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ON THE RELATION OF VARIOUS RELIABILITY MEASURES TO
EACH OTHER AND TO GAME THEORETIC VALUES

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ON THE RELATION OF VARIOUS RELIABILITY MEASURES
TO EACH OTHER AND TO GAME THEORETIC VALUES^{* +}

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Abstract

A variety of measures have recently been proposed for measuring the relative importance of individual components in the overall reliability of a system. Several of these seemingly different measures are very closely related under the conditions typically assumed in the reliability literature. The measures are also closely related to the probabilistic values of game theory; although the game theoretic literature predates the reliability literature by up to two decades, the similarities have apparently not been previously observed or exploited.

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Introduction

In many systems, whether or not the failing of a particular component causes the entire system to fail depends on what other components have previously failed. For example, in a nuclear power generating facility, certain components are redundant in the sense that they may all fail without causing the system to fail. In designing, modifying, and protecting systems, it becomes useful to assess the relative contribution, or the relative importance of the reliability, of each component to the overall reliability of the entire system.

A number of seemingly different measures for the importance of individual components have been proposed; these include those of Barlow and Proschan [2], Birnbaum [3], and Fussell [6]. Although these measures have, on occasion been discussed together (e.g., Lambert [7]) they have apparently never been critically compared. We will show that several of these measures are simple transformations of each other if, and only if, certain commonly-assumed conditions are satisfied.

The above reliability measures are closely related to the concept of probabilistic values in game theory; probabilistic values measure the relative contributions of the players to the outcome of the game. Although the work by Banzhaf [1] and Shapley [9] predates the corresponding reliability literature by up to two decades, the connection has apparently not been observed before now. We will use certain game theoretic results, translated into the context of reliability, to critically compare a variety of measures for the importance of individual components in the reliability of a system.

The following two assumptions on $P_t(S)$ are for future reference; we will not necessarily make these assumptions.

Assumption 1 (Independence of Failures): $P_t(S) = \prod_{i \in S} F_i(t) \prod_{i \notin S} (1-F_i(t))$,

where $F_i(t)$ is shorthand for $F_{\{i\}}(t)$.

Assumption 2 (Symmetry of Failures): $P_t(S)$ is a function solely of $|S|$ (i.e., the number of components in S); components are interchangeable for purposes of computing $P_t(S)$.

Sensitivity Measures

The first class of measures considered are of the form "what is the probability that the system status changes when the characteristics of a particular subset of components are altered?" Although measures are typically defined for individual components in systems with each $Q(S)$ either zero or one, the definitions may easily be stated more generally. In particular, consider the following measures.

$$M_{1,t}(X) = \sum_S P_t(S) [Q(S) - Q(S \setminus X)]$$

$$M_{2,t}(X) = \sum_S P_t(S) [Q(S \cup X) - Q(S)]$$

$$M_{3,t}(X) = \sum_S P_t(S) [Q(S \cup X) - Q(S \setminus X)] = M_{1,t}(X) + M_{2,t}(X)$$

$$M_{4,t}(X) = dH(t)/dF_X(t). \text{ (This is defined only if } H(t) \text{ is a function of } F_X(t)\text{).}$$

The first may be interpreted as the probability that the system status at time t changes if all the non-functioning components in X are repaired

at time t ; this measure corresponds to that of Fussell [5]. Alternatively, it is the change in the probability that the system functions at time t when the components in the set X are made fail-proof. This appears an appropriate measure for those defending a system against attack or trying to improve system reliability by selectively upgrading some components.

The second measure may be interpreted as the probability that the system status at time t changes if all the functioning components in X are broken at time t . Such a measure might be of interest to anyone planning to sabotage a system. The measures of Banzhaf [1], Barlow and Proschan [2], and Shapley [9] are obtained by making specific choices for the $P_t(S)$ in the second measure.

Finally, the last two measures may be interpreted as the sensitivity of the reliability of the system to whether or not all the components of the set X are functioning at time t . The third and fourth measure correspond to that of Birnbaum [3].

Although the above measures have been stated as reliability measures, they are closely related to the game theoretic "probabilistic values" v_i ($i=1, 2, \dots, n$) defined by $\sum_S P(S) [Q(SU_i) - Q(S)]$, where the $P(S)$'s may be any non-negative numbers summing to one. A game theoretic question analogous to that of reliability is: given the value of each coalition of players, what is the contribution of any one player to the overall value. For example, in a voting situation, the value of a coalition is zero or one; a coalition's value is one if and only if the members of the coalition, working together, can assure the passage of a bill. The importance of a particular individual depends on how many votes he has (or controls) and how often his votes can influence the outcome.

