LIMIT THEOREMS FOR FUNCTIONALS OF SUMS THAT CONVERGE TO FRACTIONAL STABLE MOTIONS

By

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Limit Theorems for Functionals of Sums that Converge to Fractional Stable Motions

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Abstract. Consider $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, where c_j are constants and ξ_j are iid random variables belonging to the domain attraction of a strictly stable law with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^{k} X_j$. Under certain conditions on c_j, it is known that for a suitable slowly varying function $\kappa_1(n)$ and for a suitable constant $0 < H < 1$, $\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]} \Longrightarrow$ to a fractional stable motion (indexed by H). In addition, it is known that if $f(y)$ is such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$, then $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy$, where L_t^x is the local time of the fractional stable motion at x upto time t.

In this paper we obtain three further results, motivated by asymptotic inference for certain nonlinear time series models. First, we show that if in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then when $1/3 < H < 1$ (which probably cannot be relaxed), $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \Longrightarrow W\sqrt{bL_1^0}$, where W is standard normal, independent of L_1^0, and b is a constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$. (A continuous time version of this result holds also.)

Now let, for $\nu \geq 1$, $\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j} \eta_j$ where $(\xi_j, \eta_j), -\infty < j < \infty$, are iid with ξ_j as before and $E[\eta_1] = 0$, $E[\eta_1^2] < \infty$ and $E[|\eta_1 \xi_1|] < \infty$. Then if $1/3 < H < 1$ as above but possibly $\int_{-\infty}^{\infty} f(y)\, dy \neq 0$, we show that $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \omega_k \Longrightarrow W\sqrt{b^* L_1^0}$. The constant b^* in the limit will be similar to that of b in the first result.

It is further shown that $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k, S_{k+1}, ..., S_{k+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$ for all $0 < H < 1$ and for all suitable $f(x_0, ..., x_r)$, $r \geq 1$, where $f_*(x) = E[f(x, x + S_1, ..., x + S_r)]$.

These convergencies are also shown to hold jointly with certain other random quantities.

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1 INTRODUCTION

Consider a sequence $\xi_j, -\infty < j < \infty$, of iid random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$. We recall that this is equivalent to the statement that for a suitable slowly varying function $\kappa(n)$,

$$t \longmapsto \left(n^{1/\alpha}\kappa(n)\right)^{-1} \sum_{j=1}^{[nt]} \xi_j \stackrel{fdd}{\Longrightarrow} Z_\alpha(t), \ t > 0, \tag{1}$$

where $\{Z_\alpha(t), t > 0\}$ is an α-stable Levy motion, that is, has stationary independent increments such that, for each $0 < t < \infty$,

$$E[e^{iuZ_\alpha(t)}] = \begin{cases} e^{-t|u|^\alpha\left(1-i\beta\,\mathrm{sign}(u)\,\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-t|u|} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \leq 1$. (Here and in the rest of the paper, the notation $\stackrel{fdd}{\Longrightarrow}$ signifies the convergence in distribution of random processes in the sense of convergence in distribution of all finite dimensional distributions.) For the details of the above statement, see for instance Ibragimov and Linnik (1965, Chapter 2, Section 6) or Bingham et al (1987, page 344.). Note that this definition of strict α-stability for the case $\alpha = 1$ differs from the usual one in that we take the skewness parameter β to be 0. When $\alpha = 2$, $Z_2(t)$ becomes the Brownian Motion with variance 2.

In addition, in Theorems 1 and 2 below we shall also assume that

$$\text{When } \alpha = 2, \quad E[\xi_1] = 0 \text{ and } E\left[\xi_1^2\right] < \infty. \tag{2}$$

Now consider the linear process

$$X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}, \ k \geq 1, \tag{3}$$

where ξ_j, $-\infty < j < \infty$, are as earlier with index $0 < \alpha \leq 2$, and c_j, $j \geq 0$, are constants. Let

$$S_k = \sum_{j=1}^{k} X_j.$$

Under suitable conditions (specified in Section 2 below) on the constants c_j it is known that for a suitable H, $0 < H < 1$, and for a slowly varying $\kappa_1(n)$ the process

$$\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

where the limit $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is a *Linear Fractional Stable Motion* (LFSM). It is defined by

$$\Lambda_{\alpha,H}(t) = a \int_{-\infty}^{0} \left\{ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} Z_\alpha(du) + a \int_0^t (t-u)^{H-1/\alpha} Z_\alpha(du)$$

if $H \neq 1/\alpha$, and

$$\Lambda_{\alpha,H}(t) = Z_\alpha(t) \quad \text{if } H = 1/\alpha$$

where a is a non-zero constant and $\{Z_\alpha(t), t \in R\}$ is an α-stable Levy motion, taken to be $Z_\alpha(t)$ as defined earlier for $0 < t < \infty$, and for $-\infty < t < 0$, it is taken to be $Z_\alpha(t) = Z^*_\alpha(-t)$ with $\{Z^*_\alpha(u), 0 < u < \infty\}$ an independent copy of $\{Z_\alpha(u), 0 < u < \infty\}$. See Samorodnitsky and Taqqu (1994) for the details of LFSM.

Note that when $H = 1/\alpha$, the restriction $0 < H < 1$ reduces to $1 < \alpha \leq 2$. When $\alpha = 2$, the LFSM reduces to the *Fractional Brownian Motion.*

Now let $f(y)$ be a function such that $\int \left(|f(y)| + |f(y)|^2 \right) dy < \infty$. Then, under certain further restriction on the distribution of ξ_1, it follows from Jeganathan (2004a, Theorems 2 and 3) that

$$n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy,$$

where L_t^x is the *local time* of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t. See Jeganathan (2004a) for the existence and other details of the local time of the LFSM.

In this paper the first main result (Theorem 1 in Section 2) includes the result that if the restrictions

$$\int |f(y)|^i\, dy < \infty,\ i = 1,2,3,4, \qquad \int_{-\infty}^{\infty} |y f(y)|\, dy < \infty, \tag{4}$$

$$\int_{-\infty}^{\infty} f(y)\, dy = 0, \tag{5}$$

and

$$\frac{1}{3} < H < 1$$

hold, then

$$\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \Longrightarrow W \sqrt{b L_1^0} \tag{6}$$

where W has the standard normal $(0,1)$ distribution independent of L_1^0, and b is a non-negative constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$.

We remark that the restriction $\frac{1}{3} < H < 1$ *probably cannot be relaxed* because it cannot be relaxed in the continuous time version of (6). (We shall briefly indicate this continuous time version in Remark 3 below in Section 2.)

This result is known for the random walk case $S_k = \sum_{j=1}^{k} \xi_j$ (that is, the case $c_j = 0$ for all $j \geq 1$, $c_0 = 1$), see Borodin and Ibragimov (1995, Theorem 3.3 of Chapter IV). For the symmetric Bernoulli random walk case, it was originally discovered by Dobrushin (1955). But note however that many of the structural simplifications available in the random walk case (for example the fact that $S_{l+k} - S_k$ is independent of S_k and has the same distribution as that of S_l) are not available for the present case.

Next let, for some integer $\nu \geq 1$,

$$\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j}\eta_j = \eta_k + d_1\eta_{k-1} + ... + d_{\nu-1}\eta_{k-\nu+1}, \tag{7}$$

where $(\xi_j, \eta_j), -\infty < j < \infty$, are iid ($\xi_j$ are as before) with

$$E[\eta_1] = 0, \quad E\left[\eta_1^2\right] < \infty \quad \text{and} \quad E[|\eta_1\xi_1|] < \infty. \tag{8}$$

Then the second main result (Theorem 2, Section 2) will include the convergence

$$\sqrt{n^{-(1-H)}\kappa_1(n)} \sum_{k=1}^{n} f(S_k)\omega_k \Longrightarrow W\sqrt{b^* L_1^0} \tag{9}$$

where $f(y)$ satisfies all the conditions in (4) but now (5) need not hold, that is, possibly

$$\int_{-\infty}^{\infty} f(y)\, dy \neq 0.$$

The constant b^* in the limit will have the form similar to that of b in (6).

As far as we can determine, Theorem 2 has not been known previously, even for the random walk situation $S_k = \sum_{j=1}^{k} \xi_j$ with $\omega_k = \eta_k$.

Note that the requirement $E[|\eta_1\xi_1|] < \infty$ in (8) implicitly requires certain moment condition on ξ_1. It is satisfied when $\alpha = 2$ because then $E[\xi_1^2] < \infty$ (see (2); $E[\eta_1^2] < \infty$ already by assumption). It is also satisfied, using Cauchy-Schwarz inequality, when

$$E\left[|\eta_1|^{\frac{\gamma}{\gamma-1}}\right] < \infty \text{ for some } 1 < \gamma < \alpha \text{ when } 1 < \alpha < 2.$$

The convergence (9), which is needed in obtaining the asymptotic behavior of least squares or similar estimators in certain nonlinear time series models (Jeganathan and Phillips (2006c)), is one of the primary motivations of the present investigation. We identify

the close relationship between the convergence results (6) and (9). Though unfortunately (9) is not directly deducible from (6), we shall see that, once the relationship has been identified, its proof will use similar ideas involved in (6), and in fact some of the steps can be transported or deducible from those of (6).

As the third main result (which in some form will also be required in obtaining (6) and (9)) we show, when $0 < H < 1$, that $n^{-1}\gamma_n \sum_{l=1}^{n} f(S_l, S_{l+1}, ..., S_{l+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$ where $f_*(x) = E[f(x, x+S_1, ..., x+S_r)]$. We note that the conditions imposed on $f(x_0, ..., x_r)$ exclude the limits of the functionals such as the number of level crossings of $\sum_{j=1}^{k} X_j$; for the treatment of such functionals see Jeganathan (2004b).

The plan of the paper is as follows. The required assumptions as well as the statements of the main results will be stated in Section 2, where it is also noted that the convergencies (6) and (9) can be related to a form of a martingale CLT. (Such a relationship to a martingale CLT is implicit in Borodin and Ibragimov (1995) though the methods employed there are tied in many ways to the iid structure of the random walk case $S_k = \sum_{j=1}^{k} \xi_j$ treated there.) The proof of the Theorems 1 and 2 will then consists of the verification of the conditions of this martingale CLT, which verification will be done in Sections 3 - 5.

Notations. In addition to the $\stackrel{fdd}{\Longrightarrow}$ introduced earlier, the convergence in distribution of a sequence of random variables or random vectors will be signified as usual by $\Longrightarrow$.

As above, L_t^x will stand for the local time of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t. Throughout below we let

$$\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right].$$

For any Borel measurable function $f(y)$ with $\int |f(y)|\, dy < \infty$, $\widehat{f}(\lambda)$ stands for its Fourier transform, that is,

$$\widehat{f}(\lambda) = \int e^{i\lambda y} f(y) dy.$$

We let

$$g(j) = \begin{cases} \sum_{i=0}^{j} c_i & \text{if } j \geq 0 \\ 0 & \text{if } j < 0, \end{cases}$$

where the constants c_i are as in (3) with $c_0 = 1$.

For any real valued function $h(y)$ on $\mathbb{R}^k$ we define $M_{h,\eta}(y) = \sup\{h(u) : |u - y| \leq \eta\}$ and $m_{h,\eta}(y) = \inf\{h(u) : |u - y| \leq \eta\}$.

E_l stands for the conditional expectation given the σ-field $\sigma(\xi_j;\ j \leq l)$.

The normalizing constant $b_n = n^{1/\alpha}\kappa(n)$ (where $\kappa(n)$ is as in (1)) will be used exclusively in the sense of (49) below. Similarly γ_n will be used in the sense of (14) or (50)

below. *Throughout the paper the notation C stands for a generic constant that may take different values at different places of even the same expression in the same proof.*

2 THE MAIN RESULTS AND THE RELATION TO A MARTINGALE CLT

One of the following mutually exclusive conditions will be imposed on the coefficients c_j of the process X_k, where recall that $c_0 = 1$.

(A1) (**The case** $H \neq 1/\alpha$, $0 < H < 1$). $c_j = j^{H-1-1/\alpha}u(j)$, with $H \neq 1/\alpha$, $0 < H < 1$, where $u(j)$ is slowly varying at infinity, satisfying

$$\sum_{j=0}^{\infty} c_j = 0 \quad \text{when } H - 1/\alpha < 0. \tag{10}$$

In addition, there is an integer $l_0 > 0$ and constants c_1 and c_2 such that

$$0 < c_1 \leq \frac{u(l+j_1)}{u(l-j_2)} \leq c_2 \quad \text{for all } 0 \leq j_1, j_2 \leq [l/2] \text{ and } l \geq l_0. \tag{11}$$

(A2) (**The case** $H = 1/\alpha$, $0 < H < 1$). $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} c_j \neq 0$. In addition

$$\sup_{j \geq 1} |jc_j| < \infty. \tag{12}$$

We note that the restriction (11) is automatically satisfied if $u(j)$ is monotone in j, because of the assumption of $u(j)$ being slowly varying. For instance if $u(j)$ is nondecreasing, then $1 \leq \frac{u(l+j_1)}{u(l-j_2)} \leq \frac{u(2l)}{u(l/2)}$ when $0 \leq j_1, j_2 \leq [l/2]$, where $\frac{u(2l)}{u(l/2)} \to 4$ as $l \to \infty$. (We do not know if the monotonicity of $u(j)$ can be assumed without loss of generality, in which case the restriction (11) then holds automatically.)

Note that if (10) is violated, then the case $c_j = j^{H-1-1/\alpha}u(j)$ with $H - 1/\alpha < 0$ comes under (A2). Also it is implicit that $u(j) \neq 0$ for all sufficiently large j.

Remark 1. A motivation of the condition (A1) is what has been called a *Fractional ARIMA* model with stable innovations, a detailed discussion of which can be found for instance in Samorodnitsky and Taqqu (1994, Section 7.13, page 380). In a simplest case of this model, (3) takes the form

$$X_k = (1-B)^{-d}\,\xi_k = \sum_{j=0}^{\infty} c_j\,(-d)\,B^j\xi_k = \sum_{j=0}^{\infty} c_j\,(-d)\,\xi_{k-j} \tag{13}$$

where B is the back-shift operator $B\xi_i = \xi_{i-1}$. Here we have used the formal expansion $(1-B)^{-d} = \sum_{j=0}^{\infty} c_j\,(-d)\,B^j$, so that using Stirling's approximation,

$$c_j\,(-d) = \frac{\Gamma\,(j+d)}{\Gamma\,(d)\,\Gamma\,(j+1)} \sim \frac{1}{\Gamma\,(d)} j^{d-1} \text{ as } j \to \infty \quad \text{if } d \neq 0, -1, ...$$

where $\Gamma(.)$ stands for the gamma function, and $c_j(-d) = 0$ for $j \geq d$ if $d = 0, -1, ...$

Hence if we take $H = d + \frac{1}{\alpha}$, the condition (A1) is satisfied, including (10) because $H - \frac{1}{\alpha} < 0$ is the same as $d < 0$ and hence

$$\sum_{j=0}^{\infty} c_j(-d) = (1-x)^{-d}\Big|_{x=1} = 0 \quad (d < 0).$$

In addition, when $0 < H < 1$, the series (13) converges with probability one (see Samorodnitsky and Taqqu (1994, Theorem 7.13.1, page 381)). ■

Now let

$$\gamma_n = \begin{cases} n^H u(n)\kappa(n) & \text{if (A1) is satisfied} \\ \left(\sum_{j=0}^{\infty} c_j\right) n^{1/\alpha}\kappa(n) & \text{if (A2) is satisfied,} \end{cases} \tag{14}$$

where $\kappa(n)$ is as in (1) and $u(n)$ as in (A1). Then it is known that when (A1) is satisfied, the process $\gamma_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$, $H \neq 1/\alpha$, and similarly when $1 < \alpha \leq 2$ and (A2) is satisfied, $\gamma_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} Z_\alpha(t)$. (See for instance Kasahara and Maejima (1988, Theorems 5.1, 5.2 and 5.3)), Astrauskas (1983) and Avram and Taqqu (1986).) In view of our convention that $Z_\alpha(t) = \Lambda_{\alpha,1/\alpha}(t)$ when $1 < \alpha \leq 2$, the preceding statements will be combined in the form

$$\gamma_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

with the understanding that when (A2) is satisfied the limit is $Z_\alpha(t)$ with $1 < \alpha \leq 2$.

The statement (I) of Theorems 1 and 2 below will assume (recall $\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right]$) what is called the *Cramér's condition*

$$\limsup_{|\lambda| \to \infty} |\psi(\lambda)| < 1, \tag{15}$$

in addition to further restrictions on on $f(x)$. To introduce them, for any real valued function $h(y)$ we define

$$M_{h,\eta}(y) = \sup\{h(u) : |u - y| \leq \eta\}, \qquad m_{h,\eta}(y) = \inf\{h(u) : |u - y| \leq \eta\}.$$

Then we shall require that there is a $\eta_0 > 0$ such that for all $0 < \eta \leq \eta_0$

$$\int (M_{f,\eta}(x) - m_{f,\eta}(x))\, dx \leq C\,|\eta|^d \quad \text{for some } 0 < d \leq 1, \tag{16}$$

$$\int M_{|f|^i,\eta_0}(x)dx < \infty, \quad i = 1, 2, 3, 4, \tag{17}$$

and

$$\int \left(M_{f,\eta}(x) - m_{f,\eta}(x)\right)^2 dx \to 0 \quad \text{as } \eta \to 0. \tag{18}$$

(We note that it can be seen that if $f(x)$ is compactly supported Riemann integrable then both (17) and (18) are the consequences of the condition $\int M_{|f|,\eta_0}(x)dx < \infty$.) In the statement (II) of Theorems 1 and 2 these restrictions are removed, but only under restrictions stronger than the Cramér's condition (15). They are

$$\int |\psi(\lambda)|^2 d\lambda < \infty \tag{19}$$

and

$$\int |\lambda|^3 |\psi(\lambda)|^p d\lambda < \infty \text{ for some } p > 0. \tag{20}$$

Note that because $|\psi(\lambda)| \leq 1$, (20) entails

$$\int |\psi(\lambda)|^p d\lambda < \infty. \tag{21}$$

(This is also implied by (19) for $p \geq 2$.) Also note that (21) entails the Cramér's condition (15).

Remark 2, on the restrictions (19) and (20). Though these restrictions are not involved in the statement (I) of Theorems 1 and 2 below, we now indicate that from the point of view of statistical applications indicated earlier, they are not very restrictive. The restriction (19) entails that the Lebesgue density of the distribution of ξ_1 exists (Kawata (1972, Theorem 11.6.1)). If we denote this density by $\varphi(x)$, then $\psi(\lambda) = \widehat{\varphi}(\lambda)$ and, by Plancherel's theorem, $\int |\psi(\lambda)|^2 d\lambda = 2\pi \int |\varphi(x)|^2 dx$.

Now suppose that the preceding density $\varphi(x)$ has a *distributional derivative* $\varphi'(x)$ such that $\varphi'(x)$ induces a finite signed measure (which will in particular entail $\int |\varphi'(x)| dx < \infty$). Then it can be shown that $\widehat{\varphi}(\lambda) = i\widehat{\varphi'}(\lambda)\lambda^{-1}$ where $\widehat{\varphi'}(\lambda)$ is the Fourier transform of (the signed measure induced by) $\varphi'(x)$. (This follows from standard facts about Fourier transforms and distributional derivatives, see for instance Rudin (1991).) In this case, in addition to (19), (20) holds for $p = 5$ and hence for all $p \geq 5$. This is the case for instance when $\varphi(x)$ is suitably piecewise differentiable. As a simple example suppose that $\varphi(x) = \frac{1}{2}\mathbb{I}_{\{|x|\leq 1\}}$, the density function of the random variable uniformly distributed over the interval $[-1, 1]$. Then the corresponding distributional derivative $\varphi'(x) = -\frac{1}{2}(\delta_1(x) - \delta_{-1}(x))$, where δ_a is the Dirac delta function. ■

We are now in a position to state the results. Throughout below, and without further mentioning, the requirements (A1) and (A2) are assumed to hold, and in addition the requirement (2) is assumed to hold in Theorems 1 and 2 below but not in Theorem 3.

Theorem 1. *Assume that* $1/3 < H < 1$. *Let* $f(x)$ *be Borel measurable such that (4) and (5) hold. Then the following two statements hold.*

(I) Assume that Cramér's condition (15) hold and that (16) - (18) hold for $f(x)$. *Further, let* $h(y)$ *be Borel measurable such that* $\int \left(M_{|h|,\eta_0}(x) + M_{|h|^2,\eta_0}(x) \right) dy < \infty$ *for some* $\eta_0 > 0$ *and* $\int \left(M_{h,\eta}(x) - m_{h,\eta}(x) \right) dx \to 0$ *as* $\eta \to 0$. *Then*

$$\left(\frac{1}{\gamma_n} S_{[nt]}, \frac{\gamma_n}{n} \sum_{k=1}^{n} h\left(S_k\right), \sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f\left(S_k\right) \right) \Longrightarrow \left(\Lambda_{\alpha,H}(t), L_1^0 \int h\left(y\right) dy, W\sqrt{bL_1^0} \right),$$

where L_1^0 *is the local time at* 0 *of* $\Lambda_{\alpha,H}(t)$ *as before,* W *is standard normal independent of the process* $\Lambda_{\alpha,H}(t)$ *and*

$$0 \leq b = \frac{1}{2\pi} \int \left| \widehat{f}\left(\mu\right) \right|^2 \left(1 + 2 \sum_{r=1}^{\infty} E\left[e^{-i\mu S_r} \right] \right) d\mu < \infty.$$

(II) Assume that (19) and (20) hold. Further, let $h(y)$ *be Borel measurable such that* $\int \left(|h(x)| + |h(x)|^2 \right) dy < \infty$. *Then also the convergence in Statement (I) holds.* ■

We note that the requirements on the functions $f(x)$ and $h(x)$ in the statement (I) are stronger than those in the statement (II) but the statement (I) assumes only the Cramér's condition (15). Also note that the marginal convergencies of $\frac{\gamma_n}{n} \sum_{k=1}^{n} h\left(S_k\right)$ in the preceding statements are particular cases of those in Jeganathan (2004a, Theorem 2 and Statement (ii) of Theorem 3), from where it also follows that they hold for all $0 < H < 1$, that is, the restriction $1/3 < H < 1$ is not required. Further, in view of the next remark, the restriction $1/3 < H < 1$ in Theorem 1 cannot probably be relaxed.

