\widehat{\beta}_H$, we will not detail them here.

In the partial linear model, x_t^*, y_t^*, and u_t are unknown and so (8) and (9) are not feasible. However, x_t^* and y_t^* can be estimated by the nonparametric method given in (4) – (5). Thus, the dft of x_t^* and y_t^* can be estimated based on $\widehat{x}_t^*$ and $\widehat{y}_t^*$, which in turn may be used to construct periodogram ordinates. Let $\widehat{\beta}_p$ be a preliminary $\sqrt{n}$-consistent estimator of β, e.g. the conventional partial linear regression estimator - see Fan and Li (1999). We can then estimate the dft of u_t using the residuals $\widetilde{u}_t = \widehat{y}_t^* - \widehat{\beta}_p' \widehat{x}_t^*$ and a feasible frequency domain GLS estimator for β in the partial linear regression (1) can be obtained as in (9). This is the approach that will be followed in the sequel.

It is convenient to make the following assumptions for the analysis that follows.

Assumption B: *The spectral density of u_t is bounded away from the origin and is absolutely continuous.*

Assumption C: $K(\cdot) \in \mathcal{K}_q$, *where $\mathcal{K}_q$ is the class of even functions $k(\cdot)$: $\mathcal{R} \to \mathcal{R}$ satisfying*

$$\int u^i k(u) du = \delta_{i0}, \ \ i = 0, ..., q-1$$

$$k(u) = O((1 + |u|^{q+1+\varepsilon})^{-1}), \text{ some } \varepsilon > 0,$$

and δ_{ij} *is Kronecker's delta.*

Assumption D: z_t *admits a pdf* $f(\cdot)$ *which is strictly positive and has bounded support.*

Assumption E: $f(\cdot), m_1(\cdot) = E(x_t|\cdot)$, *and* $g(\cdot)$ *have uniformly bounded continuous partial derivatives up to order* $q + 1$, *and their* ν*-th partial derivatives are Lipschitz of degree* $q - \nu$.

Assumption F: $E[\{x_t - E(x_t|z_t)\}\{x_t - E(x_t|z_t)\}']$ *is positive definite.*

Assumption D is not strictly necessary and is mainly used for convenience. It allows us to avoid modifying the nonparametric estimator by trimming, which introduces an extra sample size varying quantity that has to be determined in practice. However, our methods can be extended to the more general case by using trimming - see Linton (1995a) for similar assumptions and discussion on this issue. For some subsequent asymptotic analysis, more assumptions will be made about the regressors and the bandwidth expansion rate. In addition, we denote for any function K and integer q, $\mu_q(K) = \int K(u)u^q du/q!$.

3 Spectral Partial Linear Regression-Asymptotic Theory

To simplify the asymptotic development, this section considers the case of iid regressors and the general case with time series regressors is left to Section 5. Accordingly, we impose the following conditions on the regressors in (1) and make an additional assumption about the bandwidth parameter h.

Assumption G: (x_t, z_t), $t = 1, 2, ...$, *are independent and identically distributed, have finite fourth moments, and are independent of* u_s, *for* $\forall$ s.

Assumption H: *As* $n \to \infty, h \to 0, nh^2 \to \infty$, *and* $nh^{4q} \to 0$.

For notational convenience, we denote $E(y_t|z_t)$ and $E(x_t|z_t)$ by $m_2(z_t)$ and $m_1(z_t)$, and corresponding nonparametric kernel estimates of them by $\widehat{m}_2(z_t)$ and $\widehat{m}_1(z_t)$. The model (1) implies that

$$\widehat{m}_2(z_t) = \beta'\widehat{m}_1(z_t) + \widehat{g}(z_t) + \widehat{u}_t,$$

where

$$\widehat{u}_t = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h})u_j/\widehat{f}(z_t), \tag{10}$$

$$\widehat{g}(z_t) = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h})g(z_j)/\widehat{f}(z_t),$$

and $\widehat{m}_2(z_t)$, $\widehat{m}_1(z_t)$, and $\widehat{f}(z_t)$ are defined by (4), (5), and (6) in Section 2. Thus, we have

$$\widehat{y}_t^* = \beta'\widehat{x}_t^* + \widehat{u}_t^*, \tag{11}$$

where

$$\widehat{u}_t^* = u_t - \widehat{u}_t + g(z_t) - \widehat{g}(z_t). \tag{12}$$

Besides u_t, the residual process $\widehat{u}_t^*$ contains a smoothed residual term $\widehat{u}_t$ and a nonparametric error term $g(z_t) - \widehat{g}(z_t)$.

Using these quantities, we construct the following frequency domain estimator of β

$$\widetilde{\beta} = \left[\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1}\right]^{-1}\left[\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{y^*}}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1}\right], \tag{13}$$

where

$$I_{\widehat{x^*}\widehat{x^*}}(\lambda_s) = w_{\widehat{x^*}}(\lambda_t)w_{\widehat{x^*}}(\lambda_t)^*, I_{\widehat{x^*}\widehat{y^*}}(\lambda_s) = w_{\widehat{x^*}}(\lambda_t)w_{\widehat{y^*}}(\lambda_t)^*,$$

and

$$\widetilde{f}_{uu}(\omega_j) = m^{-1}\sum_{\lambda_s\in B_j} L(\lambda_s - \omega_j)I_{\widetilde{u}\widetilde{u}}(\lambda_s),\ I_{\widetilde{u}\widetilde{u}}(\lambda_s) = w_{\widetilde{u}}(\lambda_s)w_{\widetilde{u}}(\lambda_s)^*, \tag{14}$$

is the kernel estimator of the residual spectrum $f_{uu}(\cdot)$. $B_j = \{\lambda : \omega_j - \frac{\pi}{2M} < \lambda \leq \omega_j + \frac{\pi}{2M}\}$ is a frequency band of width π/M centered on $\omega_j = 2\pi j/n$. Let $m = [n/2M]$, where $[\cdot]$ signifies integer part. Then each band B_j contains (approximately) m fundamental frequencies λ_s. In (14), $L(\cdot)$ is the spectral window that $L(\cdot) \in \mathcal{K}_q$, and $\frac{1}{m}\sum_{\lambda_s \in B(\omega)} L(\lambda_s - \omega) = 1$. Candidate kernel functions can be found in standard texts (e.g., Hannan 1970, Brillinger 1980, and Priestley 1981). For convenience of comparison, we let the order of magnitude of the bandwidth in spectral estimation be the same as that in the nonparametric regression, so that $M \sim 1/h$ and satisfies Assumption H.

Combining (11) and (12), we have

$$\sqrt{n}(\widetilde{\beta}-\beta) = \left[\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^*\widehat{x}^*}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1}\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^*\widehat{u}^*}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1}\right].$$

As shown in the proof of Theorem 1 given in Appendix A, the asymptotic distribution of $\widetilde{\beta}$ is derived from the following two results:

$$\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^*\widehat{x}^*}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1} \to_p \frac{1}{2\pi}\int f_{x^*x^*}(\omega)f_{uu}(\omega)^{-1}d\omega, \tag{15}$$

and

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^*\widehat{u}^*}(\omega_j)\widetilde{f}_{uu}(\omega_j)^{-1} \Rightarrow N\left(0, 2\pi\int f_{x^*x^*}(\omega)f_{uu}(\omega)^{-1}d\omega\right). \tag{16}$$

The resulting limit theory is contained in the following theorem, indicating that the proposed spectral regression estimator $\widetilde{\beta}$ is $\sqrt{n}$-consistent and has a limiting normal distribution whose variance matrix attains the GLS efficiency bound in time series regression.

Theorem 1: *Under Assumptions A to H,*

$$\sqrt{n}(\widetilde{\beta}-\beta) \Rightarrow N\left(0, 2\pi\left[\int f_{x^*x^*}(\omega)f_{uu}(\omega)^{-1}d\omega\right]^{-1}\right). \tag{17}$$

It follows that $\widetilde{\beta}$ is asymptotically equivalent to what would be the usual efficient time series regression estimator of β were $x_t^* = x_t - E(x_t|z_t)$ and $y_t^* = y_t - E(y_t|z_t)$ observable. Thus, there is no loss in efficiency from the nonparametric estimation of the nonlinear component in (1).

Under Assumption G, x_t^* is *iid*, $f_{x^*x^*}(\omega) = \frac{1}{2\pi}\Sigma_*$ where Σ_* is the variance matrix of x_t^*, and therefore the asymptotic variance (17) has the form

$$2\pi\left[\int \{2\pi f_{uu}(\omega)\}^{-1}d\omega\right]^{-1}\Sigma_*^{-1}.$$

It follows that efficient regression on (3) is in this case formally identical to least squares regression on

$$y_t^* = \beta' x_t^* + w_t,$$

where w_t is $iid(0, \sigma_w^2)$ with $\sigma_w^2 = 2\pi \left[\int \{2\pi f_{uu}(\omega)\}^{-1} d\omega \right]^{-1}$.

4 Multivariate Regressor

Our results can be extended to the case where the dimensionality of z_t is arbitrary. In this Section, we consider the partial linear regression (1) when x_t, z_t, and u_t are $\mathcal{R}^p \times \mathcal{R}^l \times \mathcal{R}$ -valued random variables. To accommodate the change in the dimension of z_t, we consider the multivariate kernel function K defined as

$$K(u) = \prod_{i=1}^{l} k(u_i), \text{ and } k \in \mathcal{K}_q.$$

The Nadaraya-Watson kernel estimators are then

$$\begin{aligned} \widehat{E(y_t|z_t)} &= \frac{1}{nh^q} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) y_j / \widehat{f}(z_t), \\ \widehat{E(x_t|z)} &= \frac{1}{nh^q} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) x_j / \widehat{f}(z_t), \end{aligned}$$

and

$$\widehat{f}(z_t) = \frac{1}{nh^q} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}).$$

Defining $\widehat{y}_t^*$ and $\widehat{x}_t^*$ by (7), and

$$\begin{aligned} \widehat{u}_t &= \frac{1}{nh^q} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) u_j / \widehat{f}(z_t), \\ \widehat{g}(z_t) &= \frac{1}{nh^q} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) g(z_j) / \widehat{f}(z_t), \end{aligned}$$

equations (11) and (12) still hold. As the dimension l of z_t increases, the relative importance of the nonparametric estimation error increases and, as in Robinson (1988) it is convenient to use higher order kernels in the construction of our estimator to achieve the necessary bias reduction. Hence, we modify Assumptions C and H as follows:

Assumption C2: *$K(u) = \prod_{i=1}^{l} k(u_i)$ and $k \in \mathcal{K}_q$, where $\mathcal{K}_q$ is the class of functions satisfying the properties in Assumption C.*

Assumption H2: *As $n \to \infty$, $h \to 0$, $nh^{2l} \to \infty$, and $nh^{4q} \to 0$.*

The asymptotic analysis for the general case parallels that of Section 3 and results similar to Lemmas 2 to 14 can be proved. We do not go through all the details here but summarize these preliminary results in Appendix B. The resulting limiting theory for $\widetilde{\beta}$ is given in the following Theorem, giving the multivariate analogue of (17).

