BOOKSTRAPPING MACROECONOMIC MODELS

By

Ray C. Fair

December 2001
Revised June 2003

COWLES FOUNDATION DISCUSSION PAPER NO. 1345

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Bootstrapping Macroeconometric Models

Ray C. Fair[*]

Revised June 2003

Abstract

This paper outlines a bootstrapping approach to the estimation and analysis of macroeconometric models. It integrates for dynamic, nonlinear, simultaneous equation models the bootstrapping approach to evaluating estimators initiated by Efron (1979) and the stochastic simulation approach to evaluating models' properties initiated by Adelman and Adelman (1959). It also estimates for a particular model the gain in coverage accuracy from using bootstrap confidence intervals over asymptotic confidence intervals.

1   Introduction

Consider a dynamic, nonlinear, simultaneous equations model of the following form:

$$f_i(y_t, y_{t-1}, \ldots, y_{t-p}, x_t, \alpha_i) = u_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \quad (1)$$

where y_t is an n–dimensional vector of endogenous variables, x_t is a vector of exogenous variables, and α_i is a vector of coefficients. The first m equations

are assumed to be stochastic, with the remaining equations identities. The vector of error terms, $u_t = (u_{1t}, \ldots, u_{mt})'$, is assumed to be *iid* with mean zero. The function f_i may be nonlinear in variables and coefficients. It is assumed that an estimator is available for obtaining consistent estimates of the coefficients.

This specification is fairly general. It includes as a special case the VAR model. It also incorporates autoregressive errors. If the original error term in equation i follows a rth order autoregressive process, say $w_{it} = \rho_{1i} w_{it-1} + \ldots + \rho_{ri} w_{it-r} + u_{it}$, then equation i in (1) can be assumed to have been transformed into one with u_{it} on the right hand side. The autoregressive coefficients $\rho_{1i} \ldots \rho_{ri}$ are incorporated into the α_i coefficient vector, and additional lagged variable values are introduced. This transformation makes the equation nonlinear in coefficients if it were not otherwise, but this adds no further complications because the model is already allowed to be nonlinear. The assumption that u_t is *iid* is thus not as restrictive as it would be if the model were required to be linear in coefficients.

This paper outlines a bootstrapping approach to the estimation and analysis of the model in (1). Two somewhat separate literatures are relevant for this topic. The bootstrap was introduced in statistics in 1979 by Efron (1979).[1] The literature that followed this classic paper stressed the use of the bootstrap for estimation and the evaluation of estimators. Earlier, however, Adelman and Adelman (1959) had introduced in economics the idea of drawing errors to analyze the properties of econometric models. The literature that followed this classic paper stressed the stochastic simulation of large scale macroeconometric models. The common procedure in this literature has been to draw errors from estimated distributions

---

[1]See Hall (1992) for the history of resampling ideas in statistics prior to Efron's paper.

under the assumption of normality, although errors can just as easily be drawn from the empirical distribution of the estimated residuals. The present paper focuses exclusively on the idea of drawing errors from the estimated residuals, which is distribution free, and it uses these draws for both estimation and analysis.

While there is by now a large literature on the use of the bootstrap in economics, most of it has focused on small time series models. Good recent reviews are Li and Maddala (1996), Horowitz (1997), Berkowitz and Kilian (2000), and Härdle, Horowitz, and Kreiss (2001). The paper closest to the present work is Freedman (1984), who considered the bootstrapping of the 2SLS estimator in a dynamic, linear, simultaneous equations model. Runkle (1987) used the bootstrap to examine impulse response functions in VAR models, and Kilian (1998) extended this work to correct for bias. There is also work on bootstrapping GMM estimators (see, for example, Hall and Horowitz (1996)), but this work is of limited relevance here because it does not assume knowledge of a complete model.

In his review of bootstrapping MacKinnon (2002) analyzes an example of a linear simultaneous equations model consisting of one structural equation and one reduced form equation. He points out (p. 14) that "Bootstrapping even one equation of a simultaneous equations model is a good deal more complicated that bootstrapping an equation in which all the explanatory variables are exogenous or predetermined. The problem is that the bootstrap DGP must provide a way to generate all of the endogenous variables, not just one of them." In this paper the process generating the endogenous variables is the complete model (1). All the nonlinear restrictions on the reduced form coefficients are accounted for.

As mentioned above, the standard procedure in the literature that followed the

Adelman and Adelman (1959) paper has been to draw errors from estimated distributions. Also, in much of this literature coefficient uncertainty has not been taken into account: coefficient estimates have been taken to be fixed. Early studies that drew from estimated error distributions and treated coefficient estimates as fixed include Nagar (1969), Evans, Klein, and Saito (1972), Fromm, Klein, and Schink (1972), Green, Leibenberg, and Hirsch (1972), Cooper and Fischer (1972), Sowey (1973), Cooper (1974), Garbade (1975), Bianchi, Calzolari, and Corsi (1976), and Calzolari and Corsi (1977). When coefficient estimates have not been taken to be fixed, they have been drawn from estimated distributions of the coefficient estimates. Studies that drew *both* error terms and coefficients include Schink (1971), Haitovsky and Wallace (1972), Cooper and Fischer (1974), Muench, Rolnick, Wallace, and Weiler (1974), Schink (1974), and Fair (1980a).

In a theoretical paper Brown and Mariano (1984) analyzed the procedure of drawing errors from the estimated residuals for a static nonlinear econometric model with fixed coefficient estimates. For the stochastic simulation results in Fair (1998) errors were drawn from the estimated residuals for a dynamic, nonlinear, simultaneous equations model with fixed coefficient estimates, and this may have been the first time this distribution free approach was used for such models.

This paper makes two contributions. The first is to integrate for dynamic, nonlinear, simultaneous equations models the bootstrap approach to evaluating estimators and the stochastic simulation approach to evaluating models' properties. The procedure in Section 4 for treating coefficient uncertainty has not been used before for these kinds of models. The second is to estimate the gain in coverage accuracy from using bootstrap confidence intervals over asymptotic intervals for

4

a particular model (called the "US model"). It will be seen that the gain is fairly large for this model.

This paper does not provide the theoretical restrictions on the model in (1) that are needed for the bootstrap procedure to be valid. Assumptions beyond *iid* errors and the existence of a consistent estimator are needed, but these have not been worked out in the literature for the model considered here. This paper simply assumes that the model meets whatever restrictions are sufficient for the bootstrap procedure to be valid. Its contribution is to apply the procedure to the model in (1) and to estimate the gain in coverage accuracy assuming the procedure is valid. It remains to be seen what restrictions are needed beyond *iid* errors and a consistent estimator. It is the case, however, that the bootstrap works well regarding coverage accuracy when the US model is taken to be the truth. Given this, it seems likely that the US model falls within the required conditions for validity.