Remark 3 We note that the continuous time analogues of Theorem 1, in the forms of generalizations of the appropriate results in for instance Papanicolaou, Strook and Varadhan (1977), Yor (1983) and Rosen (1991), do not follow directly from Theorem 1. The reason is that in the method employed in the present paper the central limit phenomenon is involved at two different levels. One at the familiar level of the partial sum S_k itself, but another at the level of the partial sum of $f(S_k)$ themselves. Despite this one would tend to believe that suitable versions of continuous time analogues will hold. For instance, the following continuous time analogue of Theorem 1 can be proved by adopting essentially the same arguments of the present paper. *If* $\frac{1}{3} < H < 1$ *and if* $f(x)$ *satisfies the requirements*

(4) and (5), then

$$\kappa^{-\frac{1-H}{2}} \int_0^\kappa f\left(\Lambda_{\alpha,H}(t)\right) dt \Longrightarrow W\sqrt{bL_1^0}$$

where W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ and

$$b = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \left|\widehat{f}\,(\mu)\right|^2 E\left[e^{-i\mu\Lambda_{\alpha,H}(t)}\right] d\mu dt.$$

This constant has an alternative form

$$b = 2\int_0^\infty \int_{-\infty}^\infty E\left[f\left(x\right) f\left(x + \Lambda_{\alpha,H}(t)\right)\right] dxdt.$$

Now suppose that the distribution of $\Lambda_{\alpha,H}(t)$ is *symmetric around* 0 (which is not assumed in the preceding statement), and let $\phi\left(|y|\right)$ be the probability density function of $\Lambda_{\alpha,H}(1)$. Then it can be seen that the constant b has also the form

$$b = -2c\int_{-\infty}^\infty \int_{-\infty}^\infty f\left(x\right) |x-y|^{\frac{1}{H}-1} f\left(y\right) dydx,$$

where

$$c = \int_0^\infty \left(\frac{1}{t^H}\phi\left(0\right) - \frac{1}{t^H}\phi\left(\frac{1}{t^H}\right)\right) dt.$$

Note that c is finite only when $\frac{1}{3} < H < 1$.

We shall present elsewhere the further details of the preceding statement, as well as the asymptotic behavior of $L_t^{\varepsilon x} - L_t^0$ as $\varepsilon \to 0$, generalizing the results in Yor (1983) and Rosen (1991).

Regarding these results, it may be noted that the restriction $1/3 < H < 1$ cannot be relaxed, as can be seen from the known regularity properties of L_1^x with respect to the space variable x when L_1^x is the local time of the fractional Brownian motion (see Geman and Horowitz (1980, Table 2)).

As noted earlier, Theorem 2 below has not been known previously, even for the situation $S_k = \sum_{j=1}^k \xi_j$ with $\omega_k = \eta_k$. Its possible continuous time versions in some specific forms have also been unknown. ■

To state the next statement (recall $g(j) = \sum_{i=0}^j c_i$), define (the integer ν is as in (7))

$$\Phi_r\left(\mu\right) = \begin{cases} E\left[\omega_\nu e^{-i\mu\sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}}\right] E\left[\omega_\nu e^{-i\mu\sum_{j=0}^{\nu-1}(g(j+r)-g(j))\xi_{\nu-j}}\right] & \text{if } r \geq \nu \\ E\left[\omega_\nu\omega_{\nu+r} e^{-i\mu\left(\sum_{j=0}^{\nu+r-1} g(\nu+r-j)\xi_{\nu+r-j} - \sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}\right)}\right] & \text{if } 1 \leq r < \nu, \end{cases}$$

and, letting $g(j) = 0$ for $j < 0$,

$$\Psi_r\left(\mu\right) = \prod_{j=1+\nu, j\neq r, \ldots, j\neq r+\nu}^\infty \psi\left(-\left(g(j) - g(j-r)\right)\mu\right).$$

Theorem 2. *Suppose that all the assumptions in* either one *of the statements (I) or (II) of Theorem 1 hold, except that now possibly*

$$\int f(y)\,dy \neq 0.$$

Let the sequence ω_k *be as in (7) with* η_k *satisfying (8).*

Then

$$\left(\frac{1}{\gamma_n}S_{[nt]}, \frac{\gamma_n}{n}\sum_{k=1}^{n} h(S_k), \sqrt{\frac{\gamma_n}{n}}\sum_{k=1}^{n} f(S_k)\,\omega_k\right) \Longrightarrow \left(\Lambda_{\alpha,H}(t), L_1^0\int h(y)\,dy, W\sqrt{b^* L_1^0}\right),$$

where, with $\Phi_r(\mu)$ *and* $\Psi_r(\mu)$ *as defined above,*

$$0 \leq b^* = \frac{1}{2\pi}\int \left|\widehat{f}(\mu)\right|^2 \left(E\left[\omega_\nu^2\right] + 2\sum_{r=1}^{\infty}\Phi_r(\mu)\,\Psi_r(\mu)\right) d\mu < \infty.$$

■

Note that in the case $\nu = 1$ (and hence $\omega_k = \eta_k$), we have for $r \geq 1$,

$$\Phi_r(\mu)\,\Psi_r(\mu) = E\left[\eta_1 e^{-i\mu\xi_1}\right] E\left[\eta_1 e^{-i\mu(g(r)-g(0))\xi_1}\right] \prod_{j=1, j\neq r}^{\infty} \psi\left(-\left(g(j)-g(j-r)\right)\mu\right).$$

Also if we take $\eta_1 \equiv 1$, this reduces to $\Phi_r(\mu)\,\Psi_r(\mu) = E\left[e^{-i\mu S_r}\right]$, as is to be expected.

Theorem 3. *(I). Assume that (19) holds. Let* $f(x_0, ..., x_r)$, $r \geq 1$, *be such that*

$$\int |f(x_0,...,x_r)|^i\,dx_0...dx_r < \infty, \;\; i = 1, 2, \;\; \int\left(\int |f(x_0,...,x_r)|^2\,dx_r\right)^{\frac{1}{2}} dx_0...dx_{r-1} < \infty. \tag{22}$$

Then, for all $0 < H < 1$,

$$\frac{\gamma_n}{n}\sum_{l=1}^{n} f(S_l, S_{l+1}, ..., S_{l+r}) \Longrightarrow L_1^0\int_{-\infty}^{\infty} f_*(x)\,dx$$

where

$$f_*(x) = E\left[f(x, x+S_1, ..., x+S_r)\right].$$

(II). The preceding convergence holds also when (19) is relaxed to (15), provided (22) is assumed to hold when $M_{|f|,\eta}(x_0, ..., x_r)$ *is involved, for some* $\eta > 0$, *in place of* $f(x_0, ..., x_r)$, *and*

$$\int \left(M_{f,\eta}(x_0,...,x_r) - m_{f,\eta}(x_0,...,x_r)\right) dx_0...dx_r \to 0 \quad \text{as } \eta \to 0.$$

■

As noted earlier, the restriction (22) excludes the situation such as the number of level crossings of $\sum_{j=1}^{k} X_j$, see Jeganathan (2006b) for the treatment of such functionals.

Note that the restrictions (20) and $H > 1/3$ are not involved in Theorem 3. Also note that the limit in Theorem 3 involves $f(x)$ only in terms of $\int_{-\infty}^{\infty} f_*(x)\,dx$. Further note that in the case $f(x_0,...,x_r) = f_0(x_0)...f_r(x_r)$, the conditions in (22) hold when $\int |f_l^i(x))|\,dx < \infty$, $i = 1, 2$, $l = 0, ..., r$.

RELATION TO A MARTINGALE CLT. We next relate Theorems 1 and 2 to a martingale CLT. For this purpose, fix an integer $l_0 \geq 2$ and corresponding to Theorem 1 define, for each positive integer m,

$$\zeta_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l), \quad k \geq 1, \tag{23}$$

$$R_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} f(S_l), \quad k \geq 1. \tag{24}$$

Similarly, corresponding to Theorem 2 define (with ω_l as in (7))

$$\zeta^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l)\,\omega_l, \quad k \geq 1, \tag{25}$$

$$R^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} f(S_l)\,\omega_l, \quad k \geq 1. \tag{26}$$

In these definitions we follow the usual convention that a sum is to be interpreted as 0 if it is with respect to an empty index set. Note that

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l) = \sum_{k=1}^{m} (\zeta_{nmk} + R_{nmk}), \tag{27}$$

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l)\,\omega_l = \sum_{k=1}^{m} (\zeta^*_{nmk} + R^*_{nmk}). \tag{28}$$

We shall show that below (Lemma 9), for each l_0,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\left|\sum_{k=1}^{m} R_{nmk}\right| + \left|\sum_{k=1}^{m} R^*_{nmk}\right| > \epsilon\right] = 0 \quad \text{for all } \epsilon > 0, \tag{29}$$

and therefore, the respective limiting behaviors of (27) and (28) will be the same as those of $\sum_{k=1}^{m}\zeta_{nmk}$ and $\sum_{k=1}^{m}\zeta_{nmk}^{*}$ (For this same reason, and for notational convenience, the dependence on l_0 is not explicitly indicated.)

In Sections 3 - 5 below we establish that there is an integer $l_0 > 1$ such that the following facts hold (recall that E_l stands for the conditional expectation given $\sigma(\xi_j;\ j \le l)$).

(R1) There is a nonrandom $\Delta(n,m)$ such that

$$\sum_{k=1}^{m}\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right| \le \Delta(n,m) \to 0 \quad \text{as } n\to\infty, \text{ for each } m.$$

(R2)

$$\sum_{k=1}^{m}E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^{2}\right] \Longrightarrow bL_1^0$$

as $n\to\infty$ first and then $m\to\infty$, where the constant b is as specified in Theorem 1.

Recall that the convergence in distribution of a sequence of distribution functions is metrizable, for example by the Lévy distance (see for instance Loève (1963, page 215)). Then the preceding convergence means that the distribution of $\sum_{k=1}^{m}E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^{2}\right]$ converges in such a metric to that of bL_1^0 as $n\to\infty$ first and then $m\to\infty$.

(R3)

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=1}^{m}E\left[\zeta_{nmk}^{4}\right]=0.$$

The next condition (R4) pertains only to the case $\alpha=2$. To state it define

$$\chi_{nmk}=\frac{1}{\sqrt{n}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\xi_l. \tag{30}$$

(R4) *When* $\alpha=2$ (in which case we have $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$, see (2))

$$\limsup_{n\to\infty}P\left[\sum_{k=1}^{m}\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\chi_{nmk}\right]\right|>\varepsilon\right]=0 \text{ for each } m \text{ and } \varepsilon>0.$$

- **(R*1)** - **(R*2)**: *In the case of Theorem 2, we shall verify the preceding conditions with* ζ_{nmk}^{*} *in place of* ζ_{nmk}*, in which case the corresponding conditions will be referred to as (R*1), (R*2), (R*3) and (R*4).*

Note that the preceding conditions involve iterated limits in the sense that the limits are taken as $n \to \infty$ first and then $m \to \infty$. To proceed further it is convenient to note that they can be restated in an alternative form involving only the index n that goes to ∞. For this purpose recall that if $h(n,m)$ is a nonrandom function of n and m such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} |h(n,m)| = 0$$

then one can find a sequence $m_n \uparrow \infty$ such that

$$h(n,m_n) \to 0.$$

If $G(n,m)$ is random, then note that $G(n,m) \xrightarrow{p} 0$ as $n \to \infty$ first and then $m \to \infty$, that is,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[|G(n,m)| \geq \eta] = 0 \quad \text{for all } \eta > 0,$$

is equivalent to $\lim_{m\to\infty} \limsup_{n\to\infty} E[\min(|G(n,m)|, 1)] = 0$, and therefore, taking $h(n,m) = E[\min(|G(n,m)|, 1)]$, there is a sequence $m_n \uparrow \infty$ such that $E[\min(|G(n,m_n)|, 1)] \to 0$, which is equivalent to

$$G(n,m_n) \xrightarrow{p} 0.$$

Thus (noting that the convergence in (R2) can be restated in terms of a suitable metric), (R1) - (R4) entail that there is a sequence $m_n \uparrow \infty$ such that

$$\sum_{k=1}^{m_n} \left| E_{\left[n\frac{k-1}{m_n}\right]}[\zeta_{nm_nk}] \right| + \sum_{k=1}^{m_n} E\left[\zeta_{nm_nk}^4\right] \xrightarrow{p} 0, \tag{31}$$

$$\sum_{k=1}^{m_n} \left| E_{\left[n\frac{k-1}{m_n}\right]}[\zeta_{nm_nk}\chi_{nm_nk}] \right| \xrightarrow{p} 0 \quad (\text{for } \alpha = 2) \tag{32}$$

and

$$\sum_{k=1}^{m_n} E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}^2\right] \Longrightarrow bL_1^0 \tag{33}$$

In the same way, the conditions (R*1) - (R*4) imply that (31) - (33) hold with ζ_{nmk} replaced by ζ_{nmk}^*.

We are now in a position to present the proof of Theorem 1, when (R1) - (R4) hold. First, for convenience, we let

$$\zeta_{m_nk} = \zeta_{nm_nk}, \quad \chi_{m_nk} = \chi_{nm_nk}, \quad k = 1, ..., m_n.$$

Next, for the purpose of the proof, we

- *extend the array* $\zeta_{m_n k}$, $1 \leq k \leq m_n$, *to all* $k \geq 1$, *by taking* $\{\zeta_{m_n k}; k = m_n + 1, ...\}$ *to be an array of iid Gaussian* $\left(0, \frac{1}{m_n}\right)$ *random variables, independent of* $\{\xi_j; -\infty < j < \infty\}$.

Further, we use the **notation** $E_{m_n,l}$ for the conditional expectation given the σ-field

$$\digamma_{m_n l} = \begin{cases} \sigma\left(\xi_j,\ j \leq \left[n\frac{l}{m_n}\right]\right) & \text{if } 1 \leq l \leq m_n \\ \sigma\left(\xi_j,\ j \leq \left[n\frac{l}{m_n}\right] \text{ and } \zeta_{m_n k}, m_n + 1 \leq k \leq l\right) & \text{if } l > m_n. \end{cases}$$

Explicitly,

$$E_{m_n,l}\left[\ .\ \right] = E\left[\ .\ \middle|\ \digamma_{m_n l}\right].$$

With this extension, (31) and (32) take the strengthened forms, for any $0 < \gamma < 1$,

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{m_n,k-1}\left[\zeta_{m_n k}\right]\right| \to 0, \tag{34}$$

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} E\left[\zeta_{m_n k}^4\right] \to 0, \tag{35}$$

and

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{m_n,k-1}\left[\zeta_{m_n k}\chi_{m_n k}\right]\right| \xrightarrow{p} 0 \quad (\text{for } \alpha = 2). \tag{36}$$

Now, define the martingale differences

$$\zeta'_{m_n k} = \zeta_{m_n k} - E_{m_n,k-1}\left[\zeta_{m_n k}\right], \quad k = 1, 2, ...$$

with respect to the σ-fields $\digamma_{m_n k}, k = 1, 2, ...$It is easily seen, in view of (34), that

$$\text{(35) and (36) hold with } \zeta_{m_n k} \text{ replaced by } \zeta'_{m_n k}. \tag{37}$$

In addition, if we define

$$T_{m_n}(q) = \sum_{k=1}^{q} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\right|^2\right] = \sum_{k=1}^{q} \left\{E_{m_n,k-1}\left[\zeta_{m_n k}^2\right] - \left(E_{m_n,k-1}\left[\zeta_{m_n k}\right]\right)^2\right\},$$

then, in view of (33) and (34) and because $\zeta_{m_n k}$, $k = m_n + 1, ...$ are iid Gaussian $\left(0, \frac{1}{m_n}\right)$, for any $s \geq 1$,

$$T_{m_n}(sm_n) \Longrightarrow bL_1^0 + s - 1, \quad s \geq 1. \tag{38}$$

Now for each fixed $t > 0$, define

$$\tau_{m_n}(t) = \inf\{q \geq 1 : T_{m_n}(q) \geq t\}.$$

Note that

$$\tau_{m_n}(t) = m_n \quad \text{if } t = T_{m_n}(m_n). \tag{39}$$

We have

$$\{\tau_{m_n}(t) \leq l\} = \{T_{m_n}(l) \geq t\} \in \mathcal{F}_{m_n,l-1}, \quad l = 1, 2, ...,$$

so that for each n and $t > 0$,

$$\tau_{m_n}(t) \text{ is a stopping time with respect to the } \sigma\text{-fields } \mathcal{F}_{m_n,l-1}, l = 1, 2, ...$$

Note that for any positive integer J, $P\left[\frac{\tau_{m_n}(t)}{m_n} > J\right] \leq P[T_{m_n}(Jm_n) \leq t]$ and hence, in view of (38),

$$P\left[\frac{\tau_{m_n}(t)}{m_n} > J\right] \to 0 \quad \text{if } J > t - 1. \tag{40}$$

We thus have shown, in view of (34) - (36), (40) and because $m_n \uparrow \infty$,

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}[\zeta_{m_n k}] \xrightarrow{p} 0, \tag{41}$$

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\right|^4\right] \xrightarrow{p} 0 \tag{42}$$

and

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\chi_{m_n k}\right|\right] \xrightarrow{p} 0 \quad (\text{for } \alpha = 2) \tag{43}$$

Further, because of (35), (37) and (40),

$$E_{m_n,\tau_{m_n}(t)-1}\left[\left|\zeta'_{m_n,\tau_{m_n}(t)}\right|^2\right] \xrightarrow{p} 0. \tag{44}$$

Hence, because

$$T_{m_n}(\tau_{m_n}(t)) \geq t \geq T_{m_n}(\tau_{m_n}(t) - 1) = T_{m_n}(\tau_{m_n}(t)) - E_{m_n,\tau_{m_n}(t)-1}\left[\left|\zeta'_{m_n,\tau_{m_n}(t)}\right|^2\right],$$

$$T_{m_n}(\tau_{m_n}(t)) \xrightarrow{p} t. \tag{45}$$

Now let

$$W_n(t) = \sum_{k=1}^{\tau_{m_n}(t)} \zeta'_{m_n k}.$$

By making the convention that the sum $\sum_{k=1}^{\tau_{m_n}(t)}$ is empty when $t < 0$, we may assume for convenience that $W_n(t)$ is defined for all $-\infty < t < \infty$. Similarly, let $W(t)$ be the Brownian motion for $0 \leq t < \infty$ and $W(t) \equiv 0$ for $t < 0$. We then have

Lemma 4. *Let $W(t)$ be as above, and as before let $Z_\alpha(t)$ be the α-stable motion. Then, for $0 < \alpha \leq 2$ and for every $M > 0$ and every integer $l > 0$,*

$$t \longmapsto \left(\frac{1}{n^{1/\alpha}\kappa(n)} \sum_{j=-nl}^{[nt]} \xi_j, W_n(t) \right) \Longrightarrow (Z_\alpha(t) - Z_\alpha(-l), W(t)) \quad \text{in } D_{\mathbb{R}^2}[-l, M],$$

where ($\kappa(n)$ as in (1) and) the processes $W(t)$ and $Z_\alpha(t)$ are independent. Here " $\Longrightarrow$ in $D_{\mathbb{R}^2}[-l, M]$ " signifies the convergence in distribution in the Skorokhod space $D_{\mathbb{R}^2}[-l, M]$.

Proof. The proof consists of reducing the situation to that of Jeganathan (2006a, Theorem 1 and Remarks 5 and 8). **First suppose that $0 < \alpha < 2$.** Note that, with

$$\chi_{m_n k} = \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{l=\left[n\frac{k-1}{m_n}\right]+1}^{\left[n\frac{k}{m_n}\right]} \xi_l, \qquad (\kappa(n) \text{ as in (1)}),$$

we have $\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} = \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{j=-nl}^{\left[n\frac{[m_n t]}{m_n}\right]} \xi_j$. Hence, using (1) and noting that l is an integer,

$$\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} - \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{j=-nl}^{[nt]} \xi_j \xrightarrow{p} 0.$$

Therefore, by (1),

$$\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} \xRightarrow{fdd} Z_\alpha(t) - Z_\alpha(-l).$$

In addition, because $\frac{m_n}{n} \to 0$,

$$\sup_{-m_n l+1 \leq k \leq [m_n M]} P[|\chi_{m_n k}| > \varepsilon] \to 0.$$

The preceding two facts will imply that the conditions (C1) - (C4) of Jeganathan (2006a, Section 2.1) (with the stopping time $k_n(t)$ there taken to be $[m_n t]$) hold for the array $\{\chi_{m_n k}, k = -m_n(l+1), ...\}$ (of independent random variables), with the limit $B_t \equiv 0$ in Condition (C2), see Loève(1963, Section 22.4, Central Convergence Criterion, page 311). Here $B_t \equiv 0$ explicitly means $\sum_{k=-m_n l+1}^{[m_n L]} \sigma^2_{\chi_{m_n k}}(\tau) \xrightarrow{p} 0$ for all $L > 0$ where $\sigma^2_{\chi_{m_n k}}(\tau)$ is the

truncated variance as defined in Jeganathan (2006a) or as in Loève(1963, Condition (ii) of the Central Convergence Criterion, page 311). It is clear that this implies, in view of (40), $\sum_{k=-m_nl+1}^{[m_nM]\vee\tau_{m_n}(M)}\sigma^2_{\chi_{mnk}}(\tau)\xrightarrow{p}0$.

It is also clear from (41), (42) and (45) that the conditions (D1) - (D5) of that paper (with the stopping time $k_n(t)$ there taken to be $\tau_{m_n}(t)$) hold for the the array $\left\{\zeta'_{m_nk}, k=1,...\right\}$, with the limiting triplets (A^*_t, B^*_t, L^*_t) such that $A^*_t\equiv 0\equiv L^*_t$ and $\sum_{k=1}^{\tau_{m_n}(t)}\sigma^2_{\zeta'_{m_nk}}(\tau)\xrightarrow{p}B^*_t\equiv t$. In addition, using (33) - (35) and the fact $\sum_{k=1}^{[m_nM]}E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\right|^2\right]=\sum_{k=1}^{m_n}E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\right|^2\right]+\frac{[m_nM]-m_n}{m_n}$ (recall that $\{\zeta_{m_nk}; k=m_n+1,...\}$ are iid Gaussian $\left(0,\frac{1}{m_n}\right)$), it can be seen that $\sum_{k=1}^{[m_nM]}\sigma^2_{\zeta'_{m_nk}}(\tau)$ is bounded in probability.