In order to relate the various reliability measures, it is useful to assume that $H(t)$ is a function of $F_i(t)$ for each i . Although this assumption rules out many forms of dependencies among the failure distributions of different components, the assumption is consistent with many reliability models.

The various reliability measures are illustrated in Figure 1. The first three measures correspond to intervals, while the fourth (and also the third) correspond to slopes. This figure, together with some results of Owen [8] on multilinear extensions of games, suggests the following theorem.

Theorem 1: When $H(t)$ is a function of $F_i(t) \forall i$, then the following four conditions are equivalent.

1. $P_t(S) = \prod_{i \in S} F_i(t) \prod_{i \notin S} (1-F_i(t)) \forall t.$

2. For any function Q , the corresponding $H(t)$ is given by

$$H(t) = \sum_S Q(S) \prod_{i \in S} F_i(t) \prod_{i \notin S} (1-F_i(t)) \forall t.$$

3. For any function Q , the corresponding $H(t)$ is linear in each $F_i(t)$, and $H(t) = Q(S)$ whenever $F_i(t) = 0 \forall i \notin S$ and $F_i(t) = 1 \forall i \in S$.

4. For any function Q , the corresponding measures satisfy

$$M_{1,t}(i)/F_i(t) = M_{2,t}(i)/(1-F_i(t)) = M_{3,t}(i) = M_{4,t}(i) \forall i, t.$$

Proof: It is clear that $1 \Rightarrow 2 \Rightarrow 3$. It follows from Figures 1 and 2 that $3 \Leftrightarrow 4$. Owen proves that $3 \Rightarrow 2$. Finally, $2 \Rightarrow 1$ follows from comparing the second condition to the definition $H(t) = \sum_S Q(S) P_t(S)$ for all Q .

The first of the equivalent conditions is the independence of component failures defined earlier; this assumption is explicitly made in most of the cited works on reliability measures. The independence assumption is however quite restrictive in practical problems. For example, it rules out the possibility that a particular component (eg. a lock on a vault) must typically

fail before some other component can fail (e.g., before a safe deposit box in the vault can be burgled).

Although the independence assumption is commonly made, this relation between the four measures has, apparently, not been previously observed. Note, however, that although the measures are closely related, this relationship is not necessarily obvious from numerical results. Since the numerators in the fourth condition depend on i , the corresponding vectors $\underline{M}_{j,t} = (M_{j,t}(1), M_{j,t}(2), \dots, M_{j,t}(n))$ ($j = 1, 2, 3, 4$) of measures are not proportional to each other.

In cases where the functions $F_i(t)$ are unknown, Birnbaum [3] suggests evaluating $dH(t)/dF_i(t)$ at $F_i(t) = 1/2 \forall i$. This is equivalent to assuming that each of the other components has an independent fifty-fifty chance not functioning. This approach seems somewhat ad hoc; in the next section it is shown that this particular symmetric choice of values for $F_i(t)$ at which to evaluate $H(t)$ cannot hold for all t in a symmetric model.

Under independence of failures, setting $F_i(t) = 1/2 \forall i$ in the fourth condition results in the same measure obtained by setting $P_t(S) = 1/2^{(n-1)} \forall S$ in the first or second measure, and is precisely the game theoretic Banzhaf [1] value when Q is a zero-one function. Dubey and Shapley [5] observe that the Banzhaf value is equivalent to the Chow [4] parameters characterizing networks with threshold functions; since there are not, in general, threshold functions for the class of reliability problems under consideration, the relation to Chow parameters will not be treated further here.

Causal Measures

An alternative measure for the importance of a particular component is the probability that the system fails due to the failing of this particular component. In such a measure, a component contributes to the fallibility of the system only when it is the proverbial "straw which breaks the camel's back." A component which always fails before the system fails (but is never the last component to fail before the system fails) contributes nothing to system failures and is assigned a zero weight in such causal measures.

Consider the following two causal measures.

$$M_{5,t}(i) = \sum_S \Pr(S_t=S \mid i \text{ fails at time } t^+) [Q(S \cup i) - Q(S)]$$

$$M_{6,t}(i) = \sum_S \Pr(S_t=S \mid i \text{ repaired at time } t^+) [Q(S) - Q(S \setminus i)],$$

where " i fails at time t^+ " is to be interpreted as " $i \in S_t$, but $S_{t+e} = S_t \cup i$ for all sufficiently small positive e ," and " i repaired at time t^+ " is to be interpreted as " $i \in S_t$, but $S_{t+e} = S_t \setminus i$ for all sufficiently small positive e ." Note that this implicitly assumes zero probability of more than one failure and/or repair occurring simultaneously.