Theorem 2: *Under Assumptions A, B, C2, D, E, F, G, and H2,*

$$\sqrt{n}(\widetilde{\beta} - \beta) \Rightarrow N(0, 2\pi \left[\int f_{x^*x^*}(\omega) f_{uu}(\omega)^{-1} d\omega \right]^{-1}). \tag{18}$$

Remark: The original Hannan estimator $\widehat{\beta}_H$ can be analyzed in a similar way. Under similar conditions, it can be shown that

$$\begin{aligned}
\sqrt{n}(\widehat{\beta} - \beta) &= \left[\frac{1}{2M} \sum_{j=-M+1}^{M} \widehat{f}_{\widehat{x}^*\widehat{x}^*}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} \right]^{-1} \left[\sqrt{\frac{m}{2M}} \sum_{j=-M+1}^{M} \widehat{f}_{\widehat{x}^*\widehat{u}^*}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} \right] \\
&= \left[\frac{1}{2M} \sum_{j=-M+1}^{M} \widehat{f}_{\widehat{x}^*\widehat{x}^*}(\omega_j) f_{uu}(\omega_j)^{-1} \right]^{-1} \left[\sqrt{\frac{m}{2M}} \sum_{j=-M+1}^{M} \widehat{f}_{\widehat{x}^*\widehat{u}^*}(\omega_j) f_{uu}(\omega_j)^{-1} \right] \\
&\quad + o_p(1)
\end{aligned}$$

where $\omega_j = \pi j/M$, and

$$\begin{aligned}
\frac{1}{M} \sum_{j=-M+1}^{M} \widehat{f}_{x^*(m_1 - \widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} &= o_p(1), \\
\frac{1}{2M} \sum_{j=-M+1}^{M} \widehat{f}_{(m_1 - \widehat{m}_1)(m_1 - \widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} &= o_p(1),
\end{aligned}$$

$$\sqrt{\frac{m}{2M}}\sum_{j=-M+1}^{M}\widehat{f}_{(m_1-\widehat{m}_1)u}(\omega_j)f_{uu}(\omega_j)^{-1} = o_p(1),$$
$$\sqrt{\frac{m}{2M}}\sum_{j=-M+1}^{M}\widehat{f}_{\widehat{x}^*\widehat{u}}(\omega_j)f_{uu}(\omega_j)^{-1} = o_p(1),$$
$$\sqrt{\frac{m}{2M}}\sum_{j=-M+1}^{M}\widehat{f}_{\widehat{x}^*(g-\widehat{g})}(\omega_j)\widehat{f}_{uu}(\omega_j)^{-1} = o_p(1).$$

Thus $\widetilde{\beta}_H$ has the same asymptotic distribution as $\widetilde{\beta}$.

5 Dependent Regressors

This Section extends our method to partial linear regression models with serially correlated regressors. Such an extension allows for temporal dependence in both the residuals and the regressors. However, to develop the limit theory in this case, assumptions are needed to control the temporal dependence in (x_t, z_t).

To aid the analysis we introduce a new concept of weak dependence that better suits the present application. In particular, given the nature of our model and the frequency domain regression methods being employed, it is convenient for the asymptotic analysis to use dependence conditions that are based directly on the density functions rather than conventional mixing conditions.

To begin, we introduce the following notation. The joint density function of $(z_t, z_{t+a_1}, \cdots, z_{t+a_r})$ is denoted $p_{a_1,\cdots,a_r}(\cdot,\cdots,\cdot)$ and the marginal density of z_t is denoted by $f(\cdot)$, as before. For a function $p(t_1,\cdots,t_k)$ of k-arguments, $\nabla^{(\alpha_1,\cdots,\alpha_\kappa)}_{\nu_1,\cdots,\nu_\kappa}p(t_1,\cdots,t_k)$ is used to denote its $(\alpha_1+\cdots+\alpha_\kappa)$-th partial derivative with respect to the $\nu_1,\cdots,$ and ν_κ arguments, i.e. $\partial^{\alpha_1+\cdots+\alpha_\kappa}p(t_1,\cdots,t_k)/\partial t_{\nu_1}^{\alpha_1}\cdots\partial t_{\nu_\kappa}^{\alpha_\kappa}$. We use the following concept of asymptotic regularity.

Definition: *A function $H(x,y)$ is asymptotically regular with respect to x if there exist $\tau>0$ and some function $\psi(\cdot)$ such that, as $|x|\to\infty$,*

$$H(x,y) = (1+|x|^{-\tau})\psi(y). \tag{19}$$

τ is called the index of asymptotic regularity.

In place of Assumptions G and H in Section 3, we make use of the following conditions, which allow for autocorrelation in the regressors.

Assumption G′: (i) $\{x_t, z_t\}, t = 1, 2, ...,$ *are stationary with finite fourth moments, and are independent of* u_s, *for* $\forall\ s$.

(ii) The joint probability density functions $p_{a_1,\cdots,a_r}(\cdot,\cdots,\cdot)$ *of the process* z_t *satisfy the following conditions:*

(a) For $r \leq 3$ *and any integrable function* φ_j *that is independent of* a_j, $j = 1, \cdots r$,

$$\int \frac{p_{a_1,\cdots,a_r}(t_1,\cdots,t_r,t_{r+1})}{f(t_1)\cdots f(t_r)}\varphi_j(t_1,\cdots,t_r,t_{r+1})dt_1\cdots dt_{r+1}$$

is asymptotically regular of index τ *with respect to* a_j.

(b)For $0 \leq \alpha_i \leq q$, $i = 1, \cdots, \kappa$, *and* $0 \leq \alpha_1 + \cdots + \alpha_\kappa \leq q$,

$$\int \frac{\nabla^{(\alpha_1,\cdots,\alpha_\kappa)}_{\nu_1,\cdots,\nu_\kappa} p_{a_1,\cdots,a_r}(t_1,\cdots,t_r,t_{r+1})}{f(t_1)\cdots f(t_r)}\varphi_j(t_1,\cdots,t_r,t_{r+1})dt_1\cdots dt_{r+1}$$

is asymptotically regular of index τ *with respect to* a_j.

(iii) $x_t^* = x_t - E(x_t|z_t)$ *follows a stationary linear process that satisfies the summability conditions of Assumption A.*

Assumption H′: *As* $n \to \infty$, $h \to 0$, $nh^{2l} \to \infty$, *and* $n^{(\tau+3)/(\tau+1)}h^{4q} \to 0$, *where* $\tau > 1/3$ *is the index of asymptotic regularity.*

Assumption G′(ii) is similar to conventional mixing conditions in the existing literature in that it requires the temporal dependence between z_t and z_{t+a} to decrease as the temporal distance, a, between observations increases. However, unlike conventional mixing conditions, dependence between z_t and z_{t+a} is measured through the behavior of the joint density function. Intuitively, Assumption G′(ii) assumes that as the time distance a goes to ∞, the joint density of (z_t, z_{t+a}) will be asymptotically regular with respect to a. In other words, it can be expressed in the form $(1+|a|^{-\tau})\psi$, where ψ is independent of a. The requirement in Assumption H′ that $\tau > 1/3$ can be relaxed to $\tau > \epsilon$ with $\epsilon > 0$ arbitrarily close to zero. However, in this case, expansions to higher order terms will be needed to prove the asymptotic result.

Under Assumption G′(ii), for any integrable function φ that is independent of a, and for $0 \le \alpha_1 \le q$, $0 \le \alpha_2 \le q$, and $0 \le \alpha_1 + \alpha_2 \le q$, we have that

$$\int \frac{p_{a,b}(t,t,t)\varphi(t)}{f(t)^2}dt,\ \int \frac{p_{a,b,c}(t,s,t,s)\varphi(t,s)}{f(t)f(s)}dsdt$$

and

$$\int \nabla_{3,4}^{(\alpha_1,\alpha_2)} p_{a,b,c}(t,s,t,s)\varphi(t,s)/[f(t)f(s)]dt$$

are asymptotically regular of index τ with respect to a. Such results will be used in the proof of Theorem 3. We give some illustrations of this weak dependence concept before developing the limit theory for the partial linear regression estimator in the general case.

Example 1: If $\{z_t\}$ is an iid sequence, then $p_{a,b,c}(t_1,t_2,t_3,t_4) = \prod_{j=1}^4 f(t_j)$, and

$$\int \frac{p_{a,b,c}(t,s,t,s)\varphi(t,s)}{f(t)f(s)}dsdt = \int f(t)f(s)\varphi(t,s)dsdt$$

is independent of a, and is therefore asymptotically regular with respect to a for any τ.

Example 2: If z_t is a stationary Gaussian AR(1) process, say, $z_t = \alpha z_{t-1} + u_t$,with $u_t = iidN(0,1)$, and $|\alpha| < 1$, the correlation between z_t and z_{t+a} decreases exponentially. Thus $\int \frac{p_{a,b,c}(t,s,t,s)}{f(t)f(s)}dtds = (1 + O(|a|^{-\tau}))\psi(b,c)$ for any $\tau > 0$. Notice that the joint p.d.f. of $(z_{t+a}, z_t, z_{t+b}, z_{t+c})$ is

$$p_{a,b,c}(Z) = (2\pi)^{-2}\,|\Omega|^{-1/2}\exp\{-Z'\Omega^{-1}Z/2\},$$

with

$$\Omega = \begin{bmatrix} 1 & \alpha^{|a|} & \alpha^{|b-a|} & \alpha^{|c-a|} \\ \alpha^{|a|} & 1 & \alpha^{|b|} & \alpha^{|c|} \\ \alpha^{|b-a|} & \alpha^{|b|} & 1 & \alpha^{|b-c|} \\ \alpha^{|c-a|} & \alpha^{|c|} & \alpha^{|b-c|} & 1 \end{bmatrix}.$$