Section 2 discusses the initial estimation and introduces the US model. Section 3 then discusses the use of the bootstrap to evaluate coefficient estimates, and it uses the US model to estimate coverage accuracy. Section 4 discusses the use of the bootstrap to analyze models' properties, and Section 5 considers various extensions, including the extension to models with rational expectations. The bootstrap procedure is applied in Section 6 to the US model.

2   Initial Estimation and Example

Let α denote the vector of all the unknown coefficients in the model, $\alpha = (\alpha_1', \ldots, \alpha_m')'$, and let u denote the vector of errors for all the available periods,

$u = (u_1', \ldots, u_T')'$, where u_t is defined in Section 1. It is assumed that a consistent estimate of α is available, denoted $\hat{\alpha}$. This could be, for example, the 2SLS or 3SLS estimate of α. Given this estimate and the actual data, u can be estimated. Let $\hat{u}$ denote the estimate of u after the residuals have been centered at zero.[2] Statistics of interest can be computed. These can include t-statistics of the coefficient estimates and possible χ^2 statistics for various hypotheses. For the result in Section 6 the Andrews-Ploberger test statistic is examined, which tests for structural change.[3] τ will be used to denote the vector of estimated statistics of interest.

The example that is used for the empirical work is the US model in Fair (1994). There are 29 stochastic equations, about 100 identities, and 164 coefficients to estimate, counting autoregressive coefficients for the errors. The model is dynamic, nonlinear, and simultaneous. The version used here is on the website: http://fairmodel.econ.yale.edu.[4] The estimation period is 1954:1–2002:3, 195 quarterly observations, and the estimation method is 2SLS. This version does not have rational expectations.

---

[2]Freedman (1981) has shown that the bootstrap can fail for an equation with no constant term if the residuals are not centered at zero. For all the results reported in this paper centering has been done. From model (1), $\hat{u}_{it}$, an element of $\hat{u}$, is $f_i(y_t, y_{t-1}, \ldots, y_{t-p}, x_t, \hat{\alpha}_i)$ except for the adjustment that centers the residuals at zero.

[3]See Andrews and Ploberger (1994).

[4]One of the demand for money equations in the model is estimated under the assumption of a fourth order autorgressive error, and the sum of the autoregressive coefficients is close to one. For the work in this paper this equation was dropped, leaving 29 stochastic equations rather than 30. This equation is not important in the model because the short term interest rate is determined by an estimated interest rate rule of the Fed.

3 Distribution of the Coefficient Estimates

3.1 The Bootstrap Procedure

The bootstrap procedure for evaluating estimators for the model in (1) is:

1. For a given trial j, draw u_t^{*j} from $\hat{u}$ with replacement for $t = 1, \ldots, T$. Use these errors and $\hat{\alpha}$ to solve the model (1) dynamically for $t = 1, \ldots, T$.[5] Treat the solution values as actual values and estimate α by the consistent estimator (2SLS, 3SLS, or whatever). Let $\hat{\alpha}^{*j}$ denote this estimate. Compute also the test statistics of interest, and let τ^{*j} denote the vector of these values.

2. Repeat step 1 for $j = 1, \ldots, J$.

Step 2 gives J values of each element of $\hat{\alpha}^{*j}$ and τ^{*j}. Using these values, confidence intervals for the coefficient estimates can be computed (see below). Also, for the originally estimated value of any test statistic, one can see where it lies on the distribution of the J values.

Note that each trial generates a new data set. Each data set is generated using the same coefficient vector ($\hat{\alpha}$), but in general the data set has different errors for a period from those that existed historically. Note also that since the drawing is with replacement, the same error vector may be drawn more than once in a given trial, while others may not be drawn at all. All data sets are conditional on the actual values of the endogenous variables prior to period 1 and on the actual values of the exogenous variables for all periods.

---

[5]This is just a standard dynamic simulation, where instead of using zero values for the error terms the drawn values are used.

3.2 Estimating Coverage Accuracy

Three confidence intervals are empirically examined here.[6] Let β denote a particular coefficient in α. Let $\hat{\beta}$ denote the base estimate (2SLS, 3SLS, or whatever) of β, and let $\hat{\sigma}$ denote its estimated asymptotic standard error. Let $\hat{\beta}^{*j}$ denote the estimate of β on the jth trial, and let $\hat{\sigma}^{*j}$ denote the estimated asymptotic standard error of $\hat{\beta}^{*j}$. Let t^{*j} equal the t-statistic $(\hat{\beta}^{*j} - \hat{\beta})/\hat{\sigma}^{*j}$. Assume that the J values of t^{*j} have been ranked, and let t_r^* denote the value below which r percent of the values of t^{*j} lie. Finally, let $|t^{*j}|$ denote the absolute value of t^{*j}. Assume that the J values of $|t^{*j}|$ have been ranked, and let $|t^*|_r$ denote the value below which r percent of the values of $|t^{*j}|$ lie. The first confidence interval is simply $\hat{\beta} \pm 1.96\hat{\sigma}$, which is the 95 percent confidence interval from the asymptotic normal distribution. The second is $(\hat{\beta} - t_{.975}^*\hat{\sigma}, \hat{\beta} - t_{.025}^*\hat{\sigma})$, which is the equal-tailed percentile-t interval. The third is $\hat{\beta} \pm |t^*|_{.950}\hat{\sigma}$, which is the symmetric percentile-t interval.

The following Monte Carlo procedure is used to examine the accuracy of the three intervals. This procedure assume that the data generating process is the model (1) with true coefficients $\hat{\alpha}$.

    a. For a given repetition k, draw u_t^{**k} from $\hat{u}$ with replacement for $t = 1, \ldots, T$. Use these errors and $\hat{\alpha}$ to solve the model (1) dynamically for $t = 1, \ldots, T$. Treat the solution values as actual values and estimate α by the consistent estimator (2SLS, 3SLS, or whatever). Let $\hat{\alpha}^{**k}$ denote this estimate. Use this estimate and the solution values from the dynamic simulation to compute the residuals, u, and center them at zero. Let $\hat{u}^{**k}$ denote the estimate of u after the residuals have been centered at zero.[7]

---

[6]See Li and Maddala (1996), pp. 118-121, for a review of the number of ways confidence intervals can be computed using the bootstrap. See also Hall (1988).

