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Error Bounds and Asymptotic Expansions for Toeplitz Product Functionals of Unbounded Spectra*

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Abstract

This paper establishes error orders for integral limit approximations to traces of powers (to the p'th order) of products of Toeplitz matrices. Such products arise frequently in the analysis of stationary time series and in the development of asymptotic expansions. The elements of the matrices are Fourier transforms of functions which we allow to be bounded, unbounded, or even to vanish on $[-\pi, \pi]$, thereby including important cases such as the spectral functions of fractional processes. Error rates are also given in the case in which the matrix product involves inverse matrices. The rates are sharp up to an arbitrarily small $\varepsilon > 0$. The results improve on the $o(1)$ rates obtained in earlier work for analogous products. For the $p = 1$ case, an explicit second order asymptotic expansion is found for a quadratic functional of the autocovariance sequences of stationary long memory time series. The order of magnitude of the second term in this expansion is shown to depend on the long memory parameters. It is demonstrated that the pole in the first order approximation is removed by the second order term, which provides a substantially improved approximation to the original functional.

Key words and phrases: Asymptotic expansion, higher cumulants, long memory, singularity, spectral density, Toeplitz matrix.

1 Introduction

Let $f(x)$ and $g(x)$ be integrable real symmetric functions on $[-\pi, \pi]$. Let R_n and A_n be $n \times n$ Toeplitz matrices with entries $(R_n)_{j,k} = r_{|j-k|}$ and $(A_n)_{j,k} = a_{|j-k|}$, satisfying

$$\begin{aligned} r_n &= \int_{-\pi}^{\pi} e^{inx} f(x) dx \\ a_n &= \int_{-\pi}^{\pi} e^{inx} g(x) dx. \end{aligned}$$

Let p be a fixed, arbitrary and positive integer. Define

$$S_{n,p} = \frac{1}{n} tr(R_n A_n)^p,$$

$$L_p = (2\pi)^{2p-1} \int_{-\pi}^{\pi} \{f(x)g(x)\}^p dx \tag{1}$$

and set

$$\Delta_{n,p} = |S_{n,p} - L_p| \,. \tag{2}$$

The problem of bounding quantities of the form of (2) has a long history in the literature, dating back at least to Grenander and Szego (1951). When $f(x)$ and $g(x)$ are consistent with the conditions

$$\sum_{j=-\infty}^{\infty} |a_j||j| < \infty \tag{3}$$

and

$$\sum_{j=-\infty}^{\infty} |r_j||j| < \infty, \tag{4}$$

Taniguchi (1983) proved that $\Delta_{n,p} = O\left(n^{-1}\right)$. The conditions (3)-(4) hold, for instance, when $f(x)$ and $g(x)$ are spectral densities of short memory processes, such as those associated with ARMA models. These conditions do not hold for long-memory processes. In that case, when $f(x)$ and $g(x)$ satisfy

$$|f(x)| = O(|x|^{-\alpha}) \text{ as } |x| \to 0; \alpha < 1 \tag{5}$$

and

$$|g(x)| = O(|x|^{-\beta}) \text{ as } |x| \to 0; \beta < 1, \tag{6}$$

Fox and Taqqu (1987, henceforth FT) proved that under the condition

$$p\,(\alpha + \beta) < 1 \tag{7}$$

$\Delta_{n,p} = o\,(1)$. This bound is not sharp and we expect the error $\Delta_{n,p}$ to be dependent on the parameters α and β governing the singularity of f and g. The present paper shows this to be so. In particular, for $f(x)$ and $g(x)$ satisfying (5)-(6) and under condition (7)

$$\begin{aligned} \Delta_{n,p} &= O\left(n^{-1+p(\alpha+\beta)+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta>0, \\ \Delta_{n,p} &= O\left(n^{-1+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta\leq 0. \end{aligned} \tag{8}$$

In establishing the $o(1)$ rate for $\Delta_{n,p}$, FT use a probabilistic approach. Specifically, FT expressed $S_{n,p}$ as

$$S_{n,p} = \frac{1}{n}\int_{U_\pi} P_n(y)Q(y)dy, \tag{9}$$

where

$$\begin{aligned} U_\pi &= [-\pi,\pi]^{2p} \\ P_n(y) &= \sum_{j_1=0}^{n-1}\cdots\sum_{j_{2p}=0}^{n-1} e^{i(j_1-j_2)y_1}e^{i(j_2-j_3)y_2}\cdots e^{i(j_{2p}-j_1)y_{2p}} \\ Q(y) &= f(y_1)g(y_2)\cdots g(y_{2p}). \end{aligned} \tag{10}$$

Defining

$$\mu_n(E) = \frac{1}{(2\pi)^{2p-1}n}\int_E P_n(y)\,dy, E\subset U_\pi, \tag{11}$$

FT showed that for any set $E\subset U_\pi$ on which $Q(y)$ is bounded,

$$\lim_{n\to\infty}\int_E Q(y)\,d\mu_n(y) = \int_E Q(y)\,d\mu(y),$$

where μ is a Lebesgue measure on U_π, concentrated on the diagonal D of U_π, and D is defined as $D=\{y\in\mathbb{R}^{2p}: y_1=y_2=\cdots=y_{2p}\}$. The use of this weak convergence argument is imaginative but has the limitation that the best that can be achieved is an $o(1)$ rate.

The present work makes use of some of FT's results, particularly the power counting theory, but keeps the original algebraic form of the problem. In doing so, we are able to obtain the $O\left(n^{-1+p(\alpha+\beta)+\varepsilon}\right)$ rate given in (8), which is sharp up to the n^ε-factor. When the product involves inverse matrices such as

$$SI_{n,p} = \frac{1}{n}tr\left\{\prod_{j=1}^{p}\left(R_n(f_j)^{-1}R_n(g_j)\right)\right\},$$

with f_j and g_j satisfying (5)-(6), Dahlhaus (1989) used the Whittle approximation to $R_n(f)^{-1}$ in conjunction with Theorem 1 of FT to show that the quantity

$$\Delta I_{n,p} = \left|SI_{n,p} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\prod_{j=1}^{p}\frac{g_j(x)}{f_j(x)}\right\}dx\right|$$

is $o(1)$, under the condition[1] $p(\beta-\alpha)<1$. The present paper shows that

$$\Delta I_{n,p}=O\left(n^{\max\{-1/2+\varepsilon,-1+p(\beta-\alpha)+\varepsilon\}}\right),\forall\varepsilon>0,\text{ if }\beta-\alpha>0$$

and

$$\Delta I_{n,p}=O\left(n^{-1/2+\varepsilon}\right),\forall\varepsilon>0,\text{ if }\beta-\alpha\leq 0.$$

The aforementioned results are particularly useful in the context of estimation of spectral density parameters of stationary Gaussian processes, possibly with unbounded or vanishing spectra at the origin. Such cases arise in the study of long-memory and anti-persistent time series. In this context, cumulants of log-likelihood derivatives can be shown to be finite sums of terms of the form $SI_{n,p}$ or $S_{n,p}$ and the bound on the error of their integral limits is useful in studies of high-order theory for maximum likelihood estimators (MLE's), on which some recent work has been done by Lieberman et al. (2002), Lieberman and Phillips (2001) and Andrews and Lieberman (2002).