If we partition Ω as

$$\begin{bmatrix} 1 & \Omega'_{a*} \\ \Omega_{a*} & \Omega_{**} \end{bmatrix}$$

and let $Z = (s,t,t,s)$, then

$$\begin{aligned} & \int \frac{p_{a,b,c}(Z)}{f(t)f(s)}dtds \\ = \quad & (2\pi)^{-1}\,|\Omega|^{-1/2}\int\int \exp\left\{-\frac{1}{2}\left[\frac{(s-\Omega'_{a*}\Omega_{**}^{-1}Z_*)^2}{1-\Omega'_{a*}\Omega_{**}^{-1}\Omega_{a*}} + Z'_*\Omega_{**}^{-1}Z_* - t^2 - s^2\right]\right\}dtds. \end{aligned}$$

where

$$Z_* = (t, t, s)'.$$

It can be verified that

$$-\frac{1}{2}\left[\frac{(s-\Omega_{a*}'\Omega_{**}^{-1}Z_*)^2}{1-\Omega_{a*}'\Omega_{**}^{-1}\Omega_{a*}} + Z_*'\Omega_{**}^{-1}Z_* - t^2 - s^2\right]$$

can be written as

$$-\frac{1}{2}(t,s)\Sigma\begin{pmatrix} t \\ s \end{pmatrix} + \frac{1}{2}\alpha^{|a|}\left(\beta t^2 + 2\gamma ts + \eta s^2\right)$$

where Σ is a positive definite matrix, and $\beta t^2 + 2\gamma ts + \eta s^2$ is a quadratic function of (t, s). Notice that $|\alpha| < 1$ and

$$|\Omega|^{-1/2} = (1 + O(\alpha^{|a|}))\psi(b, c).$$

Thus, when a is large,

$$\int \frac{p_{a,b,c}(Z)}{f(t)f(s)} dtds = (1 + O(\alpha^{|a|/2}))\psi(b, c) = (1 + O(|a|^{-\tau}))\psi(b, c).$$

Similarly, for any integrable function φ, it can be verified that

$$\int \frac{p_{a,b,c}(Z)}{f(t)f(s)}\varphi(t,s)dtds = (1 + O(|a|^{-\tau}))\psi(b, c)$$

as required, and this holds for any $\tau > 0$.

Theorem 3: *Under Assumptions A, B, C, D, E, F, G′, and H′*

$$\sqrt{n}(\widetilde{\beta} - \beta) \Rightarrow N(0, 2\pi\left[\int f_{x^*x^*}(\omega)f_{uu}(\omega)^{-1}d\omega\right]^{-1}).$$

Theorem 3 extends the earlier limit theory to the case of weakly dependent regressors under assumption G′. Its proof is similar to that of Theorem 1 relies on similar supporting results to Lemmas 2-14, which continue to hold under Assumptions G′ and H′.

Asymptotic normality of $\widetilde{\beta}$ facilitates construction of test statistics for inference about β in the usual manner. For instance, the regression Wald test of $H_0 : R\beta = r$, where r is a $q \times 1$ vector and R is an $q \times p$ matrix, is simply

$$W_n = (R\widetilde{\beta} - r)'\left[R\widehat{\Sigma}^{-1}R'\right]^{-1}(R\widetilde{\beta} - r)$$

where

$$\widehat{\Sigma} = \sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^* \widehat{x}^*}(\omega_j) \widetilde{f}_{uu}(\omega_j)^{-1},$$

and, under H_0 and the same conditions as Theorem 3, $W_n \to \chi_q^2$.

6 Conclusion

This paper shows that partial linear regression models with time series errors and weakly dependent regressors can be efficiently estimated using frequency domain techniques. The approach is simply to remove the nonparametric component in the conventional manner by kernel regression and then apply efficient time series regression (or feasible GLS in the frequency domain) to an empirical version of the transformed equation (3). The resulting estimates are $\sqrt{n}$ consistent and asymptotically efficient in the sense that they have the same limit theory as GLS estimates based directly on (3). A novel aspect of our approach is the mechanism used to measure and control for weak dependence in the regressors. This mechanism places conditions directly on certain functionals of the joint density of the time series and seems likely to be useful in other econometric applications with serially correlated data.

7 Appendix

In the first subsection of this Appendix, we state Lemmas 1-15 that are used in the proofs of the Theorems. In the second subsection, we prove Theorems 1 to 3. In the third subsection, we prove Lemmas 1-15.

7.1 Lemmas

Lemma 1: $\widetilde{f}_{uu}(\omega_j) = \bar{f}_{uu}(\omega_j) + o_p(\frac{1}{\sqrt{n}})$ *uniformly in* ω_j.

Lemma 2: $I_{\widehat{u}\widehat{u}}(\omega) = O_p(n^{-1}h^{-1})$, *for any* $\omega = 2\pi s/n$, $s \neq 0$.

Lemma 3: $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s) = O_p(n^{-1}h + h^{2q})$, *for any* $\lambda_s = 2\pi s/n$, $s \neq 0$.

Lemma 4: $I_{u\widehat{u}}(\lambda_s) = O_p(n^{-1})$, *for any* $\lambda_s = 2\pi s/n$, $s \neq 0$.

Lemma 5: $I_{u(g-\widehat{g})}(\lambda_s) = O_p(h^q + n^{-1/2}h^{1/2})$, *for any* $\lambda_s = 2\pi s/n$, $s \neq 0$.

Lemma 6: $I_{\widehat{u}(g-\widehat{g})}(\lambda_s) = o_p(n^{-1/2}h^q + n^{-1}h^{q/2})$, *for any* $\lambda_s = 2\pi s/n$, $s \neq 0$.

Lemma 7: $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 8: $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 9: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 10: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 11: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 12: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 13: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} = o_p(1)$.

Lemma 14: $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*u^*}(\omega_j) f_{uu}(\omega_j)^{-1} \Rightarrow N(0, 2\pi \int f_{x^*x^*}(\omega) f_{uu}(\omega)^{-1} d\omega)$.

Lemma 15. *For* $\omega = 2\pi s/n$, $s \neq 0$, $\sum_{t=1}^{n} t^{-\tau} e^{i\omega t} = O(n^{1/(1+\tau)})$.

7.2 Proof of Theorems

Proof of Theorem 1 It suffices to prove (15) and (16). We first consider $\widetilde{f}_{uu}(\omega_j) = m^{-1}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widetilde{u}\widetilde{u}}(\lambda_s)$. Notice that $\widetilde{u}_t = \widehat{u}_t^* - (\widehat{\beta}_p - \beta)'\widehat{x}_t^*$, by result of Lemma 1 in Appendix A we have

$$\widetilde{f}_{uu}(\omega_j) = \bar{f}_{uu}(\omega_j) + o_p(\frac{1}{\sqrt{n}}),$$

for any ω_j, where

$$\bar{f}_{uu}(\omega_j) = \frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{u}^* \widehat{u}^*}(\lambda_s).$$

By definition, $\widehat{u}_t^*$ contains the true residual term, u_t, the locally smoothed residuals, $\widehat{u}_t$, and the nonparametric estimation error, $\widehat{g}(z_t) - g(z_t)$. In order to obtain (15) and (16), we have to show that the periodogram averages based on $\widehat{u}_t$ and $g(z_t) - \widehat{g}(z_t)$ are small in order of magnitude. The periodogram of $\widehat{u}_t^*$ can be decomposed as follows

$$I_{\widehat{u}^* \widehat{u}^*}(\lambda_s) = I_{uu}(\lambda_s) + I_{\widehat{u}\widehat{u}}(\lambda_s) + I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s) - 2I_{u\widehat{u}}(\lambda_s) + 2I_{u(g-\widehat{g})}(\lambda_s) - 2I_{\widehat{u}(g-\widehat{g})}(\lambda_s).$$

The estimator $\bar{f}_{uu}(\omega_j)$ can thus be further written as

$$\bar{f}_{uu}(\omega_j) = \widehat{f}_{uu}(\omega_j) + \varphi_1(\omega_j) + \varphi_2(\omega_j) + \varphi_3(\omega_j) + \varphi_4(\omega_j) + \varphi_5(\omega_j),$$

where

$$\widehat{f}_{uu}(\omega_j) = \frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{uu}(\lambda_s),$$

and

$$\begin{aligned}
\varphi_1(\omega_j) &= \frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{u}\widehat{u}}(\lambda_s), \\
\varphi_2(\omega_j) &= \frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s), \\
\varphi_3(\omega_j) &= -\frac{2}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{u\widehat{u}}(\lambda_s), \\
\varphi_4(\omega_j) &= \frac{2}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{u(g-\widehat{g})}(\lambda_s), \\
\varphi_5(\omega_j) &= -\frac{2}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{u}(g-\widehat{g})}(\lambda_s).
\end{aligned}$$

The leading term $\widehat{f}_{uu}(\omega_j)$ is the conventional kernel estimator of $f_{uu}(\omega_j)$ based on known u_t. The asymptotic properties of this estimator are well documented in the literature. The other terms in $\bar{f}_{uu}(\omega_j)$ involve errors arising from nonparametric estimation of the conditional expectations. Lemma 2 to Lemma 6 in Appendix A give the orders of magnitude of $I_{\widehat{u}\widehat{u}}(\lambda_s)$, $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s)$, $I_{u\widehat{u}}(\lambda_s)$, $I_{u(g-\widehat{g})}(\lambda_s)$, and $I_{\widehat{u}(g-\widehat{g})}(\lambda_s)$. In addition, uniform consistency and convergence rates for $\widehat{f}_{uu}(\omega_j)$ of the type

$$\max_j \left| \widehat{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right| = O_p(M^{-q}) + O_p(m^{-1/2+\epsilon}),$$

where ϵ is any small positive number, are also well developed in the literature – e.g., Brillinger (1980, Theorem 7.7.4). Masry (1996) also gives uniform rates of convergence for nonparametric estimators for stationary processes. Using arguments similar to those in Masry (1996) and Brillinger (1980), uniform results of the nonparametric kernel smoothed quantities $\varphi_i(\omega_j)$, $i = 1, ..., 5$, can be obtained. The calculations are somewhat more complicated in view of the present partial linear regression context but do not involve anything essentially new and are not reproduced here. To avoid having to deal with two bandwidth parameters, we may choose M to be of the same order of magnitude as $1/h$. With this simplification, we can then show that

$$\max_j \left| \widetilde{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right| = O_p(M^{-q}) + O_p(m^{-1/2+\epsilon}). \tag{20}$$

By a geometric expansion, we obtain

$$\begin{aligned} &\frac{1}{n} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j) \widetilde{f}_{uu}(\omega_j)^{-1} \\ = \quad &\frac{1}{n} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j) f_{uu}(\omega_j)^{-1} && (21) \\ &+\frac{1}{n} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\widetilde{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right] && (22) \\ &+\frac{1}{n} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j) f_{uu}(\omega_j)^{-2} \widetilde{f}_{uu}(\omega_j)^{-1} \left[\widetilde{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right]^2, && (23) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) \widetilde{f}_{uu}(\omega_j)^{-1} \\ = \quad &\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-1} && (24) \\ &+\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\widetilde{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right] && (25) \\ &+\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2} \widetilde{f}_{uu}(\omega_j)^{-1} \left[\widetilde{f}_{uu}(\omega_j) - f_{uu}(\omega_j) \right]^2. && (26) \end{aligned}$$