[7]From model (1), $\hat{u}_{it}^{**k}$, an element of $\hat{u}^{**k}$, is $f_i(y_t^{**k}, y_{t-1}^{**k}, \ldots, y_{t-p}^{**k}, x_t, \hat{\alpha}_i^{**k})$ except for the adjustment that centers the residuals at zero, where y_{t-h}^{**k} is the solution value of y_{t-h} from the dynamic simulation ($h = 0, 1, \ldots, p$).

b. Perform steps 1 and 2 in Section 3.1, where $\hat{u}^{**k}$ replaces $\hat{u}$ and $\hat{\alpha}^{**k}$ replaces $\hat{\alpha}$. Compute from these J trials the three confidence intervals discussed above, where $\hat{\beta}^{**k}$ replaces $\hat{\beta}$ and $\hat{\sigma}^{**k}$ replaces $\hat{\sigma}$. Record for each interval whether or not $\hat{\beta}$ is outside of the interval.

c. Repeat steps a and b for $k = 1, \ldots, K$.

After completion of the K repetitions, one can compute for each coefficient and each interval the percent of the repetitions that $\hat{\beta}$ was outside the interval. For, say, a 95 percent confidence interval, the difference between the computed percent and 5 percent is the error in coverage probability.

This procedure was used on the US model to examine coverage accuracy. For this work both J and K were taken to be 350, for a total of 122,500 times the model was estimated (by 2SLS). There were 847 solution failures out of the 122,500 trials, and these failures were skipped. The job took about 40 hours on a 1.7 Ghz PC, about one second per estimation. The results are summarized in Table 1. Rejection rates are presented for 12 of the coefficients in the model. The average for the 12 coefficients is presented as well as the average for all 164 coefficients in the model. The standard deviation for the 164 coefficients is also presented.

The average rejection rate over the 164 coefficients is .085 for the asymptotic interval, which compares to .063 and .056 for the two bootstrap intervals. The asymptotic distribution thus rejects too often, and the bootstrap distributions are fairly accurate. Although not shown in Table 1, the results are similar if 90 percent confidence intervals are used. In this case the asymptotic rejection rate averaged across the 164 coefficients is .145 (standard deviation of .055). The corresponding values for the two bootstrap intervals are .113 (standard deviation of .030) and .107 (standard deviation of .029). As noted in Section 1, given the good bootstrap

9

Table 1
Estimated Coverage Accuracy for the US Model

	Percent of Rejections using 95 Percent Confidence Intervals		
	a	b	c
CS equation			
ldv	.140	.066	.066
income	.100	.049	.057
CN equation			
ldv	.123	.066	.066
income	.126	.063	.043
CD equation			
ldv	.143	.051	.066
income	.131	.086	.071
PF equation			
ldv	.074	.057	.049
PIM	.069	.040	.040
UR	.043	.037	.040
RS equation			
ldv	.074	.080	.066
inflation	.089	.077	.069
UR	.051	.057	.051
Average (12)	.097	.061	.057
Average (164)	.085	.063	.056
SD (164)	.045	.022	.020

Notes:
a: Asymptotic confidence interval.
b: Bootstrap equal-tailed percentile-t interval.
c: Bootstrap symmetric percentile-t interval.
Average (12) = Average for the 12 coefficients.
Average (164) = Average for all 164 coefficients.
SD (164) = Standard deviation for all 164 coefficients
ldv: lagged dependent variable, CS: consumption of services,
CN: consumption of nondurables, CD: consumption of durables,
PF: private nonfarm deflator, RS: three-month Treasury bill rate,
PIM: import price deflator, UR: unemployment rate.

results it seems likely that the US model falls within the required conditions for validity of the bootstrap.

4 Analysis of Models' Properties

The bootstrap procedure is extended in this section to evaluating properties of models like (1). The errors are drawn from the residuals, which is contrary to what has been done in the previous literature except for Fair (1998). Also, as in Section 3.1, the coefficients are estimated on each trial. In the previous literature the coefficient estimates either have been taken to be fixed or have been drawn from estimated distributions.

When examining the properties of models, one is usually interested in a period smaller than the estimation period. Assume that the period of interest is s through S, where $s \geq 1$ and $S \leq T$. The bootstrap procedure for analyzing properties is:

1. For a given trial j, draw u_t^{*j} from $\hat{u}$ with replacement for $t = 1, \ldots, T$. Use these errors and $\hat{\alpha}$ to solve the model (1) dynamically for $t = 1, \ldots, T$. Treat the solution values as actual values and estimate α by the consistent estimator (2SLS, 3SLS, or whatever). Let $\hat{\alpha}^{*j}$ denote this estimate. Discard the solution values; they are not used again.

2. Draw u_t^{*j} from $\hat{u}$ with replacement for $t = s, \ldots, S$.[8] Use these errors and $\hat{\alpha}^{*j}$ to solve the model (1) dynamically for $t = s, \ldots, S$. Record the solution value of each endogenous variable for each period. This simulation uses the actual (historical) values of the variables prior to period s, not the values used in computing $\hat{\alpha}^{*j}$.

3. Multiplier experiments can be performed. First solve the model for period s through S using $\hat{\alpha}^{*j}$ and the drawn errors. Record the solution values of the endogenous variables. Then change one or more exogenous variables and solve again. The difference between the second solution value and the first for a given endogenous variable and period is the model's estimated effect of the change. Record these differences.

---

[8]If desired, these errors can be the same errors drawn in step 1 for the s through S period. With a large enough number of trials, whether one does this or instead draws new errors makes a trivial difference. It is assumed here that new errors are drawn.

11

4. Repeat steps 1, 2, and 3 for $j = 1, \ldots, J$.

5. Step 4 gives J values of each endogenous variable for each period. It also gives J values of each difference for each period if a multiplier experiment has been performed.

A distribution of J predicted values of each endogenous variable for each period is now available to examine. One can compute, for example, various measures of dispersion, which are estimates of the accuracy of the model. Probabilities of specific events happening can also be computed. If, say, one is interested in the event of two or more consecutive periods of negative growth in real output in the s through S period, one can compute the number of times this happened in the J trials. If a multiplier experiment has been performed, a distribution of J differences for each endogenous variable for each period is also available to examine. This allows the uncertainty of policy effects in the model to be examined.[9]

If the coefficient estimates are taken to be fixed, then step 1 above is skipped. The same coefficient vector $(\hat{\alpha})$ is used for all the solutions. Although in much of the stochastic simulation literature coefficient estimates have been taken to be fixed, this is not in the spirit of the bootstrap literature. From a bootstrapping perspective, the obvious procedure to follow after the errors have been drawn is to first estimate the model and then examine its properties, which is what the above procedure does.