Thus all the requirements specified in Jeganathan (2006a, Remarks 5 and 8) are satisfied. This proves the lemma when $0<\alpha<2$**.**

In the case $\alpha=2$, (36) entails that the condition (E2) in Jeganathan (2006a, as modified in Remarks 5) holds. Hence, similar to the case $0<\alpha<2$ above, the proof for this case also follows. This completes the proof of the Lemma. ■

We now come back to the proof of Theorem 1. Because Lemma 4 is true for every $l>0$, it entails (keeping in mind the conditions (A1) and (A2), see Kasahara and Maejima (1988))

$$\left(\gamma_n^{-1}S_{[nt]}, W_n(t)\right)\overset{fdd}{\Longrightarrow}\left(\Lambda_{\alpha,H}(t), W(t)\right)$$

where the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ are independent. Further, in Section 5 (see the Remark 4) below it is shown (or see Jeganathan (2004a, Proposition 6 and Lemmas 7 and 8)) that

$$T_n=\sum_{k=1}^{m_n}E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\right|^2\right]$$

is approximated by a functional of the process $\gamma_n^{-1}S_{[nt]}$ such that T_n converges in distribution if $\gamma_n^{-1}S_{[nt]}\overset{fdd}{\Longrightarrow}\Lambda_{\alpha,H}(t)$. We then have

$$\left(\gamma_n^{-1}S_{[nt]}, W_n(t), T_n\right)\overset{fdd}{\Longrightarrow}\left(\Lambda_{\alpha,H}(t), W(t), bL^0_1\right). \tag{46}$$

(Here note that in general one does not have the convergence of $\gamma_n^{-1}S_{[nt]}$ in the Skorokhod space.)

The next step is to obtain the convergence of $\left(\gamma_n^{-1}S_{[nt]}, W_n(T_n)\right)$, for which we shall need to use, in addition to (46), the fact that the convergence of $W_n(t)$ takes place in Skorokhod space (see the Lemma 4). To present the details, let, with q a positive integer and $J>0$,

$$0=\tau_{q0}<\tau_{q1}<...\tau_{q,q-1}<\tau_{qq}=J$$

be such that

$$\sup_{1\leq i\leq q}|\tau_{qi}-\tau_{q,i-1}|\to 0 \text{ as } q\to\infty.$$

Define

$$T_{n,q,J}=\begin{cases}\tau_{qi} & \text{if } \tau_{qi}\leq T_n<\tau_{q,i+1},\ i=0,1,...q-1,\\ J & \text{if } T_n\geq J.\end{cases}$$

Letting

$$T=bL_1^0,$$

define $T_{q,J}$ analogously. Now, taking $\tau_{q,q+1}=\infty$,

$$\{W_n(T_{n,q,J})\leq v\}=\cup_{i=0}^{q}\{W_n(\tau_{qi})\leq v,\tau_{qi}\leq T_n<\tau_{q,i+1}\}$$

where $\{W_n(\tau_{qi})\leq v,\tau_{qi}\leq T_n<\tau_{q,i+1}\}$ are disjoint, and hence, for $0\leq u_1\leq ...\leq u_k<\infty$ and for any reals $d_j, j=1,...,k$,

$$\begin{aligned}&P\left(W_n(T_{n,q,J})\leq v,\gamma_n^{-1}S_{[nu_j]}\leq d_j,j=1,...,k\right)\\ =\ &P\left(\cup_{i=0}^{q}\left\{W_n(\tau_{qi})\leq v,\tau_{qi}\leq T_n<\tau_{q,i+1},\gamma_n^{-1}S_{[nu_j]}\leq d_j,j=1,...,k\right\}\right)\\ =\ &\sum_{i=0}^{q}P\left(W_n(\tau_{qi})\leq v,\tau_{qi}\leq T_n<\tau_{q,i+1},\gamma_n^{-1}S_{[nu_j]}\leq d_j,j=1,...,k\right).\end{aligned}$$

One can assume without loss of generality that $\tau_{q1},...\tau_{qq}$ are continuity points of T. Then (46) together with the preceding identity entail that

$$\begin{aligned}&P\left(W_n(T_{n,q,J})\leq v,\gamma_n^{-1}S_{[nu_j]}\leq d_j,j=1,...,k\right)\\ \to\ &\sum_{i=0}^{q}P\left(W(\tau_{qi})\leq v,\tau_{qi}\leq T<\tau_{q,i+1},\Lambda_{\alpha,H}(u_j)\leq d_j,j=1,...,k\right)\\ =\ &P\left(W(T_{q,J})\leq v,\Lambda_{\alpha,H}(u_j)\leq d_j,j=1,...,k\right).\end{aligned}$$

In other words, we have

$$\left(W_n(T_{n,q,J}),\gamma_n^{-1}S_{[nt]}\right)\overset{fdd}{\Longrightarrow}\left(W(T_{q,J}),\Lambda_{\alpha,H}(t)\right).$$

(Note that $T_{q,J}$ is a function of L_1^0, which, being a functional of $\Lambda_{\alpha,H}(t)$, is independent of $W(t)$ by Lemma 4.) In addition, because $W_n(t)\Longrightarrow W(t)$ in the Skorkhod space $D[0,M]$ with $W(t)\in C[0,M]$ for every $M>0$, we have

$$\lim_{h\to 0}\limsup_{n\to\infty}P\left[\sup_{|t-s|\leq h,t,s\in[0,M]}|W_n(t)-W_n(s)|>\varepsilon\right]=0$$

for all $\varepsilon > 0$ and all $M > 0$. Hence

$$\lim_{J\to\infty}\lim_{q\to\infty}\limsup_{n\to\infty} P\left[\left|W_n\left(T_{n,q,J}\right) - W_n\left(T_n\right)\right| > \varepsilon\right] = 0.$$

Similarly

$$\lim_{J\to\infty}\lim_{q\to\infty} P\left[\left|W\left(T_{q,J}\right) - W\left(T\right)\right| > \varepsilon\right] = 0.$$

It follows that

$$\left(W_n\left(T_n\right), \gamma_n^{-1} S_{[nt]}\right) \Longrightarrow \left(W(T), \Lambda_{\alpha,H}(t)\right).$$

Noting that $\tau_{m_n}\left(T_n\right) = m_n$ (see (39)) so that $W_n\left(T_n\right) = \sum_{k=1}^{m_n} \zeta_{m_n k}$, and in view of the independence of the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ so that the distribution of $(W(T), \Lambda_{\alpha,H}(t))$ is the same as that of $\left(W\sqrt{bL_1^0}, \Lambda_{\alpha,H}(t)\right)$ where W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ (recall $T = bL_1^0$), the preceding convergence takes the form

$$\left(\sum_{k=1}^{m_n} \zeta_{m_n k}, \gamma_n^{-1} S_{[nt]}\right) \Longrightarrow \left(W\sqrt{bL_1^0}, \Lambda_{\alpha,H}(t)\right) \tag{47}$$

(Recall that $\sum_{k=1}^{m_n} \zeta_{m_n k} = \sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f\left(S_k\right)$.) Now in Section 5 (see the Remark 4) below (or in Jeganathan (2004a, Proposition 6 and Lemmas 7 and 8)) it is shown that $\frac{\gamma_n}{n} \sum_{k=1}^{n} h\left(S_k\right)$ occurring the statement of Theorem 1 is approximated by a functional of the process $\gamma_n^{-1} S_{[nt]}$ such that the former converges in distribution to $L_1^0 \int h\left(y\right) dy$ if $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$. Thus the convergence (47) holds jointly with $n^{-1}\gamma_n \sum_{k=1}^{n} h\left(S_k\right)$. This being the conclusion of Theorem 1, the proof is completed. ■

3 SOME PRELIMINARIES

In this section we first present some preliminaries for the purpose of verification of the requirements (R1) - (R4) and (R*1) - (R*4). In this section itself we shall illustrate the intent of these preliminaries by verifying the conditions (R1) and (R*1).

To begin with recall the fact that ξ_1 belongs to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$, in the sense of Section 1 above, means in particular (see Ibragimov and Linnik (1965, Theorem 2.6.5, page 85)) that, *for all u in some neighborhood of 0,*

$$\psi\left(u\right) = E\left[e^{iu\xi_1}\right] = \begin{cases} e^{-|u|^\alpha G(|u|)\left(1 - i\beta\,\mathrm{sign}(u)\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-|u|G(|u|)} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \leq 1$, where $G(u)$ is slowly varying as $u \to 0$. In particular there are constants $\eta > 0$ and $d > 0$ such that

$$\left|\psi\left(u\right)\right| \leq e^{-d|u|^\alpha G(|u|)} \quad \text{for all } |u| \leq \eta. \tag{48}$$

In addition, if one lets

$$b_n^{-1} = \inf\left\{u > 0 : u^\alpha G(u) = n^{-1}\right\},$$

then $b_n^\alpha \backsim nG(b_n^{-1})$ as $n \to \infty$, and in (1) one can take $\kappa(n) \backsim G^{\frac{1}{\alpha}}(b_n^{-1})$, so that we henceforth assume for convenience that $\kappa(n)$ in (1) and the above b_n are such that

$$b_n = n^{\frac{1}{\alpha}} G^{\frac{1}{\alpha}}(b_n^{-1}) = n^{\frac{1}{\alpha}} \kappa(n). \tag{49}$$

See for instance Bingham et al (1987, page 344) for the details of these facts. Then note that, (14) takes the form

$$\gamma_n = \begin{cases} n^{H-1/\alpha} u(n) b_n & \text{if the condition (A1) is satisfied} \\ (\sum_{j=0}^\infty c_j) b_n & \text{if the condition (A2) is satisfied.} \end{cases} \tag{50}$$

The following result is essentially well-known, and we supply its proof for completeness.

Lemma 5. *Let η be as in (48) and b_n be as in (49). Let κ_j be integers such that for some integer $j_0 > 0$ and a constant $C > 0$,*

$$\kappa_j \geq Cj \quad \text{for all } j \geq j_0. \tag{51}$$

Then for every $0 < c < \alpha$ there is a constant $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq Ce^{-a|\lambda|^c} \quad \text{for all } |\lambda| \leq \eta b_j,\ j \geq 1. \tag{52}$$

Further, if the Cramér's condition $limsup_{|\lambda|\to\infty} |\psi(\lambda)| < 1$ holds, then for every $\delta > 0$ there is a $0 < \rho < 1$ such that

$$\sup_{|\lambda| \geq \delta b_j} \left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} = \sup_{|\mu| \geq \delta} |\psi(\mu)|^{\kappa_j} \leq C\rho^j \quad \text{for all } j \geq 1. \tag{53}$$

Proof. According to (48), $\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j |\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)}$ for all $|\lambda| \leq \eta b_j$. Therefore we first recall a bound for $b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)$ for all sufficiently large j.

According to Potter's inequality (see Bingham et al (1987, Theorem 1.5.6, Statement (ii), page 25), for every $\delta > 0$ there is a $B > 0$ such that $|\frac{G(x)}{G(y)}| \leq B \max\{(x/y)^\delta, (x/y)^{-\delta}\}$ for all $x > 0, y > 0$. In particular $\left|\frac{G\left(b_j^{-1}\right)}{G\left(|\lambda| b_j^{-1}\right)}\right| \leq B \max\{|\lambda|^\delta, |\lambda|^{-\delta}\}$. Because $\max\{|\lambda|^\delta, |\lambda|^{-\delta}\} = |\lambda|^\delta$ if $|\lambda| \geq 1$, it then follows from (49) that there is a j_1 such that

$$b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right) \geq B^{-1} j^{-1} |\lambda|^{-\delta} \quad \text{for all } j \geq j_1 \text{ and } |\lambda| \geq 1.$$

Therefore, by (48), for every $0 < c < \alpha$ there is a $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j |\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)} \leq e^{-a|\lambda|^c} \quad \text{for all } 1 \leq |\lambda| \leq \eta b_j,\ j \geq j_2$$

where $j_2 = \max(j_0, j_1)$ (j_0 as in (51)). On the other hand, if $j \leq j_2$,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq 1 = e^{a|\eta b_j|^c} e^{-a|\eta b_j|^c} \leq \left(\max_{j \leq j_2} e^{a|\eta b_j|^c}\right) e^{-a|\lambda|^c} \quad \text{for all } |\lambda| \leq \eta b_j,\ j \leq j_2.$$

Further,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq 1 = ee^{-1} \leq ee^{-|\lambda|^c} \quad \text{if } |\lambda| \leq 1,\ j \geq 1.$$

Hence the proof of the first part follows from the preceding three inequalities.

Regarding the second part note that the Cramér's condition involved is equivalent to the statement that for every $\delta > 0$, there is a $0 < \tau = \tau(\delta) < 1$ such that

$$\sup_{|\lambda| \geq \delta} |\psi(\lambda)| \leq \tau < 1.$$

Hence the second statement follows, completing the proof of the lemma. ■

The following consequences of Lemma 5 will be used below. First, for any $\kappa \geq 0$,

$$\int_{\{|\lambda| \leq \eta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{[l/2]} d\lambda \leq C \int |\lambda|^\kappa e^{-a|\lambda|^c} d\lambda \leq C, \tag{54}$$

using the Statement (i) of Lemma 5.

Next let l_0 be such that for some $0 < \gamma < 1$, $[l/2] - p \geq [l\gamma]$ for all $l \geq l_0$, where p is as in (20). Then, for any $\delta > 0$ and $0 \leq \kappa \leq 3$, using the Statement (ii) of Lemma 5 and using (20),

$$\begin{aligned} &\int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{[l/2]} d\lambda \\ \leq\ & C\rho^l \int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^p d\lambda = C\rho^l b_l^{1+\kappa} \int |\lambda|^\kappa |\psi(\lambda)|^p d\lambda \leq C\rho_*^l, \quad l \geq l_0, \end{aligned} \tag{55}$$

for some constant $0 < \rho_* < 1$.

We shall also need to use the next inequality, which is a direct consequence of Hölder's inequality, see for instance Hewitt and Stromberg (1965, page 200, Exercise (13.26)). For convenience of reference we state it as a lemma.

Lemma 6. *For any functions $\varphi_i(u) : R^k \to R, i = 1, ..., q,$*

$$\int \prod_{i=1}^q |\varphi_i(u)|\, du \leq \prod_{i=1}^q \left(\int |\varphi_i(u)|^q\, du\right)^{\frac{1}{q}},\ q \geq 1.$$

By replacing $|\varphi_i(u)|$ by $|\ell(u)|^{1/q} |\varphi_i(u)|$ in this inequality, we also have

$$\int |\ell(u)| \prod_{i=1}^q |\varphi_i(u)|\, du \leq \prod_{i=1}^q \left(\int |\ell(u)|\, |\varphi_i(u)|^q\, du\right)^{1/q}, \quad q \geq 1. \tag{56}$$

We now state one consequence of this, which will be used later. For this purpose note that, when (A1) holds,

$$g(j) = \sum_{s=0}^{j} c_s \sim C j^{H-1/\alpha} u(j), \quad j \to \infty.$$

(Note that in the case $H - 1/\alpha < 0$, the requirement $\sum_{j=0}^{\infty} c_j = 0$ (see (10)) is invoked here.) Therefore the requirement (11) on $u(j)$ holds for $g(j)$ also, that is, there is an integer $l_0 > 0$ and constants c_1 and c_2 such that $g(l) \neq 0$ and

$$0 < c_1 \leq \frac{g(l + j_1)}{g(l - j_2)} \leq c_2 \quad \text{for all } 0 \leq j_1, j_2 \leq [l/2]$$

for all $l \geq l_0$. This also entails that, recalling that $\gamma_l = l^{H-1/\alpha} u(l) b_l$ so that $\frac{\gamma_l}{b_l |g(q)|} = \frac{l^{H-1/\alpha} u(l)}{|g(q)|} \sim \left| \frac{g(l)}{g(q)} \right|$, there is an l_0 such that for all $l \geq l_0$,

$$0 < D_1 \leq \frac{\gamma_l}{b_l |g(q)|} \leq D_2 \quad \text{for } [l/2] \leq q < l. \tag{57}$$

Also note that, for p as in (20), there is an l_0 such that for some $0 < \gamma < 1$,

$$l - [l/2] - p \geq [l\gamma] \quad \text{for all } l \geq l_0.$$

Then, for $\delta > 0$ such that $D_1^{-1}\delta = \eta$ with η as in the Statement (i) of Lemma 5, we have for $l \geq l_0$ and $\kappa \geq 0$,

$$\begin{aligned}
& \int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^{\kappa} \prod_{j=0}^{l} \left| \psi\left(\lambda \frac{g(j)}{\gamma_l} \right) \right| d\lambda \\
\leq & \int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^{\kappa} \prod_{j=[l/2]+1}^{l} \left| \psi\left(\lambda \frac{g(j)}{\gamma_l} \right) \right| d\lambda, \quad \text{using } |\psi(\lambda)| \leq 1 \\
\leq & \prod_{j=[l/2]+1}^{l} \left(\int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^{\kappa} \left| \psi\left(\lambda \frac{g(j)}{\gamma_l} \right) \right|^{l-[l/2]} d\lambda \right)^{\frac{1}{l-[l/2]}} \quad \text{by (56)} \\
= & \prod_{j=[l/2]+1}^{l} \left(\left| \frac{\gamma_l}{g(j) b_l} \right|^{1+\kappa} \int_{\left\{ \left| \frac{\gamma_l}{g(j) b_l} \lambda \right| \leq \delta b_l \right\}} |\lambda|^{\kappa} \left| \psi\left(\frac{\lambda}{b_l} \right) \right|^{l-[l/2]} d\lambda \right)^{\frac{1}{l-[l/2]}} \\
\leq & D_2^{1+\kappa} \int_{\left\{ |\lambda| \leq D_1^{-1} \delta b_l \right\}} |\lambda|^{\kappa} \left| \psi\left(\frac{\lambda}{b_l} \right) \right|^{l-[l/2]} d\lambda \leq C, \quad l \geq l_0, \text{ by (54) and (57).}
\end{aligned} \tag{58}$$

In the same way, for every $\delta > 0$, $0 \leq \kappa \leq 3$ and $l \geq l_0$,

$$\int_{\{|\lambda| > \delta b_l\}} |\lambda|^{\kappa} \prod_{j=0}^{l-1} \left| \psi\left(g(j) \frac{\lambda}{\gamma_l} \right) \right| d\lambda \leq C \int_{\left\{ |\lambda| > D_2^{-1} \delta b_l \right\}} |\lambda|^{\kappa} \left| \psi\left(\frac{\lambda}{b_l} \right) \right|^{l-[l/2]} d\lambda \leq C\rho^l, \quad l \geq l_0, \tag{59}$$

using (55), where $0 < \rho < 1$. In addition, noting that $g(0) = 1$ and $|\psi(\lambda)| \leq 1$, for any constants u_l, v_l, h_l such that $\min_{0\leq l\leq l_0} |u_l| > 0$ and $\min_{0\leq l\leq l_0} |v_l| > 0$, we have for $0 \leq l \leq l_0$

$$\begin{aligned} & \max_{l\leq l_0} \left| \int \prod_{q=0}^{l} \left| \psi\left(u_l \frac{\lambda}{g(q)} \right) \right| \left| \widehat{f}(v_l\lambda - h_l) \right| d\lambda \right|^2 \\ \leq & \max_{l\leq l_0} \left| \int \left| \psi(u_l\lambda)\, \widehat{f}(v_l\lambda - h_l) \right| d\lambda \right|^2 \leq \max_{l\leq l_0} \frac{1}{|u_l v_l|} \int |\psi(\lambda)|^2 d\lambda \int \left| \widehat{f}(\lambda) \right|^2 d\lambda \leq C, \end{aligned} \tag{60}$$

where we have used (19) and the fact $\int \left| \widehat{f}(\lambda) \right|^2 d\lambda = 2\pi \int |f(x)|^2 dx < \infty$.

In the context of the statements (I) of Theorems 1 and 2, as well as for the statement (II) of Theorem 3, under the Cramér's condition (15), we shall use a certain smoothing device. To state it, let μ be a finite measure on $\mathbb{R}^k$. Let η be a positive number and K_η be a probability measure on the real line $\mathbb{R}$ satisfying

$$K_\eta(\{x : |x| \leq \eta\}) = 1.$$

Let $h_1, ..., h_k$ be real valued functions on $\mathbb{R}$ such that the product $M_{|h_1|,\eta_1}(x_1)...M_{|h_k|,\eta_k}(x_k)$ is integrable with respect to μ. (Here $M_{h,\eta}(x)$, as well as $m_{h,\eta}(x)$ used below are as defined earlier in Section 2.) Then clearly

$$\begin{aligned} \int h(x_1)...h(x_k) d\mu(x_1, ..., x_k) & \geq \int m_{h_1,\eta_1}(x_1)...m_{h_k,\eta_k}(x_k) d(\mu * K_{\eta_1} * ... * K_{\eta_k})(x_1, ..., x_k) \\ & \leq \int M_{h_1,\eta_1}(x_1)...M_{h_k,\eta_k}(x_k) d(\mu * K_{\eta_1} * ... * K_{\eta_k})(x_1, ..., x_k), \end{aligned} \tag{61}$$

where $*$ stands for the convolution. The probability measure K_η here will be chosen such that its characteristic function $\widehat{K}_\eta(\lambda)$ satisfies

$$|\widehat{K}_\eta(\lambda)| \leq C \exp\{-(\eta|\lambda|)^{1/2}\} \tag{62}$$

for all real λ, where C is a constant (independent of η). This is possible in view of Bhattacharya and Ranga Rao (1976, Corollary 10.4, page 88), where K_η is used extensively as a smoothing device.

Now, similar to (59), we have for every $\kappa > 0$ and for $l \geq l_0$,

$$\begin{aligned} & \int_{\{|\lambda|>\delta b_l\}} |\lambda|^\kappa \prod_{j=0}^{l-1} \left| \psi\left(g(j) \frac{\lambda}{\gamma_l} \right) \right| \left| \widehat{K}_\eta\left(\frac{\lambda}{\gamma_l} \right) \right| d\lambda \\ \leq & C\rho^l \prod_{j=[l/2]+1}^{l} \left(\int |\lambda|^\kappa \left| \widehat{K}_\eta\left(\frac{\lambda}{g(j)\, b_l} \right) \right| d\lambda \right)^{\frac{1}{l-[l/2]}} \leq C\rho^l \left(\frac{\gamma_l}{\eta} \right)^{1+\kappa}, \end{aligned} \tag{63}$$

where we have used (62), together with $\left|\frac{\gamma_l}{g(j)b_l}\right| \geq D_1 > 0$, see (57). Note that (63) is true for all $l \geq 1$ because the left hand side is bounded by

$$\int_{\{|\lambda|>\delta b_l\}} |\lambda|^{\kappa} \left|\widehat{K}_{\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C\left(\frac{\gamma_l}{\eta}\right)^{1+\kappa} \leq C\rho^{-l_0}\rho^{l}\left(\frac{\gamma_l}{\eta}\right)^{1+\kappa} \qquad \text{for } 1 \leq l \leq l_0.$$

Similarly, because in addition $\left|\widehat{K}_{\eta}(\lambda)\right| \leq C$ and in view of (58), we also have, for every $\kappa > 0$,

$$\int_{\{|\lambda|\leq\delta b_l\}} |\lambda|^{\kappa} \prod_{j=0}^{l-1} \left|\psi\left(g(j)\frac{\lambda}{\gamma_l}\right)\right| \left|\widehat{K}_{\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C \qquad l \geq 1. \tag{64}$$

It is important to note that (63) and (64) do not invoke the restrictions (19 and (20).