Under independence of failures, the probabilities in $M_{5,t}(i)$ simplify to $P_t(S)/(1-F_i(t))$, while the probabilities in $M_{6,t}(i)$ simplify to $P_t(S)/F_i(t)$. This observation, together with Theorem 1, yields the following result.

Theorem 2: If there is independence of failures, then $M_{6,t}(i) = M_{5,t}(i) = M_{4,t}(i) = M_{3,t}(i) = M_{2,t}(i)/(1-F_i(t)) = M_{1,t}(i)/F_i(t)$.

Barlow and Proschan [2] consider systems without repair and define the importance M^* of component i as the probability that the failing of i causes the system to fail. Although the authors assume both symmetry and independence of failures, the measure can be defined for more general models.

Under independence of failures, the expected number of system failures caused by component i during the time interval T is $\int_{t \in T} M_{5,t}(i) dF_i(t)^+$, where $dF_i(t)^+$ denotes the positive part of $dF_i(t)$. If there are no repairs, then this expectation is equal to M^* . However, under independence of failures, Theorem 2 may be used to relate M^* to the sensitivity measures.

In models with no repairs and symmetry of failures, it is easy to verify that each of the $n!$ possible orders in which the n components may fail are equally likely to occur. Thus, the probability that the failure of component i is preceded by the failures of all the elements of S is $s!(n-s-1)!/n!$, where s denotes the number of elements in S . This observation results in the following generalization of the characterization of M^* by Barlow and Proschan.

Theorem 3: If there is symmetry of failures and there are no repairs, then

$$M^* = \sum_S [s!(n-s-1)!/n!] [Q(SU_i) - Q(S)]$$

The above value of M^* is also the game theoretic Shapley [9] value; the unique game theoretic value satisfying certain axioms, including one of symmetry.

If $M_{3,t}(i)$ or $M_{4,t}(i)$ is independent of t in a model without repairs and with symmetric and independent failures, then these measures are equal to $\sum_S [s!(n-s-1)!/n!] [Q(SU_i) - Q(S)]$. In particular, as Weber [10] observes for game theoretic probabilistic values, $M_{3,t}(i)$ and $M_{4,t}(i)$ cannot be equal to the Banzhaf [1] value (or, equivalently, the Birnbaum [3] measure evaluated at $F_i(t) = 1/2 \forall i$) for all times t in models with symmetric, independent failures and no repairs.

Finally, an example shows that Theorem 3 need not hold if there are repairs. In particular, consider a three component system with $Q(\{1,2,3\}) = Q(\{1,3\}) = Q(\{2,3\}) = 1$, and $Q(S) = 0$ for all other S . Assume that the "uptimes" (time from completion of repair until the next failure) of the components are identical independent exponential random variables. Likewise, the "downtimes" (time from failure until completion of repair) are identical independent exponential random variables; the uptimes and downtimes are also assumed to be independent of each other. Thus, the failures and repairs are symmetric and independent.

Characterize the state of the system by the subset of failed components; there are eight possible states. The memoryless quality of exponential distributions results in a Markov process for the transitions from one state to another. This process is depicted graphically in Figure 3. The probability that the system goes from state i to state j conditional on the system being in state i are indicated in the figure as a function of p ; p is the fraction of time a component is failed, or more precisely, the mean downtime divided by the sum of the mean uptime and the mean downtime.

The above transition probabilities imply the following state probabilities: $P_t(S) = P_{|S|} \forall t$, where $P_0 = (1-p)^2/2$, $P_1 = (1+p)(1-p)/6$, $P_2 = p(2-p)/6$, and $P_3 = p^2/2$. Thus, the relative probabilities that the failing of component i causes the system to fail are the probabilities that the system goes from state $\{3\}$ to state $\{1,3\}$, the probability that the system goes from state $\{3\}$ to state $\{2,3\}$, and the probability that the system goes from state $\{1,2\}$ to state $\{1,2,3\}$, from state $\{2\}$ to state $\{2,3\}$, or from state $\{1\}$ to state $\{1,3\}$. Thus, it follows that the probability component i failing causes the system to fail, conditional on the system failing, is $(1-p)/(4-3p)$, $(1-p)/(4-3p)$,

and $(2-p)/(4-3p)$, respectively, for $i = 1, 2,$ and 3 .