In addition to the new error order results, for the $p=1$ case we derive explicit second-order expansions to $S_{n,1}$. The expansions reveal how the asymptotic integral formula (1) breaks down as an approximation. In particular, the integral limit formula has a singularity at the boundary of the parameter space and the second order term in the asymptotic expansion removes this singularity in the limiting integral approximation, thereby leading to a substantially improved approximation.

The plan for the remainder of the paper is as follows. In Section 2 we establish an $O(n^{-1+\varepsilon})$ order for $\Delta_{n,p},\forall\varepsilon>0$, in the case where f and g are continuously differentiable. This rate is inferior to Taniguchi's (1983) direct $O(n^{-1})$ rate but is obtained under somewhat weaker conditions. The proof uses the power counting theory discussed by FT and is a building block for the case in which f and g satisfy (5)-(6). The latter is treated in Section 3 and the results are applied to obtain bounds on Toeplitz products of the form of $SI_{n,p}$. Section 5 derives an explicit second order expansion to the product trace in the $p=1$ case and demonstrates the existence of a removable singularity in the integral limit. A numerical illustration is provided. Section 6 gives some examples and Section 7 concludes.

2 Bounds in the Short Memory Case

This section provides a bound on $\Delta_{n,p}$ in the short memory case. The following result (see Taniguchi and Kakizawa, 2000) is due to Taniguchi (1983).

Lemma 1 *Under conditions (3)-(4),* $\Delta_{n,p}=O(n^{-1})$.

Theorem 2 below gives a related result that is not as sharp but which holds under the following condition for which (3) and (4) are clearly sufficient.

[1] Dahlhaus states the condition $p(\beta-\alpha)<\frac{1}{2}$ but it appears that $p(\beta-\alpha)<1$ is needed.

A1 *There exist constants* K_i, *with* $0 < K_i < \infty, i = 1, 2, 3, 4,$ *such that*

$$\sup_{x\in[-\pi,\pi]} |f(x)| < K_1, \sup_{x\in[-\pi,\pi]} |g(x)| < K_2, \sup_{x\in[-\pi,\pi]} |f'(x)| < K_3, \sup_{x\in[-\pi,\pi]} |g'(x)| < K_4.$$

Theorem 2 *Under A1,* $\Delta_{n,p} = O\left(n^{-1+\varepsilon}\right), \forall\varepsilon > 0.$

The proof of Theorem 2 provides a step toward the result given in the next section on the bound in the long-memory case. The proofs of Theorems 2 and 5 make use of power counting theory and it is helpful to adopt the notation of FT. Consider the function $P : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$, defined as

$$P(x) = |L_1(x)|^{b_1} \cdots |L_m(x)|^{b_m},$$

where $L_j(x) = M_j(x) + \theta_j, j = 1, ..., m,$ the b_j's and θ_j's are real constants and the M_j's are linear functionals on $\mathbb{R}^n$. Define $T = \{L_1, ..., L_m\}$. For any $W \subset T$, let $s(W) = T \cap span\{W\}$ and define $d(P, W) = |W| + \sum_{\{j:L_j\in s(W)\}} b_j$. A set $W = \{L_{i_1}, ..., L_{i_k}\}$ is said to be strongly independent if $M_{i_1}, ..., M_{i_k}$ are linearly independent. Denote by S the set of L_j's in T for which all $b_j > 0$. Let $U_t = \{x \in \mathbb{R}^n : -t \leq x_i \leq t, i = 1, ..., n\}$. We use the following result (Theorem 3.1 of FT).

Lemma 3 *Suppose that* $d(P, W) > 0$ *for every strongly independent set* $W \subset S$. *Then* $\int_{U_t} P(x)\, dx < \infty, \forall t > 0$.

Proof of Theorem 2 In what follows K denotes a generic positive constant. Fixing $p \geq 1$, it is easy to see from (10) that

$$\frac{1}{(2\pi)^{2p-1}n} \int_{[-\pi,\pi]^{2p-1}} P_n(y)dy_2 \cdots dy_{2p} = 1.$$

Hence,

$$(2\pi)^{2p-1} \int_{-\pi}^{\pi} \{f(y)g(y)\}^p dy = \frac{1}{n} \int_{U_\pi} P_n(y)Q(y_1, \cdots, y_1)dy. \tag{12}$$

>From (1), (2), (9) and (12),

$$\begin{aligned}\Delta_{n,p} &= \frac{1}{n} \left| \int_{U_\pi} P_n(y)\{Q(y) - Q(y_1, \ldots, y_1)\}dy \right| \\ &\leq \frac{1}{n} \int_{U_\pi} |P_n(y)\{Q(y) - Q(y_1, \ldots, y_1)\}|\, dy.\end{aligned}$$

Now, (10) can be rearranged as

$$\begin{aligned}P_n(y) &= \left(\sum_{j_1=0}^{n-1} e^{i(y_1-y_{2p})j_1} \right) \left(\sum_{j_2=0}^{n-1} e^{i(y_2-y_1)j_2} \right) \cdots \left(\sum_{j_{2p}=0}^{n-1} e^{i(y_{2p}-y_{2p-1})j_{2p}} \right) \\ &= h_n^*(y_1 - y_{2p})h_n^*(y_2 - y_1) \cdots h_n^*(y_{2p} - y_{2p-1}),\end{aligned} \tag{13}$$

say. It is easily shown (FT, p. 237) that on $[-2\pi, 2\pi]$

$$|h_n^*(z)| \leq 4h_n(z), \tag{14}$$

where

$$h_n(z) = \min(|z + \delta(z)|^{-1}, n),$$

and $\delta(z)$ is the alternator

$$\delta(z) = \begin{cases} 2\pi, \text{ if } -2\pi \leq z < -\pi \\ 0, \text{ if } -\pi \leq z < \pi \\ -2\pi, \text{ if } \pi \leq z \leq 2\pi \end{cases}. \tag{15}$$

For any $\eta \in (0,1)$ we also have

$$h_n(z) \leq h_{n,\eta}(z) \equiv n^\eta |z + \delta(z)|^{\eta-1}, -2\pi \leq z \leq 2\pi. \tag{16}$$

Under Assumption A1 we can expand $Q(y)$ around $y = (y_1, \ldots, y_1)$ as

$$Q(y) - Q(y_1, \ldots, y_1) = \sum_{j=2}^{2p} \frac{\partial Q(\tilde{y})}{\partial y_j}(y_j - y_1), \tag{17}$$

where $\tilde{y} = (y_1, c_2, \ldots, c_{2p})$ and $|c_j - y_1| \leq |y_j - y_1|$, $j = 2, \ldots, 2p$. Further,

$$|y_j - y_1| \leq \sum_{k=2}^{j} |y_k - y_{k-1}|. \tag{18}$$

Note that U_π is a finite union of intersections of sets of the form

$$\begin{aligned} R_{jk} &= \left\{y \in \mathbb{R}^{2p} : y_j - y_{j-1} \in a_{jk}\right\} \cap U_\pi, j = 2, ..., 2p, \\ R_{1k} &= \left\{y \in \mathbb{R}^{2p} : y_{2p} - y_1 \in a_{1k}\right\} \cap U_\pi, \\ a_{j1} &= [-2\pi, -\pi), a_{j2} = [-\pi, \pi), a_{j3} = [\pi, 2\pi], j = 1, ..., 2p. \end{aligned} \tag{19}$$