To prove Theorem 1, we show that, under our assumptions, (21) converges in probability to $(2\pi)^{-1} \int f_{x^*x^*}(\omega) f_{uu}(\omega)^{-1} d\omega$, (24) converges weakly to $N\left(0, 2\pi \int f_{x^*x^*}(\omega) f_{uu}(\omega)^{-1} d\omega\right)$, and (22), (23), (25) and (26) are $o_p(1)$. Notice that, under Assumption B, $f_{uu}(\omega)$ is bounded away from the origin, and under our bandwidth assumptions, by (20) we can show that (22), (23) and (26) are $o_p(1)$. The result for (25) is more complicated. First we decompose this term in the following way:

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widetilde{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
\approx\ &\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\sum_{\nu=1}^{5}\left[\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2}\varphi_\nu(\omega_j)\right].
\end{aligned}$$

Then, we verify that each of these terms are $o_p(1)$. The verification of the magnitudes of these terms is similar in each case, so we illustrate the proof for only one of these cases. We write

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
=\ &\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]\\
&+\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right].
\end{aligned}$$

For the leading term,

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2}\left[\widehat{f}_{uu}(\omega_j)-f_{uu}(\omega_j)\right]$$

$$= \frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) \left(I_{uu}(\lambda_s) - f_{uu}(\lambda_s) \right) \right] \quad (27)$$

$$+ \frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) \left(f_{uu}(\lambda_s) - f_{uu}(\omega_j) \right) \right] \quad (28)$$

and we shall analyze each of them. The first term is the "variance" term, under Assumptions A and G, notice that

$$\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) \left(I_{uu}(\lambda_s) - f_{uu}(\lambda_s) \right) \right]$$

has mean zero and its second moment is approximately

$$\begin{aligned}
& \frac{1}{n} \sum_{j=-n/2+1}^{n/2} E\left[I_{x^*x^*}(\omega_j) I_{uu}(\omega_j) \right] f_{uu}(\omega_j)^{-4} \left[\frac{1}{m^2} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j)^2 f_{uu}(\lambda_s)^2 \right] \\
\approx\ & \frac{1}{n} \sum_{j=-n/2+1}^{n/2} f_{x^*x^*}(\omega_j) f_{uu}(\omega_j)^{-1} \left[\frac{1}{m^2} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j)^2 \right] \\
\approx\ & \frac{1}{m} \left\{ \frac{1}{n} \sum_{j=-n/2+1}^{n/2} f_{x^*x^*}(\omega_j) f_{uu}(\omega_j)^{-1} \left[\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j)^2 \right] \right\} \\
=\ & O(m^{-1}),
\end{aligned}$$

and thus the term (27) is of order $o_p(1)$. The second term (28) contains the bias effect in $\widehat{f}_{uu}(\omega_j)$:

$$\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) \left(f_{uu}(\lambda_s) - f_{uu}(\omega_j) \right) \approx -M^{-q} k_q f_{uq}(\omega_j),$$

where k_q is the characteristic exponent associated with the kernel function L (e.g., Hannan 1970) and

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2} \left[\frac{1}{m} \sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) \left(f_{uu}(\lambda_s) - f_{uu}(\omega_j) \right) \right] \\
\approx\ & M^{-q} k_q \left[\frac{1}{\sqrt{n}} \sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-2} f_{uq}(\omega_j) \right] \\
=\ & O_p(M^{-q}).
\end{aligned}$$

Now we move to (21) and (24). Notice that $\widehat{x}_t^* = x_t^* + [m_1(z_t) - \widehat{m}_1(z_t)]$ and $\widehat{u}_t^* = u_t - \widehat{u}_t + g(z_t) - \widehat{g}(z_t)$. We have the following decomposition:

$$\frac{1}{n} \sum_{j=-n/2+1}^{n/2} I_{\widehat{x}^*\widehat{x}^*}(\omega_j) f_{uu}(\omega_j)^{-1}$$

$$= \frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*x^*}(\omega_j) f_{uu}(\omega_j)^{-1} + \frac{2}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1}$$

$$+\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1},$$

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-1}$$

$$= \frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-1} + \frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j) f_{uu}(\omega_j)^{-1}$$

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} + \frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1}$$

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-1} + \frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-1}.$$

It is shown by Lemma 7 to Lemma 14 that the errors caused by the preliminary estimation are all negligible $o_p(1)$ terms. Thus, we have

$$\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{x^*}}(\omega_j) f_{uu}(\omega_j)^{-1} = \frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*x^*}(\omega_j) f_{uu}(\omega_j)^{-1} + o_p(1),$$

$$\frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{\widehat{x^*}\widehat{u^*}}(\omega_j) f_{uu}(\omega_j)^{-1} = \frac{1}{\sqrt{n}}\sum_{j=-n/2+1}^{n/2} I_{x^*u}(\omega_j) f_{uu}(\omega_j)^{-1} + o_p(1),$$

and the results (15) and (16) follow. ■

Proof of Theorem 2: The proof of Theorem 2 parallels that of Theorem 1. In particular, it can be shown that the following results hold.

- $I_{\widehat{u}\widehat{u}}(\omega) = O_p(n^{-1}h^{-l})$, for any $\omega = 2\pi s/n$, $s \neq 0$.
- $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s) = O_p(n^{-1}h^{l} + h^{2q})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{u\widehat{u}}(\lambda_s) = O_p(n^{-1})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{u(g-\widehat{g})}(\lambda_s) = O_p(h^q + n^{-1/2}h^{l/2})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{\widehat{u}(g-\widehat{g})}(\lambda_s) = o_p(n^{-1/2}h^q + n^{-1}h^{q/2})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.

- $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(n^{-1}h^{2q})$
- $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(n^{-1}h^{2q} + n^{-1}h^{l})$
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = O(n^{-1/2}h^{-l/2})$
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(h^{q} + n^{-1/2}h^{l/2})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(n^{-1/2}h^{q/2} + h^{q})$
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(h^{q})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} = O_p(n^{1/2}h^{2q}+n^{-1/2}h^{q-l}+h^{q+l})$.

And thus the result of Theorem 2 follows. ■

Proof of Theorem 3 The logic in proving Theorem 3 is the same as those for Theorem 1 described in Section 2 and again similar results as Lemmas 2 to 14 have to be established. We use Assumptions G′ and H′ to control the weak dependence and the moment of quantities in Lemmas 2 to 14. Corresponding to Lemmas 2-14, we prove that

- $I_{\widehat{u}\widehat{u}}(\omega) = O_p(n^{-\tau/(1+\tau)} + n^{-1}h^{-1})$, for any $\omega = 2\pi s/n$, $s \neq 0$.
- $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s) = O_p(n^{-1}h + h^{2q})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{\widehat{u}\widehat{u}}(\omega) = O_p(n^{-\tau/(1+\tau)} + n^{-1}h^{-l})$, for any $\omega = 2\pi s/n$, $s \neq 0$.
- $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s) = O_p(n^{-1}h^{l} + h^{2q}n^{1/(1+\tau)})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{u\widehat{u}}(\lambda_s) = O_p(n^{-\tau/(1+\tau)})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{u(g-\widehat{g})}(\lambda_s) = O_p(h^{q}n^{1/2(1+\tau)} + n^{-1/2}h^{l/2})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $I_{\widehat{u}(g-\widehat{g})}(\lambda_s) = o_p(n^{-1/2}h^{q} + n^{-1}h^{q/2})$, for any $\lambda_s = 2\pi s/n$, $s \neq 0$.
- $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{x^*(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(n^{-\tau/(1+\tau)}h^{2q})$.
- $\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(n^{-\tau/(1+\tau)}h^{2q} + n^{-1}h^{l})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = O(n^{-1/2}h^{-l/2})$.

- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j)f_{uu}(\omega_j)^{-1} = O_p(h^q n^{1/2(1+\tau)} + n^{-1/2}h^{l/2})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j)f_{uu}(\omega_j)^{-1} = O_p(n^{-1/2}h^{q/2} + h^q n^{1/2(1+\tau)})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j)f_{uu}(\omega_j)^{-1} = O_p(h^q n^{-\tau/2(1+\tau)})$.
- $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j)\widehat{f}_{uu}(\omega_j)^{-1} = O_p(n^{(\tau+3)/2(1+\tau)}h^{2q}+n^{-1/2}h^{q-l}+h^{q+l})$.

For example, for $I_{\widehat{u}\widehat{u}}(\omega)$, we show that the leading term

$$\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{p=1}^{n} f(z_t)^{-1}f(z_p)^{-1}\left[\frac{1}{nh}\sum_{j=1}^{n}K(\frac{z_t-z_j}{h})u_j\right]\left[\frac{1}{nh}\sum_{l=1}^{n}K(\frac{z_p-z_l}{h})u_l\right]e^{i\omega t}e^{-i\omega p}$$

is of order $O_p(n^{-\tau/(1+\tau)})$. As the proof of Lemma 2, the first moment of this term can be decomposed into the summation $A+B+C+D$, where

$$\begin{aligned}
A &= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} \\
&\quad f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_r c_{r+l-j}\right), \\
B &= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n} \\
&\quad f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_j}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_r^2\right), \\
C &= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-2}K(\frac{z_t-z_j}{h})K(\frac{z_t-z_l}{h})\left(\sum_{r=0}^{\infty}c_r c_{r+l-j}\right), \\
D &= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} f(z_t)^{-2}K(\frac{z_t-z_j}{h})^2\left(\sum_{r=0}^{\infty}c_r^2\right).
\end{aligned}$$

Now we verify the order of magnitude of these terms.