---

[9]The use of stochastic simulation to estimate event probabilities was first discussed in Fair (1993b), where the coefficient estimates were taken to be fixed and errors were drawn from estimated distributions. Estimating the uncertainty of multiplier or policy effects in nonlinear models was first discussed in Fair (1980b), where both errors and coefficients were drawn from estimated distributions.

5 Extensions

Bias Correction

Since 2SLS and 3SLS estimates are biased, it may be useful to use the bootstrap procedure to correct for bias. This is especially true for estimates of lagged dependent variable coefficients. It has been known since the work of Orcutt (1948) and Hurwicz (1950) that least squares estimates of these coefficients are biased downwards even when there are no right hand side endogenous variables.

In the present context a bias-correction procedure using the bootstrap is as follows.

1. From step 2 in Section 3.1 there are J values of each coefficient available. Compute the mean value for each coefficient, and let $\bar{\alpha}$ denote the vector of the mean values. Let $\gamma = \bar{\alpha} - \hat{\alpha}$, the estimated bias. Compute the coefficient vector $\hat{\alpha} - \gamma$ and use the coefficients in this vector to adjust the constant term in each equation so that the mean of the error terms is zero. Let $\tilde{\alpha}$ denote $\hat{\alpha} - \gamma$ except for the constant terms, which are as adjusted. $\tilde{\alpha}$ is then taken to be the unbiased estimate of α. Let θ denote the vector of estimated biases: $\theta = \hat{\alpha} - \tilde{\alpha}$.

2. Using $\tilde{\alpha}$ and the actual data, compute the errors. Denote the error vector as $\tilde{u}$. ($\tilde{u}$ is centered at zero because of the constant term adjustment in step 1.)

3. The steps in Section 4 can now be performed where $\tilde{\alpha}$ replaces $\hat{\alpha}$ and $\tilde{u}$ replaces $\hat{u}$. The only difference is that after the coefficient vector is estimated by 2SLS, 3SLS, or whatever, it has θ subtracted from it to correct for bias. In other words, subtract θ from $\hat{\alpha}^{*j}$ on each trial.[10]

---

[10]One could for each trial do a bootstrap to estimate the bias—a bootstrap within a bootstrap. The base coefficients would be $\hat{\alpha}^{*j}$ and the base data would be the generated data on trial j. This is expensive, and an approximation is simply to use θ on each trial. This is the procedure used by Kilian (1998) in estimating confidence intervals for impulse responses in VAR models. Kilian (1998) also does, when necessary, a stationary correction to the bias correction to avoid pushing stationary impulse response estimates into the nonstationary region. This type of adjustment is not pursued here.

The example in Section 6 examines the sensitivity of some of the results to the bias correction.

Optimal Control

At the point where multiplier experiments are discussed above, optimal control experiments can also be performed. Assume that the period of interest is s through S and that the objective is to maximize the expected value of W, where W is

$$W = \sum_{t=s}^{S} g_t(y_t, x_t) \tag{2}$$

Let z_t be the vector of control variables, where z_t is a subset of x_t, and let z be the vector of all the control values: $z = (z_s, \ldots, z_S)$. Under the assumption of certainty equivalence, the control problem is solved at the beginning of period s by setting the errors for period s and beyond equal to zero. If this is done, then for each value of z one can compute a value of W by first solving the model for $y_s, \ldots, y_S$ and then using these values along with the values for $x_s, \ldots, x_S$ to compute W in (2). Stated this way, the optimal control problem is choosing variables (the elements of z) to maximize an *unconstrained* nonlinear function. By substitution, the constrained maximization problem is transformed into the problem of maximizing an unconstrained function of the control variables:

$$W = \Phi(z) \tag{3}$$

where Φ stands for the mapping $z \longrightarrow y_s, \ldots, y_S, x_s, \ldots, x_S \longrightarrow W$. Given this setup, the problem can be turned over to a nonlinear maximization algorithm like Davidon-Fletcher-Powell (DFP). For each iteration of the algorithm, the derivatives

of Φ with respect to the elements of z, which are needed by the algorithm, can be computed numerically. An algorithm like DFP is generally quite good at finding the optimum for a typical control problem.[11]

At each trial j one can solve this problem. Let z_s^{*j} be the computed optimal value of z_s on trial j. This is the value that would be implemented for period s by the control authority.[12] At the end one has a distribution of the J values of z_s^{*j}, which can be examined. Note that z_s^{*j} varies across trials only because the coefficient estimates vary. The errors that are drawn for trial j for periods s through S don't matter because of the use of certainty equivalence. The distribution of the z_s^{*j} values thus indicates how sensitive the control values are to the uncertainty in the coefficient estimates.

Rational Expectations

Consider model (1) with rational expectations:

$$f_i(y_t, y_{t-1}, \ldots, y_{t-p}, E_{t-1}y_t, E_{t-1}y_{t+1}, \ldots, E_{t-1}y_{t+h}, x_t, \alpha_i) = u_{it},$$
$$(i = 1, \ldots, n), \quad (t = 1, \ldots, T), \tag{4}$$

where E_{t-1} is the conditional expectations operator based on the model and on information through period $t - 1$.

The bootstrap procedure requires initial consistent estimates of the α_i. It also requires, of course, the ability to solve the model given a set of coefficient estimates. Various modifications of the 2SLS estimator are available for estimating equations

---

[11]See Fair (1974) for various applications of this procedure.

[12]The control problem also calculates the optimal values for periods $s + 1$ through S, but in practice these would never have to be implemented because a new problem could be solved at the beginning of period $s + 1$ after period s was realized. This is the "open-loop feedback" approach.

with rational expectations, and so one of these could be used.[13] There are also a number of methods for solving rational expectations models like (4). One method that generally works well is the "extended path" (EP) method in Fair and Taylor (1983, 1990). The solution methods assume that agents form their expectations at the beginning of period t by setting the errors for period t and beyond to zero and then solving the model. Estimation and solution methods are thus available for allowing the above bootstrap procedure to be used for models with rational expectations.

When estimating or solving (4) for, say, periods 1 through T, data beyond period T are needed, and so the period analyzed must end before the actual end of the historical data. It should also be noted that if a single equation estimation method is used, the expectations used by the estimation method are not the expectations that one gets when the overall model is solved after the coefficients have been estimated. In other words, the expectations used by the estimation method are not model consistent. This means that one has to be careful in computing the errors ($\hat{u}$) after the coefficients are estimated ($\hat{\alpha}$ computed). For example, the errors for period 1 are computed by first solving the model to get the expectations. This is done by using the historical data prior to period 1 and setting the errors for period 1 and beyond to zero. Once the expectations are computed, the errors for period 1 are computed using these expectation values and the actual values of the endogenous variables for period 1. The process is then repeated for period 2, where the expectations are computed using the historical data prior to period 2 and setting the errors for period 2 and beyond to zero. The process continues through

---

[13] See Fair (1993a) for a review of these estimators.