It follows from (13)-(18) that

$$\begin{aligned} \Delta_{n,p} \leq & \frac{K}{n^{1-2p\eta}} \sum \sum_{j=2}^{2p} \sum_{k=2}^{j} \int_{U_\pi} |y_k - y_{k-1}||y_1 - y_{2p} + \delta(y_1 - y_{2p})|^{\eta-1} \\ & \times \prod_{l=2}^{2p} |y_l - y_{l-1} + \delta(y_l - y_{l-1})|^{\eta-1} dy. \end{aligned}$$

where the outermost (finite) summation is over all possible configurations implied by (19). Make the change of variables

$$\begin{aligned} x_1 &= y_1 \\ x_l &= y_l - y_{l-1}, l = 2, \ldots, 2p \end{aligned}$$

and note that $y_{2p} - y_1 = \sum_{m=2}^{2p} x_m$. The Jacobian of transformation is unity and the transformed integration region is

$$U'_\pi = \left\{ x \in \mathbb{R}^{2p} : -\pi \leq x_1 \leq \pi, -\pi \leq x_1 + x_2 \leq \pi, \ldots, -\pi \leq \sum_{i=1}^{2p} x_i \leq \pi. \right\}.$$

The last integral becomes

$$\frac{K}{n^{1-2p\eta}} \sum \sum_{j=2}^{2p} \sum_{k=2}^{j} \int_{U'_\pi} |x_k| \left| -\sum_{m=2}^{2p} x_m + \delta_1 \right|^{\eta-1} \prod_{l=2}^{2p} |x_l + \delta_l|^{\eta-1} dx. \tag{20}$$

Let

$$U_{2p\pi} = \left\{ x \in \mathbb{R}^{2p} : -2p\pi \leq x_i \leq 2p\pi, i = 1, \ldots, 2p \right\}.$$

It is clear that $U'_\pi \subset U_{2p\pi}$, hence (20) is at most

$$\frac{K}{n^{1-2p\eta}} \sum \sum_{j=2}^{2p} \sum_{k=2}^{j} \int_{U_{2p\pi}} |x_k| \left| -\sum_{m=2}^{2p} x_m + \delta_1 \right|^{\eta-1} \prod_{l=2}^{2p} |x_l + \delta_l|^{\eta-1} dx. \tag{21}$$

For the k in (21), we set

$$\begin{aligned} M_1 &= -\sum_{m=2}^{2p} x_m, M_j = x_j, j = 2, ..., 2p, M_{2p+1} = x_k, \\ L_1 &= M_j + \delta_j, j = 1, ..., 2p, L_{2p+1} = M_{2p+1}. \end{aligned}$$

In this case $T = \{L_1, \ldots L_{2p+1}\}$ and $S = T \backslash \{L_{2p+1}\}$. For any set $W = \{L_{\tau_1}, ..., L_{\tau_r}\}$ with $\{\tau_1, ..., \tau_r\} \subset \{1, ..., 2p\} \setminus \{k\}$ and $r < 2p - 1$,

$$d(p, W) = r + r(\eta - 1) = r\eta > 0.$$

For any W including either L_k or L_{2p+1} and with $|W| = r < 2p - 1$, $s(W)$ includes both L_k and L_{2p+1} and so

$$d(p, W) = r + r(\eta - 1) + 1 = r\eta + 1 > 0.$$

Consider now a set W including either L_k or L_{2p+1} and with $|W| = 2p - 1$. Here $S(W) = T$ and

$$d(p, W) = 2p - 1 + 2p(\eta - 1) + 1 = 2p\eta > 0.$$

Note that positivity of $d(p, W)$ is assured in this case due to the unit exponent on $|x_k|$, a term arising due to the bound on $|Q(y) - Q(y_1, \ldots, y_1)|$. Finally, the case $W = S$ yields

$$d(p, W) = 2p + 2p(\eta - 1) + 1 = 2p\eta + 1 > 0.$$

By Lemma 3, the integral in (21) is finite. Thus, $\Delta_{n,p} \leq Kn^{-(1-2p\eta)}$. Since we may choose η arbitrarily close to zero, $\Delta_{n,p} = O(n^{-1+\varepsilon}), \forall\varepsilon > 0$. $\square$

3 Bounds in the Long Memory case

We make the following assumptions.

A2 *The functions* $f(x)$ *and* $g(x)$ *are symmetric, real valued, continuously differentiable at all* $x \neq 0$ *and there exist* $0 < c_1 < \infty$, $0 < c_2 < \infty$ *such that*

$$\begin{aligned} |f(x)| &\leq c_1 |x|^{-\alpha}, \alpha < 1, \\ |g(x)| &\leq c_2 |x|^{-\beta}, \beta < 1, \end{aligned}$$

$\forall x \in [-\pi, \pi]$.

A3 *For all* $t > 0, \exists M_{t1}$ *and* M_{t2} *such that*

$$\sup_{|x|>t} |f'(x)| \leq M_{t1} \text{ and } \sup_{|x|>t} |g'(x)| \leq M_{t2}.$$

Lemma 4 *(Theorem 1, FT) Under A2 and the condition* $p(\alpha + \beta) < 1$, $\lim_{n\to\infty} \Delta_{n,p} = 0$.

The proof of Lemma 4 in FT relies on a weak convergence argument. The approach is to partition U_π into three disjoint sets, E_t, F_t, G, satisfying

$$\begin{aligned} E_t &= U_\pi \setminus \{W \cup U_t\} \\ F_t &= U_t \setminus W \\ G &= U_\pi \cap W, \end{aligned} \tag{22}$$

where

$$\begin{aligned} W &= U_{i=1}^{2p} W_i, \\ W_i &= \left\{ y \in \mathbb{R}^{2p} : |y_i| \leq \frac{|y_{i+1}|}{2} \right\}, i = 1, \ldots, 2p-1, \\ W_{2p} &= \left\{ y \in \mathbb{R}^{2p} : |y_{2p}| \leq \frac{|y_1|}{2} \right\} \end{aligned}$$

and

$$U_t = [-t, t]^{2p}, 0 < t \leq \pi.$$

Then, the derivation proceeds as follows: (i) $Q(y)$ is shown to be bounded on E_t. (ii) μ_n, defined in (11), is shown to converge weakly to Lebesgue measure μ concentrated on the diagonal D of U_π. (iii) Weak convergence establishes that

$$\lim_{n\to\infty} \int_{E_t} Q(y)\, d\mu_n(y) = \int_{E_t} Q(y)\, d\mu(y) = \int_{t \leq |y| \leq \pi} \{f(y)\, g(y)\}^p\, dy, 0 < t \leq 1. \tag{23}$$

(iv) The proof is completed by verifying that

$$\lim_{t\to 0}\limsup_{n\to\infty}\frac{1}{n}\int_{F_t}P_n(y)Q(y)dy=0$$

and

$$\lim_{n\to\infty}\int_G P_n(y)Q(y)dy=0.$$

The crux of the argument is (23) and relies on weak convergence, leading to the $o(1)$ convergence of Lemma 4.

The following result uses an algebraic rather than probabilistic approach to provide more explicit information on the convergence rate.