$$\begin{aligned}
A &= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} \\
&\quad f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_r c_{r+l-j}\right) \\
&= \sigma_\varepsilon^2 E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega(t-p)}\left(\sum_{r=0}^{\infty}c_r c_{r+l-j}\right)
\end{aligned}$$

$$
\begin{aligned}
& \int\int\int\int \frac{p_{(t-p),(t-j),(t-l)}(z_t, z_p, z_j, z_l)}{f(z_t)f(z_p)} K(\frac{z_t - z_j}{h})K(\frac{z_p - z_l}{h}) dz_l dz_t dz_p dz_j \\
= \; & \sigma_\varepsilon^2 E \frac{1}{2\pi n^3} \sum_{t=1}^{n} \sum_{t\neq p, p=1}^{n} \sum_{j=1}^{n} \sum_{l\neq j, l=1}^{n} e^{i\omega(t-p)} \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
& \int\int\int\int \frac{p_{(t-p),(t-j),(t-l)}(z_j + hu_t, z_l + hu_p, z_j, z_l)}{f(z_j + hu_t)f(z_l + hu_p)} K(u_t)K(u_p) dz_l dz_j du_t du_p \\
\approx \; & \sigma_\varepsilon^2 E \frac{1}{2\pi n^3} \sum_{t=1}^{n} \sum_{t\neq p, p=1}^{n} \sum_{j=1}^{n} \sum_{l\neq j, l=1}^{n} e^{i\omega(t-p)} \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
& \int\int \frac{p_{(t-p),(t-j),(t-l)}(z_j, z_l, z_j, z_l)}{f(z_j)f(z_l)} dz_l dz_j \int K(u_t) du_t \int K(u_p) du_p.
\end{aligned}
$$

Under Assumption G$'$, if t and p are distant, then for some function δ,

$$
\int\int \frac{p_{(t-p),(t-j),(t-l)}(z_j, z_l, z_j, z_l)}{f(z_j)f(z_l)} dz_l dz_j = \left(1 + (t-p)^{-\tau}\right) \delta(t-j, t-l),
$$

so that

$$
\begin{aligned}
A \; \approx \; & \sigma_\varepsilon^2 E \frac{1}{2\pi n^3} \sum_{t=1}^{n} \sum_{t\neq p, p=1}^{n} \sum_{j=1}^{n} \sum_{l\neq j, l=1}^{n} \left(1 + (t-p)^{-\tau}\right) \delta(t-j, t-l) e^{i\omega(t-p)} \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
= \; & \sigma_\varepsilon^2 E \frac{1}{2\pi n^3} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{l\neq j, l=1}^{n} \delta(t-j, t-l) \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \sum_{t\neq p, p=1}^{n} \left(1 + (t-p)^{-\tau}\right) e^{i\omega(t-p)} \\
\approx \; & \sigma_\varepsilon^2 E \frac{1}{2\pi} n^{-3+1/(1+\tau)} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{l\neq j, l=1}^{n} \delta(t-j, t-l) \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
= \; & O(n^{-\tau/(1+\tau)}),
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{p=1}^{n} \left(1 + |t-p|^{-\tau}\right) e^{i\omega(t-p)} \; = \; & \sum_{p=1}^{n} e^{i\omega(t-p)} + \sum_{p=1}^{n} |t-p|^{-\tau} e^{i\omega(t-p)} \\
= \; & \sum_{p=1}^{n} |t-p|^{-\tau} e^{i\omega(t-p)} \\
= \; & O(n^{1/(1+\tau)}),
\end{aligned}
$$

by Lemma 15.

Similarly,

$$
\begin{aligned}
B \; = \; & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) E \frac{1}{2\pi n^3 h^2} \sum_{t=1}^{n} \sum_{p=1}^{n} \sum_{j=1}^{n} f(z_t)^{-1} f(z_p)^{-1} K(\frac{z_t - z_j}{h}) K(\frac{z_p - z_j}{h}) e^{i\omega(t-p)} \\
= \; & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{p\neq t, p=1}^{n} \sum_{j=1}^{n}
\end{aligned}
$$

$$\left[\int\int\int \frac{p_{(t-p),(t-j)}(z_t,z_p,z_j)}{f(z_t)f(z_p)}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_j}{h})dz_tdz_pdz_j\right]e^{i\omega(t-p)}$$

$$= \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)\frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{p\neq t,p=1}^{n}\sum_{j=1}^{n}$$

$$\left[\int\int\int \frac{p_{(t-p),(t-j)}(z_j+hu_t,z_j+hu_p,z_j)}{f(z_j+hu_t)f(z_j+hu_p)f(z_j)f(z_j)}K(u_t)K(u_p)du_tdu_pdz_j\right]e^{i\omega(t-p)}$$

$$\approx \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)\frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{p\neq t,p=1}^{n}\sum_{j=1}^{n}$$

$$\left[\int \frac{p_{(t-p),(t-j)}(z_j,z_j,z_j)}{f(z_j)f(z_j)}dz_j\right]\int\int K(u_t)K(u_p)du_tdu_pe^{i\omega(t-p)}$$

$$\approx \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)\frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{j=1}^{n}\delta(t-j)\sum_{p=1}^{n}\left(1+|t-p|^{-\tau}\right)e^{i\omega(t-p)}$$

$$\approx \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)\frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{j=1}^{n}\delta(t-j)O(n^{1/(1+\tau)})$$

$$= O(n^{-\tau/(1+\tau)}),$$

and,

$$C = \sigma_\varepsilon^2E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n}f(z_t)^{-2}K(\frac{z_t-z_j}{h})K(\frac{z_t-z_l}{h})\left(\sum_{r=0}^{\infty}c_rc_{r+l-j}\right)$$

$$= O(n^{-1}),$$

and

$$D = \sigma_\varepsilon^2E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}f(z_t)^{-2}K(\frac{z_t-z_j}{h})^2\left(\sum_{r=0}^{\infty}c_r^2\right)$$

$$= \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}\int\int K(\frac{z_t-z_j}{h})^2f(z_t)^{-1}f(z_j)dz_tdz_j$$

$$= \sigma_\varepsilon^2\left(\sum_{r=0}^{\infty}c_r^2\right)E\frac{1}{2\pi n}\frac{1}{n^2h}\sum_{t=1}^{n}\sum_{j=1}^{n}\left[\int K(u)^2\right]Ef(z_t)^{-1}$$

$$= O(\frac{1}{nh}).$$

In a similar way, it can be verified that the second moment is $O_p(n^{-2\tau/(1+\tau)}+n^{-2}h^{-2})$.

For $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s)$, the leading terms are of order $O_p(n^{-1}h+h^{2q}n^{1/(1+\tau)}))$. Consider

$$E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}e^{i\omega t}e^{-i\omega p}$$

$$f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l)),$$

when t, p, j, l are different,

$$\begin{aligned}
& E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-1}f(z_p)^{-1} \\
& K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l))e^{i\omega t}e^{-i\omega p} \\
= & \frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega(t-p)} \\
& \int\int\int\int \frac{p_{(t-p),(t-j),(t-l)}(z_t,z_p,z_j,z_l)}{f(z_t)f(z_p)}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h}) \\
& (g(z_t)-g(z_j))(g(z_p)-g(z_l))dz_ldz_tdz_pdz_j \\
= & \frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega(t-p)} \\
& \int\int\int\int \frac{p_{(t-p),(t-j),(t-l)}(z_t,z_p,z_t+hu_j,z_p+hu_l)}{f(z_t)f(z_p)} \\
& K(u_j)K(u_l)(g(z_t)-g(z_t+hu_j))(g(z_p)-g(z_p+hu_l))du_ldu_jdz_tdz_p.
\end{aligned}$$

Notice that by Taylor expansion

$$\begin{aligned}
& p_{(t-p),(t-j),(t-l)}(z_t,z_p,z_t+hu_j,z_p+hu_l) \\
= & \sum_{\alpha,\beta}\nabla_{3,4}^{(\alpha,\beta)}p_{(t-p),(t-j),(t-l)}(z_t,z_p,z_t,z_p)h^{\alpha+\beta}u_j^{\alpha}u_l^{\beta},
\end{aligned}$$

$$\begin{aligned}
g(z_t+hu_j)-g(z_t) &= \sum_{a=1}^{q}\frac{1}{a!}g^{(a)}(z_t)u_j^ah^a+o_p(h^q) \\
g(z_p+hu_l)-g(z_p) &= \sum_{b=1}^{q}\frac{1}{b!}g^{(b)}(z_p)u_l^bh^b+o_p(h^q),
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi n^3}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega(t-p)}\sum_{\alpha,\beta,a,b} \\
& \int\int\int\int\frac{1}{a!b!}\frac{\nabla_{3,4}^{(\alpha,\beta)}p_{(t-p),(t-j),(t-l)}(z_t,z_p,z_t,z_p)}{f(z_t)f(z_p)}g^{(a)}(z_t)g^{(b)}(z_p)h^{\alpha+\beta+a+b} \\
& K(u_j)K(u_l)u_j^{\alpha+a}u_l^{\beta+b}du_ldu_jdz_tdz_p
\end{aligned}$$

$$= \frac{h^{2q}}{2\pi n^3}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega(t-p)}\sum_{\alpha+a=q,\beta+b=q}\frac{1}{a!b!}$$

$$\int\int \frac{\nabla_{3,4}^{(\alpha,\beta)} p_{(t-p),(t-j),(t-l)}(z_t, z_p, z_t, z_p)}{f(z_t)f(z_p)} g^{(a)}(z_t)g^{(b)}(z_p)dz_t dz_p$$

$$\int \qquad K(u_j)u_j^q du_j \int K(u_l)u_l^q du_l,$$

$$\begin{aligned} &= \frac{h^{2q}}{2\pi n^3}\mu_q(K)^2 \sum_{t=1}^n \sum_{t\neq p,p=1}^n \sum_{j=1}^n \sum_{l\neq j,l=1}^n e^{i\omega(t-p)} \sum_{\alpha+a=q,\beta+b=q} \frac{1}{a!b!}\left(1+(t-p)^{-\tau}\right)\delta(t-j,t-l) \\ &= O(h^{2q}n^{1/(1+\tau)}). \end{aligned}$$

Similarly,

$$\begin{aligned} & E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^n \sum_{t\neq p,p=1}^n \sum_{j=1}^n f(z_t)^{-1}f(z_p)^{-1} \\ & K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_j}{h})(g(z_p)-g(z_j))e^{i\omega t}e^{-i\omega p} \\ = \; & O(h^{2q}n^{1/(1+\tau)}), \end{aligned}$$

and

$$\begin{aligned} & E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^n \sum_{j=1}^n \sum_{l\neq j,l=1}^n \\ & f(z_t)^{-2}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_t-z_l}{h})(g(z_t)-g(z_l)) \\ = \; & O(h^{2q}), \end{aligned}$$

when $t=p$, $j=l$,

$$E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^n\sum_{j=1}^n f(z_t)^{-2}K(\frac{z_t-z_j}{h})^2(g(z_t)-g(z_j))^2 = O(n^{-1}h).$$