If p is close to one, then most of the system components are likely to be non-functional; under such cases, a system failure is most likely to have been caused by a transition from state $\{1,2\}$ to state $\{1,2,3\}$. Indeed, as p tends to one, the conditional probabilities that component i causes system failure tends to $0, 0,$ and 1 respectively. Alternatively, if p is very small, then the system is likely to be in a state with few failed components and system failures are likely to have been caused by any one of the three components failing. In particular, as p tends towards zero, the conditional probability that component i causes the system to fail tends to $1/4, 1/4,$ and $1/2$ respectively. Note that in the above example with symmetric and independent failures and repairs, the relative importance of the three components (measured in terms of which component's failing causes the system to fail) depends on the parameter p and cannot satisfy the conclusion of Theorem 3 for arbitrary p . Thus, the assumption of no repairs is necessary for Theorem 3.

Conclusion

A number of apparently different measures for the importance of an individual component to the reliability of a system are examined in this paper. By defining all the measures within the same, sufficiently general, model, some insight is gained into the different probability questions corresponding to the different measures. It is, however, shown that under the (common) assumption of independence of failures, the four "sensitivity" measures are very closely related to each other. It is also shown that under the (common) assumption of symmetric failure rate distributions, the "causal" measures must take a particular form, and that they are also, over appropriate time intervals, equal to one of the previously mentioned sensitivity measures; in addition, it is noted that Birnbaum's suggestion for evaluating his sensitivity measure in cases of insufficient information is not the, essentially unique, time independent measure corresponding to models with symmetric and independent failures.

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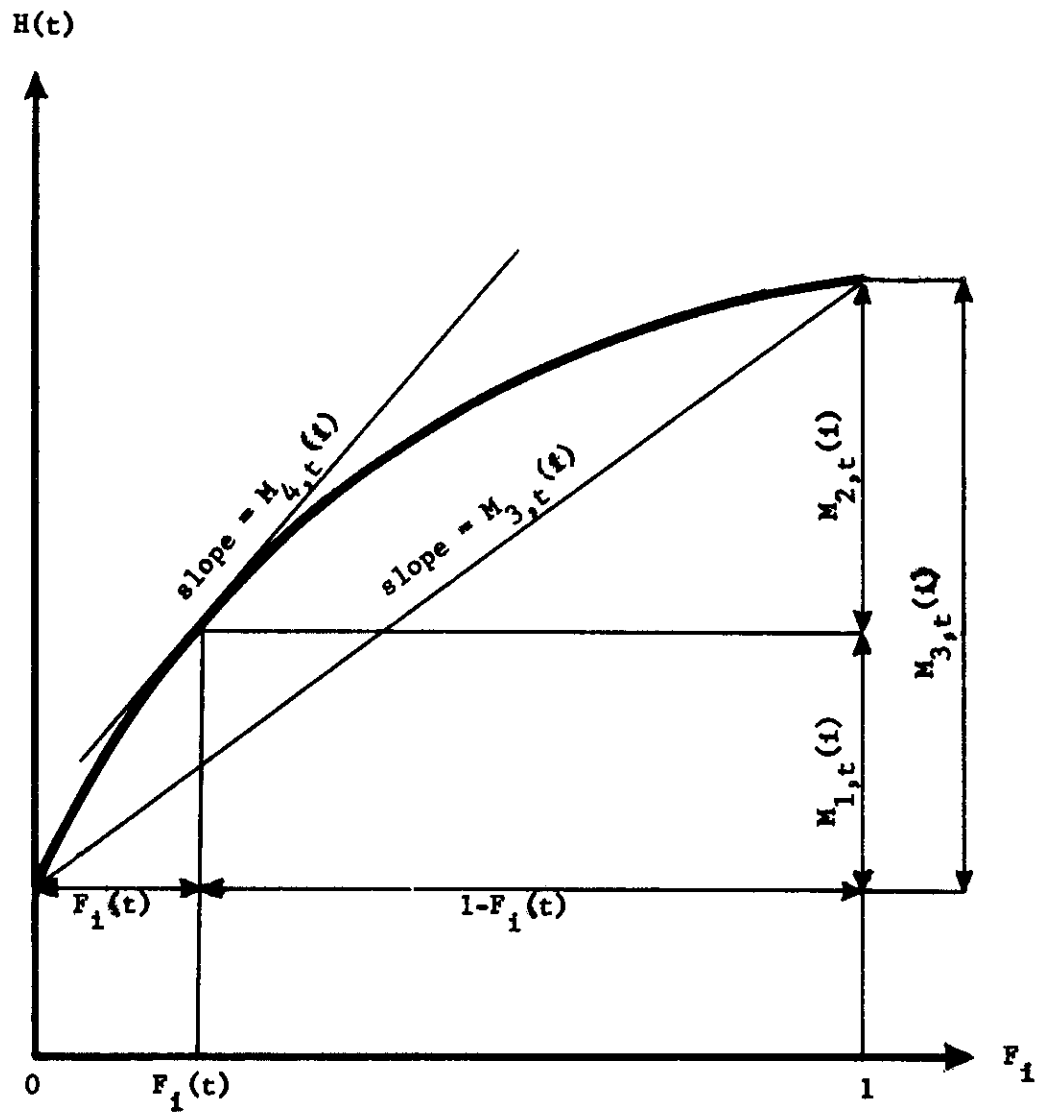