Theorem 5 *Under A2-A3 and the condition* $p(\alpha+\beta)<1$,

$$\begin{aligned}\Delta_{n,p} &= O\left(n^{-1+p(\alpha+\beta)+\varepsilon}\right), \forall\varepsilon>0,\ \textit{if}\ \alpha+\beta>0,\\ \Delta_{n,p} &= O\left(n^{-1+\varepsilon}\right), \forall\varepsilon>0,\ \textit{if}\ \alpha+\beta\le 0.\end{aligned}$$

Proof Since $U_\pi=E_t\cup F_t\cup G$,

$$\begin{aligned}\Delta_{n,p} &= \frac{1}{n}\left|\int_{E_t\cup F_t\cup G}P_n(y)\{Q(y)-Q(y_1,\ldots,y_1)\}dy\right|\\ &\le \int_{E_t}|P_n(y)\{Q(y)-Q(y_1,\ldots,y_1)\}|dy\\ &\quad+\frac{1}{n}\int_{F_t}|P_n(y)Q(y)|dy\\ &\quad+\frac{1}{n}\int_{F_t}|P_n(y)Q(y_1,\ldots,y_1)|dy\\ &\quad+\frac{1}{n}\int_G|P_n(y)Q(y)|dy\\ &\quad+\frac{1}{n}\int_G|P_n(y)Q(y_1,\ldots,y_1)|dy\ .\end{aligned}\tag{24}$$

Note that from p. 237 of FT, $|y_j|>t/2^{2p-1}, j=1,\ldots,2p$, on E_t. Hence, by Assumptions A2-A3, Q and $\partial Q/\partial y_j$ are bounded on $E_t, j=1,\ldots,2p$. Thus, the first integral on the rhs of (24) is less than or equal to (21), which is $O(n^{-1+\varepsilon})$, $\forall\,\varepsilon>0$. To deal with the second integral, note that from (22) $F_t=U_t\cap W^c$ and that on W^c,

$$|y_1|>\frac{|y_2|}{2}>\frac{|y_3|}{4}>\cdots>\frac{|y_{2p}|}{2^{2p-1}}>\frac{|y_1|}{2^{2p}}.$$

Under Assumption A2,

$$|Q(y)|<K|y_1|^{-\alpha}|y_2|^{-\beta}\cdots|y_{2p}|^{-\beta}$$

which is less than or equal to $K|y_1|^{-p(\alpha+\beta)}$ on W^c. Thus

$$\frac{1}{n}\int_{F_t}|P_n(y)Q(y)|dy \le \frac{K}{n}\int_{U_t\cap W^c}|y_1|^{-p(\alpha+\beta)}|P_n(y)|dy. \tag{25}$$

Make the change of variables

$$y_i = \prod_{k=1}^{i} z_k, i = 1, ..., 2p. \tag{26}$$

The transformed integration range is $U_t' \cap W^{c\prime}$, where

$$U_t' = \left\{ z \in \mathbb{R}^{2p} : -t \le z_1 \le t, -t \le z_1 z_2 \le t, ..., -t \le \prod_{i=1}^{2p} z_i \le t. \right\}$$

and

$$W^{c\prime} = \left\{ z \in \mathbb{R}^{2p} : |z_1| > \frac{1}{2}|z_1 z_2| > \frac{1}{2^{2p-1}}\left|\prod_{i=1}^{2p} z_i\right| > \frac{1}{2^{2p}}|z_1|. \right\}$$

The Jacobian of transformation is $|z_1||z_1z_2|\cdots|z_1\cdots z_{2p-1}|$, which is at most $K|z_1|^{2p-1}$ on $W^{c\prime}$. We can choose t sufficiently small such that $\delta(z) = 0$ for all possible z. Using (13)-(16), (25) is less than or equal to

$$\begin{aligned}
& \frac{K}{n^{1-2p\eta}}\int_{(U_t\cap W^c)'}|z_1|^{2p-1-p(\alpha+\beta)}\left|\prod_{i=1}^{2p} z_i - z_1\right|^{\eta-1}|z_1z_2 - z_1|^{\eta-1} \\
& \times |z_1z_2z_3 - z_1z_2|^{\eta-1}\cdots\left|\prod_{i=1}^{2p} z_i - \prod_{i=1}^{2p-1} z_i\right|^{\eta-1} dz \\
\le\ & \frac{K}{n^{1-2p\eta}}\int_{(U_t\cap W^c)'}|z_1|^{2p-1-p(\alpha+\beta)}|z_1|^{\eta-1}\left|\prod_{i=2}^{2p} z_i - 1\right|^{\eta-1}|z_1|^{\eta-1}|z_2-1|^{\eta-1} \\
& \times |z_1z_2|^{\eta-1}|z_3-1|^{\eta-1}\cdots\left|\prod_{i=2}^{2p-1} z_i\right|^{\eta-1}|z_{2p}-1|^{\eta-1}dz \\
\le\ & \frac{K}{n^{1-2p\eta}}\int_{(U_t\cap W^c)'}|z_1|^{2p-1-p(\alpha+\beta)+2p(\eta-1)}\left|\prod_{i=2}^{2p} z_i - 1\right|^{\eta-1} \\
& \times |z_2-1|^{\eta-1}\cdots|z_{2p}-1|^{\eta-1}dz \\
=\ & \frac{K}{n^{1-2p\eta}}\int_{(U_t\cap W^c)'}|z_1|^{-1-p(\alpha+\beta)+2p\eta}\left|\prod_{i=2}^{2p} z_i - 1\right|^{\eta-1} \qquad (27)\\
& \times |z_2-1|^{\eta-1}\cdots|z_{2p}-1|^{\eta-1}dz,
\end{aligned}$$

the last inequality following from the bounds implied by $W^{c'}$. This integral is clearly finite under the conditions $-1-p(\alpha+\beta)+2p\eta>-1$ and $1>\eta>0$, i.e.,

$$2p\eta > p\left(\alpha+\beta\right) \text{ and } 1>\eta>0. \tag{28}$$

It follows from (27)-(28) that

$$\frac{1}{n}\int_{F_t}|P_n(y)Q(y)|dy \leq Kn^{-1+p(\alpha+\beta)+\varepsilon}, \forall\varepsilon>0, \text{ if } \alpha+\beta>0 \text{ and } p\left(\alpha+\beta\right)<1$$

and

$$\frac{1}{n}\int_{F_t}|P_n(y)Q(y)|dy \leq Kn^{-1+\varepsilon}, \forall\,\varepsilon>0, \text{ if } \alpha+\beta\leq 0.$$

The third integral on the rhs of (24) is handled in an analogous way. To deal with the fourth integral, we note that the order of the integral will follow from the order of

$$\frac{1}{n}\int_{U_\pi\cap W_1}|P_n(y)Q(y)|dy\,,$$

see pp. 236–237 of FT. By Proposition 6.1 of FT, if $\alpha+\beta>0$, then

$$\frac{1}{n}\int_{U_\pi\cap W_1}|P_n(y)Q(y)|dy = O\left(n^{-1+p(\alpha+\beta)+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta>0$$

and

$$\frac{1}{n}\int_{U_\pi\cap W_1}|P_n(y)Q(y)|dy = O\left(n^{-1+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta\leq 0.$$

A similar result follows for the last integral in (24). Thus, under Assumptions A2-A3 and the condition $p\left(\alpha+\beta\right)<1$, we obtain

$$\begin{aligned}\Delta_{n,p} &= O\left(n^{-1+p(\alpha+\beta)+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta>0,\\ \Delta_{n,p} &= O\left(n^{-1+\varepsilon}\right), \forall\varepsilon>0, \text{ if } \alpha+\beta\leq 0.\end{aligned}$$

□

4 Products involving Matrix Inverses

Next, we provide error bounds on limiting approximations of traces of matrix products of the form $SI_{n,p}$, where f_j and g_j satisfy A2-A3, with exponents α and β, respectively. The following limit result is due to Dahlhaus (1989, Theorem 5.1).

