THE COMPUTATION OF COUNTERFACTUAL EQUILIBRIA IN HOMOTHETIC WALRASIAN ECONOMIES

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The Computation of Counterfactual Equilibria in Homothetic Walrasian Economies*

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Abstract

We propose a nonparametric test for multiple calibration of numerical general equilibrium models, and we present an effective algorithm for computing counterfactual equilibria in homothetic Walrasian economies, where counterfactual equilibria are solutions to the Walrasian inequalities.

Keywords: Applied general equilibrium analysis, Walrasian inequalities, Calibration

JEL Classification: C63, C68, D51, D58

1 Introduction

Numerical specifications of applied microeconomic general equilibrium models are inherently indeterminate. Simply put, there are more unknowns (parameters) than equations (general equilibrium restrictions). Calibration of parameterized numerical general equilibrium models resolves this indeterminacy using market data from a "benchmark year"; parameter values are gleaned from the empirical literature on production functions and demand functions and the general equilibrium restrictions. The calibrated model allows the simulation and evaluation of alternative policy prescriptions, such

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as changes in the tax structure, by using Scarf's algorithm or one of its variants to compute counterfactual equilibria. Not surprisingly, the legitimacy of calibration as a methodology for specifying numerical general equilibrium models is the subject of an ongoing debate within the profession, ably surveyed by Dawkins et al. (2002). In their survey, they briefly discuss multiple calibration. That is, choosing parameter values for numerical general equilibrium models consistent with market data for two or more years. It is the implications of this notion that we explore in this paper.

Our approach to counterfactual analysis derives from Varian's unique insight that nonparametric analysis of demand or production data admits extrapolation, i.e., "given observed behavior in some economic environments, we can forecast behavior in other environments," Varian (1982, 1984). The forecast behavior in applied general equilibrium analysis is the set of counterfactual equilibria.

Here is an example inspired by the discussion of extrapolation in Varian (1982), illustrating the nonparametric formulation of decidable counterfactual propositions in demand analysis. Suppose we observe a consumer choosing a finite number of consumption bundles x_i at market prices p_i, i.e., $(p_1, x_1), (p_2, x_2), ..., (p_n, x_n)$. If the demand data is consistent with utility maximization subject to a budget constraint, i.e., satisfies GARP, the generalized axiom of revealed preference, then there exists a solution of the Afriat inequalities, U, that rationalizes the data, i.e., if $p_i \cdot x \leq p_i \cdot x_i$ then $U(x_i) \geq U(x)$ for $i = 1, 2, ..., n$, where U is concave, continuous, monotone and nonsatiated (Afriat, 1967; Varian, 1983). Hence we may pose the following question for any two unobserved consumption bundles $\bar{x}$ and $\hat{x}$: Will $\bar{x}$ be revealed preferred to $\hat{x}$ for **every** solution of the Afriat inequalities? An equivalent formulation is the counterfactual proposition: $\bar{x}$ is not revealed preferred to $\hat{x}$ for some price vector p and some utility function U, a solution of the Afriat inequalities.

This proposition can be expressed in terms of the solution set for the following family of polynomial inequalities: The Afriat inequalities for the augmented data set $(p_1, x_1), (p_2, x_2), ..., (p_n, x_n)$, $(p, \hat{x})$ and the inequality $p \cdot \bar{x} > p \cdot \hat{x}$, where p is unobserved. If these inequalities are solvable, then the stated counterfactual proposition is true. If not, then the answer to our original question is yes. Notice that n of the Afriat inequalities are quadratic in the unobservables, i.e., the product of the marginal utility of income at $\hat{x}$ and the price vector p.

We extend the analyses of Brown and Matzkin (1996) and Brown and Shannon (2000), where the Walrasian and dual Walrasian inequalities are derived, to encompass the computation and evaluation of counterfactual equilibria in homothetic Walrasian economies.