As a further preliminary, we next introduce a decomposition for S_k which will be repeatedly used throughout below. In this section itself we shall illustrate the intent of this decomposition, as well as the Lemmas 5 and 6, by verifying the conditions (R1) and (R*1). Recall that

$$S_k = \sum_{l=-\infty}^{0} (g(k-l) - g(-l))\xi_l + \sum_{l=1}^{k} g(k-l)\xi_l,$$

where recall that $g(j) = \sum_{s=0}^{j} c_s$. The indicated decomposition is

$$S_k = S_{k,j} + S^*_{k,j}, \;\; 1 \leq j \leq k-1, \tag{65}$$

where

$$S_{k,j} = \sum_{l=-\infty}^{0} (g(k-l) - g(-l))\xi_l + \sum_{l=1}^{k-j} g(k-l)\xi_l \tag{66}$$

and

$$S^*_{k,j} = \sum_{l=k-j+1}^{k} g(k-l)\xi_l = \sum_{q=0}^{j-1} g(q)\,\xi_{k-q}.$$

Here it is important to note that

$$S_{k,j} \text{ and } S^*_{k,j} \text{ are independent.}$$

In addition note that the marginal distribution of $S^*_{k,j}$ is the same as that of $\sum_{i=0}^{j-1} g(i)\,\xi_i$.

Next, in order to deal with ζ^*_{nmk} (see (25), we have (recall that E_j stands for the conditional expectation given $\{\xi_k, k \leq j\}$)

$$E_{k-\nu}\left[f(S_k)\,\omega_k\right] = f_*(S_{k,\nu}) \tag{67}$$

where $S_{k,\nu}$ is as in (65), $\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j}\eta_j$ is as in (7) and

$$\begin{aligned} f_*(x) &= E\left[f\left(x + \sum_{j=k-\nu+1}^{k} g(k-j)\xi_j\right) \sum_{j=k-\nu+1}^{k} d_{k-j}\eta_j\right] \\ &= E\left[f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right) \sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right]. \end{aligned}$$

In the verifications of the conditions (R*1) - (R*4) for the variables ζ^*_{nmk}, the function $f_*(x)$ will essentially take the role of $f(x)$ of Theorem 1, and therefore we need to check that $f_*(x)$ satisfies the conditions (4) and (5) with $f_*(x)$ involved in place of $f(x)$. We state this fact separately.

Lemma 7. *Let $f(x)$ be such that the restriction in (4) hold (but (5) need not hold). Then both the requirements (4) and (5) hold for $f_*(x)$, that is, they hold with $f_*(x)$ involved in place of $f(x)$.*

In addition, when the extra restrictions (16) - (18) in the statement (I) of Theorems 1 and 2 hold for $f(x)$, the same hold for $f_*(x)$ also.

Proof. First note that

$$\begin{aligned} \int |f_*(x)|\,dx &\leq \int E\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right| \left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|\right] dx \\ &= E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right| \int \left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right| dx\right] \\ &= \left(\int |f(x)|\,dx\right) E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|\right] < \infty, \quad \text{by (8)}. \end{aligned} \tag{68}$$

Next, by Cauchy-Schwarz inequality and using $E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|^2\right] \leq C$ (see (8)),

$$f_*^2(x) \leq E\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right] E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|^2\right] \leq CE\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right].$$

This also implies, noting $|f_*(x)|^3 = |f_*^2(x)|^{\frac{3}{2}}$,

$$|f_*(x)|^3 \leq C\left(E\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right]\right)^{\frac{3}{2}} \leq CE\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^3\right].$$

The same holds for $|f_*(x)|^4$. Hence, for $i = 2, 3, 4$,

$$\begin{aligned}\int |f_*(x)|^i \, dx &\leq C \int E\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu - j)\xi_j\right)\right|^i\right] dx \\ &= CE\left[\int \left|f\left(x + \sum_{j=1}^{\nu} g(\nu - j)\xi_j\right)\right|^i dx\right] \\ &= CE\left[\int |f(x)|^i \, dx\right] = C\int |f(x)|^i \, dx < \infty.\end{aligned}$$

The arguments in (68) also entails that, noting that $E\left[\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right] = 0$,

$$\int f_*(x)\, dx = \left(\int f(x)\, dx\right) E\left[\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right] = 0.$$

Next,

$$\int |x f_*(x)|\, dx \leq E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right| \int \left(|x| + \left|\sum_{j=1}^{\nu} g(\nu - j)\xi_j\right|\right) |f(x)|\, dx\right] < \infty$$

by (4) and (8).

The second statement of lemma can be easily verified, by noting for instance that

$$M_{f_*,\eta}(y) = E\left[M_{f,\eta}\left(y + \sum_{j=1}^{\nu} g(\nu - j)\xi_j\right) \sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right].$$

This completes the proof. ■

Note that for the above $M_{f_*,\eta}(y)$, we also have

$$\int M_{f_*,\eta}(y) dy = 0. \tag{69}$$

For the next result we note that, using the condition $\int_{-\infty}^{\infty} |yf(y)|\, dy < \infty$, we have

$$\left|\widehat{f}(\lambda_1) - \widehat{f}(\lambda_2)\right| \leq C|\lambda_1 - \lambda_2|.$$

Now (4) entails that $\widehat{f}(0) = \int_{-\infty}^{\infty} f(y)\, dy = 0$. Thus $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$. We also have $\left|\widehat{f}(\lambda)\right| \leq C$ using $\int |f(y)|\, dy < \infty$. Thus, when (4) and (5) hold,

$$\left|\widehat{f}(\lambda)\right| \leq C \min(|\lambda|, 1). \tag{70}$$

Further, corresponding to $M_{f,\eta}(x)$, though $\int_{-\infty}^{\infty} M_{f,\eta}(y)dy \neq 0$, we have

$$\left|\widehat{M_{f,\eta}}(\lambda) - \widehat{f}\,(\lambda)\right| \leq \int \left(M_{f,\eta}(x) - f(x)\right) dx \leq C\,|\eta|^d$$

using the restriction (16), and hence

$$\left|\widehat{M_{f,\eta}}(\lambda)\right| \leq \left|\widehat{f}\,(\lambda)\right| + \left|\widehat{M_{f,\eta}}(\lambda) - \widehat{f}\,(\lambda)\right| \leq C\min\left(|\lambda|\,, 1\right) + C\,|\eta|^d\,. \tag{71}$$

By Lemma 7, this holds for $M_{f_*,\eta}(y)$ also (but note however we also have (69)).

Lemma 8. *There is a $0 < \rho < 1$ and a positive integer l_0 such that*

$$\sup_{j\geq 0} |E_j\left[f\left(S_{j+l}\right)\right]| \leq \frac{C}{\gamma_l^2} \quad \textit{for all } l \geq l_0,$$

where recall that E_j stands for the conditional expectation given $\{\xi_k, k \leq j\}$.

Under the conclusion of Lemma 7, the same bound holds for $\sup_{j\geq 0} |E_j\left[f\left(S_{j+l}\right)\omega_{j+l}\right]|$ if $l_0 > \nu$, where ω_{j+l} and ν are as in (7).

Proof. First consider the situation of the statement (II) of Theorems 1 and 2, where (58) and (59) hold. We have $f\left(y\right) = \frac{1}{2\pi}\int e^{-i\lambda y}\widehat{f}\,(\lambda)\,d\lambda$. Hence, using (65),

$$f\left(S_{j+l}\right) = \frac{1}{2\pi}\int e^{-i\lambda\left(S_{j+l,l}+S^*_{j+l,l}\right)}\widehat{f}\,(\lambda)\,d\lambda.$$

Therefore, because $S_{j+l,l}$ and $S^*_{j+l,l}$ are independent,

$$\begin{aligned} |E_j\left[f\left(S_{j+l}\right)\right]| &\leq \frac{C}{\gamma_l}\int \left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{j+l,l}}\right]\right| \left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \\ &= \frac{C}{\gamma_l}\int \left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right| \prod_{q=0}^{l-1}\left|\psi\left(\frac{g\left(q\right)}{\gamma_l}\lambda\right)\right| d\lambda, \end{aligned}$$

where we have used $\left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{j+l,l}}\right]\right| = \left|\prod_{i=0}^{l-1}\psi\left(\frac{\lambda g(i)}{\gamma_l}\right)\right|$.

Now let l_0 be such that (58) and (59) hold. Then using (70), if $l \geq l_0$,

$$\int \left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right| \prod_{q=0}^{l-1}\left|\psi\left(\frac{g\left(q\right)}{\gamma_l}\lambda\right)\right| d\lambda \leq \frac{C}{\gamma_l} + C\rho^l \leq \frac{C}{\gamma_l}, \quad l \geq l_0,$$

using $0 < \rho < 1$. This gives the inequality in the statement of the lemma.

Now consider the situation of the statement (I) of Theorems 1 and 2. According to (61),

$$|E_j\left[f\left(S_{j+l}\right)\right]| \leq \max\left(|E_j\left[M_{f,\eta}\left(S_{j+l}+V_\eta\right)\right]|\,, |E_j\left[m_{f,\eta}\left(S_{j+l}+V_\eta\right)\right]|\right),$$

where V_η has the distribution K_η and is independent of S_{j+l} (and E_j stands for the conditional expectation given $\{\xi_k, k \leq j\}$ and V_η). The same arguments above give

$$|E_j\left[M_{f,\eta}\left(S_{j+l}+V_\eta\right)\right]| \leq \frac{C}{\gamma_l}\int \left|\widehat{M_{f,\eta}}\left(\frac{\lambda}{\gamma_l}\right)\right| \prod_{q=0}^{l-1}\left|\psi\left(\frac{g\left(q\right)}{\gamma_l}\lambda\right)\right| \left|K_\eta\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda$$

for all sufficiently small $\eta > 0$. Using (71), together with (63) and (64), we then obtain

$$|E_j\left[M_{f,\eta}\left(S_{j+l}+V_\eta\right)\right]| \leq \frac{C}{\gamma_l^2}+\frac{C}{\gamma_l}\rho^l\left(\frac{\gamma_l}{\eta}\right)^{1+\kappa}+C\frac{|\eta|^d}{\gamma_l}.$$

The same inequality holds for $|E_j\left[m_{f,\eta}\left(S_{j+l}+V_\eta\right)\right]|$. Choosing $\eta = \gamma_l^{-\frac{1}{d}}$, and noting $0 < \rho < 1$, we obtain the required inequality of the lemma. Hence the first part of the lemma follows.

Regarding the second part, it is implied by the first part of the lemma and by the conclusions of Lemma 7, because we have $E_j\left[f\left(S_{j+l}\right)\omega_{j+l}\right] = E_j\left[E_{j+l-\nu}\left[f\left(S_{j+l}\right)\omega_{j+l}\right]\right]$, where $E_{j+l-\nu}\left[f\left(S_{j+l}\right)\omega_{j+l}\right] = f_*\left(S_{j+l,\nu}\right)$ with $f_*\left(x\right)$ as in Lemma 7. (Note that S_{j+l} and $S_{j+l,\nu}$ have the same structure and hence the conclusion of the first part of the lemma for S_{j+l} holds for $S_{j+l,\nu}$ also.) This completes the proof. ■

We next verify (29).

Lemma 9. *(29) holds for each positive integer l_0 (recall that R_{nmk} depends on l_0). More specifically,*

$$\max_{1\leq k\leq m}\left(|R_{nmk}|+|R^*_{nmk}|\right) = O_p\left(\sqrt{\frac{\gamma_n}{n}}\right),$$

where recall that $\frac{\gamma_n}{n} \to 0$.

Proof. First suppose that $k \geq 2$. We have (using the notation $|f|\left(x\right) = |f\left(x\right)|$)

$$E\left[|R_{nmk}|\right] \leq \sqrt{\frac{\gamma_n}{n}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} E\left[|f|\left(S_l\right)\right].$$

Now, according to the arguments of Lemma 8 with $j = 0$ and with $|f|\left(S_l\right)$ in place of $f\left(S_l\right)$ (note that $\left|\widehat{|f|}\left(\lambda\right)\right| \leq \int |f|\left(y\right)dy < \infty$), there is an n_0 and a constant $C > 0$ (both independent of $k \geq 2$) such that

$$\max_{\left[n\frac{k-1}{m}\right]<l\leq\left[n\frac{k-1}{m}\right]+l_0-1} E\left[|f|\left(S_l\right)\right] \leq C \quad \text{for all } n \geq n_0.$$

Thus $E\left[|R_{nmk}|\right] \leq C\sqrt{\frac{\gamma_n}{n}}$ for all $n \geq n_0$ and $k \geq 2$.

In the case $k = 1$, note that $\left(\sqrt{\frac{\gamma_n}{n}}\right)^{-1} R_{nm1} = \sum_{l=1}^{l_0-1} f\left(S_l\right)$, which is a fixed random variable and hence is of order $O_p\left(1\right)$.

In view of Lemma 7 and similar to Lemma 8, the preceding arguments for R_{nmk} apply for R^*_{nmk} also. Hence the lemma follows. ■

We next verify (R1) and (R*1) as a consequence of Lemma 8.

Verification of (R1) and (R*1): First consider (R1) corresponding to ζ_{nmk}, where ζ_{nmk} is as defined in (23). We have, by the first part of Lemma 8, there is an l_0 (independent of $k \geq 1$) such that

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right| \leq \sqrt{\frac{\gamma_n}{n}} \sum_{l=l_0}^{\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]} \left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right| \leq C\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2}.$$

Here recall that $\gamma_n = n^{-H}u(n)$, where $u(n)$ is slowly varying.

Hence if $1/2 \leq H < 1$, it is clear that $\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2} \to 0$.

In the case $0 < H < 1/2$, we have $\sum_{l=1}^{n} \frac{1}{\gamma_l^2} \sim C\frac{n}{\gamma_n^2}$, so that

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2} \sim C\sqrt{n}\gamma_n^{-\frac{3}{2}} = Cn^{-\frac{3H-1}{2}} \left(u(n)\right)^{-\frac{3}{2}}.$$

Because $1/3 < H < 1$, this converges to 0, and hence (R1) is verified. In the same way (R*1) is verified using the second part of Lemma 8. ■

4 VERIFICATION OF (R2), (R*2) AND (R4)

We first consider (R2) and then we shall indicate the modifications required for (R*2). We have

$$\begin{aligned} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right] &= \frac{\gamma_n}{n} \sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right] \\ &\quad + 2\frac{\gamma_n}{n} \sum_{l=l_0}^{n_{mk}} \sum_{r=1}^{n_{mk}-l} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \end{aligned}$$

where and throughout below

$$n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right].$$

Clearly, (R2) is a consequence of the next Lemmas 10 and 11 and Propositions 12 and 13.

Lemma 10. *For each $1 \leq k \leq m$,*

$$\max_{1\leq k\leq m} \frac{\gamma_n}{n} \sum_{l=l_0}^{n_{mk}} \sum_{r=q}^{n_{mk}} E\left[\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right|\right] \to 0$$

as $n \to \infty$ first and then $q \to \infty$.

Lemma 11.

$$\sum_{l=1}^{\infty}\int\left|\psi_{S_l}(\mu)\right|\left|\widehat{f}(\mu)\right|^2 d\mu<\infty,$$

where $\psi_{S_l}(\mu)$ is the characteristic function of S_l. In particular the quantity b defined in Theorem 1 is finite.

Noting that (R2) involves $\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}^2]$, the preceding two lemmas allow us to concentrate, for each $q\geq 1$, on the limit of

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]$$
$$+2\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=l_0}^{n_{mk}}\sum_{r=1}^{q} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$$

as $n\to\infty$ first and then $m\to\infty$. This purpose is served by the next two results.

Proposition 12. *(I). Suppose that (19) holds and that $\int |f(x)|^2\,dy<\infty$. Then*

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\Longrightarrow \widehat{f^2}(0)\,L_1^0$$

as $n\to\infty$ first and then $m\to\infty$.

(II). Suppose that (15) holds and assume that $\int M_{|f|^2,\eta_0}(x)dy<\infty$ for some $\eta_0>0$ and $\int\left(M_{f^2,\eta}(x)-m_{f^2,\eta}(x)\right)dx\to 0$ as $\eta\to 0$. Then the convergence in the statement (I) above holds.

Here, regarding the limit $\widehat{f^2}(0)\,L_1^0$ note that

$$\widehat{f^2}(0)=\int f^2(x)\,dx=\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2 d\mu,$$

where the last equality is obtained by Plancherel's theorem.

In the situation of Theorem 1 discussed above, the next result for the case $w(u,v)=f(u)\,f(v)$ will be required; the general case will be used to obtain Theorem 3 (for $r=1$), as well as to verify (R*2).

Proposition 13. *(I). Assume that (19) holds. Let $w(u,v)$ be such that*

$$\int\int|w(x,y)|^i\,dxdy<\infty,\ \ i=1,2,\qquad \int\left(\int|w(x,y)|^2\,dy\right)^{\frac{1}{2}}dx<\infty.$$

Then for each $r\geq 1$,

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\Longrightarrow\left(\frac{1}{2\pi}\int\psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)L_1^0$$

as $n \to \infty$ first and then $m \to \infty$.

(II). The preceding convergence holds also when (19) is relaxed to (15), provided the restrictions on $w(u,v)$ in the statement (I) are assumed to hold when $M_{|w|,\eta}(x,y)$, for some $\eta > 0$, is involved in place of $w(x,y)$, together with the restriction

$$\int \left(M_{w,\eta}(x,y) - m_{w,\eta}(x,y)\right) dxdy \to 0 \quad \text{as } \eta \to 0.$$

Here it can be easily seen that

$$\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu = \int E\left[w(x,x+S_r)\right] dx.$$

In the case $w(u,v) = f(u)\,f(v)$, the restrictions in the above statement (I) reduce to $\int \left(|f(x)| + |f(x)|^2\right) dx < \infty$ and a similar reduction for the statement (II).

Regarding the proofs we start with

Proof of Lemma 11. In view of (65), $|\psi_{S_l}(\mu)| \leq \left|\prod_{j=0}^{l-1} \psi\left(g(j)\,\mu\right)\right|$. Also $\int \left|\widehat{f}(\mu)\right|^2 d\mu < \infty$. Hence it is enough to show that

$$\sum_{l=l_0}^{\infty} \int \prod_{j=0}^{l} \left|\psi\left(g(j)\,\mu\right)\right| \left|\widehat{f}(\mu)\right|^2 d\mu = \sum_{l=l_0}^{\infty} \frac{1}{\gamma_l} \int \left|\prod_{j=0}^{l} \psi\left(g(j)\,\frac{\mu}{\gamma_l}\right)\right| \left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu \tag{72}$$

is finite, for a suitable l_0. Because $\left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right| \leq C\left|\frac{\mu}{\gamma_l}\right|$ (see (70)), we have using (58) and (59) (with $\kappa = 0$ and with the role of (21) is now being played by $\int \left|\widehat{f}(\mu)\right|^2 d\mu < \infty$),

$$\int \left|\prod_{j=0}^{l-1} \psi\left(g(j)\,\frac{\mu}{\gamma_l}\right)\right| \left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu \leq \frac{C}{\gamma_l^2} + C\rho^l \leq \frac{C}{\gamma_l^2}, \quad l \geq l_0,$$

for a suitable l_0 and for some $0 < \rho < 1$. Hence, (72) is bounded by $C\sum_{l=l_0}^{\infty} \frac{1}{\gamma_l^3}$, where note that $\sum_{l=l_0}^{\infty} \frac{1}{\gamma_l^3} < \infty$ when the assumed restriction $3H > 1$ holds. Hence the proof. ■

For the proofs of the remaining statements, we need to introduce some preliminaries. First recall from (65) that

$$S_{\left[n\frac{k-1}{m}\right]+l} = S_{\left[n\frac{k-1}{m}\right]+l,l} + \sum_{j=1}^{l} g(l-j)\,\xi_{\left[n\frac{k-1}{m}\right]+j}$$

where recall

$$S_{\left[n\frac{k-1}{m}\right]+l,l} = \sum_{j=-\infty}^{0} \left(g\left(\left[n\frac{k-1}{m}\right] + l - j\right) - g(1-j)\right)\xi_j + \sum_{j=1}^{\left[n\frac{k-1}{m}\right]} g\left(\left[n\frac{k-1}{m}\right] + l - j\right)\xi_j.$$

Here note that the right hand side involves the array $\left\{\xi_j : -\infty < j \leq \left[n\frac{k-1}{m}\right]\right\}$ which does not depend on l. We observe that

- The vectors $\left\{S_{\left[n\frac{k-1}{m}\right]+l,l}; 1<l\leq n_{mk}\right\}$ and $\left\{\sum_{j=1}^{l} g\left(l-j\right)\xi_{\left[n\frac{k-1}{m}\right]+j}; 1<l\leq n_{mk}\right\}$ are independent. Further the distribution of $\left\{\sum_{j=1}^{l} g\left(l-j\right)\xi_{\left[n\frac{k-1}{m}\right]+j}; 1<l\leq n_{mk}\right\}$ is the same as that of $\{T_l; 1<l\leq n_{mk}\}$ where

$$T_l=\sum_{j=1}^{l} g\left(l-j\right)\xi_j.$$

Hence we can write

$$\begin{aligned} & E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \\ = \; & E\left[w\left(y_1+T_l,y_2+T_{l+r}\right)\right]_{(y_1,y_2)=\left(S_{\left[n\frac{k-1}{m}\right]+l,l},S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}\right)}. \end{aligned} \tag{73}$$

Letting, for any $0\leq\nu_n<l$ (ν_n will also be allowed to tend to ∞ appropriately),

$$T_{nl}^{*}=\sum_{j=1}^{l-\nu_n} g(l-j)\xi_j, \qquad T_{nl,r}^{*}=\sum_{j=1}^{l-\nu_n} g(l+r-j)\xi_j,$$

we have

$$T_l=T_{nl}^{*}+\sum_{j=l-\nu_n+1}^{l} g(l-j)\xi_j, \qquad T_{l+r}=T_{nl,r}^{*}+\sum_{j=l-\nu_n+1}^{l+r} g(l+r-j)\xi_j.$$

(Note that T_{nl}^{*} and $T_{nl,r}^{*}$ depend on ν_n.) Hence, letting $\widehat{w}\left(\lambda,\mu\right)$ for the corresponding Fourier transform of $w\left(x_1,x_2\right)$, we have for any $0\leq\nu_n<l$,

$$\begin{aligned} & \left(2\pi\right)^2 E\left[w\left(y_1+T_l,y_2+T_{l+r}\right)\right] \\ = \; & \int e^{-i\lambda y_1-i\mu y_2} E\left[e^{-i\lambda T_l-i\mu T_{l+r}}\right]\widehat{w}\left(\lambda,\mu\right)d\lambda d\mu \\ = \; & \int e^{-i\lambda y_1-i\mu y_2} E\left[e^{-i(\lambda+\mu)T_{nl}^{*}-i\mu\left(T_{nl,r}^{*}-T_{nl}^{*}\right)}\right] E\left[e^{-i\lambda\left(T_l-T_{nl}^{*}\right)-i\mu\left(T_{l+r}-T_{nl,r}^{*}\right)}\right]\widehat{w}\left(\lambda,\mu\right)d\lambda d\mu \\ = \; & \frac{1}{\gamma_n}\int e^{-i\frac{\lambda}{\gamma_n}y_1-i\mu(y_2-y_1)} E\left[e^{-i\lambda\frac{T_{nl}^{*}}{\gamma_n}-i\mu\left(T_{nl}^{*}-T_{nl,r}^{*}\right)}\right] E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^{*}\right)-i\mu\left(T_{l+r}-T_{nl,r}^{*}-T_l+T_{nl}^{*}\right)}\right] \\ & \times\widehat{w}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)d\lambda d\mu. \end{aligned} \tag{74}$$

Now (recall $g(j)=0$ if $j<0$)

$$\begin{aligned} & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^{*}\right)-i\mu\left(T_{l+r}-T_{nl,r}^{*}-T_l+T_{nl}^{*}\right)}\right] \\ = \; & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^{*}\right)-i\mu\sum_{j=l-\nu_n+1}^{l+r}(g(l+r-j)-g(l-j))\xi_j}\right] \\ = \; & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^{*}\right)-i\mu\sum_{j=l-\nu_n+1}^{l}\left(c_{l+1-j}+\ldots+c_{l+r-j}\right)\xi_j}\right]\prod_{j=0}^{r-1}\psi\left(-g\left(j\right)\mu\right), \end{aligned}$$

where and throughout below we let

$$c_j = 0 \text{ for } j < 0.$$

Similarly

$$E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\left(T^*_{nl,r}-T^*_{nl}\right)}\right] = \prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu\left(c_{j+1}+...+c_{j+r}\right)\right).$$

Hence

$$\begin{aligned}
&\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\left(T^*_{nl}-T^*_{nl,r}\right)}\right]E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]\right| \\
\leq\ &\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu\left(c_{j+1}+...+c_{j+r}\right)\right)\right|\prod_{j_1=0}^{r-1}\left|\psi\left(-g\left(j_1\right)\mu\right)\right|. \qquad (75)
\end{aligned}$$

With these preliminaries, we now consider the proof of Proposition 13 through a series of steps. (The proof of Lemma 10 will be given in the next section because it involves computations similar to those in the verification of (R3).) In order to state and prove the first step, we need the following result.