Lemma 6 *For f_j and g_j satisfying A2-A3 with exponents α and β, and under the condition $p\left(\alpha-\beta\right)<1$, $\lim_{n\to\infty}\Delta I_{n,p}=0$.*

This lemma is useful in finding the order of magnitude of cumulants of Gaussian log-likelihood derivatives and is an important tool in the development of asymptotic expansions for the Gaussian MLE for fractional processes - see Lieberman, Rousseau and Zucker (2002). In proving Lemma 6, Dahlhaus used two main arguments: (i) the Whittle approximation to $R_n(f)^{-1}$, viz., $R_n\left(\{4\pi^2 f\}^{-1}\right)$; and (ii) FT's Theorem 1(a). The following theorem improves on this result by providing an explicit rate for the convergence of $\Delta I_{n,p}$.

Theorem 7 *For f_j and g_j satisfying Assumptions A2-A3 with exponents $\alpha, \beta < 1$, and under the condition $p(\alpha - \beta) < 1$,*

$$\Delta I_{n,p} = O\left(n^{\max\{-1/2+\varepsilon, -1+p(\beta-\alpha)+\varepsilon\}}\right), \forall \varepsilon > 0, \text{ if } \beta - \alpha > 0$$

and

$$\Delta I_{n,p} = O\left(n^{-1/2+\varepsilon}\right), \forall \varepsilon > 0, \text{ if } \beta - \alpha \leq 0.$$

Proof By the triangle inequality,

$$\begin{aligned} \Delta I_{n,p} \leq\ & \left| \frac{1}{n} tr \left\{ \prod_{j=1}^{p} \left(R_n(f_j)^{-1} R_n(g_j) \right) \right\} - \frac{1}{n} tr \left\{ \prod_{j=1}^{p} \left(R_n\left(\left(4\pi^2 f_j\right)^{-1} \right) R_n(g_j) \right) \right\} \right| \\ & + \left| \frac{1}{n} tr \left\{ \prod_{j=1}^{p} \left(R_n\left(\left(4\pi^2 f_j\right)^{-1} \right) R_n(g_j) \right) \right\} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \prod_{j=1}^{p} \frac{g_j(x)}{f_j(x)} \right\} dx \right|. \quad (29) \end{aligned}$$

It is clear from the proof of Theorem 5.1 of Dahlhaus (1989) that the first term of (29) is at most $Kn^{-1/2+\varepsilon}, \forall \varepsilon > 0$. To deal with the second term, we note that by Theorem 5, if $p(\beta - \alpha) < 1$ and $\beta - \alpha > 0$, the second member is at most $Kn^{-1+p(\beta-\alpha)+\varepsilon}, \forall \varepsilon > 0$. If $p(\beta - \alpha) < 1$ and $\beta - \alpha \leq 0$, the second member is at most $Kn^{-1+\varepsilon}$, and the stated result follows. □

5 Second Order Expansions for $S_{n,1}$

This section provides an explicit second order expansion for $S_{n,p}$ when $p = 1$. The main result in this section is as follows.

Theorem 8 *Let $f(x)$ and $g(x)$ be the spectral density functions of two long memory times series defined as*

$$f(x) = \frac{\sigma^2/2\pi}{|1-e^{ix}|^{2d_r}}, g(x) = \frac{\sigma^2/2\pi}{|1-e^{ix}|^{2d_a}}, \text{ with } d_r, d_a \in [0, \frac{1}{2}).$$

Under the condition $d_r + d_a < \frac{1}{2}$,

$$\begin{aligned} S_{n,1} &= \frac{1}{n} tr\left[R_n(f) A_n(g))\right] \\ &= 2\pi \int_{-\pi}^{\pi} f(x) g(x) dx - \frac{a(d_r, d_a)}{n^{1-2d_r-2d_a}} + o\left(\frac{1}{n^{1-2d_r-2d_a}}\right), \end{aligned} \tag{30}$$

where

$$a(d_r, d_a) = \frac{2\sigma^4}{\pi^2} \sin(\pi d_r) \sin(\pi d_a) \Gamma(1-2d_r) \Gamma(1-2d_a) \left[\frac{1}{1-2d_r-2d_a} + \frac{1}{d_r+d_a}\right].$$

Proof Observe that

$$\begin{aligned} S_{n,1} &= \sum_{h=-n+1}^{n-1} r_h a_{-h} w_h, \text{ for } w_h = 1 - \frac{|h|}{n} \\ &= \sum_{h=-\infty}^{\infty} r_h a_{-h} - \sum_{|h| \geq n} r_h a_{-h} - \frac{1}{n} \sum_{h=-n+1}^{n-1} |h| r_h a_{-h} \\ &= \int_{-\pi}^{\pi} \left[\sum_{h=-\infty}^{\infty} r_h e^{-ixh}\right] g(x) dx - \sum_{|h| \geq n} r_h a_{-h} - \frac{1}{n} \sum_{h=-n+1}^{n-1} |h| r_h a_{-h} \\ &= 2\pi \int_{-\pi}^{\pi} f(x) g(x) dx - I_A - I_B, \text{ say} \end{aligned} \tag{31}$$

The autocovariance r_h is

$$\begin{aligned} r_h &= \frac{(-1)^h \Gamma(1-2d_r) \sigma^2}{\Gamma(h-d_r+1) \Gamma(1-h-d_r)} \\ &= \frac{(-1)^h \sigma^2 \Gamma(1-2d_r) \sin[\pi(h+d_r)] \Gamma(h+d_r)}{\pi \Gamma(h-d_r+1)} \\ &= \frac{\sigma^2 \Gamma(1-2d_r) \sin[\pi d_r] \Gamma(h+d_r)}{\pi \Gamma(h-d_r+1)}, \end{aligned}$$

which uses the reflection formula $\Gamma(1-z)\Gamma(z) = \pi / \sin(\pi z)$. From the asymptotic expansion of the gamma function for large $h > 0$, we have

$$r_h = \frac{\sigma^2 \Gamma(1-2d_r) \sin[\pi d_r]}{\pi h^{1-2d_r}} \left[1 + O\left(\frac{1}{h}\right)\right].$$

Hence, for $d_r + d_a < \frac{1}{2}$,

$$I_A = \sum_{|h| \geq n} r_h a_{-h} = 2 \sum_{h \geq n} r_h a_h$$

$$
\begin{aligned}
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)}{\pi^2}\sum_{h\geq n}\frac{\sin[\pi d_r]\sin[\pi d_a]}{h^{2-2d_r-2d_a}}\left[1+O\left(\frac{1}{h}\right)\right] \\
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)}{\pi^2}\sin[\pi d_a]\sin[\pi d_a]\int_n^\infty \frac{1}{h^{2-2d_r-2d_a}}dh\,[1+o(1)] \\
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)}{\pi^2}\frac{\sin[\pi d_a]\sin[\pi d_a]}{n^{1-2d_r-2d_a}(1-2d_r-2d_a)}[1+o(1)], \qquad (32)
\end{aligned}
$$

as $n\to\infty$.