Figure 1

Relation of Measures for General System Reliability Function

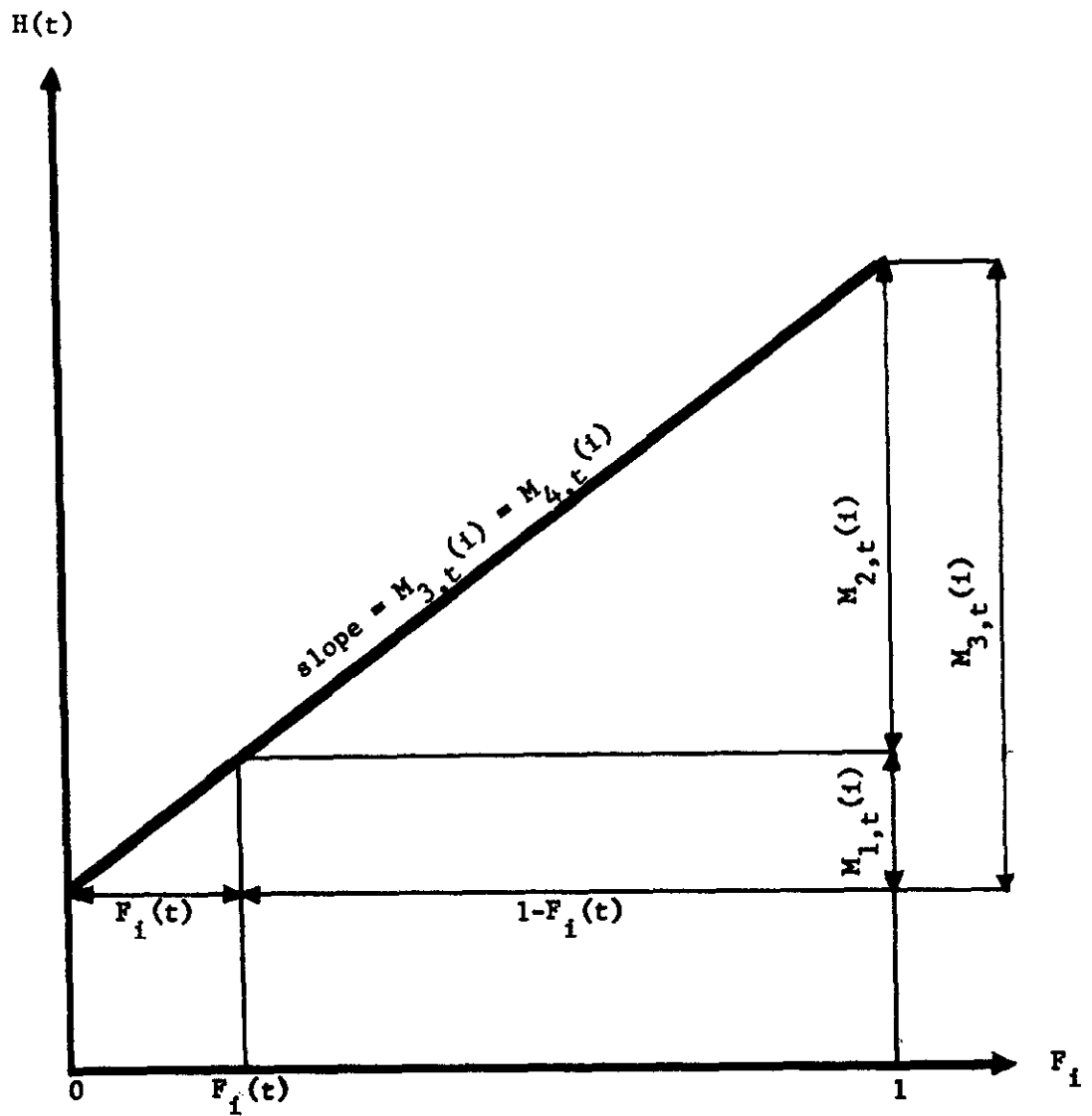


Figure 2