Brown and Matzkin (1996) characterized the Walrasian model of competitive market economies for data sets consisting of a finite number of observations on market prices, income distributions and aggregate demand. The Walrasian inequalities, as they are called here, are defined by the Afriat inequalities for individual demand and budget constraints for each consumer; the Afriat inequalities for profit maximization over a convex aggregate technology; and the aggregation conditions that observed aggregate demand is the sum of unobserved individual demands. The Brown–Matzkin theorem states that market data is consistent with the Walrasian model if and only if the Walrasian inequalities are solvable for the unobserved utility levels, marginal utilities of income and individual demands. Since individual demands are assumed to be unobservable, the Afriat inequalities for each consumer are quadratic in the unobservables, i.e., the product of the marginal utilities of income and individual demands.[1]

A decision method for this system of Walrasian inequalities constitutes a specification test for multiple calibration of numerical general equilibrium models, i.e., the market data is consistent with the Walrasian model if and only if the Walrasian inequalities are solvable. In our section on algorithms, we give an effective deterministic algorithm for this decision problem. For every $\varepsilon > 0$, the algorithm computes a finite ε-net of solutions. The algorithm is based on the following observation: There is a finite set of candidate marginal utilities of income (one per agent per observation) such that every set of consumption bundles admitting a solution of the Afriat inequalities with strictly quadratically concave utilities, actually admits a solution with strictly concave utilities with one of our candidate marginal utilities of income. Moreover, this solution is the solution of a linear program. The Walrasian inequalities are solvable if and only if for all sufficiently small ε, the corresponding linear programs are solvable.

An important point is that this set of candidates has cardinality $(1/\varepsilon)^{NT}$ where $\varepsilon > 0$ is a parameter, N is the number of observations and T the number of agents. Hence the algorithm will run in time bounded by a function which is polynomial in the **number of commodities** and exponential only in N and T. In situations involving a large number of commodities and a small N, T, this is very efficient. Note that trade between countries observed over a small number of periods is an example.

[1] The Afriat inequalities for competitive profit maximizing firms are linear given market data — see Varian (1984). Hence we limit our discussion to the nonlinear Afriat inequalities for consumers.

A more challenging problem is the computation of counterfactual equilibria. Fortunately, a common restriction in applied general equilibrium analysis is the assumption that consumers are maximizing homothetic utility functions subject to their budget constraints and firms have homothetic production functions. A discussion of the Afriat inequalities for cost minimization and profit maximization for firms with homothetic production functions can be found in Varian (1984). Afriat (1981) and subsequently Varian (1983) derived a family of inequalities in terms of utility levels, market prices and incomes that characterize consumer's demands if utility functions are homothetic. We shall refer to these inequalities as the homothetic Afriat inequalities.

Following Shoven and Whalley (1992), see page 107, we assume that we observe **all** the exogenous and endogenous market variables in the benchmark equilibrium data sets, used in the calibration exercise. As an example, suppose there is only one benchmark data set, then the homothetic strict Afriat inequalities for each consumer are of the form:[2]

$$\begin{array}{lcl} U^1 \leq \lambda^2 p^2 \cdot x^1 & \text{and} & U^2 \leq \lambda^1 p^1 \cdot x^2 \\ U^1 = \lambda^1 I^1 & & U^2 = \lambda^2 I^2 \end{array}$$

where we observe p^1, x^1 and I^1. Given λ^1 and λ^2 we have a linear system of inequalities in the unobserved U^1, U^2, x^2, p^2 and I^2. A similar set of inequalities can be derived for cost minimizing or profit maximizing firms with production functions that are homogenous of degree one.

As an application of our approach, we revisit the Harberger model of capital income taxation as exposited in Shoven and Whalley (1992). We discuss the simulation and evaluation of a change in the taxation of capital in a homothetic two-sector general equilibrium model, assuming the market data available in multiple calibration of numerical two-sector general equilibrium models.