Next, take any integer $L>1$ such that $\frac{1}{L}+\frac{L}{n}\to 0$ and write

$$
\begin{aligned}
I_B &= \frac{1}{n}\sum_{h=-n+1}^{n-1}|h|\,r_h a_{-h} = \frac{1}{n}\sum_{|h|<n}|h|\,r_h a_h = \frac{2}{n}\sum_{h=1}^{n} h r_h a_h \\
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)\sin[\pi d_r]\sin[\pi d_a]}{n\pi^2}\left[\sum_{h=L}^{n}\frac{h}{h^{2-2d_r-2d_a}}\left[1+O\left(\frac{1}{h}\right)\right]\right. \\
&\quad \left. +\sum_{h=1}^{L-1}\frac{h\Gamma(h+d_r)\Gamma(h+d_a)}{\Gamma(h-d_r+1)\Gamma(h-d_a+1)}\right] \\
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)\sin[\pi d_r]\sin[\pi d_a]}{n\pi^2}\left[\frac{n^{2d_r+2d_a}}{2d_r+2d_a}+O\left(L^{2d_r+2d_a}\right)\right] \\
&= \frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)\sin[\pi d_r]\sin[\pi d_a]}{\pi^2 n^{1-2d_r-2d_a}(2d_r+2d_a)}[1+o(1)]. \qquad (33)
\end{aligned}
$$

We deduce from (31), (32) and (33) that

$$
\begin{aligned}
S_{n,1} &= 2\pi\int_{-\pi}^{\pi} f(x)\,g(x)\,dx \\
&\quad -\frac{2\sigma^4\Gamma(1-2d_r)\Gamma(1-2d_a)}{\pi^2}\frac{\sin[\pi d_r]\sin[\pi d_a]}{n^{1-2d_r-2d_a}}\left[\frac{1}{1-2d_r-2d_a}+\frac{1}{2d_r+2d_a}\right] \\
&\quad +o\left(\frac{1}{n^{1-2d_r-2d_a}}\right),
\end{aligned}
$$

giving the stated second order approximation. □

Remarks

(a) Fig. 1 shows the computed values of the asymptotic form, the second-order expansion (30), and the exact value of $S_{n,1}$, for a range of $d=d_r+d_a$ with $d_r=d_a$ over $d\in[0,0.5)$ and with $\sigma^2=1$. Observe that the asymptotic form is undefined when $d_r=d_a=\frac{1}{4}$, and the adequacy of the asymptotic approximation deteriorates rapidly as $d\to\frac{1}{2}$. On the other hand, while the coefficient $a(d_r,d_a)$ in the second term of the approximation (30) also becomes infinitely large as

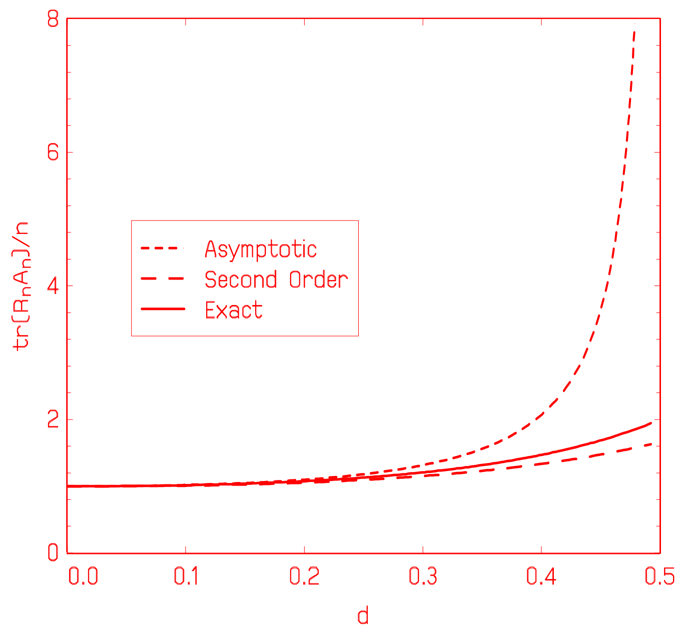

Figure 1: Exact Value and Approximations to $\frac{1}{n} tr\left[R_n A_n\right]$ for $n = 50$

$d_r, d_a \longrightarrow \frac{1}{4}$, the second order approximation (30) appears to do well over the full range of values, showing only a small deterioration as $d \longrightarrow \frac{1}{2}$. There is some tendency for the second order term to overcorrect for the large error in the asymptotic approximation as $d \longrightarrow \frac{1}{2}$. The exact value of $S_{n,1}$ remains bounded over the domain $d_r, d_a \in [0, \frac{1}{4}]$ for finite n.

(b) We analyze the behavior of the approximations as $d_r, d_a \longrightarrow \frac{1}{4}$. Setting $\sigma^2 = 1$, the first-order asymptotic formula has a pole when $d = d_r + d_a = \frac{1}{2}$. In particular,

$$2\pi \int_{-\pi}^{\pi} f(x) g(x)\, dx = \frac{\sigma^4}{2\pi} \int_{-\pi}^{\pi} \left|1 - e^{ix}\right|^{-2d} dx = \sigma^4 \frac{\Gamma(1-2d)}{\Gamma(1-d)^2},$$

the variance of a time series with long memory parameter d. Then, using the Laurent expansion of the gamma function $\Gamma(1-2d)$ around its pole at $d = \frac{1}{2}$, we have

$$\Gamma(1-2d) = \frac{1}{1-2d} + \psi(1) + \frac{1}{2}(1-2d)\left[\frac{\pi^2}{3} + \psi^2(1) - \psi'(1)\right] + O\left((1-2d)^2\right), \tag{34}$$

where ψ is the psi function and $\psi(1) = -\gamma$, Euler's constant. It follows that

$$2\pi \int_{-\pi}^{\pi} f(x) g(x)\, dx = \frac{\sigma^4}{\pi(1-2d)} + O(1), \tag{35}$$

as $d \to \frac{1}{2}$. On the other hand, the asymptotic behavior of the second order term $a(d_r, d_a)$ as $d_r, d_a \to \frac{1}{4}$ is readily seen to be

$$a(d_r, d_a) = \frac{2\sigma^4\pi}{\pi^2}\sin^2\left[\frac{\pi}{4}\right]\left[\frac{1}{1-2d}+1+O(1)\right] = \frac{\sigma^4}{\pi(1-2d)}+O(1).$$

Thus, the pole in the first order approximation is removed by the second order term, so that the approximation

$$2\pi\int_{-\pi}^{\pi} f(x)\,g(x)\,dx - \frac{a(d_r, d_a)}{n^{1-2d_r-2d_a}}$$

is bounded as $d_r, d_a \to \frac{1}{4}$. This good behavior explains why the second order approximation produces a good approximation that does uniformly well over $d_r, d_a \in [0, 0.25]$, including the limits of the domain.