Similarly, we can verify the orders of magnitude for the other terms. ■

7.3 Proofs of Lemmas

Proof of Lemma 1: By definition

$$\begin{aligned} \widetilde{u}_t &= \widehat{y}_t^* - \widehat{\beta}_p' \widehat{x}_t^* \\ &= \widehat{y}_t^* - \beta'\widehat{x}_t^* - (\widehat{\beta}_p-\beta)'\widehat{x}_t^* \\ &= u_t - \widehat{u}_t + g(z_t) - \widehat{g}(z_t) - (\widehat{\beta}_p-\beta)'\widehat{x}_t^*, \\ &= \widehat{u}_t^* - (\widehat{\beta}_p-\beta)'\widehat{x}_t^*, \end{aligned}$$

where $\widehat{\beta}_p$ is a preliminary $\sqrt{n}-$estimator of β. Thus

$$\begin{aligned} I_{\widetilde{u}\widetilde{u}}(\lambda_s) &= w_{\widetilde{u}}(\lambda_t)w_{\widetilde{u}}(\lambda_t)^* \\ &= I_{\widehat{u^*}\widehat{u^*}}(\lambda_s) - 2(\widehat{\beta}_p - \beta)' I_{\widehat{x^*}\widehat{u^*}}(\lambda_s) + (\widehat{\beta}_p - \beta)' I_{\widehat{x^*}\widehat{x^*}}(\lambda_s)(\widehat{\beta}_p - \beta) \\ &= I_{\widehat{u^*}\widehat{u^*}}(\lambda_s) + o_p(1). \end{aligned}$$

and

$$\begin{aligned} \widetilde{f}_{uu}(\omega_j) &= \frac{1}{m}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widetilde{u}\widetilde{u}}(\lambda_s) \\ &= \frac{1}{m}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{u^*}\widehat{u^*}}(\lambda_s) \\ &\quad -2(\widehat{\beta}_p - \beta)' \left[\frac{1}{m}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{x^*}\widehat{u^*}}(\lambda_s)\right] \\ &\quad +(\widehat{\beta}_p - \beta)' \left[\frac{1}{m}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{x^*}\widehat{x^*}}(\lambda_s)\right](\widehat{\beta}_p - \beta) \\ &= \overline{f}_{uu}(\omega_j) + o_p(\frac{1}{\sqrt{n}}) \end{aligned}$$

where

$$\overline{f}_{uu}(\omega_j) = \frac{1}{m}\sum_{\lambda_s \in B_j} L(\lambda_s - \omega_j) I_{\widehat{u^*}\widehat{u^*}}(\lambda_s).$$

■

Proof of Lemma 2: By definition

$$I_{\widehat{u}\widehat{u}}(\omega) = \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{p=1}^{n} \widehat{u}_t \widehat{u}_p e^{i\omega t} e^{-i\omega p} \tag{29}$$

and

$$\widehat{u}_t = \frac{1}{nh}\sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) u_j / \widehat{f}(z_t). \tag{30}$$

Plugging (30) into (29), the leading term in $I_{\widehat{u}\widehat{u}}(\omega)$ is

$$\frac{1}{2\pi n^3 h^2}\sum_{t=1}^{n}\sum_{p=1}^{n} f(z_t)^{-1} f(z_p)^{-1}\left[\sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) u_j\right]\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h}) u_l\right] e^{i\omega t} e^{-i\omega p}. \tag{31}$$

We show that (31) is of order $O_p(n^{-1}h^{-1})$ by verification of the order of its moments. Start with the first moment:

$$
\begin{aligned}
& E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{p=1}^{n} f(z_t)^{-1}f(z_p)^{-1}\left[\sum_{j=1}^{n}K(\frac{z_t-z_j}{h})u_j\right]\left[\sum_{l=1}^{n}K(\frac{z_p-z_l}{h})u_l\right]e^{i\omega t}e^{-i\omega p} \\
= \; & E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \\
& f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})\left(\sum_{r=0}^{\infty}c_r\varepsilon_{j-r}\right)\left(\sum_{k=0}^{\infty}c_k\varepsilon_{l-k}\right)e^{i\omega t}e^{-i\omega p} \\
= \; & E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \\
& f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})\sum_{r=0}^{\infty}c_rc_{r+l-j}\varepsilon_{j-r}^2e^{i\omega t}e^{-i\omega p} \\
= \; & \sigma_\varepsilon^2E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \\
& f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_rc_{r+l-j}\right) \\
= \; & \sigma_\varepsilon^2E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} \\
& f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_l}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_rc_{r+l-j}\right) \\
& +\sigma_\varepsilon^2E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n} \\
& f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})K(\frac{z_p-z_j}{h})e^{i\omega t}e^{-i\omega p}\left(\sum_{r=0}^{\infty}c_r^2\right) \\
& +\sigma_\varepsilon^2E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-2}K(\frac{z_t-z_j}{h})K(\frac{z_t-z_l}{h})\left(\sum_{r=0}^{\infty}c_rc_{r+l-j}\right) \\
& +\sigma_\varepsilon^2E\frac{1}{2\pi n^3h^2}\sum_{t=1}^{n}\sum_{j=1}^{n} f(z_t)^{-2}K(\frac{z_t-z_j}{h})^2\left(\sum_{r=0}^{\infty}c_r^2\right) \\
= \; & A+B+C+D.
\end{aligned}
$$

The order of these terms is now verified. In particular,

$$
A = \sigma_\varepsilon^2E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n}
$$

$$
\begin{aligned}
& f(z_t)^{-1} f(z_p)^{-1} K(\frac{z_t - z_j}{h}) K(\frac{z_p - z_l}{h}) e^{i\omega t} e^{-i\omega p} \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
= \ & \sigma_\varepsilon^2 E \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{t \neq p, p=1}^{n} \sum_{j=1}^{n} \sum_{l \neq j, l=1}^{n} \\
& \int\int\int\int K(\frac{z_t - z_j}{h}) K(\frac{z_p - z_l}{h}) f(z_j) f(z_l) dz_l dz_t dz_p dz_j e^{i\omega t} e^{-i\omega p} \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
= \ & O(n^{-1}),
\end{aligned}
$$

and,

$$
\begin{aligned}
B \ = \ & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) E \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{p=1}^{n} \sum_{j=1}^{n} f(z_t)^{-1} f(z_p)^{-1} K(\frac{z_t - z_j}{h}) K(\frac{z_p - z_j}{h}) e^{i\omega(t-p)} \\
= \ & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{p \neq t, p=1}^{n} \sum_{j=1}^{n} \left[\int\int\int K(\frac{z_t - z_j}{h}) K(\frac{z_p - z_j}{h}) f(z_j) dz_t dz_p dz_j \right] e^{i\omega(t-p)} \\
= \ & O(n^{-1})
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
C \ = \ & \sigma_\varepsilon^2 E \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{j=1}^{n} \sum_{l \neq j, l=1}^{n} f(z_t)^{-2} K(\frac{z_t - z_j}{h}) K(\frac{z_t - z_l}{h}) \left(\sum_{r=0}^{\infty} c_r c_{r+l-j} \right) \\
= \ & O(n^{-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
D \ = \ & \sigma_\varepsilon^2 E \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{j=1}^{n} f(z_t)^{-2} K(\frac{z_t - z_j}{h})^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) \\
= \ & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) E \frac{1}{2\pi n} \frac{1}{(nh)^2} \sum_{t=1}^{n} \sum_{j=1}^{n} \int\int K(\frac{z_t - z_j}{h})^2 f(z_t)^{-1} f(z_j) dz_t dz_j \\
= \ & \sigma_\varepsilon^2 \left(\sum_{r=0}^{\infty} c_r^2 \right) E \frac{1}{2\pi n} \frac{1}{n^2 h} \sum_{t=1}^{n} \sum_{j=1}^{n} \left[\int K(u)^2 \right] E f(z_t)^{-1} \\
= \ & O(\frac{1}{nh}).
\end{aligned}
$$

Thus, the first moment of (31) is of order $(nh)^{-1}$. In a similar fashion, it can be verified that the second moment of (31) is $O_p(n^{-2}h^{-2})$. ■

Proof of Lemma 3: The order of magnitude for $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s)$ can be verified in a similar way to Lemma 2. Substituting

$$
\widehat{g}(z_t) = \frac{1}{nh} \sum_{j=1}^{n} K(\frac{z_t - z_j}{h}) g(z_j) / \widehat{f}(z_t),
$$

into $I_{(g-\widehat{g})(g-\widehat{g})}(\lambda_s)$ we have

$$I_{(g-\widehat{g})(g-\widehat{g})}(\omega) \sim \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{p=1}^{n} f(z_t)^{-1}f(z_p)^{-1}e^{i\omega t}e^{-i\omega p}$$
$$\left[\frac{1}{nh}\sum_{j=1}^{n}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))\right]\left[\frac{1}{nh}\sum_{l=1}^{n}K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l))\right].$$

Again, by a calculation of moments, we show that it is of order $O_p(n^{-1}h+h^{2q})$. For example, the first moment can be calculated as follows

$$E\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{p=1}^{n} f(z_t)^{-1}f(z_p)^{-1}e^{i\omega t}e^{-i\omega p}$$
$$\left[\frac{1}{nh}\sum_{j=1}^{n}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))\right]\left[\frac{1}{nh}\sum_{l=1}^{n}K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l))\right]$$
$$= E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} e^{i\omega t}e^{-i\omega p}$$
$$f(z_t)^{-1}f(z_p)^{-1}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l)),$$

$$= E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-1}f(z_p)^{-1}$$
$$K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_l}{h})(g(z_p)-g(z_l))e^{i\omega t}e^{-i\omega p}$$
$$+E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n} f(z_t)^{-1}f(z_p)^{-1}$$
$$K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_p-z_j}{h})(g(z_p)-g(z_j))e^{i\omega t}e^{-i\omega p}$$
$$+E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n}$$
$$f(z_t)^{-2}K(\frac{z_t-z_j}{h})(g(z_t)-g(z_j))K(\frac{z_t-z_l}{h})(g(z_t)-g(z_l))$$
$$+E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} f(z_t)^{-2}K(\frac{z_t-z_j}{h})^2(g(z_t)-g(z_j))^2.$$