Lemma 14. *Let $f(x_0,...,x_r)$, $r \geq 1$, be such that $\int\left(\int\left|f(x_0,...,x_r)\right|^2 dx_r\right)^{\frac{1}{2}} dx_0...dx_{r-1} < \infty$.* Then

$$\sup_{\lambda_0,...\lambda_{r-1},c}\int\left|\widehat{f}(\lambda_0,...\lambda_{r-2},\lambda_{r-1}+c\mu,\mu)\right|^2 d\mu \leq \int\left(\int\left|f(x_0,...,x_r)\right|^2 dx_r\right)^{\frac{1}{2}} dx_0...dx_{r-1}.$$

In particular for $w\left(x,y\right)$ as in the statement (I) of Proposition 13,

$$\sup_{c,\lambda}\int\left|\widehat{w}\left(\lambda+c\mu,\mu\right)\right|^2 d\mu \leq \int\left(\int\left|w\left(x,y\right)\right|^2 dy\right)^{\frac{1}{2}} dx \leq C.$$

Proof. We have by definition

$$\begin{aligned}
&\widehat{f}(\lambda_0,...\lambda_{r-2},\lambda_{r-1}+c\mu,\mu) \\
=\ &\int e^{i\lambda_0x_0+...+i\lambda_{r-2}x_{r-2}+i(\lambda_{r-1}+c\mu)x_{r-1}+i\mu x_r}f(x_0,...,x_r)dx_0...dx_r \\
=\ &\int e^{i\mu x_r}\left\{\int e^{i\lambda_0x_0+...+i\lambda_{r-1}x_{r-1}}f(x_0,...,x_{r-1},x_r-cx_{r-1})dx_0...dx_{r-1}\right\}dx_r.
\end{aligned}$$

Then by Plancherel's theorem, for each $\lambda_0, ...\lambda_{r-1}, c$,

$$\begin{aligned}
&\int \left|\widehat{f}(\lambda_0, ...\lambda_{r-2}, \lambda_{r-1} + c\mu, \mu)\right|^2 d\mu \\
= &\int \left|\int e^{i\lambda_0 x_0 + ... + i\lambda_{r-1}x_{r-1}} f(x_0, ..., x_{r-1}, x_r - cx_{r-1}) dx_0 ... dx_{r-1}\right|^2 dx_r \\
\leq &\int \left|\int |f(x_0, ..., x_{r-1}, x_r - cx_{r-1})|^2 dx_r\right|^{1/2} dx_0 ... dx_{r-1} \\
= &\int \left(\int |f(x_0, ..., x_{r-1}, x_r)|^2 dx_r\right)^{1/2} dx_0 ... dx_{r-1},
\end{aligned}$$

where in obtaining the inequality we have used the generalized Minskowski inequality (see for instance Folland (1984, page 186)). This proves the result. ■

In the statements in the next result we collect some of the approximations we need; in the statement (II) we use the quantities

$$M^*_{w,\eta}(y_1, y_2) = \sup\{w(y_1, u) : |u - y_2| \leq \eta\}, \quad m^*_{w,\eta}(y_1, y_2) = \inf\{w(y_1, u) : |u - y_2| \leq \eta\},$$

and as before K_η is the probability measure concentrated on $\{u : |u| \leq \eta\}$. Note that $M^*_{w,\eta}(y_1, y_2) \leq M_{w,\eta}(y_1, y_2)$ and $m^*_{w,\eta}(y_1, y_2) \geq m_{w,\eta}(y_1, y_2)$.

Lemma 15. *Let $w(x, y)$ be as in Proposition 13, and let $T^*_{nl,r}$ and T^*_{nl}, defined previously, correspond to $2\nu_n < [n\delta]$, $0 < \delta < 1$. Then the following two statements hold (recall $T_l = \sum_{j=1}^{l} g(l-j)\xi_j$)*

(I): Suppose that (19) holds. Let $R_n(y_1, y_2, a, \delta)$ be the difference between

$$(2\pi)^2 \frac{\gamma_n}{n} \sum_{l=[n\delta]+1}^{n} E[w(y_1 + T_l, y_2 + T_{l+r})] \tag{76}$$

and

$$\begin{aligned}
&\frac{1}{n} \sum_{l=[n\delta]+1}^{n} \int_{\{|\mu|\leq a, |\lambda|\leq a\}} U_n(\lambda, \mu, y_1, y_2) E\left[e^{-i\lambda \frac{T^*_{nl}}{\gamma_n} - i\mu\left(T^*_{nl,r} - T^*_{nl}\right)}\right] \\
&\times E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T^*_{nl}\right) - i\mu\left(T_{l+1} - T^*_{nl,r} - T_l + T^*_{nl}\right)}\right] \widehat{w}\left(\frac{\lambda}{\gamma_n} - \mu, \mu\right) d\lambda d\mu
\end{aligned} \tag{77}$$

where

$$U_n(\lambda, \mu, y_1, y_2) = e^{-i\lambda\gamma_n^{-1} y_1 - i\mu(y_2 - y_1)}.$$

Then

$$\lim_{a\to\infty} \limsup_{n\to\infty} \left(\sup_{y_1, y_2} |R_n(y_1, y_2, a, \delta)|\right) = 0 \quad \text{for each } \delta > 0.$$

(II): Suppose that (15) holds (instead of (19)). Let V_η be a r.v. with distribution K_η, independent of (T_l, T_{l+r}). Consider (76) with $E\left[M^*_{w,\eta}(y_1+T_l, y_2+T_{l+r}+V_\eta)\right]$ in place of $E\left[w(y_1+T_l, y_2+T_{l+r})\right]$ and (77) with $\widehat{M^*_{w,\eta}}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\widehat{K}_\eta(\mu)$ in place of $\widehat{w}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)$. Let $R_n(y_1,y_2,a,\delta,\eta)$ be the difference between these two. Then

$$\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|R_n(y_1,y_2,a,\delta,\eta)|\right)=0 \quad \text{for each } \delta,\ \eta>0.$$

The same holds when $m^*_{w,\eta}$ is involved in place of $M^*_{w,\eta}$.

Proof. First consider the statement (I), under (19). Note that (76) involves the left hand side of the identity (74). Further when in (77) the $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ is replaced by $\int_{R^2}$, it reduces to that involving the right hand side of (74). Therefore the difference $R_n(y_1,y_2,a,\delta)$ in the statement (I) of the lemma is simply the same as (77) but with the integral $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ replaced by the $\int_{\{|\mu|\le a,|\lambda|\le a\}^c}$, where $\{|\mu|\le a,|\lambda|\le a\}^c$ stands for the complement of $\{|\mu|\le a,|\lambda|\le a\}$. For notational simplification, we treat the case $r=1$. Then, using (75) and noting that $|\psi(\lambda)|\le 1$, $|U_n(\lambda,\mu,y_1,y_2)|\le C$, and $\{|\mu|\le a,|\lambda|\le a\}^c\subset\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}$, we have

$$\begin{aligned}&|R_n(y_1,y_2,a,\delta)|\\ \le\ &\frac{1}{n}\sum_{l=[n\delta]+1}^{n}\int_{\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\right|\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|d\lambda d\mu\end{aligned}$$

where we have let

$$F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)=\psi(-\mu)\,\widehat{w}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right).$$

Note that $\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|\le\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|$ because $\nu_n<[n\delta]/2\le[l/2]$.

Now using (56),

$$\begin{aligned}&\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\prod_{j=[l/2]}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|d\lambda d\mu\\ \le\ &\prod_{j=[l/2]}^{l-1}\left(\int_{\{|\mu|>a,|\lambda|<\infty\}}F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|^{l-[l/2]}d\lambda d\mu\right)^{\frac{1}{l-[l/2]}}.\end{aligned}\tag{78}$$

Here note that (making the change of variable $\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1} \longmapsto \frac{\lambda}{b_{[n\delta]}}$)

$$\begin{aligned}
&\int_{\{|\mu|>a,|\lambda|<\infty\}} F\left(\frac{\lambda}{\gamma_n} - \mu, \mu\right) \left|\psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1}\right)\right|^{l-[l/2]} d\lambda d\mu \\
= &\frac{\gamma_n}{|g(j)|\, b_l} \int_{\{|\mu|>a,|\lambda|<\infty\}} \left|F\left(\frac{\lambda}{g(j) b_l} - \mu + \mu \frac{c_{j+1}}{g(j)}, \mu\right)\right| \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda d\mu \\
\leq &C \frac{\gamma_n}{\gamma_l} Q_n(a) \int \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda \leq C Q_n(a) \frac{\gamma_n}{\gamma_l}
\end{aligned}$$

where

$$Q(a) = \max_{[n\delta]\leq j\leq n} \sup_v \int_{\{|\mu|>a\}} \left|F\left(v - \mu + \mu \frac{c_{j+1}}{g(j)}, \mu\right)\right| d\mu$$

and we have used the facts $\max_{[l/2]\leq j\leq l} \frac{\gamma_l}{|g(j)| b_l} \leq C$ (see (57)) and $\int \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda \leq C$ (see (54) and (55)). Note that

$$\begin{aligned}
\int_{\{|\mu|>a\}} \left|F\left(\nu - \mu + \mu \frac{c_{j+1}}{g(j)}, \mu\right)\right| d\mu &\leq \sqrt{\int \left|\widehat{w}\left(\nu - \mu + \mu \frac{c_{j+1}}{g(j)}, \mu\right)\right|^2 d\mu \int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu} \\
&\leq C \left(\int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu\right)^{1/2} \qquad (79)
\end{aligned}$$

where in the last step we have used Lemma 14. Thus

$$\begin{aligned}
&\int_{\{|\mu|>a,|\lambda|<\infty\}} \left|F\left(\frac{\lambda}{\gamma_n} - \mu, \mu\right) \prod_{j=[l/2]}^{l-1} \psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1}\right)\right| d\lambda d\mu \\
\leq &C \frac{\gamma_n}{\gamma_l} \left(\int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu\right)^{1/2}.
\end{aligned}$$

We also have $\prod_{j=\nu_n}^{l-1} \left|\psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1}\right)\right| \leq \prod_{j=[[n\delta]/2]}^{[n\delta]} \left|\psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1}\right)\right|$ because $\nu_n < [n\delta]/2 < l/2$. Hence in the same way as above

$$\begin{aligned}
&\int_{\{|\mu|\leq a,|\lambda|>a\}} \left|F\left(\frac{\lambda}{\gamma_n} - \mu, \mu\right) \prod_{j=[[n\delta]/2]}^{[n\delta]} \psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1}\right)\right| d\lambda d\mu \\
\leq &C Q_n^*(a) \int_{\{|\lambda|>d_n a - e_n\}} \left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda
\end{aligned}$$

where

$$e_n = a b_{[n\delta]} \max_{[[n\delta]/2]\leq j\leq [n\delta]} |c_{j+1}| \text{ and } d_n = \min_{[[n\delta]/2]\leq j\leq [n\delta]} \frac{\gamma_n}{|g(j)|\, b_{[n\delta]}}$$

and

$$\begin{aligned} Q_n^*(a) &= \max_{[[n\delta]/2]\leq j\leq[n\delta]} \sup_{\nu} \int_{\{|\mu|\leq a\}} \left| F\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu \\ &\leq C\left(\int_{\{|\mu|\leq a\}} |\psi(\mu)|^2 d\mu\right)^{1/2} \leq C, \qquad \text{similar to (79).} \end{aligned}$$

Note that $d_n \geq d > 0$ for some $d > 0$ (see (57)). In addition $e_n \to 0$. To see this, assume for simplicity that $b_n \sim n^{\frac{1}{\alpha}}$, and $c_j \sim j^{H-1-\frac{1}{\alpha}}$ in the case of assumption (A1). Noting that $H-1-\frac{1}{\alpha} < 0$, we then have $e_n \sim Cn^{H-1}$. In the case of Assumption (A2), we have $|e_n| \leq Cn^{\frac{1}{\alpha}-1}$ where $\frac{1}{\alpha}-1<0$ because $1<\alpha\leq 2$.

Hence there is an n_0 such that $\{|\lambda| > d_n a - e_n\} \subset \left\{|\lambda| > \frac{d}{2}a\right\}$ for all $n \geq n_0$. Hence using (54) and (55)

$$\int_{\{|\lambda|>d_n a-e_n\}} \left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda \leq C\int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]},$$

where $0<\rho<1$. Thus $|R_n(y_1,y_2,a,\delta)|$ is bounded by

$$\leq C\left(\int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu\right)^{1/2}\left(\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\right) + C\int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]}$$

for all $n \geq n_0$. In view of (19) and the fact $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \leq C$, this completes the proof of the first statement.

The proof of the statement (II) is the same as that of the first, except that the role of $\psi(\mu)$ (for instance in (79)) is now played by $\widehat{K}_\eta(\mu)$. This completes the proof of the lemma. ■

Lemma 16. *Let $K_n(y_1,y_2,a,\delta)$, $\delta>0$, be the difference between*

$$(2\pi)^2 \frac{\gamma_n}{n}\sum_{l=l_0}^{n} E\left[w\left(y_1+T_l, y_2+T_{l+r}\right)\right] \tag{80}$$

and

$$\frac{1}{n}\sum_{l=[n\delta]}^{n}\int_{\{|\mu|\leq a,|\lambda|\leq a\}} U_n(\lambda,\mu,y_1,y_2)\, E\left[e^{-i\lambda\gamma_n^{-1}T_l}\right]\psi_{S_r}(-\mu)\,\widehat{w}(-\mu,\mu)\,d\lambda d\mu \tag{81}$$

where $U_n(\lambda,\mu,y_1,y_2) = e^{-i\lambda\gamma_n^{-1}y_1 - i\mu(y_2-y_1)}$ as in the statement (I) Lemma 15 (and $\psi_{S_r}(\mu) = E\left[e^{i\mu S_r}\right]$ as before). Then

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|K_n(y_1,y_2,a,\delta)|\right) = 0.$$

Similarly $(2\pi)^2 \frac{\gamma_n}{n} \sum_{l=l_0}^{n} E\left[M^*_{w,\eta}(y_1 + T_l, y_2 + T_{l+r} + V_\eta)\right]$, *corresponding to the statement (II) of Lemma 15, has the approximation given by (81) but with* $\widehat{M^*_{w,\eta}}(-\mu,\mu)\,\widehat{K}_\eta(\mu)$ *involved in place of* $\widehat{w}(-\mu,\mu)$. *The same holds for* $(2\pi)^2 \frac{\gamma_n}{n} \sum_{l=l_0}^{n} E\left[m^*_{w,\eta}(y_1 + T_l, y_2 + T_{l+r} + V_\eta)\right]$.

Proof. It is enough to prove the approximation in the first statement. According to (74) and (75), we have

$$\begin{aligned} & |E[w(y_1 + T_l, y_2 + T_{l+r})]| \\ \leq\ & \frac{C}{\gamma_l} \int \prod_{j=[l/2]}^{l-1} \left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - \mu\left(c_{j+1} + ... + c_{j+r}\right)\right)\right| |\psi(-\mu)| \left|\widehat{w}\left(\frac{\lambda}{\gamma_l} - \mu, \mu\right)\right| d\lambda d\mu. \end{aligned}$$

According to the arguments contained in the in the first part of the preceding proof of Lemma 15, this is bounded by $\frac{C}{\gamma_l}$. Thus

$$\sup_{y_1,y_2} \left| \frac{\gamma_n}{n} \sum_{l=1}^{[n\delta]} E\left[f(y_1 + T_l) f(y_2 + T_{l+r})\right] \right| \leq C \frac{\gamma_n}{n} \sum_{l=1}^{[n\delta]} \frac{1}{\gamma_l}.$$

Clearly this converges to 0 as $n \to \infty$ first and then $\delta \to 0$.

Hence, in view of Lemma 15, letting $R^*_n(y_1, y_2, a, \delta)$ for the difference between (80) and (77),

$$\lim_{\delta \to 0} \lim_{a \to \infty} \limsup_{n \to \infty} \left(\sup_{y_1,y_2} |R^*_n(y_1, y_2, a, \delta)| \right) = 0.$$

Therefore, letting $R^{**}_n(y_1, y_2, a, \delta)$ for the difference between (77) and (81), it is enough to show that

$$\limsup_{n \to \infty} \left(\sup_{y_1,y_2} |R^{**}_n(y_1, y_2, a, \delta)| \right) = 0 \text{ for each } a, \delta. \tag{82}$$

Note that without loss of generality, we can assume that ν_n, upon which $T^*_{nl,r}$ and T^*_{nl} of Lemma 15 depend, is such that $\nu_n \to \infty$ and $\frac{\nu_n}{n} \to 0$. Then, because $T_l - T^*_{nl}$ and $\sum_{s=0}^{\nu_n - 1} g(s)\,\xi_s$ have the same distribution,

$$\sup_{[n\delta] \leq l \leq n} P\left(\gamma_n^{-1} |T_l - T^*_{nl}| > \epsilon\right) = P\left(\left| \gamma_n^{-1} \sum_{s=0}^{\nu_n - 1} g(s)\,\xi_s \right| > \epsilon \right) \to 0,$$

where we have used the fact that $\gamma_{\nu_n}^{-1} \sum_{s=0}^{\nu_n - 1} g(s)\,\xi_s$ converges in distribution and $\gamma_n^{-1}\gamma_{\nu_n} \to 0$. Hence

$$\sup_{|\lambda| \leq a, |\mu| \leq a, [n\delta] \leq l \leq n} \left| E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T^*_{nl}\right) - i\mu\left(T_{l+1} - T^*_{nl,r} - T_l + T^*_{nl}\right)} \right] - E\left[e^{-i\mu\left(T_{l+r} - T^*_{nl,r} - T_l + T^*_{nl}\right)} \right] \right| \to 0.$$

Further, noting that

$$E\left[e^{-i\mu\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right] = \prod_{j=0}^{\nu_n+r-1} \psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)$$

and

$$\prod_{j=0}^{\infty} \psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right) = \psi_{S_r}\left(-\mu\right) \tag{83}$$

we have (with r being fixed), because $\nu_n \to \infty$,

$$\sup_{|\mu|\leq a}\left|E\left[e^{-i\mu\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]-\psi_{S_r}\left(-\mu\right)\right| \to 0.$$

Now $\left|T^*_{nl,r}-T^*_{nl}\right| = \left|\sum_{j=\nu_n}^{l-1}(c_{j+1}+...+c_{j+r})\xi_j\right|$. Let $0<\tau<\alpha$ be suitably close to α such that $\sum_{j=\nu_n}^{\infty}|c_j|^\tau \to 0$. Then

$$\begin{aligned}\sup_{[n\delta]\leq l<\infty} P\left(\left|T^*_{nl,r}-T^*_{nl}\right|>\varepsilon\right) &= \sup_{[n\delta]\leq l<\infty} P\left(\left|\sum_{j=\nu_n}^{l-1}(c_{j+1}+...+c_{j+r})\xi_j\right|>\varepsilon\right)\\ &\leq Cr\sum_{j=\nu_n}^{\infty}|c_j|^\tau \to 0,\end{aligned} \tag{84}$$

where the inequality is obtained using for instance Avram and Taqqu (1986, Lemma 1, Section 3, page 408)). Hence

$$\sup_{|\lambda|\leq b,|\mu|\leq a,[n\delta]\leq l\leq n}\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\left(T^*_{nl,r}-T^*_{nl}\right)}\right]-E\left[e^{-i\lambda\frac{T_l}{\gamma_n}}\right]\right| \to 0 \text{ as } n\to\infty$$

Hence (82) follows. This completes the proof of the Lemma. ■

The preceding Lemma 16 leads to the next statement where we define

$$\begin{aligned}S\left(\frac{k-1}{m},\frac{t}{m}\right) &= c\int_{-\infty}^{0}\left\{\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z_\alpha(du)\\ &\quad +c\int_0^{\frac{k-1}{m}}\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}Z_\alpha(du)\end{aligned} \tag{85}$$

and

$$T\left(t\right) = \int_0^t (t-u)^{H-1/\alpha}\, Z_\alpha(du). \tag{86}$$

Note that

$$S\left(\frac{k-1}{m},0\right) = \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right).$$

Lemma 17. *For each integer $m \geq 1$,*

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$$

where $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ and $T(t)$ are as defined above in (85) and (86).