(c) We may also develop asymptotics as $d \to \frac{1}{2}$ along different sequences for d_a and d_r. For instance, suppose $d = d_r + d_a \to \frac{1}{2}$ with $d_r \to \frac{1}{2}$ and $d_a \to 0$. Then the representation (35) for the first order asymptotic term clearly continues to apply. On the other hand, using (34) we have

$$\begin{aligned} a(d_r, d_a) &= \frac{2\sigma^4\pi d_a}{\pi^2}\sin\frac{\pi}{2}\left(\frac{1}{1-2d_r}\right)\left(\frac{1}{1-2d}\right)+O(1) \\ &= \frac{2\sigma^4\pi\left(\frac{1}{2}-d_r\right)}{\pi^2}\left(\frac{1}{1-2d_r}\right)\left(\frac{1}{1-2d}\right)+O(1) \\ &= \frac{\sigma^4}{\pi(1-2d)}+O(1), \end{aligned}$$

and second order equivalence continues to hold at the limit of the domain of d_r, d_a. More generally, along an arbitrary ray for which $d = d_r + d_a \to \frac{1}{2}$ we have

$$\begin{aligned} a(d_r, d_a) &= \frac{2\sigma^4\Gamma(2d_a)\Gamma(1-2d_a)}{\pi^2}\sin\left[\frac{\pi}{2}(1-2d_a)\right]\sin[\pi d_a]\left(\frac{1}{1-2d}\right)+O(1) \\ &= \frac{2\sigma^4\pi}{\pi^2\sin(2\pi d_a)}\sin\left[\frac{\pi}{2}(1-2d_a)\right]\sin[\pi d_a]\left(\frac{1}{1-2d}\right)+O(1) \\ &= \frac{\sigma^4\cos\left(\frac{\pi}{2}-2\pi d_a\right)}{\pi\sin(2\pi d_a)}\left(\frac{1}{1-2d}\right)+O(1) = \frac{\sigma^4}{\pi(1-2d)}+O(1), \end{aligned}$$

and again the equivalence holds.

(d) The second order approximation (30) is new. It reveals how the first order limit formula breaks down, shows that the second order term removes the singularity in the limit approximation and provides what appears to be a substantially improved approximation.

6 Applications

These results have applications to the distribution theory of Gaussian MLE's. For a zero mean, stationary, long-memory process with covariance matrix $\Sigma_n(f_\theta)$, the Gaussian likelihood is given by

$$L_n(\theta; x) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\log \det\Sigma_n(f_\theta) - \frac{1}{2}x_n'\Sigma_n^{-1}(f_\theta)x_n,$$

where $x_n = (X_1, ..., X_n)'$ and $f_\theta(\lambda)$ is the spectral density. The latter depends on a vector of unknown parameters $\theta \in \Theta \subset \mathbb{R}^m$, satisfying

$$f_\theta(\lambda) \sim |\lambda|^{-\alpha(\theta)} A_\theta(\lambda) \text{ as } \lambda \to 0,$$

with $0 < \alpha(\theta) < 1$ and $A_\theta(\lambda)$ slowly varying at 0. For a given set of subscripts $\nu = (r_1 \ldots r_q)$, denote the log likelihood derivative (LLD) of order q by $L_\nu = \partial^q L/\partial\theta_{r_1} \cdots \partial\theta_{r_q}$. LLD's have the form

$$L_\nu = x' B_\nu(\theta)x - F_\nu(\theta),$$

where

$$B_\nu(\theta) = -\frac{1}{2}\frac{\partial^q \Sigma_n^{-1}(f_\theta)}{\partial\theta_{r_1}\cdots\partial\theta_{r_q}} = \sum_{k=1}^{b_\nu} a_k \left[\prod_{j=1}^{p_k} \Sigma_n^{-1}(f_\theta)\Sigma_n(g_{\theta,j})\right] \Sigma_n^{-1}(f_\theta),$$

the a_k's are constants, the g_j's are derivatives of the spectral density with respect to θ and

$$F_\nu(\theta) = \sum_{k=1}^{b_\nu} a_k tr \left[\prod_{j=1}^{p_k} \Sigma_n^{-1}(f_\theta)\Sigma_n(g_{\theta,j})\right],$$

e.g., see Lieberman, Rousseau and Zucker (2002). The cumulants of the LLD's are finite sums of terms proportional to

$$tr\left[\prod_{j=1}^{p} \Sigma_n^{-1}(f_\theta)\Sigma_n(g_{\theta,j})\right].$$

For a long-memory process, the $g_{\theta,j}(\lambda)$ are $O\left(|\lambda|^{\alpha-\delta}\right), \forall\delta > 0$. Hence, by theorem 7

$$\left|\frac{1}{n}tr\left[\prod_{j=1}^{p} \Sigma_n^{-1}(f_\theta)\Sigma_n(g_{\theta,j})\right] - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\prod_{j=1}^{p}\frac{g_{\theta,j}(\lambda)}{f_\theta(\lambda)}\right)d\lambda\right| = O\left(n^{-1/2+\delta}\right), \forall\delta > 0. \tag{36}$$

This result is important in establishing high-order theory for Gaussian MLE's of spectral parameters when there may be a singularity in the spectra.

For the Gaussian ARFIMA$(0, d, 0)$ model with $d \in (0, 1/2)$ and unit error variance, Lieberman and Phillips (2001) derived Edgeworth expansions to the distribution of the normalized MLE, $\hat{\delta}_n = \sqrt{n}\left(\hat{d}_n - d_0\right)$, where $\hat{d}_n$ is the MLE and d_0 is the true value of d. The spectral density of the process in this case is

$$f_d(\lambda) = \frac{1}{2\pi}\left|1 - e^{i\lambda}\right|^{-2d}.$$

Set $\alpha = 2d$. Clearly, $f_d(\lambda)$ satisfies A2-A3. Let

$$\begin{aligned} \left(\Sigma^{-1}\Sigma^*\right)_{d,d} &= \left(\Sigma^{-1}\dot{\Sigma}\right)^2; \left(\Sigma^{-1}\Sigma^*\right)_{d,dd} = \Sigma^{-1}\dot{\Sigma}\Sigma^{-1}\ddot{\Sigma}; \\ \left(\Sigma^{-1}\Sigma^*\right)_{(2d,d,d-3d,dd)} &= 2\left(\Sigma^{-1}\dot{\Sigma}\right)^3 - 3\Sigma^{-1}\dot{\Sigma}\Sigma^{-1}\ddot{\Sigma}, \end{aligned}$$

and so forth, where

$$\dot{\Sigma} = (\partial/\partial d)\,\Sigma, \ddot{\Sigma} = \left(\partial^2/\partial d^2\right)\Sigma.$$

Define

$$C^*_{n,1}(d) = -\frac{tr\left(\left(\Sigma^{-1}\Sigma^*\right)_{(d,d,d-d,dd)}\right)}{tr\left(\left(\Sigma^{-1}\Sigma^*\right)_{(d,d)}\right)},$$

$$C^*_{n,3}(d) = \frac{1}{12n} tr\left(\left(\Sigma^{-1}\Sigma^*\right)_{(2d,d,d-3d,dd)}\right)$$

and

$$\kappa_{n,1,1}(d) = \frac{1}{2n} tr\left(\left(\Sigma^{-1}\Sigma^*\right)_{(d,d)}\right).$$

The 'exact' Edgeworth expansion for the density of $\hat{\delta}_n$ is

$$\tilde{h}^{(1)}_{\hat{\delta}_n}(u; d) = \phi\left(u; \kappa^{-1}_{n,1,1}(d)\right)\{1 + \frac{1}{\sqrt{n}}[C^*_{n,1}(d)\,u + C^*_{n,3}(d)\,u^3]\}. \tag{37}$$