16

period T. Once $\hat{u}$ is computed, the bootstrap procedure can proceed as in Sections 3 and 4.

6 An Example

In this section the overall bootstrap procedure is applied to the US model, where the estimation period is 1954:1–2002:3 and the estimation method is 2SLS.

The calculations were run in one large batch job. The main steps were:

1. Estimate the 29 equations by 2SLS for 1954:1–2002:3. Compute standard errors of the coefficient estimates, and perform the Andrews-Ploberger (1994) (AP) test on selected equations. Using the 2SLS estimates and zero values for the errors, solve the model dynamically for 2000:4-2002:3 and perform a multiplier experiment for this period. Using the actual data and the 2SLS estimates, compute the 29-dimensional error vectors for 1954:1–2002:3 (195 vectors).

2. Do the following 2000 times: 1) draw with replacement 195 error vectors from the residual vectors for 1954:1–2002:3, 2) using the drawn errors and the 2SLS estimates from step 1, solve the model dynamically for 1954:1–2002:3 to get new data, 3) using the new data, estimate the model by 2SLS, compute t-statistics for the coefficient estimates, and perform the AP tests, 4) reset the data prior to 2000:4 to the actual data, 5) draw with replacement 8 error vectors from the residual vectors for 2000:4–2002:3, 6) using the new 2SLS estimates and the drawn errors, solve the model dynamically for 2000:4–2002:3 and perform the multiplier experiment for this period.

3. Step 2 gives for each equation 2000 values of each coefficient estimate, t-statistic, and AP statistic. It also gives 2000 predicted values of each endogenous variable for each quarter within 2000:4–2002:3 and 2000 differences for each endogenous variable and each quarter from the multiplier experiment. These values can be analyzed as desired. Some examples are given below. Steps 4-6 that follow are the bias-correction calculations.

4. From the 2000 values for each coefficient, compute the mean and then subtract the mean from twice the 2SLS coefficient estimate from step 1. Use

17

these values to adjust the constant term in each equation so that the mean of the error terms is zero. Using these coefficients (including the adjusted constant terms), record the differences between the 2SLS coefficient estimates from step 1 and these coefficients. Call the vector of these values the "bias-correction vector." Using the new coefficients and zero values for the errors, solve the model dynamically for 2000:4–2002:3 and perform the multiplier experiment for this period. Using the actual data and the new coefficients, compute the 29-dimensional error vectors for 1954:1–2002:3 (195 vectors).

5. Do the following 2000 times: 1) draw with replacement 195 error vectors from the residual vectors from step 4 for 1954:1–2002:3, 2) using the drawn errors and the coefficients from step 4, solve the model dynamically for 1954:1–2002:3 to get new data, 3) using the new data, estimate the model by 2SLS and adjust the estimates for bias using the bias-correction vector from step 4, 4) reset the data prior to 2000:4 to the actual data, 5) draw with replacement 8 error vectors from the residual vectors from step 4 for 2000:4–2002:3, 6) using the new coefficient estimates and the drawn errors, solve the model dynamically for 2000:4–2002:3 and perform the multiplier experiment for this period.

6. Step 5 gives 2000 predicted values of each endogenous variable for each quarter within 2000:4–2004:1 and 2000 differences for each endogenous variable and each quarter from the multiplier experiment.

The same sequence of random numbers was used for the regular calculations (steps 1-3) as was used for the bias-correction calculations (steps 4-6). This lessens stochastic simulation error in comparisons between the two sets of results. If the model failed to solve for a given trial (either for the 1954:1–2002:3 period or the 2000:4–2002:3 period), the trial was skipped. No failures occurred for the regular calculations, but there were 5 failures out of the 2000 trials for the bias-correction calculations. Each trial takes about one second on a 1.7 Ghz PC using the Fair-Parke (1995) program.

Table 2 presents some results from step 2 for the coefficient estimates. Results for 12 coefficients from 5 equations are presented. The 5 equations are three consumption equations, a price equation, and an interest rate rule. The coefficients are for the lagged dependent variable in each equation, income in each consumption equation, the price of imports and the unemployment rate in the price equation, and inflation and the unemployment rate in the interest rate rule. These are some of the main coefficients in the model.

The first three columns show the 2SLS estimate, the mean from the 2000 trials, and the ratio of the two. As expected, the mean is smaller than the 2SLS estimate for all the lagged dependent variable coefficients: the 2SLS estimates of these coefficients are biased downwards. The smallest ratio is 0.966, a bias of 3.4 percent.

Column 4 gives the asymptotic confidence intervals; column 5 gives the confidence intervals using the equal-tailed percentile-t interval; and column 6 gives the symmetric percentile-t interval using the absolute values of the t-statistics. These columns show that the asymptotic intervals tend to be narrower than the bootstrap intervals. In 19 of the 24 cases the left value for the asymptotic interval is larger than the left value for the bootstrap interval, and in 19 of the 24 cases the right value for the asymptotic interval is smaller than the right value for the bootstrap interval. The asymptotic intervals will thus tend to reject more often than the bootstrap intervals. It was seen in Section 3.2 that the asymptotic interval rejects too often.

Table 2
Confidence Intervals for Selected Coefficients

	(1) $\hat{\beta}$	(2) $\bar{\beta}$	(3) (2)/(1)	(4) a	(5) b	(6) c
CS equation						
ldv	0.7873	0.7609	0.966	0.7215	0.7449	0.7031
				0.8531	0.8827	0.8716
income	0.1058	0.1163	1.099	0.0613	0.0458	0.0516
				0.1504	0.1415	0.1601
CN equation						
ldv	0.7823	0.7565	0.967	0.7219	0.7442	0.7026
				0.8427	0.8718	0.8621
income	0.0973	0.1134	1.165	0.0575	0.0393	0.0461
				0.1372	0.1241	0.1486
CD equation						
ldv	0.3294	0.3720	1.129	0.2226	0.1755	0.1913
				0.4362	0.3979	0.4675
income	0.1077	0.1218	1.131	0.0701	0.0532	0.0591
				0.1453	0.1291	0.1564
PF equation						
ldv	0.8806	0.8715	0.990	0.8487	0.8580	0.8426
				0.9125	0.9246	0.9186
PIM	0.0480	0.0477	0.994	0.0440	0.0442	0.0438
				0.0520	0.0525	0.0522
UR	-0.1780	-0.1787	1.004	-0.2238	-0.2239	-0.2266
				-0.1322	-0.1280	-0.1293
RS equation						
ldv	0.9092	0.9026	0.993	0.8834	0.8870	0.8812
				0.9349	0.9398	0.9371
inflation	0.0803	0.0848	1.057	0.0549	0.0520	0.0538
				0.1056	0.1023	0.1067
UR	-0.1128	-0.1123	0.995	-0.1699	-0.1716	-0.1713
				-0.0558	-0.0545	-0.0543

Notes:

a: $\hat{\beta} - 1.96\hat{\sigma}$ b: $\hat{\beta} - t^*_{.975}\hat{\sigma}$ c: $\hat{\beta} - |t^*|_{.950}\hat{\sigma}$
$\hat{\beta} + 1.96\hat{\sigma}$ $\hat{\beta} - t^*_{.025}\hat{\sigma}$ $\hat{\beta} + |t^*|_{.950}\hat{\sigma}$

$\hat{\beta}$ = 2SLS estimate; $\hat{\sigma}$ = estimated asymptotic standard error of $\hat{\beta}$.