Relation of Measures for Linear System Reliability Function

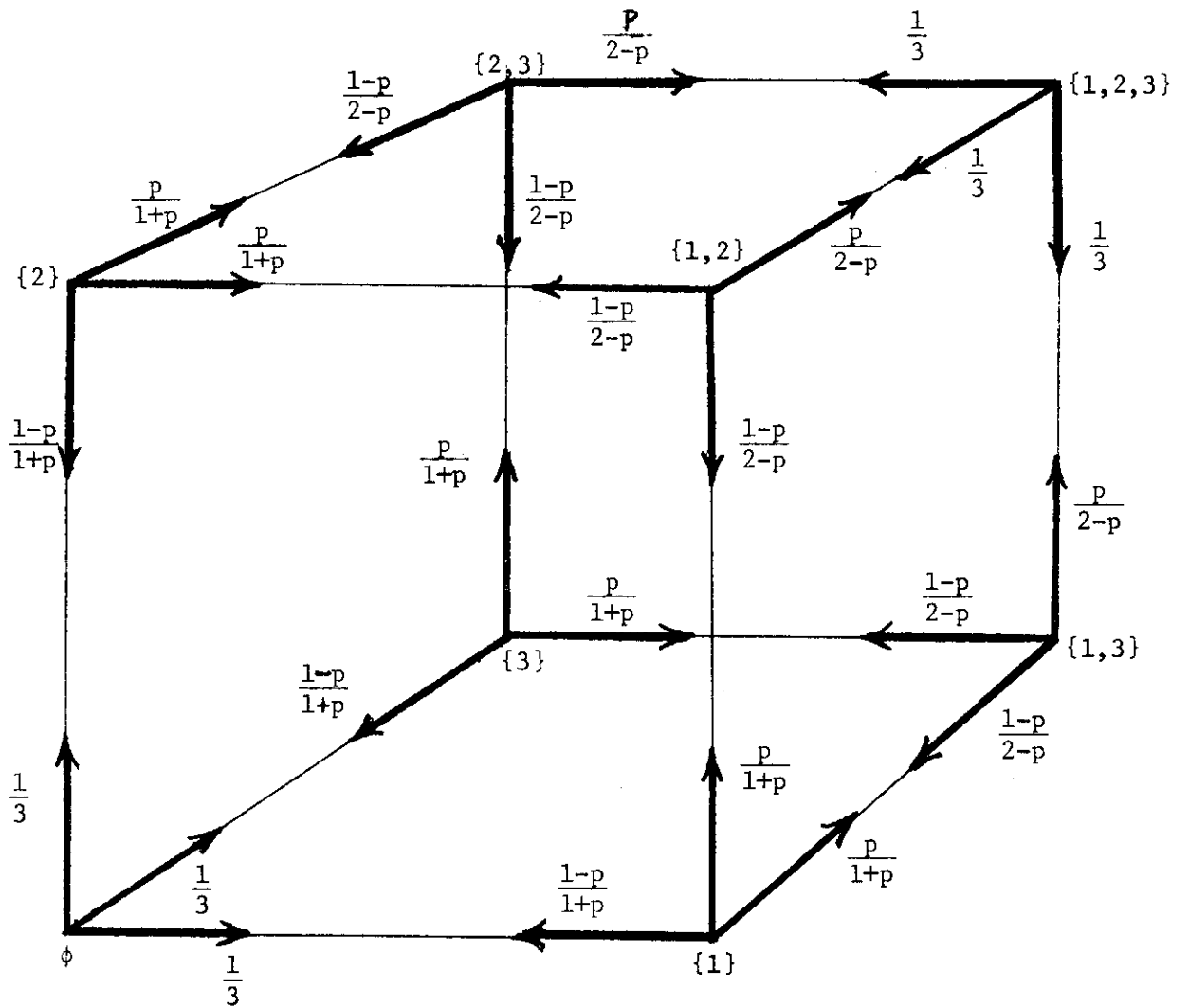


FIGURE 3. Markov Process Corresponding to Example.