2 Economic Models

We consider an economy with L commodities and T consumers. Each agent has $\mathbb{R}^L_+$ as her consumption set. We restrict attention to strictly positive market prices $S = \{p \in \mathbb{R}^L_{++} : \sum_{i=1}^L p_i = 1\}$. The Walrasian model assumes that consumers have utility functions $u_t : \mathbb{R}^L_+ \to \mathbb{R}$, income I_t and that

[2] Here we assume utility functions are homogenous of degree one.

aggregate demand $\bar{x} = \sum_{t=1}^{T} x_t$, where

$$u_t(x_t) = \max_{\substack{\text{s.t. } p \cdot x \leq I_t \\ x \geq 0}} u_t(x).$$

Suppose we observe a finite number N of profiles of income distributions $\{I_t^r\}_{t=1}^T$, market prices p^r and aggregate demand $\bar{x}^r$, where $r = 1, 2, ..., N$, but we do not observe the utility functions or demands of individual consumers. When are these data consistent with the Walrasian model of aggregate demand? The answer to this question is given by the following theorems of Brown and Matzkin (1996) and Brown and Shannon (2000).

Theorem 1 (Brown and Matzkin) *There exist nonsatiated, continuous, strictly concave, monotone utility functions* $\{u_t\}_{t=1}^T$ *and* $\{x_t^r\}_{t=1}^T$*, such that* $u_t(x_t^r) = \max_{p^r \cdot x \leq I_t^r} u_t(x)$ *and* $\sum_{t=1}^T x_t^r = \bar{x}^r$*, where* $r = 1, 2, ..., N$*, if and only if* $\exists$ $\{\hat{u}_t^r\}$*,* $\{\lambda_t^r\}$ *and* $\{x_t^r\}$ *for* $r = 1, ..., N;\ t = 1, ..., T$ *such that*

$$\hat{u}_t^r < \hat{u}_t^s + \lambda_t^s p^s \cdot (x_t^r - x_t^s) \ (r \neq s = 1, ..., N;\ t = 1, ..., T) \tag{1}$$

$$\lambda_t^r > 0,\ \hat{u}_t^r > 0 \ \textit{and} \ \ x_t^r \geq 0 \ (r = 1, ..., N;\ t = 1, ..., T) \tag{2}$$

$$p^r \cdot x_t^r = I_t^r \ (r = 1, ..., N;\ t = 1, ..., T) \tag{3}$$

$$\sum_{t=1}^{T} x_t^r = \bar{x}^r \ (r = 1, ..., N) \tag{4}$$

(1) and (2) constitute the strict Afriat inequalities; (3) defines the budget constraints for each consumer; and (4) is the aggregation condition that observed aggregate demand is the sum of unobserved individual consumer demand. This family of inequalities is called here the (strict) Walrasian inequalities.[3] The observable variables in this system of inequalities are the I_t^r, p^r and $\bar{x}^r$, hence this is a nonlinear family of polynomial inequalities in unobservable utility levels $\hat{u}_t^r$, marginal utilities of income λ_t^r and individual consumer demands x_t^r.

Brown and Shannon (2000) proposed an equivalent family of polynomial inequalities in terms of the dual strict Afriat inequalities which we find more useful for our analysis of calibration.

The dual strict Afriat inequalities for each consumer t can be expressed as follows:

$$\hat{v}_t^r > \hat{v}_t^s - \lambda_t^s I_t^s x_t^s \cdot \left(\frac{p^r}{I_t^r} - \frac{p^s}{I_t^s} \right) \ (r \neq s = 1, ..., N;\ t = 1, ..., T) \tag{5}$$

[3] Brown and Matzkin call them the equilibrium inequalities, but there are other plausible notions of equilibrium in market economies.

$$\lambda_t^r > 0 \text{ and } x_t^r \gg 0 \ (r = 1, ..., N;\ t = 1, ..., T) \tag{6}$$

Theorem 2 (Brown and Shannon) *There exist numbers $\hat{v}_t^r$, λ_t^r and vectors x_t^r for $(r = 1, ..., N;\ t = 1, ..., T)$ satisfying the dual strict Afriat inequalities* (5) *and* (6) *if and only if there exist numbers $\hat{u}_t^r$, λ_t^r and vectors x_t^r for $(r = 1, ..., N;\ t = 1, ..., T)$ satisfying the strict Afriat inequalities* (1) *and* (2).

Hence we define the dual strict Walrasian inequalities as (3), (4), (5) and (6) where now the data is consistent with the Walrasian model of aggregate demand if and only if the dual Walrasian inequalities are solvable.