It can be verified that the leading terms in the above expectation are $O_p(n^{-1}h+h^{2q})$. For example, when t,p,j,l are different,

$$E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-1}f(z_p)^{-1}$$

$$
\begin{aligned}
& K(\frac{z_t - z_j}{h})(g(z_t) - g(z_j))K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))e^{i\omega t}e^{-i\omega p} \\
= \; & \frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} e^{i\omega t}e^{-i\omega p} \\
& E\left[f(z_t)^{-1}K(\frac{z_t - z_j}{h})(g(z_t) - g(z_j))\right] E\left[f(z_p)^{-1}K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))\right],
\end{aligned}
$$

notice that

$$
\begin{aligned}
& E\left[f(z_t)^{-1}K(\frac{z_t - z_j}{h})(g(z_t) - g(z_j))\right] \\
= \; & E\left[f(z_t)^{-1}\left\{\int K(\frac{z_t - z_j}{h})(g(z_t) - g(z_j))f(z_j)dz_j\right\}\right] \\
\sim \; & h^{q+1}E\left[\frac{g(z_t)f^{(q)}(z_t) - (gf)^{(q)}(z_t)}{f(z_t)}\right]\left[(q!)^{-1}\int K(u)u^q du\right] \\
= \; & h^{q+1}\mu_q(K)E\left[\frac{g(z_t)f^{(q)}(z_t) - (gf)^{(q)}(z_t)}{f(z_t)}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq p,p=1}^{n}\sum_{j=1}^{n}\sum_{l\neq j,l=1}^{n} f(z_t)^{-1}f(z_p)^{-1} \\
& K(\frac{z_t - z_j}{h})(g(z_t) - g(z_j))K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))e^{i\omega t}e^{-i\omega p} \\
= \; & O(h^{2q}),
\end{aligned}
$$

and when $t = p$, $j = l$,

$$
\begin{aligned}
& E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} f(z_t)^{-2}K(\frac{z_t - z_j}{h})^2(g(z_t) - g(z_j))^2 \\
= \; & E\frac{1}{2\pi n}\frac{1}{(nh)^2}\sum_{t=1}^{n} Ef(z_t)^{-2}\sum_{j=1}^{n}\int K(\frac{z_t - z_j}{h})^2(g(z_t) - g(z_j))^2 f(z_j)dz_j \\
\sim \; & E\frac{1}{2\pi n}\frac{h}{(nh)^2}\sum_{t=1}^{n} Ef(z_t)^{-2}\sum_{j=1}^{n} h^2 f(z_t)g'(z_t)^2\int K(u)^2u^2du \\
= \; & \frac{1}{2\pi}n^{-2}h\sum_{t=1}^{n} Ef(z_t)^{-1}g'(z_t)^2\int K(u)^2u^2du \\
= \; & n^{-1}h\frac{1}{2\pi}\int K(u)^2u^2duE\left[f(z)^{-1}g'(z)^2\right].
\end{aligned}
$$

The calculation of the other terms and of the second moments are similar. ■

Proof of Lemma 4: The first moment of the leading term in $I_{u\widehat{u}}(\omega)$ is

$$\begin{aligned}
& \frac{1}{2\pi n^2 h} E\left[\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{l=1}^{n} e^{i\omega_j(t-p)} f(z_p)^{-1} K(\frac{z_p - z_l}{h}) \left(\sum_{j=0}^{\infty} c_j \varepsilon_{t-j}\right)\left(\sum_{r=0}^{\infty} c_r \varepsilon_{l-r}\right)\right] \\
= & \frac{1}{2\pi n^2 h} E\left[\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{l=1}^{n} e^{i\omega_j(t-p)} f(z_p)^{-1} K(\frac{z_p - z_l}{h}) \left(\sum_{r+t-l\geq 0, r=0}^{\infty} c_r c_{r+t-l} \varepsilon_{l-r}^2\right)\right] \\
\sim & \frac{\sigma_\varepsilon^2}{2\pi n^2 h} \sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{l=1}^{n} e^{i\omega_j(t-p)} E f(z_p)^{-1}\left[\int K(\frac{z_p - z_l}{h}) f(z_l) dz_l\right]\left(\sum_{r+t-l\geq 0, r=0}^{\infty} c_r c_{r+t-l}\right) \\
= & O(n^{-1}),
\end{aligned}$$

and, similarly, the second moment of $I_{u\widehat{u}}(\omega)$ is $O(n^{-2})$. Thus, $I_{u\widehat{u}}(\lambda_s) = O_p(n^{-1})$. ■

Proof of Lemma 5: It is mean zero and so we only need to verify the second moment. The order of magnitude can be easily verified by the results of Lemmas 2 and 3. ■

Proof of Lemma 6: It can be verified that $I_{\widehat{u(g-\widehat{g})}}(\lambda_s)$ is mean zero and its second moment is of order $O(n^{-1}h^{2q} + n^{-2}h^q)$. ■

Proof of Lemma 7: By definition $I_{x^*(m_1-\widehat{m}_1)}(\omega_j) = \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{p=1}^{n} x_t^*(m_1(z_p) - \widehat{m}_1(z_p))e^{i\omega_j(t-p)}$. Substitute this into $n^{-1}\sum_{j=-n/2+1}^{n/2} I_{x^*(m_1-\widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1}$, giving

$$\sum_{j=-n/2+1}^{n/2} \frac{1}{2\pi n^3 h} f_{uu}(\omega_j)^{-1}\left[\sum_{t=1}^{n}\sum_{p=1}^{n} x_t^*\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(m_1(z_p) - \widehat{m}_1(z_p))/f(z_p)\right] e^{i\omega_j(t-p)}\right].$$

Conditional on z, it can be shown that the above term is mean zero and its second moment is

$$\frac{1}{(2\pi)^2 n^6 h^2}\sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E\sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n} x_t^* x_k^* e^{i\omega_j(t-p)} e^{i\omega_s(k-r)}$$
$$\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(m_1(z_p) - m_1(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(m_1(z_r) - m_1(z_b))/f(z_r)\right].$$

Notice that (x_t, z_t) are independent across t, conditional on z, $E(x_t^* x_k^*) = 0$, for $t \neq k$. Thus, we have

$$\frac{1}{(2\pi)^2 n^6 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E \sum_{t=1}^{n} \sum_{p=1}^{n} \sum_{r=1}^{n} (x_t^*)^2 \, e^{i\omega_j(t-p)} e^{i\omega_s(t-r)}$$
$$\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(m_1(z_p) - m_1(z_l))/f(z_p) \right] \left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(m_1(z_r) - m_1(z_b))/f(z_r) \right].$$

Again, we can verify that the order of magnitude of the leading terms in the above moment are $o(1)$. For example, when $t \neq p \neq r \neq l \neq b$, it can be verified that

$$\begin{aligned}
& \frac{1}{(2\pi)^2 n^6 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E \sum_{t=1}^{n} \sum_{p=1}^{n} \sum_{r=1}^{n} (x_t^*)^2 \, e^{i\omega_j(t-p)} e^{i\omega_s(t-r)} \\
& \left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(m_1(z_p) - m_1(z_l))/f(z_p) \right] \left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(m_1(z_r) - m_1(z_b))/f(z_r) \right] \\
= \; & \frac{1}{(2\pi)^2 n^4 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n} \sum_{p=1}^{n} \sum_{r=1}^{n} E \, (x_t^*)^2 \\
& e^{i\omega_j(t-p)} e^{i\omega_s(t-r)} h^{2q+2} \mu_q(K)^2 E^2 \left[\frac{m_1 f^{(q)}(z) - (f m_1)^{(q)}(z)}{f(z)} \right] \\
= \; & O(n^{-1} h^{2q}).
\end{aligned}$$

■

Proof of Lemma 8: Notice that

$$I_{(m_1 - \widehat{m}_1)(m_1 - \widehat{m}_1)}(\omega) = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} \left[(m_1(z_t) - \widehat{m}_1(z_t))(m_1(z_s) - \widehat{m}_1(z_s)) \right] e^{i\omega t} e^{-i\omega s},$$

and

$$\begin{aligned}
& (m_1(z_t) - \widehat{m}_1(z_t))(m_1(z_s) - \widehat{m}_1(z_s)) \\
= \; & \frac{1}{(nh)^2} \sum_{j=1}^{n} \sum_{l=1}^{n} \widehat{f}(z_t)^{-1} K(\frac{z_t - z_j}{h})(m_1(z_t) - m_1(z_j)) \widehat{f}(z_s)^{-1} K(\frac{z_s - z_l}{h})(m_1(z_s) - m_1(z_l)).
\end{aligned}$$

The leading term in

$$n^{-1} \sum_{j=-n/2+1}^{n/2} I_{(m_1 - \widehat{m}_1)(m_1 - \widehat{m}_1)}(\omega_j) f_{uu}(\omega_j)^{-1}$$

is

$$\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} e^{i\omega t}e^{-i\omega s}$$
$$f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_s)^{-1}K(\frac{z_s-z_l}{h})(m_1(z_s)-m_1(z_l)),$$

which can be decomposed into the sum of the following terms

$$\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t,s,j,l=1}^{n}\sum\sum\sum_{t\neq s\neq j\neq l} e^{i\omega t}e^{-i\omega s}f(z_t)^{-1}$$
$$K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_s)^{-1}K(\frac{z_s-z_l}{h})(m_1(z_s)-m_1(z_l))$$
$$+\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq s,s=1}^{n} e^{i\omega t}e^{-i\omega s}f(z_t)^{-1}$$
$$K(\frac{z_t-z_s}{h})(m_1(z_t)-m_1(z_s))f(z_s)^{-1}K(\frac{z_s-z_t}{h})(m_1(z_s)-m_1(z_t))$$
$$+\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{t\neq s,s=1}^{n}\sum_{j} e^{i\omega t}e^{-i\omega s}$$
$$f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_s)^{-1}K(\frac{z_s-z_j}{h})(m_1(z_s)-m_1(z_j))$$
$$+\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j} f(z_t)^{-2}$$
$$K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))K(\frac{z_t-z_l}{h})(m_1(z_t)-m_1(z_l))$$
$$+\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} f(z_t)^{-1}$$
$$K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j)).$$