$\bar{\beta}$ = mean of the values of $\hat{\beta}^{*j}$, where $\hat{\beta}^{*j}$ is the estimate of β
    on the jth trial.

t^*_r = value below which r percent of the values of t^{*j} lie,
    where $t^{*j} = (\hat{\beta}^{*j} - \hat{\beta})/\hat{\sigma}^{*j}$,
    where $\hat{\sigma}^{*j}$ is the estimated asymptotic standard error of $\hat{\beta}^{*j}$.

$|t^*|_r$ = value below which r percent of the values of $|t^{*j}|$ lie.

See Table 1 for variable notation.

Table 3
Results for the AP Tests

Eq.	# of coefs.	AP	Bootstrap			Asymptotic		
			1%	5%	10%	1%	5%	10%
CS	9	21.18	17.47	13.84	12.15	11.16	8.96	7.77
CN	9	14.67	14.50	12.16	10.64	11.16	8.96	7.77
CD	9	12.76	16.48	12.76	11.23	11.16	8.96	7.77
IH	7	7.17	13.25	10.62	9.35	9.50	7.31	6.28
PF	6	12.77	10.72	8.07	6.85	8.70	6.51	5.58

Notes:
IH: Housing investment; see Table 1 for other notation.
Sample period: 1954:1–2002:3.
Period for possible break: 1970:1–1979:4.
Value of $\lambda = 2.29$.
Asymptotic values from Andrews and Ploberger (1994), Table 1.

Table 3 presents results for the AP test for five equations: the three consumption equations, a housing investment equation, and the price equation.[14] The overall sample period is 1954:1–2002:3, and the period for a possible break was taken to be 1970:1-1979:4. (An advantage of the AP test is that the possible break point can be specified to be within a period rather than a particular quarter.) Table 3 gives for each equation the computed AP value, the bootstrap confidence values, and the asymptotic confidence values. The asymptotic confidence values are taken from Table 1 in Andrews and Ploberger (1994). The value of λ in the AP notation for the present results is 2.29. The bootstrap confidence values for an equation are computed using the 2000 values of the AP statistic. The 5 percent value, for example, is the value above which 100 of the AP values lie.

There is a clear pattern in Table 3, which is that the asymptotic confidence values are too low. They lead to rejection of the null hypothesis of stability too

---

[14]The test was not performed for the interest rate rule because the equation is already estimated under the assumption of a change in Fed behavior in the 1979:4–1982:3 period.

often. Relying on the asymptotic values for the AP test thus appears to be too harsh.

Table 4 presents results for the simulations for 2000:4–2002:3. Results for four variables are presented: the log of real GDP, the log of the GDP deflator, the unemployment rate, and the three-month Treasury bill rate. Four sets of results are presented: with and without coefficient uncertainty and with and without bias correction.[15] Consider the first set of results (upper left corner) in Table 4. The first column gives the deterministic prediction (based on setting the error terms to zero and solving once), and the second gives the median value of the 2000 predictions. These two values are close to each other, which means there is little bias in the deterministic prediction. The third column gives the difference between the median predicted value and the predicted value below which 15.87 percent of the values lie, and the fourth column gives the difference between the predicted value above which 15.87 percent of the values lie and the median value. For a normal distribution these two differences are the same and equal one standard error. Computing these differences is one possible way of measuring forecast uncertainty in the model. The same differences are presented for the other three sets of results in Table 4.

Three conclusions can be drawn from the results in Table 4. First, the left and right differences are fairly close to each other. Second, the differences with no coefficient uncertainty are only slightly smaller than those with coefficient

---

[15]The results without coefficient uncertainty were obtained in a separate batch job. This batch job differed from the one outlined at the beginning of this section in that in part 6) of step 2 the 2SLS estimates from step 1 are used, not the new 2SLS estimates. Also, in part 6) of step 5 the coefficients from step 4 are used, not the new coefficient estimates. For this job there were no solution failures for the regular calculations and 3 failures for the bias-correction calculations.

Table 4
Simulation Results for 2000:4–2002:3

Var.	h	$\hat{Y}$	$Y_{.5}$	left	right	$Y_{.5}$	left	right
						No		
			Coefficient Uncertainty			**Coefficient Uncertainty**		
					No Bias Correction			
gdrp	1	7.746	7.745	0.562	0.569	7.746	0.506	0.486
	4	7.748	7.746	1.423	1.434	7.748	1.248	1.240
	8	7.778	7.774	1.719	1.712	7.777	1.445	1.522
gdpd	1	4.681	4.681	0.275	0.322	4.681	0.277	0.291
	4	4.700	4.700	0.591	0.621	4.700	0.513	0.589
	8	4.718	4.717	0.886	0.931	4.717	0.734	0.786
UR	1	4.146	4.152	0.365	0.344	4.167	0.363	0.369
	4	4.445	4.488	0.745	0.757	4.491	0.687	0.651
	8	4.642	4.748	0.863	0.956	4.683	0.819	0.821
RS	1	5.970	5.974	0.545	0.538	5.987	0.584	0.485
	4	5.155	5.068	1.196	1.200	5.102	1.112	1.162
	8	5.002	4.829	1.428	1.455	4.969	1.327	1.359
					Bias Correction			
gdpr	1	7.746	7.746	0.539	0.571	7.746	0.516	0.515
	4	7.750	7.750	1.542	1.512	7.750	1.283	1.366
	8	7.781	7.782	2.020	2.105	7.781	1.658	1.709
gdpd	1	4.681	4.681	0.270	0.324	4.681	0.281	0.303
	4	4.699	4.699	0.609	0.630	4.699	0.513	0.585
	8	4.718	4.717	0.972	0.986	4.717	0.742	0.804
UR	1	4.173	4.224	0.384	0.358	4.195	0.347	0.346
	4	4.482	4.600	0.858	0.815	4.540	0.717	0.667
	8	4.602	4.774	1.122	1.100	4.664	0.910	0.885
RS	1	5.942	5.905	0.538	0.551	5.948	0.538	0.503
	4	5.162	5.060	1.228	1.298	5.114	1.125	1.181
	8	5.086	4.997	1.628	1.567	5.077	1.425	1.395

Notes:

h = number of quarters ahead.