Brown and Shannon (2000) in their Lemma 1 also show that any solution of the dual strict Afriat inequalities gives rise to C^∞ functions $w_t : \mathbb{R}_+^{L+1} \to \mathbb{R}$ where $w_t(p, I)$ is convex in (p/I), strictly increasing in I and strictly decreasing in p such that $w_t(p_t^r, I_t^r) = \hat{v}_t^r$ and $D_{p/I} w_t(p_t^r, I_t^r) = -\lambda_t^r I_t^r x_t^r$ for $(r = 1, ..., N;\ t = 1, ..., T)$. Shannon and Zame (2002) define a smooth convex function $w_t : \mathbb{R}_+^J \to \mathbb{R}$ as (strictly) quadratically convex on a convex subset $Y \subset \mathbb{R}_+^J$ if there is a constant $k_t > 0$ such that for each $x, y \in Y$, where $x \neq y$:

$$w_t(y) > w_t(x) + Dw_t(x) \cdot (y - x) + k_t ||x - y||^2. \tag{7}$$

They point out that any smooth strictly convex function on a compact convex subset $Y \subset \mathbb{R}^J$ is (strictly) quadratically convex. Their observation follows from the second order Taylor expansion of $w_t(x)$ and the fact that $D^2 w_t(x)$ is positive definite on $\mathbb{R}^J$, for all $x \in Y$. Let $\lambda_{\min}(x) =$ minimum eigenvalue of $D^2 w_t(x)$, then $k_t = \min_{x \in Y} \lambda_{\min}(x)$. The inequality in (7) will be used to define the dual strict quadratically convex Afriat inequalities.

The case of homothetic utilities is characterized by the following theorem of Brown and Matzkin (1986).

Theorem 3 (Brown and Matzkin) *There exist nonsatiated, continuous, strictly concave homothetic monotone utility functions $\{u_t\}_{t=1}^T$ and $\{x_t^r\}_{t=1}^T$ such that $u_t(x_t^r) = \max_{p^r \cdot x \leq I_t^r} u_t(x)$ and $\sum_{t=1}^T x_t^r = \bar{x}_t^r$, where $r = 1, 2, ..., N$ if and only if $\exists \{\hat{u}_t^r\}$ and $\{x_t^r\}$ for $r = 1, ..., N$; $t = 1, ..., T$ such that*

$$\hat{u}_t^r < \hat{u}_t^s \frac{p^s \cdot x_t^r}{p^s \cdot x_t^s} \quad (r \neq s = 1, ..., N;\ t = 1, ..., T) \tag{8}$$

$$\hat{u}_t^r > 0 \mathit{and}\ x_t^r \geq 0 \quad (r = 1, ..., N;\ t = 1, ..., T) \tag{9}$$

$$p^r \cdot x_t^r = I_t^r \quad (r = 1, ..., N;\ t = 1, ..., T) \tag{10}$$

$$\sum_{t=1}^{T} x_t^r = \bar{x}^r \quad (r = 1, ..., N) \tag{11}$$

(8) *and* (9) *constitute the strict Afriat inequalities for homothetic utility functions.*

Both the Brown–Matzkin and Brown–Shannon analyses extend to production economies, where firms are price-taking profit maximizers. See Varian (1984) for the Afriat inequalities characterizing the behavior of firms in the Walrasian model of a market economy.

3 Algorithms

3.1 A Nonparametric Test for Multiple Calibration

In multiple calibration, two or more years of market data together with empirical studies on demand and production functions and the general equilibrium restrictions are used to specify numerical general equilibrium models. The maintained assumption is that the market data in each year is consistent with the Walrasian model of market economies. This assumption which is crucial to the calibration approach is never tested, as noted in Dawkins et al. (2002).

The assumption of Walrasian equilibrium in the observed markets is testable, under a variety of assumptions on consumer's tastes, using the necessary and sufficient conditions stated in Theorems 1, 2, and 3 and the market data available in multiple calibration. In particular, Theorem 3 can be used as a specification test for the numerical general equilibrium models discussed in Shoven and Whalley (1992), where it is typically assumed that utility functions are homothetic.