The same holds for $\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[M^*_{w,\eta}\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}+V_\eta\right)\right]$ but in the limit $\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{M^*_{w,\eta}}(-\mu,\mu)\,\widehat{K}_\eta(\mu)\,d\mu$ involved in place of $\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu$. The same holds for $m^*_{w,\eta}$ also. (See Lemma 15 for $M^*_{w,\eta}(x,y)$ and $m^*_{w,\eta}(x,y)$.)

Proof. We consider only the first statement because the proofs for the remaining statements are the same. Because $\frac{\gamma_{n_{mk}}}{n_{mk}}\frac{n}{\gamma_n}\sim m^{1-H}$, it is enough to show that, for each m and k,

$$\frac{\gamma_{n_{mk}}}{n_{mk}}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \tag{87}$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda. \tag{88}$$

Let (y_1,y_2) be as in (73), that is

$$(y_1,y_2)=\left(S_{\left[n\frac{k-1}{m}\right]+l,l}, S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}\right). \tag{89}$$

With this (y_1,y_2), let $R_n(a,\delta)$ be the difference between (87) and

$$\frac{1}{n_{mk}(2\pi)^2}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\leq a,|\mu|\leq a\}} U_n(\lambda,\mu,y_1,y_2)\,E\left[e^{-i\lambda\gamma^{-1}_{n_{mk}}T_l}\right]\psi_{S_r}(-\mu)\,\widehat{w}(-\mu,\mu)\,d\lambda d\mu, \tag{90}$$

where now

$$U_n(\lambda,\mu,y_1,y_2)=e^{-i\lambda\gamma^{-1}_{n_{mk}}y_1-i\mu(y_2-y_1)}.$$

It follows from Lemma 16 that, for each $\epsilon>0$,

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty} P\left(|R_n(a,\delta)|>\epsilon\right)=0.$$

Therefore it is enough to show that (90) converges in distribution to (88) by taking the limit as $n \to \infty$ first, then $a \to \infty$ and then $\delta \to 0$.

To obtain the limit as $n \to \infty$, note that $U_n(\lambda, \mu, y_1, y_2)$ above involves

$$\gamma_{n_{mk}}^{-1} y_1 = \gamma_{n_{mk}}^{-1} S_{\left[n\frac{k-1}{m}\right]+l,l} \quad \text{and} \quad y_2 - y_1 = S_{\left[n\frac{k-1}{m}\right]+l+r,l+r} - S_{\left[n\frac{k-1}{m}\right]+l,l}.$$

We have

$$S_{\left[n\frac{k-1}{m}\right]+l+r,l+r} - S_{\left[n\frac{k-1}{m}\right]+l,l} = \sum_{j=-\infty}^{\left[n\frac{k-1}{m}\right]} (c_{l+\left[n\frac{k-1}{m}\right]+1-j} + .. + c_{l+\left[n\frac{k-1}{m}\right]+r-j})\xi_j,$$

and hence, similar to (84),

$$\begin{aligned} & \sup_{\tau_n < l < \infty} P\left(\left|S_{\left[n\frac{k-1}{m}\right]+l+r,l+r} - S_{\left[n\frac{k-1}{m}\right]+l,l}\right| > \varepsilon\right) \\ = & \sup_{\tau_n < l < \infty} P\left(\left|\sum_{i=l}^{\infty} (c_{i+1} + .. + c_{i+r})\xi_i\right| > \varepsilon\right) \to 0 \quad \text{for any } \tau_n \uparrow \infty. \end{aligned}$$

Further, with $S_{mk}\left(\frac{t}{m}\right)$ and $T(t)$ as defined in (85) and (86),

$$\left(\gamma_{n_{mk}}^{-1} S_{\left[n\frac{k-1}{m}\right]+[n_{mk}t],[n_{mk}t]}, \gamma_{n_{mk}}^{-1} T_{[n_{mk}t]}\right) \overset{fdd}{\Longrightarrow} \left(m^H S\left(\frac{k-1}{m}, \frac{t}{m}\right), T(t)\right).$$

It then follows (though the preceding convergence is only $\overset{fdd}{\Longrightarrow}$), in the same way as in Jeganathan (2004a, Lemma 8), that (90) with (y_1, y_2) as in (89) converges in distribution to

$$\frac{1}{(2\pi)^2} \int_{\{|\lambda| \le a, |\mu| \le a\}} \left\{\int_\delta^1 e^{-i\lambda m^H S\left(\frac{k-1}{m}, \frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt\right\} \psi_{S_r}(-\mu)\, \widehat{w}(-\mu, \mu)\, d\lambda d\mu$$

for each a and $\delta > 0$. (Note that in obtaining this convergence only Let $K(a)$ be the difference between this and

$$\left(\frac{1}{2\pi} \int \psi_{S_r}(\mu)\, \widehat{w}(-\mu, \mu)\, d\mu\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_\delta^1 e^{-i\lambda m^H S\left(\frac{k-1}{m}, \frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt d\lambda.$$

(Here m, k and δ are fixed.)

Then noting that $\left|e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}\right| \le 1$ and $|\psi_{S_r}(\mu)| \le |\psi(\mu)|$, we have

$$\begin{aligned} (2\pi)^2 K(a) \le & \left(\int_{-\infty}^{\infty} |\widehat{w}(-\mu, \mu)|\, |\psi(\mu)|\, d\mu\right) \int_\delta^1 \int_{\{|\lambda| > a\}} \left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda dt \\ & + \left(\int_{\{|\mu| > a\}} |\widehat{w}(-\mu, \mu)|\, |\psi(\mu)|\, d\mu\right) \int_\delta^1 \int_{-\infty}^{\infty} \left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda dt. \end{aligned}$$

Now note that

$$\begin{aligned}\int_{\{|\lambda|>a\}} \left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda &\leq C\int_{\{|\lambda|>a\}} e^{-c\left|\lambda t^H\right|^\alpha} d\lambda \\ &= Ct^{-H}\int_{\{|\lambda|>at^H\}} e^{-c|\lambda|^\alpha} d\lambda \\ &\leq C\delta^{-H}\int_{\{|\lambda|>a\delta^H\}} e^{-c|\lambda|^\alpha} d\lambda = R(a), \text{ say, if } \delta \leq t \leq 1.\end{aligned}$$

Hence

$$(2\pi)^2 K(a) \leq R(a)\int_{-\infty}^{\infty} |\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu + R(0)\int_{\{|\mu|>a\}} |\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu,$$

where note that $R(a) \to 0$ as $a \to \infty$ and $R(0) < \infty$. In addition

$$\begin{aligned}\int_{\{|\mu|>a\}} |\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu &\leq \sqrt{\int |\widehat{w}(-\mu,\mu)|^2\,d\mu \int_{\{|\mu|>a\}} |\psi(\mu)|^2\,d\mu} \\ &\leq C\sqrt{\int_{\{|\mu|>a\}} |\psi(\mu)|^2\,d\mu},\end{aligned}$$

where we have used $\int |\widehat{w}(-\mu,\mu)|^2\,d\mu \leq C$, see Lemma 14. Thus $K(a) \to 0$ as $a \to \infty$. Next note that

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\int_0^{\delta} e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt d\lambda\right| &\leq C\left(\int_0^{\delta} t^{-H} dt\right)\left(\int_{-\infty}^{\infty} e^{-c|\lambda|^\alpha} d\lambda\right) \\ &\leq \frac{C\delta^{1-H}}{1-H} \to 0 \text{ as } \delta \to 0.\end{aligned}$$

This completes the proof of the lemma. ■

To complete the proof of Proposition 13, we thus require

Lemma 18.

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\int_0^1 \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] d\lambda\right] dt \Longrightarrow L_1^0 \quad \text{as } m \to \infty.$$

Proof. We first show that

$$\lim_{\delta\to 0}\limsup_{m\to\infty}\int_0^{\delta} \frac{1}{m^{1-H}} E\left[\left|\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] d\lambda\right|\right] dt = 0. \tag{91}$$

To see this note that, in view of (85), $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ is α-stable with scale parameter σ_{tmk} such that

$$\sigma_{tmk} \geq C\left|\frac{t}{m}+\frac{k-1}{m}\right|^H.$$

(See Samorodnitsky and Taqqu (1994, page 345)). Hence

$$\begin{aligned} E\left[\left|\frac{1}{m^{1-H}}\sum_{k=2}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)}d\lambda\right|\right] &\leq \frac{1}{m}\sum_{k=2}^{m}\int\left|E\left[e^{i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}\right]\right|d\lambda \\ &\leq \frac{1}{m}\sum_{k=2}^{m}\frac{1}{\sigma_{mk}}\int e^{-c|\lambda|^{\alpha}}d\lambda \\ &\leq \frac{C}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^{H}\int e^{-c|\lambda|^{\alpha}}d\lambda \leq C \end{aligned}$$

because $\frac{1}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^{H}\leq C$. Here note that in the sum $\sum_{k=2}^{m}$ the leading term corresponding to $k=1$ is left out, but for this we have, in the same way as above, noting $\sigma_{tm1}\geq C\left|\frac{t}{m}\right|^{H}$,

$$E\left[\left|\frac{1}{m^{1-H}}\int_{-\infty}^{\infty}e^{-i\lambda m^{H}S\left(0,\frac{t}{m}\right)}d\lambda\right|\right]\leq C\frac{t^{-H}}{m^{1-H}}.$$

Hence,

$$\begin{aligned} &\int_{0}^{\delta}E\left[\left|\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)}d\lambda\right|\right]dt \\ \leq\ & C\int_{0}^{\delta}\left(1+\frac{t^{-H}}{m^{1-H}}\right)dt = C\left(\frac{\delta^{1-H}}{m^{1-H}}+\delta\right). \end{aligned}$$

Hence, noting that $\left|E\left[e^{-i\lambda T(t)}\right]\right|\leq 1$, (91) follows.

Now consider

$$\begin{aligned} &\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt \\ =\ &\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_{t}\left(-m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)\right)dt \end{aligned} \tag{92}$$

where $h_{t}(y)\geq 0$ is the density function of $T(t)$, i.e.,

$$h_{t}(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda y}\widehat{h_{t}}(\lambda)\,d\lambda \quad\text{where}\quad \widehat{h_{t}}(\lambda)=E\left[e^{-i\lambda T(t)}\right].$$

Note that for each fixed t, $\left\{S\left(\frac{k-1}{m},\frac{t}{m}\right),0\leq k\leq m\right\}$ has the same structure as that of $\left\{\Lambda_{\alpha,H}\left(\frac{k}{m}\right),0\leq k\leq m\right\}$. Hence Jeganathan (2004a, Proposition 6) contains the fact that the difference between the integrand $\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_{t}\left(-m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)\right)$ in (92) and

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_{t}\left(-m^{H}\left(S\left(\frac{k-1}{m},\frac{t}{m}\right)+\varepsilon z\right)\right)e^{-z^{2}/2}dz \tag{93}$$

converges to 0 in mean-square, as $m \to \infty$ first and then $\varepsilon \to 0$. In addition it is easy to see that the arguments in Jeganathan (2004a) also give that this mean-square convergence is uniform over $\delta \leq t \leq 1$. (Note that this is a very specific case so that the steps in Jeganathan (2004a) will take a rather simple and direct form.)

Now, note that $\frac{1}{m^{1-H}} \sum_{k=1}^{m} \int h_t \left(-m^H \left(y + \varepsilon z\right)\right) e^{-z^2/2} dz$ is sufficiently smooth in y (see Jeganathan (2004a, Lemma 7)). Hence, for each $\varepsilon > 0$, it can be seen that (93) can be approximated, as $m \to \infty$, by

$$\frac{1}{m^{1-H}} \sum_{k=1}^{m} \frac{1}{\sqrt{2\pi}} \int h_t \left(-m^H \left(S\left(\frac{k-1}{m}, 0\right) + \varepsilon z\right)\right) e^{-z^2/2} dz$$

uniformly over $\delta \leq t \leq 1$, which in turn is approximated by $\frac{1}{m^{1-H}} \sum_{k=1}^{m} \frac{1}{\sqrt{2\pi}} h_t \left(-m^H S\left(\frac{k-1}{m}, 0\right)\right)$ as before as $m \to \infty$ first and then $\varepsilon \to 0$.

Noting that $S\left(\frac{k-1}{m}, 0\right) = \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)$, we thus have approximated (92) by

$$\int_{\delta}^{1} \frac{1}{m^{1-H}} \sum_{k=1}^{m} h_t \left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) dt,$$

which in turn is approximated as before, as $m \to \infty$ first and then $\delta \to 0$, by

$$\begin{aligned} & \int_{0}^{1} \frac{1}{m^{1-H}} \sum_{k=1}^{m} h_t \left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) dt \\ = \; & \frac{1}{m^{1-H}} \sum_{k=1}^{m} g\left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) \Longrightarrow \left(\int g\left(y\right) dy\right) L_1^0 = L_1^0 \end{aligned}$$

where $g\left(y\right) = \int_0^1 h_t\left(y\right) dt$. Note that $\int g\left(y\right) dy = \int_0^1 \int h_t\left(y\right) dy dt = 1$ because $\int h_t\left(y\right) dy = 1$ for each t. In obtaining this convergence we have used Jeganathan (2004a, Theorem 4). Note that $\int g^2\left(y\right) dy \leq \int_0^1 \int h_t^2\left(y\right) dy dt \leq C \int_0^1 t^{-H} dt \leq C$. This completes the proof of the lemma. ■

Proof of Proposition 13. When (19) holds, the proof the statement (I) follows directly from the preceding lemma and the first statement of the lemmas 15 -17. Regarding the proof the statement (II) under (15), we have (with $M_{w,\eta}^{*}\left(x,y\right)$, $m_{w,\eta}^{*}\left(x,y\right)$ and V_η as in Lemma 15)

$$E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \begin{array}{l} \leq E_{\left[n\frac{k-1}{m}\right]}\left[M_{w,\eta}^{*}\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r} + V_\eta\right)\right] \\ \geq E_{\left[n\frac{k-1}{m}\right]}\left[m_{w,\eta}^{*}\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r} + V_\eta\right)\right] \end{array}.$$

Therefore, in view of the second statements of Lemmas 15 - 17, it only remains to show that $\int \psi_{S_r}\left(\mu\right) \widehat{M_{w,\eta}^{*}}\left(-\mu,\mu\right) \widehat{K}_\eta\left(\mu\right) d\mu \to \int \psi_{S_r}\left(\mu\right) \widehat{w}\left(-\mu,\mu\right) d\mu$ as $\eta \to 0$ and the same for

$m^*_{w,\eta}$. But, noting $\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu = 2\pi \int E[w(x,x+S_r)]\,dx$, this is easily verified using the restrictions in the statement (II), completing the proof. ■

Proof of Proposition 12. Under (19), it is implicit in the proofs of Lemmas 15 and 16 that the difference between $\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]$ and

$$\widehat{f^2}(0)\,\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{n_{mk}2\pi}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\leq a\}} e^{-i\lambda\gamma_{n_{mk}}^{-1}S_{\left[n\frac{k-1}{m}\right]+l,l}}E\left[e^{-i\lambda n_{mk}^{-H}T_l}\right]d\lambda$$

converges to 0 in probability as $n\to\infty$ first, then $a\to\infty$ and then $\delta\to 0$, which in turn converges in distribution to $\widehat{f^2}(0)\,\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$, see Lemma 17. Hence the proof under (19) follows by Lemma 18. Similarly to the preceding proof of Proposition 13, the proof under (15) also follows. ■

Having verified (R2), we now show that the same holds for (R*2) also except for some modifications.

Verification of (R*2). To indicate the required modifications, note that

$$\begin{aligned}
&E_{\left[n\frac{k-1}{m}\right]}\left[\left|\zeta^*_{nmk}\right|^2\right]\\
=\;&\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right]\\
&+2\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}\sum_{r=1}^{n_{mk}-l}E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right]
\end{aligned}$$

where recall that $\omega_q=\sum_{j=q-\nu+1}^{q}d_{q-j}\eta_j=\eta_q+d_1\eta_{q-1}+...+d_{\nu-1}\eta_{q-\nu+1}$. Also recall from (65) that $S_{\left[n\frac{k-1}{m}\right]+l}=S_{\left[n\frac{k-1}{m}\right]+l,\nu}+S^*_{\left[n\frac{k-1}{m}\right]+l,\nu}$ where the distribution of $\left(S^*_{\left[n\frac{k-1}{m}\right]+l,\nu},\omega_{\left[n\frac{k-1}{m}\right]+l}\right)$ is independent of $S_{\left[n\frac{k-1}{m}\right]+l,\nu}$ and has the same as that of $\left(S^*_{v,\nu},\omega_\nu\right)$ with $S^*_{v,\nu}=\sum_{j=0}^{\nu-1}g(j)\xi_{\nu-j}$. Hence we have

$$E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right]=E_{\left[n\frac{k-1}{m}\right]}\left[g\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}\right)\right]$$

where

$$g(x)=E\left[f^2\left(x+S^*_{\nu,\nu}\right)\omega^2_\nu\right].$$

Thus, noting that $\int g(x)\,dx=E[\omega^2_\nu]\int|f(x)|^2\,dx=E[\omega^2_\nu]\,\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2 d\mu$, together with the fact that $S_{\left[n\frac{k-1}{m}\right]+l,\nu}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l}$ so that Proposition 12 becomes essentially applicable, we have

$$\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right]\Longrightarrow\left(E[\omega^2_\nu]\,\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2 d\mu\right)L^0_1.$$

(Here and below, we have taken into account the second statement of Lemma 7, without further mentioning.)

To deal with the remaining sums, suppose that $r \geq \nu$. Let (recall $S^*_{q+r,\nu} = S_{q+r} - S_{q+r,\nu}$.)

$$S^{\#}_{q+r,\nu} = S_{q+r} - \left(S^*_{q+r,r+\nu} - S^*_{q+r,r}\right) - S^*_{q+r,\nu}.$$

Note that $\left(S_{q,\nu}, S^{\#}_{q+r,\nu}\right)$ is independent of $\left(S^*_{q,\nu}, \left(S^*_{q+r,r+\nu} - S^*_{q+r,r}\right) + S^*_{q+r,\nu}\right)$. Then

$$\begin{aligned} & E_{q-\nu}\left[f\left(S_q\right)\omega_q f\left(S_{q+r}\right)\omega_{q+r}\right] \\ = \; & E_{q-\nu}\left[f\left(S_{q,\nu} + S^*_{q,\nu}\right)\omega_q f\left(S^{\#}_{q+r,\nu} + \left(S^*_{q+r,r+\nu} - S^*_{\nu+r,r}\right) + S^*_{q+r,\nu}\right)\omega_{q+r}\right] \\ = \; & E_{q-\nu}\left[w_r\left(S_{q,\nu}, S^{\#}_{q+r,\nu}\right)\right] \end{aligned}$$

where

$$w_r\left(x,y\right) = E\left[f\left(x + S^*_{\nu,\nu}\right)\omega_\nu f\left(y + \left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right) + S^*_{\nu+r,\nu}\right)\omega_{\nu+r}\right].$$

Thus, when $r \geq \nu$ and $l > \nu$,

$$\begin{aligned} & E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right] \\ = \; & E_{\left[n\frac{k-1}{m}\right]}\left[w_r\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}, S^{\#}_{\left[n\frac{k-1}{m}\right]+l+r,\nu}\right)\right]. \end{aligned} \tag{94}$$

In the case $r < \nu$, the right hand side here takes the form

$$E_{\left[n\frac{k-1}{m}\right]}\left[w^*_r\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}, S_{\left[n\frac{k-1}{m}\right]+l+r,\nu+r}\right)\right] \tag{95}$$

with

$$w^*_r\left(x,y\right) = E\left[f\left(x + S^*_{\nu,\nu}\right)\omega_\nu f\left(y + S^*_{\nu+r,r+\nu}\right)\omega_{\nu+r}\right].$$

Now, in the case $r \geq \nu$, $\left(S^*_{\nu,\nu}, \left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)\right)$ is independent of $S^*_{\nu+r,\nu}$, and hence we have

$$\widehat{w_r}\left(\lambda,\mu\right) = \widehat{f}\left(\lambda\right)\widehat{f}\left(\mu\right) E\left[\omega_\nu e^{-i\lambda S^*_{\nu,\nu} - i\mu\left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)}\right] E\left[\omega_{\nu+r} e^{-i\mu S^*_{\nu+r,\nu}}\right].$$

Here, noting that $S^*_{\nu,\nu} = \sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}$,

$$\begin{aligned} E\left[\omega_{\nu+r} e^{-i\mu S^*_{\nu+r,\nu}}\right] &= E\left[\omega_\nu e^{-i\mu S^*_{\nu,\nu}}\right] = \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{-i\mu S^*_{\nu,\nu}}\right] \\ &= \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{-i\mu g(\nu-i)\xi_i}\right] \prod_{j=0, j\neq\nu-i}^{\nu-1} \psi\left(-g(j)\mu\right). \end{aligned}$$

Noting $S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r} = \sum_{j=0}^{\nu+r-1} g(j)\xi_{\nu+r-j} - \sum_{j=0}^{r-1} g(j)\xi_{\nu+r-j} = \sum_{j=0}^{\nu-1} g(j+r)\xi_{\nu-j}$, we similarly have

$$\begin{aligned} & E\left[\omega_\nu e^{-i\lambda S^*_{\nu,\nu} - i\mu\left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)}\right] \\ = & \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{(-i\lambda g(\nu-i) - i\mu g(\nu-i+r))\xi_i}\right] \prod_{j=0, j\neq\nu-i}^{\nu-1} \psi\left(-g(j)\lambda - g(j+r)\mu\right). \end{aligned}$$

In the case $\nu = 1$, note that the preceding two quantities give (recall $d_0 = 1$ and $g(0) = 1$)

$$\widehat{w_r}(\lambda,\mu) = \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\eta_1 e^{-i\mu\xi_1}\right] E\left[\eta_1 e^{(-i\lambda - i\mu g(r))\xi_1}\right].$$

In the case $0 < r < \nu$, we have

$$\begin{aligned} \widehat{w^*_r}(\lambda,\mu) &= \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\omega_\nu\omega_{\nu+r} e^{-i\lambda S^*_{\nu,\nu} - i\mu S^*_{\nu+r,\nu+r}}\right] \\ &= \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\omega_\nu\omega_{\nu+r} e^{-i\lambda\sum_{j=1}^{\nu} g(\nu-j)\xi_j - i\mu\sum_{j=1}^{\nu+r} g(\nu+r-j)\xi_j}\right]. \end{aligned}$$

We now consider the analogues of Proposition 13 for the sums of (94) and (95) (note that Proposition 13 involves the sum of $E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$). Note that $S_{\left[n\frac{k-1}{m}\right]+l,\nu}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l}$, and similarly both $S^{\#}_{\left[n\frac{k-1}{m}\right]+l+r,\nu}$ and $S_{\left[n\frac{k-1}{m}\right]+l+r,v+r}$ have the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l+r}$. It can be seen from the proof of Proposition 13 that in both cases (94) and (95) the role of $\psi_{S_r}(-\mu)$ (see (83)) is now played by $\Psi_r(\mu)$ defined in Theorem 2. In addition note that both $\widehat{w_r}(\lambda,\mu)$ and $\widehat{w^*_r}(\lambda,\mu)$ contain the factor $\widehat{f}(\lambda)\,\widehat{f}(\mu)$, which will serve the purpose of $\widehat{w}(\lambda,\mu)\,\psi(\mu)$ in the proof of Proposition 13. We thus see that for each $r \geq 1$,

$$\begin{aligned} & \frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right] \\ \Longrightarrow\ & \left(\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2\Psi_r(\mu)\,\Phi_r(\mu)\,d\mu\right) L_1^0 \end{aligned}$$

as $n \to \infty$ first and then $m \to \infty$. Here $\Psi_r(\mu)$ is as defined earlier in Theorem 2, and $\left|\widehat{f}(\mu)\right|^2\Phi_r(\mu) = \widehat{w_r}(-\mu,\mu)$ in the case $r \geq \nu$ and $\left|\widehat{f}(\mu)\right|^2\Phi_r(\mu) = \widehat{w^*_r}(-\mu,\mu)$ in the case $r < \nu$. Specifically

$$\Phi_r(\mu) = \begin{cases} E\left[\omega_\nu e^{i\mu S^*_{\nu,\nu} - i\mu\left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)}\right] E\left[\omega_\nu e^{-i\mu S^*_{\nu,\nu}}\right] & \text{if } r \geq \nu \\ E\left[\omega_\nu\omega_{\nu+r} e^{i\mu\sum_{j=1}^{\nu} g(\nu-j)\xi_j - i\mu\sum_{j=1}^{\nu+r} g(\nu+r-j)\xi_j}\right] & \text{if } 1 \leq r < \nu. \end{cases}$$

(This $\Phi_r(\mu)$ coincides with that involved in Theorem 2.)