$\hat{Y}$ = predicted value from deterministic simulation.

Y_r = value below which r percent of the values of Y^j lie, where Y^j is the predicted value on the jth trial.

left = $Y_{.5} - Y_{.1587}$, right = $Y_{.8413} - Y_{.5}$, units are percentage points.

gdpr: log of real GDP; gdpd: log of GDP deflator; see Table 1 for others.

uncertainty, and so most of the predictive uncertainty is due to the additive errors. Third, the bias-correction results are fairly similar to the non bias-correction ones, which suggests that bias is not a major problem in the model. In most cases the uncertainty estimates are larger for the bias-correction results.

Table 5 presents results for the multiplier experiment. The experiment was an increase in real government purchases of goods of one percent of real GDP for 2000:4–2002:3. The format of Table 5 is similar to that of Table 4, where the values are multipliers[16] rather than predicted values. The first column gives the multiplier computed from deterministic simulations, and the second gives the median value of the 2000 multipliers. As in Table 3, these two values are close to each other. The third column gives the difference between the median multiplier and the multiplier below which 15.87 percent of the values lie, and the fourth column gives the difference between the multiplier above which 15.87 percent of the values lie and the median multiplier. These two columns are measures of the uncertainty of the government spending effect in the model.

Three conclusions can be drawn from the results in Table 5. First, the left and right differences are fairly close to each other. Second, the differences are fairly small relative to the size of the multipliers, and so the estimated policy uncertainty is fairly small for a government spending change. Third, the bias-correction results are similar to the non bias-correction ones, which again suggests that bias is not a major problem in the model.

---

[16]The word 'multiplier' is used here to refer to the difference between the predicted value of a variable after the policy change and the predicted value of the variable before the change. This difference is not strictly speaking a multiplier because it is not divided by the government spending change.

Table 5
Multiplier Results for 2000:4–2002:3

Var.	h	$\hat{d}$	$d_{.5}$	left	right	$\hat{d}$	$d_{.5}$	left	right
		No Bias Correction				**Bias Correction**			
gdpr	1	1.010	1.035	.069	.081	0.984	0.979	.065	.078
	4	1.571	1.613	.075	.088	1.530	1.530	.067	.078
	8	1.361	1.394	.080	.088	1.325	1.325	.079	.083
gdpd	1	.036	.034	.008	.009	.039	.039	.008	.008
	4	.282	.279	.045	.048	.284	.279	.044	.046
	8	.569	.578	.078	.081	.558	.514	.067	.075
UR	1	-.280	-.279	.037	.037	-.281	-.278	.039	.035
	4	-.747	-.753	.072	.063	-.742	-.742	.074	.061
	8	-.560	-.587	.072	.076	-.536	-.546	.074	.079
RS	1	.258	.261	.046	.054	.255	.251	.044	.052
	4	.753	.759	.108	.109	.750	.747	.106	.105
	8	.678	.664	.113	.117	.647	.650	.116	.124

Notes:
h = number of quarters ahead.
$\hat{Y}^a$ = predicted value from deterministic simulation, no policy change.
$\hat{Y}^b$ = predicted value from deterministic simulation, policy change.
$\hat{d} = \hat{Y}^b - \hat{Y}^a$
Y^{aj} = predicted value on the jth trial, no policy change.
Y^{bj} = predicted value on the jth trial, policy change.
$d^j = Y^{bj} - Y^{aj}$
d_r = value below which r percent of the values of d^j lie.
left = $d_{.5} - d_{.1587}$, right = $d_{.8413} - d_{.5}$, units are percentage points.
See Tables 1 and 4 for variable notation.

7 Conclusion

This paper has outlined a bootstrapping approach to the estimation and analysis of

dynamic, nonlinear, simultaneous equations models. It draws on the bootstrapping

literature initiated by Efron (1979) and the stochastic simulation literature initiated

by Adelman and Adelman (1959). The procedure in Section 4 has not been used

before for these models. The procedure is distribution free, and it allows a wide range of questions to be considered, including estimation, prediction, and policy analysis.

The results in Section 6 are for illustration only, but they are suggestive of the usefulness of the bootstrapping procedure for models like (1). Computations like those in Table 3 can be done for many different statistics. Computations like those in Table 4 can be used to compare different models, where various measures of dispersion can be considered. These measures account for both uncertainty from the additive error terms and coefficient estimates, which puts models on an equal footing if they have similar sets of exogenous variables. Computations like those in Table 5 can be done for a wide variety of policy experiments. Finally, the results in Table 1 show that the bootstrap works well for the US model regarding coverage accuracy.

References

[1] Adelman, I., and F.L. Adelman, 1959, "The Dynamic Properties of the Klein-Goldberger Model," *Econometrica*, 27, 596-625.

[2] Andrews, D.W.K., and W. Ploberger, 1994, "Optimal Tests When a Nuisance Parameter is Present Only Under the Alternative," *Econometrica*, 62,

[3] Berkowitz, J. and L. Kilian, 2000, "Recent Developments in Bootstrapping Time Series," *Econometric Reviews*, 19, 1-48.

[4] Bianchi, C., G. Calzolari, and P. Corsi, 1976, "Divergences in the Results of Stochastic and Deterministic Simulation of an Italian Non Linear Econometric Model," in L. Dekker, ed., *Simulation of Systems*, Amsterdam: North-Holland.

[5] Brown, B.W., and R.S. Mariano, 1984, "Residual-Based Procedures for Prediction and Estimation in a Nonlinear Simultaneous System," *Econometrica*, 52, 321-343.

[6] Calzolari, G., and P. Corsi, 1977, "Stochastic Simulation as a Validation Tool for Econometric Models," paper presented at IIASA Seminar, September 13-15, Laxenburg, Vienna.

[7] Cooper, J.P., 1974, *Development of the Monetary Sector, Prediction and Policy Analysis in the FRB-MIT-Penn Model*, Lexington, Mass.: D.C. Heath.

[8] _____, and S. Fischer, 1972, "Stochastic Simulation of Monetary Rules in Two Macroeconometric Models," *Journal of the American Statistical Association*, 67, 750-760.