Lieberman and Phillips (2001) showed that

$$\lim_{n\to\infty} C^*_{n,1}(d) = 0; \lim_{n\to\infty} C^*_{n,3}(d) = -\zeta(3); \lim_{n\to\infty} \kappa_{n,1,1}(d) = \frac{\pi^2}{6}. \tag{38}$$

No bounds on the orders of the errors of the limits in (38) were given. Using (37)-(38), the 'approximate' Edgeworth expansion for the density of $\hat{\delta}_n$ is

$$\tilde{h}^{(1),A}_{\hat{\delta}_n}(u) = \phi\left(u; \frac{6}{\pi^2}\right)\left\{1 - \frac{\zeta(3)}{\sqrt{n}}u^3\right\}.$$

The 'exact' and 'approximate' cdf expansions have the form

$$\begin{aligned} \tilde{H}^{(1)}_{\hat{\delta}_n}(x; d) &= \int_{-\infty}^{x} \tilde{h}^{(1)}_{\hat{\delta}_n}(u; d)du \\ &= \Phi(x\sqrt{\kappa_{n,1,1}}) \\ &\quad - \frac{1}{\sqrt{n\kappa_{n,1,1}}}\phi(x\sqrt{\kappa_{n,1,1}})\left\{\left(C^*_{n,1} + 2C^*_{n,3}\kappa^{-1}_{n,1,1}\right) + C^*_{n,3}x^2\right\}, \end{aligned} \tag{39}$$

and

$$\tilde{H}^{(1),A}_{\hat{\delta}_n}(x) = \Phi\left(x\frac{\pi}{\sqrt{6}}\right) + \frac{\sqrt{6}\zeta(3)}{\pi\sqrt{n}}\phi\left(x\frac{\pi}{\sqrt{6}}\right)\left\{\frac{12}{\pi^2} + x^2\right\}, \tag{40}$$

respectively. Lieberman and Phillips (2001) showed that for $d \in D \equiv (0, 1/2)$ and any compact $D^* \subset D$

$$\sup_{x \in R}\sup_{D^*}\left|P_{d_0}\left(\hat{\delta}_n \leq x\right) - \tilde{H}^{(1)}_{\hat{\delta}_n}(x; d)\right| = o\left(n^{-1/2}\right),$$

$$\sup_{x \in R}\sup_{D^*}\left|\tilde{H}^{(1)}_{\hat{\delta}_n}(x; d) - \tilde{H}^{(1),A}_{\hat{\delta}_n}(x)\right| = o\left(n^{-1/2}\right).$$

The results in the present paper allow a more precise determination of the errors in these expansions. For instance, applying Theorem 7, it is apparent that the differences between corresponding terms in the 'exact' and 'approximate' cdf expansions are

$$\left|C^*_{n,1}(d)\right| = O\left(n^{-1/2+\delta}\right), \forall\delta > 0 \tag{41}$$

$$\left|C^*_{n,3}(d) + \zeta(3)\right| = O\left(n^{-1/2+\delta}\right), \forall\delta > 0 \tag{42}$$

and

$$\left|\kappa_{n,1,1}(d) - \frac{\pi^2}{6}\right| = O\left(n^{-1/2+\delta}\right), \forall\delta > 0. \tag{43}$$

So the omitted term in the 'approximate' expansion (40) is contaminated by an approximation error of $O\left(n^{-1+\delta}\right)$ arising from (41)-(43). Numerical calculations in Lieberman and Phillips (2001) indicate that (39) delivers a better general approximation than (40) and so the approximation errors (41)-(43) appear to be relevant in practice.

7 Conclusion

Products of Toeplitz matrices arise commonly in the Gaussian estimation of the parameters of time series models. The evaluation of such products is needed for the development of first order asymptotics and asymptotic expansions. Results such as (36) are useful in this respect because they allow for unbounded spectra and therefore give error bounds on the approximations that apply in long memory models. In some cases, as is apparent from the analysis in Section 5, the first order theory breaks down because the limiting integral representation diverges even though the finite sample cumulant is finite. Section 5 shows that a second order asymptotic expansion successfully removes the singularity in this case and delivers a substantially improved approximation. Extension of these results to the general case seems worthwhile.

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9 Originals of Graphics

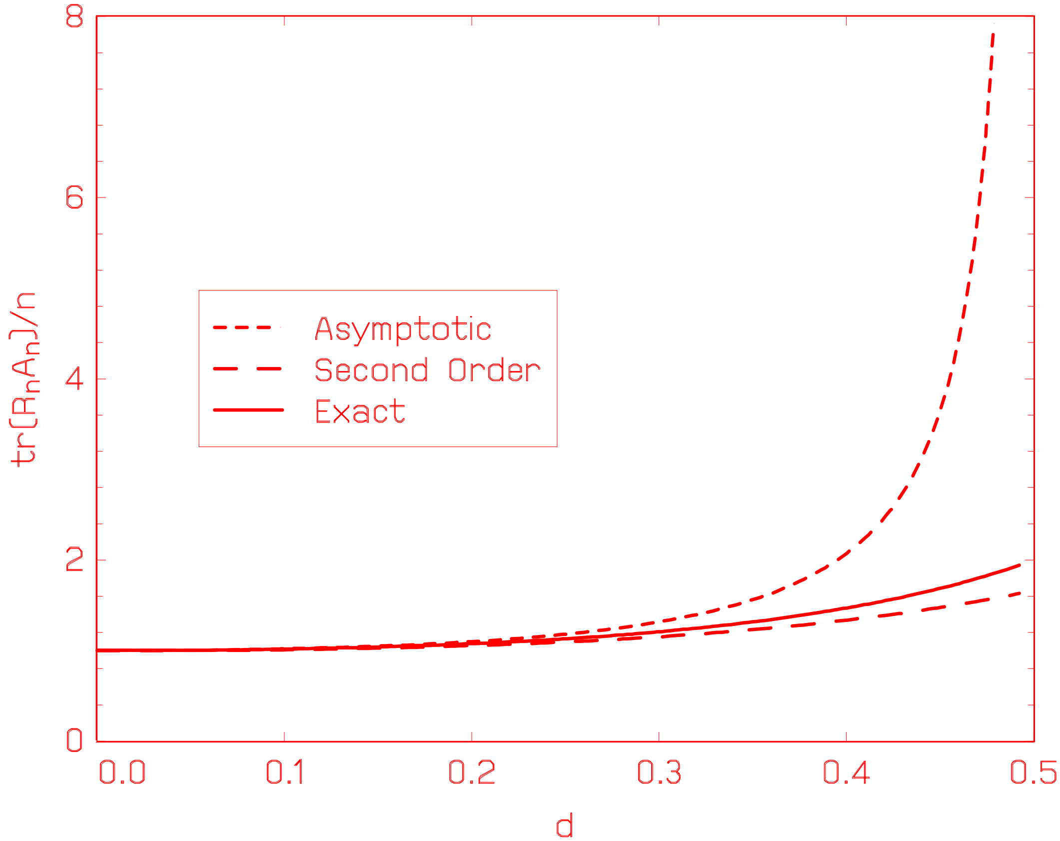

Fig. 1. Exact Value and Approximations to $\frac{1}{n} tr\,[R_n A_n]$ for $n = 50$