If we observe all the exogenous and endogenous variables, as assumed by Shoven and Whalley, then the specification test is implemented by solving the linear program, defined by (1), (2), (3), and (4) for utility levels and marginal utilities of income or in the homothetic case, solving the linear program defined by (8), (9), (10), and (11) for utility levels.

If individual demands for goods and factors are not observed then the specification test is implemented using the algorithm given in the next section.

Following Varian, we can extrapolate from the observed market data available in multiple calibration to unobserved market configurations. We simply augment the equilibrium inequalities defined by the observed data with additional polynomial inequalities characterizing possible but unobserved market configurations of utility levels, marginal utilities of income, individual demands, aggregate demands, income distributions and equilibrium prices. Counterfactual equilibria are defined as solutions to this augmented family of equilibrium inequalities.

In general, the Afriat inequalities in this system will be cubic because they involve the product of unobserved marginal utilities of income, the unobserved equilibrium prices and unobserved individual demands. This is to be contrasted with observations that include the market prices where the Afriat inequalities are only quadratic in the product of the unobserved marginal utility of income and individual demand. The important exception is the homothetic case where consumers have homothetic utilities and firms have homothetic production functions. Then the relevant Afriat inequalities are only quadratic in the unobservable terms containing the λ's.

3.2 Computing Counterfactual Equilibria

In this section, we describe an algorithm to find solutions of the dual strict Walrasian inequalities. We do so by simply showing that there is a finite set of "candidate" λ's-marginal utilities of income (one for each consumer for each observation) — such that for every solution of x's (individual consumption bundles), one of the candidate λ's works with the x's. First we assume that the observables $\bar{x}^r$, p^r, I_t^r for $r = 1, 2, ..., N$, $t = 1, 2, ..., T$ are given and further, we assume that the dual strict Afriat inequalities were generated by utilities belonging to a C^2-compact family of smooth strictly convex indirect utilities on a compact cube containing $\bar{x}^r$ in its interior for $r = 1, 2, ..., N$. We write the dual strict quadratically convex Afriat inequalities in this case with the substituition $z_t^s = \lambda_t^s I_t^s$, which is made to simplify the computation below :

$$w_t^r > w_t^s - z_t^s x_t^s \cdot \left(\frac{p^r}{I_t^r} - \frac{p^s}{I_t^s} \right) + k \left\| \frac{p^r}{I_t^r} - \frac{p^s}{I_t^s} \right\|^2 \quad (r, s = 1, ..., N;\ t = 1, ..., T) \tag{12}$$

$$z_t^r > 0 \text{ and } x_t^r \gg 0 \quad (r = 1, ..., N;\ t = 1, ..., T). \tag{13}$$

Let $\varepsilon_1 = \min_{\substack{r,s,t \\ r \neq s}} ||p^r/I_t^r - p^s/I_t^s||$.

Lemma 1. *Suppose* $\hat{w}_t^r, \hat{z}_t^r$ and $\hat{x}_t^r$ *solve the dual strict quadratically convex Walrasian inequalities* (3), (4), (12), (13). *Then for any* z_t^r *with* $\hat{z}_t^r \leq z_t^r \leq$

$\hat{z}_t^r + k\varepsilon_1^2 \; \forall t, r$ *we have that* $\hat{v}_t^r = \hat{w}_t^r$, $\lambda_t^r = z_t^r/I_t^r$, $\hat{x}_t^r$ *satisfy the dual strict Walrasian inequalities* (3), (4), (5), (6).