[9] _____, 1974, "Monetary and Fiscal Policy in the Fully Stochastic St. Louis Econometric Model," *Journal of Money, Credit and Banking*, 6, 1-22.

[10] Efron, B., 1979, "Bootstrap Methods: Another Look at the Jackknife," *The Annals of Statistics*, 7, 1-26.

[11] Evans, M.K., L. Klein, and M. Saito, 1972, "Short-Run Prediction and Long-Run Simulation of the Wharton Model," in B.G. Hickman, ed., *Econometric Models of Cyclical Behavior*, New York: Columbia University Press, 949-1139.

[12] Fair, R.C., 1974, "On the Solution of Optimal Control Problems as Maximization Problems," *Annals of Economic and Social Measurement*, 3, 135-154.

[13] _____, 1980a, "Estimating the Expected Predictive Accuracy of Econometric Models," *International Economic Review*, 21, 355-378.

[14] _____, 1980b, "Estimating the Uncertainty of Policy Effects in Nonlinear Econometric Models," *Econometrica*, 48, 1381-1391.

[15] _____, 1993a, "Testing the Rational Expectations Hypothesis in Macroeconometric Models," *Oxford Economic Papers*, 45, 169-190.

[16] _____, 1993b, "Estimating Event Probabilities in Macroeconometric Models," in J.H. Stock and M.W. Watson, eds., *Business Cycles, Indicators, and Forecasting*, Chicago: The University of Chicago Press, 157-176.

[17] _____, 1994, *Testing Macroeconometric Models*, Cambridge, MA: Harvard University Press.

[18] _____, 1998, "Estimated Stabilization Costs of the EMU," *National Institute Economic Review*, 164, 90-99.

[19] _____, and William R. Parke, 1995, *The Fair-Parke Program for the Estimation and Analysis of Nonlinear Econometric Models*, available at http://fairmodel.econ.yale.edu.

[20] Fair, R.C., and John B. Taylor, 1983, "Solution and Maximum Likelihood Estimation of Dynamic Rational Expectations Models," *Econometrica*, 51, 1169-1185.

[21] _____, 1990, "Full Information Estimation and Stochastic Simulation of Models with Rational Expectations," *Journal of Applied Econometrics*, 5, 381-392.

[22] Freedman, D., 1981, "Bootstrapping Regression Models," *The Annals of Statistics*, 9, 1218-1228.

[23] _____, 1984, "On Bootstrapping Two-Stage Least-Squares Estimates in Stationary Linear Models," *The Annals of Statistics*, 12, 827-842.

[24] Fromm, G., L.R. Klein, and G.R. Schink, 1972, "Short- and Long-Term Simulations with the Brookings Model," in B.G. Hickman, ed., *Econometric Models of Cyclical Behavior*, New York: Columbia University Press, 201-292.

[25] Garbade, K.D., 1975, *Discretionary Control of Aggregate Economic Activity*, Lexington, Mass.: D.C. Heath.

[26] Green, G.R., M. Liebenberg, and A.A. Hirsch, 1972, "Short- and Long-Term Simulations of the OBE Econometric Model," in B.G. Hickman, ed., *Econometric Models of Cyclical Behavior*, New York: Columbia University Press, 25-123.

[27] Haitovsky, Y., and N. Wallace, 1972, "A Study of Discretionary and Nondiscretionary Monetary and Fiscal Policies in the Context of Stochastic Macroeconometric Models," in V. Zarnowitz, ed., *The Business Cycle Today*, New York: Columbia University Press.

[28] Hall, P., 1988, "Theoretical Comparison of Bootstrap Confidence Intervals," *The Annals of Statistics*, 16, 927-953.

[29] _____, 1992, *The Bootstrap and Edgeworth Expansion*, New York: Springer-Verlag.

[30] _____, and J.L. Horowitz, 1996, "Bootstrap Critical Values for Tests Based on Generalized-Method-of-Moments Estimators," *Econometrica*, 64, 891-916.

[31] Härdle, W., J. Horowitz, and J. Kreiss, 2001, "Bootstrap Methods for Time Series," mimeo.

[32] Horowitz, J.L., 1997, "Bootstrap Methods in Econometrics: Theory and Numerical Performance," Chapter 7 in D.M. Kreps and K.F. Wallis, *Advances in Economics and Econometrics: Theory and Applications*, Volume III, Cambridge: Cambridge University Press, 188-222.

[33] Hurwicz, L., 1950, "Least-Squares Bias in Time Series," in T.C. Koopmans, ed., *Statistical Inference in Dynamic Economic Models*," Cowles Commission Monograph No. 10, New York: John Wiley & Sons, Inc., 365-383.

[34] Kilian, L., 1998, "Small-Sample Confidence Intervals for Impulse Response Functions," *The Review of Economics and Statistics*, 80, 218-230.

[35] Li, H., and G.S. Maddala, 1996, "Bootstrapping Time Series Models," *Econometric Reviews*, 15, 115-158.

[36] MacKinnon, J.G., 2002, "Bootstrap Inference in Econometrics," presidential address at the 2002 Annual Meeting of the Canadian Economics Association.

[37] Muench, T., A. Rolnick, N. Wallace, and W. Weiler, 1974, "Tests for Structural Change and Prediction Intervals for the Reduced Forms of the Two Structural Models of the U.S.: The FRB-MIT and Michigan Quarterly Models," *Annals of Economic and Social Measurement*, 3, 491-519.

[38] Nagar, A.L., 1969, "Stochastic Simulation of the Brookings Econometric Model," in J.S. Duesenberry, G. Fromm, L.R. Klein, and E. Kuh, eds., *The Brookings Model: Some Further Results*, Chicago: Rand McNally.

[39] Orcutt, G.H., 1948, "A Study of the Autoregressive Nature of the Time Series Used for Tinbergen's Model of the Economic System of the United States, 1919-1932," *Journal of the Royal Statistical Society*, Series B, 1-45.

[40] Runkle, D.E., 1987, "Vector Autoregressions and Reality," *Journal of Business & Economic Statistics*, 5, 437-442.

[41] Schink, G.R., 1971, "Small Sample Estimates of the Variance-Covariance Matrix Forecast Error for Large Econometric Models: The Stochastic Simulation Technique," Ph.D. dissertation, University of Pennsylvania.

[42] _____, 1974, "Estimation of Small Sample Forecast Error for Nonlinear Dynamic Models: A Stochastic Simulation Approach," mimeo.

[43] Sowey, E.R., 1973, "Stochastic Simulation For Macroeconomic Models: Methodology and Interpretation," in A.A. Powell and R.W. Williams, eds., *Econometric Studies of Macro and Monetary Relations*, Amsterdam: North-Holland.