Regarding Lemma 10 we shall see in the next section that its proof, under the conditions of Theorem 1, depends crucially on the fact that $\left|\widehat{f}(\lambda)\,\widehat{f}(\mu)\right| \leq C\,|\mu|\,|\lambda|$, which holds under

the conditions of Theorem 1, see (70). In the present case the role of $\widehat{f}(\lambda)\widehat{f}(\mu)$ is played by $\widehat{w_r}(\lambda,\mu)$ (it is enough to restrict to the case $r \geq \nu$), for which we now obtain the bound

$$|\widehat{w_r}(\lambda,\mu)| \leq C\left(|\mu|\,|\lambda| + |\mu|^2\right). \tag{96}$$

To see this, assume for convenience that $\nu = 1$. Then (see above)

$$\begin{aligned} |\widehat{w_r}(\lambda,\mu)| &= \left|\widehat{f}(\lambda)\widehat{f}(\mu)\,E\left[\eta_1 e^{-i\mu\xi_1}\right]E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right| \\ &\leq C\left|E\left[\eta_1 e^{-i\mu\xi_1}\right]E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right| \end{aligned}$$

where, using $E[\eta_1] = 0$,

$$\left|E\left[\eta_1 e^{-i\mu\xi_1}\right]\right| = \left|E\left[\eta_1\left(e^{-i\mu\xi_1}-1\right)\right]\right| \leq |\mu|\,E\left[|\eta_1\xi_1|\right] \leq C|\mu|$$

and similarly $\left|E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right| \leq C(|\lambda|+|\mu|)$.

We shall see later that (96) will give the analogue of Lemma 10, see the arguments at the end of the proof of Lemma 10 in Section 5 below. ■

We next verify (R4) (where $\alpha = 2$ and hence $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$, see (2)).

Verification of (R4): For notational convenience, we take $\gamma_r = r^H$ and $g(r) \sim Cr^{H-1/2}$. Then (recall from (30) that $\chi_{nmk} = \frac{1}{\sqrt{n}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\xi_l$)

$$\zeta_{nmk}\chi_{nmk} = n^{-\frac{1}{2}-\frac{1-H}{2}}\left(I_{1,nmk} + I_{2,nmk} + I_{3,nmk}\right) \tag{97}$$

where

$$I_{1,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=l+1}^{\left[n\frac{k}{m}\right]} f(S_l)\,\xi_r,$$

$$I_{2,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=l+1}^{\left[n\frac{k}{m}\right]} \xi_l f(S_r)$$

and

$$I_{3,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f(S_l)\,\xi_l.$$

Now

$$E_{\left[n\frac{k-1}{m}\right]}\left[f(S_l)\,\xi_l\right] = E_{\left[n\frac{k-1}{m}\right]}\left[f_1(S_{l,1})\right] \quad \text{with} \quad f_1(y) = E\left[\xi_1 f(y+\xi_1)\right],$$

where $S_{l,1}$ is as in (66). Note that $f_1(y)$ is similar to $f_*(y)$ in (67), and hence by Lemma 7 $\int f_1(y)\,dy = 0$ and similarly other restrictions in Theorem 1 stated for $f(y)$ are satisfied

for $f_1(y)$. It follows from the next section (specifically, using the left hand side of (117) with $q=1$, together with the first bound in (114)) that

$$E_{\left[n\frac{k-1}{m}\right]}\left[\left(n^{-\frac{1-H}{2}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f_1(S_{l,1})\right)^2\right] \leq C.$$

Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[n^{-\frac{1}{2}-\frac{1-H}{2}} I_{3,nmk}\right]\right| \leq Cn^{-\frac{1}{2}}. \tag{98}$$

Clearly

$$E_{\left[n\frac{k-1}{m}\right]}\left[I_{1,nmk}\right] = 0. \tag{99}$$

To deal with $I_{2,nmk}$ we have (see (65))

$$S_r = S_{r,\left[n\frac{k-1}{m}\right]} + S^*_{r,\left[n\frac{k-1}{m}\right]},$$

where recall that $S^*_{r,\left[n\frac{k-1}{m}\right]} = \sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1} g(q)\,\xi_{r-q}$ and is independent of $S_{r,\left[n\frac{k-1}{m}\right]}$. We also have $f(S_r) = \frac{1}{2\pi}\int e^{-i\lambda S_r}\widehat{f}(\lambda)\,d\lambda$. Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_l f(S_r)\right]\right| \leq \frac{1}{2\pi}\int \left|E\left[\xi_l e^{-i\lambda\sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1} g(q)\xi_{r-q}}\right]\right| \left|\widehat{f}(\lambda)\right| d\lambda,$$

and hence

$$\begin{aligned}
&\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]} f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]\right| \\
\leq\ & \frac{1}{\gamma_r}\int \left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}\right]\right| \prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\frac{\lambda}{\gamma_r}g(q)\right)\right| \left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right| d\lambda.
\end{aligned} \tag{100}$$

Now, because $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$ ((R4) pertains only to the case $\alpha=2$),

$$\left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}\right]\right| = \left|E\left[\xi_1\left(e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}-1\right)\right]\right| \leq C\frac{|\lambda|}{\gamma_r}\left|g(r-l)\right|.$$

Further $\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right| \leq C\frac{|\lambda|}{\gamma_r}$, see (70). Also $\int\prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\frac{\lambda}{\gamma_r}g(q)\right)\right| d\lambda \leq C$ by (58) and (59).

Thus, noting that $\gamma_r = r^H$ and $\sum_{l=1}^{r-1}|g(r-l)| \sim Cr^{H+1-1/2}$ because $g(s)\sim Cs^{H-1/2}$,

$$\begin{aligned}
n^{-\frac{1}{2}-\frac{1-H}{2}}\left|E_{\left[n\frac{k-1}{m}\right]}\left[I_{2,nmk}\right]\right| &= n^{-\frac{1}{2}-\frac{1-H}{2}}\left|\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]} f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]\right| \\
&\leq Cn^{-\frac{1}{2}-\frac{1-H}{2}}\sum_{r=1}^{n_{mk}}\sum_{l=1}^{r-1}\gamma_r^{-3}\left|g(r-l)\right| \\
&\leq Cn^{-\frac{1}{2}-\frac{1-H}{2}}n^{-2H+\frac{3}{2}} = Cn^{-\frac{3H-1}{2}}.
\end{aligned}$$

Because $3H-1>0$, this together with (98) and (99) complete the verification of (R4) in the situation of the statement (II) of Theorem 1.

In the case of the situation of the statement (I) of Theorem 1 also the bound (100) holds except that the factor $\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right|$ in the right hand side needs to be replaced by $\left|\widehat{K}_\eta\left(\frac{\lambda}{\gamma_r}\right)\right|\max\left(\left|\widehat{M_{f,\eta}}\left(\frac{\lambda}{\gamma_r}\right)\right|,\left|\widehat{m_{f,\eta}}\left(\frac{\lambda}{\gamma_r}\right)\right|\right)$. Hence, using (71) as in the proof of Lemma 8, it is seen that (R4) holds also in the present situation. This completes the verification of (R4). ■

We next show that the verification of (R4) entails that of (R*4).

Verification of (R*4). We have

$$E_{\left[n\frac{k-1}{m}\right]}\left[\zeta^*_{nmk}\chi_{nmk}\right]=\sqrt{\frac{\gamma_n}{n}}E_{\left[n\frac{k-1}{m}\right]}\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}f\left(S_l\right)\omega_l\right)\chi_{nmk}\right].$$

Note that $\left(f\left(S_l\right)\omega_l-E_{l-1}\left[f\left(S_l\right)\omega_l\right],\xi_l\right)$, $l\geq 1$, form martingale differences and hence

$$\begin{aligned}
&E_{\left[n\frac{k-1}{m}\right]}\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}\left(f\left(S_l\right)\omega_l-E_{l-1}\left[f\left(S_l\right)\omega_l\right]\right)\right)\chi_{nmk}\right]\\
=&\ \frac{1}{\sqrt{n}}E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}E_{l-1}\left\{\left(f\left(S_l\right)\omega_l-E_{l-1}\left[f\left(S_l\right)\omega_l\right]\right)\xi_l\right\}\right]\\
=&\ \frac{1}{\sqrt{n}}E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}E_{l-1}\left[f\left(S_l\right)\omega_l\xi_l\right]\right]
\end{aligned}$$

where in the last step we have used $E_{l-1}\left[\xi_l\right]=0$, so that $E_{l-1}\left[E_{l-1}\left[f\left(S_l\right)\omega_l\right]\xi_l\right]=0$.

Consider

$$E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}E_{l-1}\left[f\left(S_l\right)\eta_l\xi_l\right]\right]=E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}g\left(S_{l,1}\right)\right]$$

where we have used $E_{l-1}\left[f\left(S_l\right)\eta_l\xi_l\right]=g\left(S_{l,1}\right)$ for a suitable $g\left(x\right)$ (with $\int\left(\left|g\left(x\right)\right|+g^2\left(x\right)\right)dx<\infty$ in the case of the statement (II) of Theorem 2 and a similar restriction in the case of the statement (I)). Therefore $\frac{\gamma_n}{n}E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}g\left(S_{l,1}\right)\right]$ is bounded in absolute value by a constant (see (114) below), so that

$$\frac{1}{\sqrt{n}}\sqrt{\frac{\gamma_n}{n}}E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}g\left(S_{l,1}\right)\right]\leq C\frac{1}{\sqrt{n}}\sqrt{\frac{\gamma_n}{n}}\left(\frac{\gamma_n}{n}\right)^{-1}=\frac{C}{\sqrt{\gamma_n}}\to 0.$$

Thus it remains to show that (R*4) holds for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\omega_l\right]$. We shall reduce this situation to that of (R4). Recall that $\omega_l = \sum_{j=l-\nu+1}^{l} d_{l-j}\eta_j$, which sum consists of ν terms. We use induction on ν. Suppose that $\nu = 1$, that is, $\omega_l = \eta_l$. Then

$$E_{l-1}\left[f\left(S_l\right)\omega_l\right] = E_{l-1}\left[f\left(S_l\right)\eta_l\right] = f_*\left(S_{l,1}\right).$$

Here $f_*\left(x\right) = E_{l-1}\left[f\left(x+\xi_1\right)\eta_1\right]$, which satisfies all the conditions of Theorem 1 (see Lemma 7), and hence (R*4) holds for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\eta_l\right]$ when $\nu = 1$.

Now suppose that (R*4) holds for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\omega_l\right]$ when $\nu = i-1$. Then, when $\nu = i$, we have $E_{l-1}\left[f\left(S_l\right)\omega_l\right] = E_{l-1}\left[f\left(S_l\right)\left(\omega_l - \eta_l\right)\right] + E_{l-1}\left[f\left(S_l\right)\eta_l\right]$, where note that

$$E_{l-1}\left[f\left(S_l\right)\left(\omega_l - \eta_l\right)\right] = E_{l-1}\left[f\left(S_l\right)\omega^*_l\right] = \omega^*_l E_{l-1}\left[f\left(S_l\right)\right] = g\left(S_{l,1}\right)\omega^*_l.$$

Here $\omega^*_l = \omega_l - \eta_l = \sum_{j=l-i+1}^{l-1} d_{l-j}\eta_j$, and hence $g\left(S_{l,1}\right)\omega^*_l$ has the same structure as that of $f\left(S_l\right)\omega_l$ but with $\nu = i-1$ (for which we have assumed that (R*4) holds). Hence one can assume that (R*4) holds for $\zeta^*_{nmk} = \sum g\left(S_{l,1}\right)\omega^*_l$ also. We have already verified (R*4) for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\eta_l\right]$. Thus (R*4) holds for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\omega_l\right]$ when $\nu = i$. This completes the proof of the verification of (R*4) by induction. ■

5 PROOF OF LEMMA 10 AND THE VERIFICATION OF (R3) AND (R*3)

In the rest of the paper we let

$$g(j,r) = g(j+r) - g(j) = c_{j+1} + ... + c_{j+r}.$$

We first isolate some bounds on $g(j,r)$ in the next Lemma 19.

Lemma 19. *Let $\vartheta > 0$ be such that*

$$0 < \vartheta < \begin{cases} \min\left(1-H, H, \left|\frac{1}{\alpha} - H\right|, \frac{1}{\alpha}\right) & \text{if } H \neq \frac{1}{\alpha} \\ \min\left(1-\frac{1}{\alpha}, \frac{1}{\alpha}\right) & \text{if } H = \frac{1}{\alpha}. \end{cases} \tag{101}$$

Then

$$\sup_{[l/2]\leq j\leq l, q\geq 1, r\geq 1} \left| b_l \frac{g\left(j+q,r\right)}{\gamma_r} \right| \leq C l^{\vartheta} \text{ for all } 1 \leq l \leq n. \tag{102}$$

Proof. First consider the case $H \neq \frac{1}{\alpha}$, in which the requirement (A1) of Section 2 holds. Let $\delta = \frac{\vartheta}{3}$ so that (101) becomes

$$0 < 3\delta < \min\left(1-H, H, \left|\frac{1}{\alpha} - H\right|, \frac{1}{\alpha}\right). \tag{103}$$

Recall the Potter's inequality, mentioned in Lemma 5 of Section 3 above, that if $G(x)$ is slowly varying at ∞, then there is a $B > 0$ such that $|\frac{G(x)}{G(y)}| \leq B \max\{(x/y)^{\delta}, (x/y)^{-\delta}\}$ for all $x > 0, y > 0$. Therefore one can assume that

$$\left|\frac{c_i}{i^{H-1-\frac{1}{\alpha}}}\right| \leq Bi^{\delta}, \quad \left|\frac{g(i)}{i^{H-\frac{1}{\alpha}}}\right| \leq Bi^{\delta}, \quad \frac{b_r}{r^{\frac{1}{\alpha}}} \leq Br^{\delta}, \quad \frac{r^H}{\gamma_r} \leq Br^{\delta}.$$

We in particular have

$$\frac{b_l}{\gamma_r} \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta}. \tag{104}$$

Further, noting $H - 1 - \frac{1}{\alpha} + \delta < 0$ (see (103)), we have when $j \geq [l/2]$,

$$\begin{aligned} |g(j+q,r)| &= |c_{j+q+1} + .. + c_{j+q+r}| \\ &\leq C\left|(j+q+1)^{H-1-\frac{1}{\alpha}+\delta} + ... + (j+q+r)^{H-1-\frac{1}{\alpha}+\delta}\right| \\ &\leq Cr(j+q)^{H-1-\frac{1}{\alpha}+\delta} \leq Cr(\min(l,q))^{H-1-\frac{1}{\alpha}+\delta}, \quad j \geq [l/2]. \end{aligned} \tag{105}$$

Here, in obtaining the second inequality we have used $j \geq [l/2]$ and $H - 1 - \frac{1}{\alpha} + \delta < 0$.

Further, when $H - \frac{1}{\alpha} < 0$ (in which case $H - \frac{1}{\alpha} + \delta < 0$, see (103)), we have

$$\begin{aligned} |g(j+q,r)| &\leq |g(j+q)| + |g(j+q+r)| \\ &\leq C(j+q)^{H-\frac{1}{\alpha}+\delta} \leq C(\min(l,q))^{H-\frac{1}{\alpha}+\delta}, \quad j \geq [l/2], \end{aligned} \tag{106}$$

and similarly when $H - \frac{1}{\alpha} > 0$,

$$\begin{aligned} |g(j+q,r)| &\leq C(j+q+r)^{H-\frac{1}{\alpha}+\delta} \\ &\leq \begin{cases} \begin{cases} Cl^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r \leq l \\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r > l \end{cases} & H - \frac{1}{\alpha} > 0,\ q \leq l \\ \begin{cases} Cq^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r \leq q \\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r > q \end{cases} & H - \frac{1}{\alpha} > 0,\ q > l. \end{cases} \end{aligned} \tag{107}$$

First consider the situation

$$q \leq l.$$

Using (104) and (105) and noting $1 - H - 2\delta > 0$ (see (103)),

$$\left|b_l \frac{g(j+q,r)}{\gamma_r}\right| \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r l^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{l}\right)^{1-H-2\delta} r^{3\delta} \leq Cl^{3\delta}, \text{ if } r \leq l,\ j \geq [l/2].$$

In addition, using (106) and (107) and noting $H - \delta > 0$ and $\frac{1}{\alpha} - 2\delta > 0$ (see (103)), we have

$$\left|b_l \frac{g(j+q,r)}{\gamma_r}\right| \leq \begin{cases} Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} l^{H-\frac{1}{\alpha}+\delta} = Cr^{-H+\delta} l^{H-\delta} l^{3\delta} \leq Cl^{3\delta}, & H - \frac{1}{\alpha} < 0,\ r > l,\ j \leq l \\ Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = Cr^{2\delta-\frac{1}{\alpha}} l^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq Cl^{3\delta}, & H - \frac{1}{\alpha} > 0,\ r > l,\ j \leq l. \end{cases}$$

Now consider

$$q > l.$$

From (105) we have,

$$\left| b_l \frac{g\left(j+q,r\right)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r q^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{q}\right)^{1-H+\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}, \text{ if } r \leq q,\, j \geq [l/2]\,.$$

When $H - \frac{1}{\alpha} < 0$, $r > q$, we obtain from (106) that

$$\left| b_l \frac{g\left(j+q,r\right)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} q^{H-\frac{1}{\alpha}+\delta} = \left(\frac{q}{r}\right)^{H-\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}.$$

When $H - \frac{1}{\alpha} > 0$, $r > q$, we have from (107) that

$$\left| b_l \frac{g\left(j+q,r\right)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = \left(\frac{l}{r}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}$$

because $\frac{1}{\alpha} - 2\delta > 0$ (see (103)) and $l < q < r$. This completes the proof of the lemma when $H \neq \frac{1}{\alpha}$.

Now consider the case $H = \frac{1}{\alpha}$. In this case, by (12), we have $\sup_{i\geq 1} |ic_i| \leq C$. In addition $\sup_{i\geq 1} |g\left(i\right)| \leq C$ by (A2). Therefore, the inequalities (104) - (107) hold when $H = \frac{1}{\alpha}$, and hence the remaining arguments also hold with $H = \frac{1}{\alpha}$. This completes the proof of the lemma. ■

Below we assume ϑ of Lemma 19 satisfies (in addition to (101))

$$3H - 6\vartheta > 1. \tag{108}$$

This is possible in view of the restriction $3H > 1$.