Proof

$$
\begin{aligned}
&\hat{w}_t^s - z_t^s \hat{x}_t^s \left(\frac{p^r}{I_t^r} - \frac{p^s}{I_t^s} \right) \\
&\leq \hat{w}_t^s - \hat{z}_t^s \hat{x}_t^s \cdot \frac{p^r}{I_t^r} + z_t^s \frac{\hat{x}_t^s \cdot p^s}{I_t^s} \text{ since } \hat{z}_t^s \leq z_t^s \text{ and } \hat{x}_t^s \cdot p^r/I_t^r \geq 0 \\
&\leq \hat{w}_t^s - \hat{z}_t^s \hat{x}_t^s \frac{p^r}{I_t^r} + \hat{z}_t^s \frac{\hat{x}_t^s \cdot p^s}{I_t^s} + k\varepsilon_1^2 \text{ using } z_t^s \leq \hat{z}_t^s + k\epsilon_1^2 \text{ and } \hat{x}_t^s \cdot p^s \leq I_t^s \\
&\leq \hat{w}_t^s - \hat{z}_t^s \hat{x}_t^s \left(\frac{p^r}{I_t^r} - \frac{p^s}{I_t^s} \right) + k\varepsilon_1^2 < \hat{w}_t^r \text{ by hypothesis.}
\end{aligned}
$$

If the dual strict Afriat inequalities have a solution, we may scale the w and z so that after scaling, we may assume that all the w and z are between 0 and 1. Similarly for the dual strict quadratically convex Afriat inequalities (12), we may scale w, z as well as k so that after scaling we may assume that the w and z and k are between 0 and 1. We make this assumption.

Now define

$$
S_k = \left\{ z : z_t^r \text{ is an integer multiple of } k\varepsilon_1^2 \quad 0 \leq z_t^r \leq 1 + k\varepsilon_1^2 \right\}.
$$

Then, by the lemma it follows that

Lemma 2 *For any* $\hat{w}_t^r, \hat{z}_t^r \in [0,1], \hat{x}_t^r$ *solving the dual strict quadratically convex Walrasian inequalities* (12), (13), (3), (4)*, there exists a* $z \in S_k$ *such that* $\hat{w}_t^r$, $\lambda_t^r = z_t^r/I_t^r$, $\hat{x}_t^r$ *satisfy the dual strict Walrasian inequalities* — (5), (6), (3), (4).

This immediately yields an algorithm: for fixed $k \in [0,1]$, we enumerate the set S_k and then for each candidate $z \in S_k$, we now solve a **linear program** to see if the inequalities (5), (6), (3) and (4) have a solution. If there is no solution for any z , then we conclude that the system (12), (13), (3), (4) has no solution with this k. Otherwise, we would have found a solution to (5), (6), (3), (4). Note that

$$
|S_k| \leq \frac{(1 + (k\varepsilon_1^2))^{NT}}{(k\varepsilon_1^2)^{NT}} \approx \frac{1}{(k\varepsilon_1^2)^{NT}}, \tag{14}
$$

which is only exponential in NT and independent of L, the number of commodities. For each candidate z, we solve a linear program where the computational time is bounded above by a polynomial in N, T, L. Thus, when the number of observations and number of agents are small compared to the number of commodities, this is a very efficient algorithm. For each k, we execute the algorithm, reducing k by a factor of 2 each time. We stop if no solution has been found for $k = 2^{-M}$. Hence there are at most $\log_2 M$ iterations of the algorithm.

4 The Harberger Tax-Model

We begin by recalling the two-sector model of the US economy. In this model there are two types of households or consumers; two types of firms or producers; two goods; and two factors of production, labor and capital. We assume that consumers have homothetic utility functions and are endowed with the factors of production. Firms have production functions that are homogeneous of degree one, hence make zero profits in equilibrium. Following Harberger, we assume that factors are inelastically supplied.

Assuming there are two years of data available, as is the case in a typical multiple calibration exercise, the Walrasian inequalities for the two-sector model constitute a specification test for the Harberger tax-model. In both years we observe: individual demands of consumers and firms for goods and factors; each consumer's endowment of factors; the income distributions of households; and market prices of goods and factors, where labor is the numeraire good. We can now state the Walrasian inequalities for the two-sector model.