Again, it can be verified by calculation of moments that all these terms are $o_p(1)$. For example, if we calculate their first moments, when $t\neq s\neq j\neq l$,

$$E\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} e^{i\omega t}e^{-i\omega s}$$
$$f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_s)^{-1}K(\frac{z_s-z_l}{h})(m_1(z_s)-m_1(z_l))$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}$$

$$e^{i\omega t}e^{-i\omega s}h^{2q+2}\mu_q(K)^2E^2\left[\frac{m_1 f^{(q)}(z)-(m_1 f)^{(q)}(z)}{f(z)}\right]$$

$$= O(n^{-1}h^{2q}).$$

And, when $t = s \neq j = l$,

$$E\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}$$

$$f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))f(z_t)^{-1}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} E\left[\frac{(m_1(z_t)-m_1(z_j))}{f(z_t)}K(\frac{z_t-z_j}{h})\right]^2$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} E\left\{\int\left[\frac{(m_1(z_t)-m_1(z_j))}{f(z_t)}K(\frac{z_t-z_j}{h})\right]^2 f(z_j)dz_j\right\}$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n}$$

$$hE\left\{\int\left[\frac{(m_1(z_t)-m_1(z_t+u))}{f(z_t)}K(u)\right]^2 f(z_t+uh)du\right\}$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}^{n} h^3E\left\{\int\frac{(m_1'(z_t))^2}{f(z_t)}u^2K(u)^2du\right\}$$

$$= \frac{h}{n}E\left[\frac{(m_1'(z))^2}{f(z)}\right]\int u^2K(u)^2du\left[\frac{1}{2\pi n}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\right]$$

$$= O(n^{-1}h).$$

When $t = s \neq j \neq l$,

$$\frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j}$$

$$f(z_t)^{-2}K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))K(\frac{z_t-z_l}{h})(m_1(z_t)-m_1(z_l))$$

$$= \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j} Ef(z_t)^{-2}$$

$$
\begin{aligned}
&\int\int K(\frac{z_t-z_j}{h})(m_1(z_t)-m_1(z_j))K(\frac{z_t-z_l}{h})(m_1(z_t)-m_1(z_l))f(z_j)f(z_l)dz_jdz_l \\
= & \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j} h^2 \\
& E\frac{\int\int K(u)(m_1(z_t)-m_1(z_t+uh))K(v)(m_1(z_t)-m_1(z_t+vh))f(z_t+uh)f(z_t+vh)dudv}{f(z_t)^2} \\
= & \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j} \\
& h^{2q+2}E\frac{[\int K(u)u^q du]^2\left[(m_1f^{(q)}(z_t)-(m_1f)^{(q)}(z_t))\right]^2}{f(z_t)^2} \\
= & \frac{1}{2\pi n^2}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\frac{1}{(nh)^2}\mu_q(K)^2\sum_{t=1}^{n}\sum_{j=1}\sum_{l\neq j} E\left[\frac{(m_1f^{(q)}(z_t)-(m_1f)^{(q)}(z_t))}{f(z_t)^2}\right]^2 \\
= & \frac{h^{2q}}{n}\mu_q(K)^2E\left[\frac{(m_1f^{(q)}(z)-(m_1f)^{(q)}(z))}{f(z)^2}\right]^2\left[\frac{1}{2\pi n}\sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1}\right] \\
= & O(n^{-1}h^{2q}).
\end{aligned}
$$

Other terms and the second moments can be analyzed similarly and thus

$$
\frac{1}{n}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j)f_{uu}(\omega_j)^{-1} = o_p(1).
$$

■

Proof of Lemma 9: It can be verified that it has mean zero and has $O(n^{-1}h^{-1})$ second moment. ■

Proof of Lemma 10: Notice that $E(U|X,Z)=0$, so the expectation of

$$
n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j)f_{uu}(\omega_j)^{-1}
$$

is zero. Now we verify that its second moment is $o(1)$. The order of magnitude of the second moment of $n^{-1/2}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)u}(\omega_j)f_{uu}(\omega_j)^{-1}$ is determined by

$$
E\left[n^{-1}\sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j)I_{uu}(\omega_j)f_{uu}(\omega_j)^{-2}\right]
$$

$$= n^{-1} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-2} E\left[I_{(m_1-\widehat{m}_1)(m_1-\widehat{m}_1)}(\omega_j)\right] E\left[I_{uu}(\omega_j)\right].$$

It can be verified that this second moment is $O(h^{2q} + n^{-1}h)$. ■

Proof of Lemma 11: Again, we verify the moments of $n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1}$. Notice that

$$\begin{aligned}
& n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} \\
= \; & n^{-1/2} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \left[\frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{p=1}^{n} (m_1(z_t) - \widehat{m}_1(z_t)) \widehat{u}_p e^{i\omega_j (t-p)} \right] \\
= \; & n^{-3/2} \sum_{j=-n/2+1}^{n/2} \frac{1}{2\pi} f_{uu}(\omega_j)^{-1} \left[\sum_{t=1}^{n} \sum_{p=1}^{n} (m_1(z_t) - \widehat{m}_1(z_t)) \left[\frac{1}{nh} \sum_{l=1}^{n} K(\frac{z_p - z_l}{h}) u_l \right] e^{i\omega_j (t-p)} \right].
\end{aligned}$$

Using the linear process form of u_t, and by an analysis similar to that of the previous Lemmas, we can show that the leading terms in $n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1}$ are $O_p(n^{-1/2} h^{q/2})$ and $O_p(h^q)$. Thus

$$n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)\widehat{u}}(\omega_j) f_{uu}(\omega_j)^{-1} = O_p(h^q).$$

■

Proof of Lemma 12:

$$\begin{aligned}
& n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{x^*(g-\widehat{g})}(\omega_j) f_{uu}(\omega_j)^{-1} \\
= \; & n^{-3/2} \sum_{j=-n/2+1}^{n/2} \frac{1}{2\pi} f_{uu}(\omega_j)^{-1} \left[\sum_{t=1}^{n} \sum_{p=1}^{n} x_t^* (g(z_p) - \widehat{g}(z_p)) e^{i\omega_j (t-p)} \right] \\
\sim \; & n^{-5/2} \sum_{j=-n/2+1}^{n/2} \frac{1}{2\pi} f_{uu}(\omega_j)^{-1} \left[\sum_{t=1}^{n} \sum_{p=1}^{n} x_t^* \left[\frac{1}{h} \sum_{l=1}^{n} K(\frac{z_p - z_l}{h}) (g(z_p) - g(z_l)) / f(z_p) \right] e^{i\omega_j (t-p)} \right].
\end{aligned}$$

Conditional on z, it can be shown that the above term has mean zero and its second moment is

$$\frac{1}{(2\pi)^2 n^5 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E \sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n} x_t^* x_k^* e^{i\omega_j (t-p)} e^{i\omega_s (k-r)}$$
$$\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(g(z_r) - g(z_b))/f(z_r)\right].$$

Notice that (x_t, z_t) are independent across t, conditional on z, $E(x_t^* x_k^*) = 0$, for $t \neq k$. Thus, we have

$$\frac{1}{(2\pi)^2 n^5 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E \sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{r=1}^{n} (x_t^*)^2 e^{i\omega_j (t-p)} e^{i\omega_s (t-r)}$$
$$\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(g(z_r) - g(z_b))/f(z_r)\right].$$

Again, we can verify the order of magnitudes of the leading terms in the above moment calculation. For example, when $t \neq p \neq r \neq l \neq b$,

$$\begin{aligned}
&\frac{1}{(2\pi)^2 n^5 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} E \sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{r=1}^{n} (x_t^*)^2 e^{i\omega_j (t-p)} e^{i\omega_s (t-r)} \\
&\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_r - z_b}{h})(g(z_r) - g(z_b))/f(z_r)\right] \\
= \; &\frac{1}{(2\pi)^2 n^3 h^2} \sum_{s,j=-n/2+1}^{n/2} f_{uu}(\omega_s)^{-1} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n}\sum_{p=1}^{n}\sum_{r=1}^{n} E\,(x_t^*)^2 \\
&e^{i\omega_j (t-p)} e^{i\omega_s (t-r)} h^{2q+2} \mu_q(K)^2 E^2 \left[\frac{g f^{(q)}(z) - (fg)^{(q)}(z)}{f(z)}\right] \\
= \; &O(h^{2q}).
\end{aligned}$$

■

Proof of Lemma 13:

$$\begin{aligned}
&n^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1 - \widehat{m}_1)(g-\widehat{g})}(\omega_j) \widehat{f}_{uu}(\omega_j)^{-1} \\
\sim \; &n^{-1/2} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \frac{1}{2\pi n} \left[\sum_{t=1}^{n}\sum_{p=1}^{n} (m(z_t) - \widehat{m}(z_t))(g(z_p) - \widehat{g}(z_p)) e^{i\omega_j (t-p)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi n^{3/2}} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n}\sum_{p=1}^{n} e^{i\omega_j(t-p)} \left(\frac{1}{nh}\right)^2 \\
&\quad \left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/\widehat{f(z_p)}\right]\left[\sum_{b=1}^{n} K(\frac{z_t - z_b}{h})(g(z_t) - g(z_b))/\widehat{f}(z_r)\right] \\
&\sim \frac{1}{2\pi n^{3/2}} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n}\sum_{p=1}^{n} e^{i\omega_j(t-p)} \left(\frac{1}{nh}\right)^2 \\
&\quad \left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_t - z_b}{h})(g(z_t) - g(z_b))/f(z_r)\right].
\end{aligned}
$$

We can show that the leading term

$$
\begin{aligned}
&\frac{1}{2\pi n^{3/2}} \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n}\sum_{p=1}^{n} e^{i\omega_j(t-p)} \left(\frac{1}{nh}\right)^2 \\
&\left[\sum_{l=1}^{n} K(\frac{z_p - z_l}{h})(g(z_p) - g(z_l))/f(z_p)\right]\left[\sum_{b=1}^{n} K(\frac{z_t - z_b}{h})(g(z_t) - g(z_b))/f(z_r)\right]
\end{aligned}
$$

is $o_p(1)$. Thus

$$
\begin{aligned}
& En^{-1/2} \sum_{j=-n/2+1}^{n/2} I_{(m_1-\widehat{m}_1)(g-\widehat{g})}(\omega_j)\widehat{f}_{uu}(\omega_j)^{-1} \\
\sim\ & \frac{h^{2q}}{2\pi n^{3/2}}\mu_q(K)^2 E\left[\frac{gf^{(q)}(z) - (fg)^{(q)}(z)}{f(z)}\frac{mf^{(q)}(z) - (mf)^{(q)}(z)}{f(z)}\right] \\
& \sum_{j=-n/2+1}^{n/2} f_{uu}(\omega_j)^{-1} \sum_{t=1}^{n}\sum_{p=1}^{n} e^{i\omega_j(t-p)} \\
=\ & O(n^{1/2}h^{2q}).
\end{aligned}
$$

Similarly, it can be verified that the leading terms of its second moment are of order of magnitude $O(nh^{4q})$, $O(n^{-1}h^{2q-2})$, and $O(h^{2q+2})$ respectively. ■

Proof of Lemma 14: This result can be obtained as in traditional spectral regression theory (Hannan, 1970). ■

The following Lemma is useful in the proof of Theorem 3 below.

Proof of Lemma 15

$$\begin{aligned}\sum_{t=1}^{n} t^{-\tau} e^{i\omega t} &= \sum_{t=1}^{n} t^{-\tau} e^{i\omega t} \\ &= \sum_{t\leq L} t^{-\tau} e^{i\omega t} + \sum_{t>L} t^{-\tau} e^{i\omega t} \\ &= O(L) + O(nL^{-\tau}) \\ &= O(n^{1/(1+\tau)})\end{aligned}$$

where the last equality was obtained by letting $L = O(n^{1/(1+\tau)})$. ■

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