Households:

$$\hat{u}_t^r \leq \lambda_t^s p^s \cdot x_t^r \quad (r \neq s = 1,2;\ t = 1,2) \tag{15}$$

$$\hat{u}_t^r = \lambda_t^r I_t^r \quad (r = 1,2;\ t = 1,2) \tag{16}$$

$$\hat{u}_t^r > 0 \text{ and } x_t^r \geq 0 \quad (r = 1,2;\ t = 1,2) \tag{17}$$

$$p^r \cdot x_t^r = I_t^r \quad (r = 1,2;\ t = 1,2) \tag{18}$$

$$\sum_{t=1}^{2} x_t^r = \bar{x}^r \quad (r = 1,2) \tag{19}$$

Firms:

$$\hat{f}_t^r \leq \gamma_t^s q_t^s \cdot y_t^s \quad (r \neq s = 1,2;\ t = 1,2) \tag{20}$$

$$\hat{f}_t^r = \gamma_t^r J_t^r \tag{21}$$

$$\hat{f}_t^r > 0 \text{ and } y_t^r \geq 0 \quad (r = 1, 2;\ t = 1, 2) \tag{22}$$

$$q_t^r \cdot y_t^r = J_t^r \quad (r = 1, 2;\ t = 1, 2) \tag{23}$$

$$\sum_{t=1}^{2} y_t^r = \bar{y}^r \quad (r = 1, 2) \tag{24}$$

where

x_t^r is consumer t's demand for goods in period r
$\hat{u}_t^r$ is consumer t's utility level in period r
I_t^r is consumer t's income in period r
p^r is the price vector for goods in period r
$\bar{x}^r$ is the aggregate demand for goods in period r
λ_t^r is consumer t's marginal utility of income in period r

and

y_t^r is firm t's demand for factors in period r
$\hat{f}_t^r$ is firm t's output level in period r
J_t^r is firm t's cost in period r
q^r is the price vector for factor demands in period r
$\bar{y}^r$ is the aggregate supply of factors in period r
$1/\gamma_t^r$ is firm t's marginal cost in period r

Equilibrium:

$$\begin{aligned} I_t^r &= q^r \cdot \bar{y}_t^r \quad (r = 1, 2;\ t = 1, 2) \\ \bar{x}^r &= (f_t^r, f_2^r) \quad (r = 1, 2) \\ \bar{y}^t &= \sum_{t=1}^{2} \bar{y}_t^r \quad (r = 1, 2) \end{aligned}$$

$\bar{y}_t^r$ is consumer t's endowment of factors in period r.

If we now impose a per unit tax T on the use of capital by firm 1 then counterfactual equilibria are solutions to the equilibrium inequalities for the two-sector model, augmented in the following manner: The range of r in the equilibrium inequalities is now $\{1, 2, 3\}$ where $q_1^3 = (1, \alpha + T)$, $q_2^3 = (1, \alpha)$ and $\alpha > 0$ where α is the net rate of return and $\alpha + T$ is the gross rate of return — see Chapter 6 in Shoven and Whalley (1992). In addition, we require $p^3 \gg 0$ and consumer's factor endowments to be the same in periods two and three. Tax revenue is redistributed to households as lump sum payments. In each counterfactual equilibrium we compute the social loss due to the tax: $\frac{1}{2} T \Delta K_1$, where ΔK_1 is the change in demand for capital by firm 1. A decidable family of counterfactuals in this model are of the form:

The augmented equilibrium inequalities and the inequality $\frac{1}{2}T\Delta K_1 > \alpha$, where α is known and fixed. Of course, we can also compute the equivalent variation, EV, or compensating variation, CV, for each household, using the utility levels before and after the imposition of the tax on corporate capital. Given the assumption of linear homogenous utility functions:

$$CV = \frac{U^N - U^0}{U^N} \times I^N \text{ and } EV = \frac{U^N - U^0}{U^0} \times I^0,$$

where 0 denotes the equilibrium values before the tax and N denotes the equilibrium values after the tax, see Shoven and Whalley (1992, p. 125).

The homothetic Walrasian inequalities are weak inequalities, if utility functions and production functions are assumed to be homogenous of degree one. Our algorithm is only applicable to a system of strict inequalities, hence we approximate the homogenous of degree one case by a family of inequalities derived from utility and production functions assumed to be homogenous of degree r, where $r \in (0, 1)$ and r is close to 1.

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