We are now in a position to proceed with the proof of Lemma 10 and the verification of (R3). For this purpose, using (73) and using the idea in (74) we have

$$\left| E_{\left[n\frac{k-1}{m}\right]} \left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r} \right) \right] \right| \leq \int \left| E\left[e^{-i\lambda T_l - i\mu T_{l+r}} \right] \right| \widehat{w}\left(\lambda,\mu\right) d\lambda d\mu$$

(Recall $T_l = \sum_{j=1}^{l} g\left(l-j\right)\xi_j$.) We have

$$\lambda T_l + \mu T_{l+r} = \sum_{j=1}^{l} \left(\lambda g\left(l-j\right) + \mu g\left(l+r-j\right)\right)\xi_j + \sum_{j=l+1}^{l+r} \mu g\left(l+r-j\right)\xi_j,$$

where the first sum on the r.h.s. is independent of the second sum. Therefore

$$E\left[e^{-i\lambda T_l - i\mu T_{l+r}} \right] = \left(\prod_{j=0}^{l-1} \psi\left(-\lambda g(j) - \mu g\left(r+j\right)\right) \right) \left(\prod_{j_1=0}^{r-1} \psi\left(-g\left(j_1\right)\mu\right) \right).$$

Substituting this above, and making the transformation $(\lambda+\mu,\mu)\longmapsto(\lambda,\mu)$, we obtain (recall $g(j,r)=g(j+r)-g(j)$)

$$\begin{aligned}
&\left|E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right| \\
\leq\ & \int\left|\left(\prod_{j=0}^{l-1}\left|\psi\left(-\lambda g(j)-\mu g\left(j,r\right)\right)\right|\right)\left(\prod_{j_1=0}^{r-1}\left|\psi\left(-g\left(j_1\right)\mu\right)\right|\right)\right|\widehat{w}\left(\lambda-\mu,\mu\right)d\lambda d\mu \\
\leq\ & \frac{1}{\gamma_l\gamma_r}\int_{R^2}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(-\frac{\lambda g(j)}{\gamma_l}-\frac{\mu}{\gamma_r}g\left(j,r\right)\right)\right|\right) \\
&\times\left(\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g\left(j_1\right)\frac{\mu}{\gamma_r}\right)\right|\right)\left|\widehat{w}\left(\frac{\lambda}{\gamma_l}-\frac{\mu}{\gamma_r},\frac{\mu}{\gamma_r}\right)\right|d\lambda d\mu \\
=\ & \frac{1}{\gamma_l\gamma_r}\int_{R^2}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right|\right)\left(\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g\left(j_1\right)\frac{\mu}{\gamma_r}\right)\right|\right) \\
&\times\left|\widehat{w}\left(\frac{\lambda}{\gamma_l}-\frac{g\left(j,r\right)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r},\frac{\mu}{\gamma_r}\right)\right|d\lambda d\mu. \qquad (109)
\end{aligned}$$

Here note that the right hand side is nonrandom. The same bound but with $\widehat{w}\left(\lambda,\mu\right)$ replaced by

$$\left|\widehat{K}_\eta\left(\lambda\right)\right|\left|\widehat{K}_\eta\left(\mu\right)\right|\max\left(\left|\widehat{M_{w,\eta}}\left(\lambda,\mu\right)\right|,\left|\widehat{m_{w,\eta}}\left(\lambda,\mu\right)\right|\right), \qquad (110)$$

holds also.

From these bounds we now obtain

$$\left|E\left[w\left(S_l,S_{l+r}\right)\right]\right|\leq\frac{C}{\gamma_l\gamma_r}\quad\text{for all } l,r\geq 1 \qquad (111)$$

if either $\left|\widehat{w}\left(\lambda,\mu\right)\right|\leq C$ and (19) hold or $\max\left(\left|\widehat{M_{w,\eta}}\left(\lambda,\mu\right)\right|,\left|\widehat{m_{w,\eta}}\left(\lambda,\mu\right)\right|\right)\leq C$ and (15) hold.

In the case $\left|\widehat{w}\left(\lambda,\mu\right)\right|\leq C$ and (19) hold, this follows using (58) and (59), together with (60). For the other case, when $\widehat{w}\left(\lambda,\mu\right)$ in the right most side of (109) is replaced by (110) with $\max\left(\left|\widehat{M_{w,\eta}}\left(\lambda,\mu\right)\right|,\left|\widehat{m_{w,\eta}}\left(\lambda,\mu\right)\right|\right)\leq C$, the resulting bound is bounded by

$$\begin{aligned}
&\frac{1}{\gamma_l\gamma_r}\int_{R^2}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right|\right)\left(\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g\left(j_1\right)\frac{\mu}{\gamma_r}\right)\right|\right) \\
&\times\left|\widehat{K}_\eta\left(\frac{\lambda}{\gamma_l}+\frac{g\left(j,r\right)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right|\left|\widehat{K}_\eta\left(\frac{\mu}{\gamma_r}\right)\right|d\lambda d\mu, \qquad (112)
\end{aligned}$$

where, following (63) and (64), we have when $l \geq l_0$ for a suitable l_0,

$$\int_R \left|\widehat{K}_\eta\left(\frac{\lambda}{\gamma_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right| \prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right| d\lambda$$

$$\leq \; C + C\rho^l \prod_{j=[l/2]+1}^{l}\left(\int\left|\widehat{K}_\eta\left(\frac{\lambda}{g(j)\,b_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right| d\lambda\right)^{\frac{1}{l-[l/2]}} \leq C + C\rho^l\frac{\gamma_l}{\eta} \leq C,$$

because (using (57))

$$\int\left|\widehat{K}_\eta\left(\frac{\lambda}{g(j)\,b_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right| d\lambda = \int\left|\widehat{K}_\eta\left(\frac{\lambda}{g(j)\,b_l}\right)\right| d\lambda \leq C\frac{\gamma_l}{\eta}.$$

It was also noted that (63) and (64) hold for $1 \leq l \leq l_0$ also, and the same hold for first of the preceding two bounds also. Thus, $\left|E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right|$, the left most side of (109), is bounded by

$$\frac{C}{\gamma_l\gamma_r}\int_R\left|\widehat{K}_\eta\left(\frac{\mu}{\gamma_r}\right)\right|\left(\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu}{\gamma_r}\right)\right|\right) d\mu \leq \frac{C}{\gamma_l\gamma_r},$$

using arguments similar to the above. Thus (110) holds under (15) also.

Similarly, if either $|\widehat{v}(\lambda)| \leq C$ and (19) hold or $\left|\widehat{M_{v,\eta}}(\lambda)\right| \leq C$ and (15) hold, then

$$|E[v(S_l)]| \leq \frac{C}{\gamma_l} \quad \text{for all } l \geq 1.$$

Before giving the proof of Lemma 10, we note the following useful fact that follows from the preceding bound and the bound (111) (recall $n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right]$):

$$E\left[\left(\frac{\gamma_n}{n}\sum_{j=1}^{n}h(S_j) - \frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[h\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right)^2\right] \to 0 \tag{113}$$

as $n \to \infty$ first and then $m \to \infty$, for any $h(x)$ for which both $h(x)$ and $h^2(x)$ are Lebesgue integrable.

Remark 4. This fact together with the approximation contained in the proof of Proposition 12 has been used in the proof of Theorem 1 given in Section 2. In addition, essentially the same arguments will be used to deduce Theorem 3 from Proposition 13. ■

To see that (113) holds note that $\sum_{l=1}^{n_{mk}} h\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) - \sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[h\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]$, $1 \leq k \leq m$, form an array of martingale differences, and hence the expected value in (113)

is bounded by

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{k=1}^{m} E\left[\left(\sum_{l=1}^{n_{mk}} h\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right)^2\right]$$
$$\leq \left(\frac{\gamma_n}{n}\right)^2 \left\{\sum_{l=1}^{n} E\left[h^2\left(S_l\right)\right] + 2\sum_{l=1}^{n}\sum_{r=1}^{n_{mk}} \left|E\left[h\left(S_l\right) h\left(S_{l+r}\right)\right]\right|\right\}$$

where

$$\sum_{l=1}^{n} E\left[h^2\left(S_l\right)\right] \leq C\sum_{l=1}^{n}\frac{1}{\gamma_l}, \qquad \sum_{l=1}^{n}\sum_{r=1}^{n_{mk}} \left|E\left[h\left(S_l\right) h\left(S_{l+r}\right)\right]\right| \leq C\sum_{l=1}^{n}\sum_{r=1}^{n_{mk}}\frac{1}{\gamma_l\gamma_r} \tag{114}$$

where we have used the bounds $\left|E\left[h^2\left(S_l\right)\right]\right| \leq \frac{C}{\gamma_l}$ and $\left|E\left[h\left(S_l\right) h\left(S_{l+r}\right)\right]\right| \leq \frac{C}{\gamma_l\gamma_r}$ obtained above; for instance the later one is obtained from (111) by taking $w\left(x,y\right) = h\left(x\right) h\left(y\right)$.

We have $\sum_{l=1}^{n}\frac{1}{\gamma_l} \sim C\frac{n}{\gamma_n}$ and, using $n_{mk} \sim \frac{n}{m}$ and $\gamma_{n_{mk}} \sim \gamma_n m^{-H}$, $\max_{1\leq k\leq m}\sum_{r=1}^{n_{mk}}\frac{1}{\gamma_r} \sim C\frac{n}{\gamma_n}\left(\frac{1}{m}\right)^{1-H}$. Thus it follows that (113) holds. ■

Proof of Lemma 10. We first consider the proof under (19), in which case we shall apply the bound (109) with $\widehat{w}\left(\lambda,\mu\right) = \widehat{f}\left(\lambda\right)\widehat{f}\left(\mu\right)$. The fact that $\left|\widehat{f}\left(\lambda\right)\right| \leq C\left|\lambda\right|$ will now be crucially used (whereas (114) uses only $\left|\widehat{h}\left(\lambda\right)\right| + \left|\widehat{h^2}\left(\lambda\right)\right| \leq C$). Here note that, for any ϑ satisfying (101),

$$\left|\frac{\gamma_l g\left(j,r\right)}{\gamma_r g(j)}\right| = \left|\frac{\gamma_l}{b_l g(j)}\right|\left|\frac{b_l g\left(j,r\right)}{\gamma_r}\right| \leq Cl^{\vartheta}, \quad [l/2] \leq j \leq l, \;\; r \geq 1 \tag{115}$$

by (57) and Lemma 19. Therefore

$$\left|\widehat{f}\left(\frac{\lambda}{\gamma_l} + \frac{g\left(j,r\right)}{\gamma_r g(j)}\mu - \frac{\mu}{\gamma_r}\right)\widehat{f}\left(\frac{\mu}{\gamma_r}\right)\right| \leq C\left(\frac{\left|\lambda\right|}{\gamma_l} + \frac{\left|\mu\right| l^{\vartheta}}{\gamma_l} + \frac{\left|\mu\right|}{\gamma_r}\right)\frac{\left|\mu\right|}{\gamma_r}.$$

Hence (109) is bounded by (when $\widehat{w}\left(\lambda,\mu\right) = \widehat{f}\left(\lambda\right)\widehat{f}\left(\mu\right)$)

$$\frac{1}{\gamma_l\gamma_r}\int_{R^2}\left(\frac{\left|\lambda\right|}{\gamma_l} + \frac{\left|\mu\right| l^{\vartheta}}{\gamma_l} + \frac{\left|\mu\right|}{\gamma_r}\right)\frac{\left|\mu\right|}{\gamma_r}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right|\right)\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g\left(j_1\right)\frac{\mu}{\gamma_r}\right)\right| d\lambda d\mu$$
$$\leq C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \tag{116}$$

when $l, r \geq l_0$, where in obtaining the inequality we have used (58) and (59). Further using (60), the same bound (116) holds for (109) when $l \leq l_0$ and/or $r \leq l_0$ also. Thus we need to show that

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r}\left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \to 0 \tag{117}$$

as $n \to \infty$ first and then $q \to \infty$. To see that this is true, take for convenience that

$$\gamma_n = n^H \quad \text{for all } n \geq 1.$$

First note that, using the restriction $1 < 3H$,

$$\frac{\gamma_n}{n} \sum_{l=1}^{n} \sum_{r=q}^{n} \frac{1}{\gamma_l \gamma_r^3} = \left(\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \right) \sum_{r=q}^{n} \frac{1}{\gamma_r^3} \leq C q^{1-3H} \to 0 \quad \text{as } q \to \infty,$$

where we have used $\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \leq C$ and $\sum_{r=q}^{n} \frac{1}{\gamma_r^3} = \sum_{r=q}^{n} \frac{1}{r^{3H}} \leq C q^{1-3H}$. Next

$$\sum_{l=1}^{n} \frac{l^{\vartheta}}{\gamma_l^2} = \sum_{l=1}^{n} \frac{1}{l^{2H-\vartheta}} \leq \begin{cases} C \log n & \text{if } 2H - \vartheta \geq 1 \\ C n^{1-2H+\vartheta} & \text{if } 2H - \vartheta < 1. \end{cases}$$

Also, $\frac{\gamma_n}{n} = n^{H-1}$. Hence

$$\frac{\gamma_n}{n} \sum_{l=1}^{n} \sum_{r=q}^{n} \frac{l^{\vartheta}}{\gamma_l^2 \gamma_r^2} \leq \frac{\gamma_n}{n} \left(\sum_{l=1}^{n} \frac{l^{\vartheta}}{\gamma_l^2} \right)^2 \leq \begin{cases} C n^{H-1} (\log n)^2 & \text{if } 2H - \vartheta \geq 1 \\ C n^{H-1+2-4H+2\vartheta} = C n^{1-3H+2\vartheta} & \text{if } 2H - \vartheta < 1 \end{cases} \tag{118}$$

where note that $1 - 3H + 2\vartheta < 0$ in view of (108) Thus (117) holds and hence the proof of Lemma 10 is complete under the restriction (19).

Under the restriction (15), we use the same bound (109) but with $\widehat{w}(\lambda, \mu)$ replaced by

$$\left| \widehat{K}_{\eta_1}(\lambda) \right| \left| \widehat{K}_{\eta_2}(\mu) \right| \max \left(\left| \widehat{M_{f,\eta_1}}(\lambda) \right| \left| \widehat{M_{f,\eta_2}}(\mu) \right|, \left| \widehat{m_{f,\eta_1}}(\lambda) \right| \left| \widehat{m_{f,\eta_2}}(\mu) \right| \right),$$

In this case, using the arguments that follows (112), together with (71), we have

$$\begin{aligned} & \int_R \left| \widehat{M_{f,\eta_1}} \left(\frac{\lambda}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)} \mu - \frac{\mu}{\gamma_r} \right) \right| \left| \widehat{K}_{\eta_1} \left(\frac{\lambda}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)} \mu - \frac{\mu}{\gamma_r} \right) \right| \prod_{j=[l/2]}^{l-1} \left| \psi \left(\frac{\lambda g(j)}{\gamma_l} \right) \right| d\lambda \\ \leq \; & C \left(\frac{1}{\gamma_l} + \frac{|\mu| \, l^{\vartheta}}{\gamma_l} + \frac{|\mu|}{\gamma_r} \right) + C |\eta_1|^d + C \rho^l \frac{\gamma_l}{\eta}. \end{aligned}$$

Thus the left most side of (109) has the bound

$$C \left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r} + |\eta_1|^d + \rho^l \frac{\gamma_l}{\eta_1} \right) \left(\frac{1}{\gamma_r} + |\eta_2|^d + \rho^r \left(\frac{\gamma_r}{\eta_2} \right)^2 \right).$$

By choosing $\eta_1 = \gamma_l^{-\frac{1}{d}}$ and $\eta_2 = \gamma_r^{-\frac{1}{d}}$, and noting (recall $0 < \rho < 1$) that $\rho^r \left(\frac{\gamma_r}{\eta_2} \right)^2 = \rho^r \gamma_r^{2+\frac{2}{d}} \leq C \gamma_r^{-1}$ and similarly $\rho^l \frac{\gamma_l}{\eta_1} \leq C \gamma_l^{-1}$, we see that the preceding bound reduces to that in (116). This completes the proof of Lemma 10 (for the situation of Theorem 1).

Now, regarding the Lemma 10 for the situation of Theorem 2, it was indicated earlier (see the end of the Verification of (R*2), Section 4) that the only essential difference is that in place of $\widehat{w}(\lambda,\mu)=\widehat{f}(\lambda)\,\widehat{f}(\mu)$ in the above arguments, $\widehat{w_r}(\lambda,\mu)$ as defined in the Verification of (R*2) will be involved, for which we have the inequality (96). Thus, in place of (117), we need to verify that $\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r}\left\{\left(\frac{l^{\vartheta}}{\gamma_l}+\frac{1}{\gamma_r}\right)\frac{1}{\gamma_r}+\frac{1}{\gamma_r^2}\right\}\to 0$ as $n\to\infty$ first and then $q\to\infty$, but this has been done above. ■

We next verify (R3) and (R*3).

VERIFICATION OF (R3). We show that (recall $n_{mk}=\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]$)

$$E\left[\zeta_{nmk}^4\right]\le\frac{C}{n^{\delta}}+C\left(\frac{\gamma_n}{n}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\frac{1}{\gamma_l}\right)\left(\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\right),\quad\text{for some }\delta>0. \tag{119}$$

This will verify (R3), because then

$$\sum_{k=1}^{m}E\left[\zeta_{nmk}^4\right]\le\frac{Cm}{n^{\delta}}+C\left(\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\right)\left(\max_{1\le k\le m}\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\right)$$

where $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\le C$ and $\max_{1\le k\le m}\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\sim\frac{1}{1-H}\left(\frac{1}{m}\right)^{1-H}$ as $n\to\infty$.

We shall show in detail that

$$\left(\frac{\gamma_n}{n}\right)^2\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\left|E\left[f(S_l)\,f(S_{l+r})\,f^2(S_{l+r+q})\right]\right| \tag{120}$$

and

$$\left(\frac{\gamma_n}{n}\right)^2\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\sum_{s=1}^{n_{mk}}\left|E\left[f(S_l)\,f(S_{l+r})\,f(S_{l+r+q})\,f(S_{l+r+q+s})\right]\right| \tag{121}$$

is bounded by r.h.s. of (119). The same can be similarly shown to be true for the remaining analogues in the expansion of $E\left[\zeta_{nmk}^4\right]=\left(\frac{\gamma_n}{n}\right)^2E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}f(S_l)\right)^4\right]$. We shall use Lemma 19 in a manner similar to the proof of Lemmas 10 above. *In addition, we shall give the details of the verification only for the situation of the statement (II) of Theorems 1 and 2. The corresponding situation of the statement (I) can be similarly verified using the ideas in the earlier proof of Lemma 10.*

We first deal with (120). Using exactly the same ideas as in (109), we have (noting

$\left|\widehat{f^2}(\lambda)\right| \leq C$)

$$
\begin{aligned}
& (2\pi)^3 \left|E\left[f(S_l) f(S_{l+r}) f^2(S_{l+r+q})\right]\right| \\
\leq\ & \frac{C}{\gamma_l\gamma_r\gamma_q} \int \left(\prod_{j_1=[l/2]}^{l-1} \left| \psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1,r)}{\gamma_r} + \frac{\lambda_3 g(j_1+r,q)}{\gamma_q} \right) \right| \right) \\
& \times \left(\prod_{j_2=[r/2]}^{r-1} \left| \psi\left(\frac{\lambda_2 g(j_2)}{\gamma_r} + \frac{\lambda_3 g(j_2,q)}{\gamma_q} \right) \right| \right) \left(\prod_{j_3=[q/2]}^{q-1} \left| \psi\left(\frac{\lambda_3 g(j_3)}{\gamma_q} \right) \right| \right) \\
& \times \left| \widehat{f}\left(\frac{\lambda_1}{\gamma_l} - \frac{\lambda_2}{\gamma_r} \right) \right| \left| \widehat{f}\left(\frac{\lambda_2}{\gamma_r} - \frac{\lambda_3}{\gamma_q} \right) \right| d\lambda_1 d\lambda_2 d\lambda_3, \qquad (122)
\end{aligned}
$$

where recall that $g(j,r) = g(j+r) - g(j)$. We make the transformations

$$
\frac{\lambda_3 g(j_3)}{\gamma_q} \mapsto \frac{\lambda_3 g(j_3)}{\gamma_q},
$$

$$
\frac{\lambda_2 g(j_2)}{\gamma_r} + \frac{\lambda_3 g(j_2,q)}{\gamma_q} = \frac{g(j_2)}{\gamma_r}\left(\lambda_2 + \lambda_3 \frac{\gamma_r g(j_2,q)}{\gamma_q g(j_2)} \right) \mapsto \frac{\lambda_2 g(j_2)}{\gamma_r},
$$

$$
\begin{aligned}
\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1,r)}{\gamma_r} + \frac{\lambda_3 g(j_1+r,q)}{\gamma_q} &= \frac{g(j_1)}{\gamma_l}\left(\lambda_1 + \lambda_2 \frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)} + \lambda_3 \frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)} \right) \\
&\mapsto \frac{\lambda_1 g(j_1)}{\gamma_l}.
\end{aligned}
$$

Here note that, in the same way as in (115) using (59) and Lemma 19, we have

$$
\left| \frac{\gamma_r g(j_2,q)}{\gamma_q g(j_2)} \right| \leq C r^\vartheta, \quad \left| \frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)} \right| \leq C l^\vartheta, \quad \left| \frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)} \right| \leq C l^\vartheta
$$

uniformly in the variables involved. (For instance, using (59) and Lemma 19, $\left|\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right| = \left|\frac{\gamma_l}{b_l g(j_1)}\right| \left|\frac{b_l g(j_1+r,q)}{\gamma_q}\right| \leq C l^\vartheta$, $[l/2] \leq j_1 \leq l$, $r,q \geq 1$.) Therefore, in the same way (109) is bounded by (116), the right hand side in (122) is bounded by

$$
\begin{aligned}
& \frac{C}{\gamma_l\gamma_r\gamma_q} \int \left(\frac{1}{\gamma_l}\left(|\lambda_1| + |\lambda_2| l^\vartheta + |\lambda_3| l^\vartheta r^\vartheta + |\lambda_3| l^\vartheta \right) + \frac{1}{\gamma_r}\left(|\lambda_2| + |\lambda_3| r^\vartheta \right) \right) \\
& \times \left(\frac{1}{\gamma_r}\left(|\lambda_2| + |\lambda_3| r^\vartheta \right) + \frac{|\lambda_3|}{\gamma_q} \right) \left(\prod_{j_1=[l/2]}^{l-1} \left| \psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l} \right) \right| \right) \\
& \times \left(\prod_{j_2=[r/2]}^{r-1} \left| \psi\left(\frac{\lambda_2 g(j_2)}{\gamma_r} \right) \right| \right) \left(\prod_{j_3=[q/2]}^{q-1} \left| \psi\left(\frac{\lambda_3 g(j_3)}{\gamma_q} \right) \right| \right) d\lambda_1 d\lambda_2 d\lambda_3 \\
\leq\ & \frac{C}{\gamma_l\gamma_r\gamma_q} \left(\frac{l^\vartheta + l^\vartheta r^\vartheta}{\gamma_l} + \frac{r^\vartheta}{\gamma_r} \right) \left(\frac{r^\vartheta}{\gamma_r} + \frac{1}{\gamma_q} \right).
\end{aligned}
$$

Thus we need to consider

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l \gamma_r \gamma_q} \left(\frac{l^\vartheta + l^\vartheta r^\vartheta}{\gamma_l} + \frac{r^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta}{\gamma_r} + \frac{1}{\gamma_q}\right). \tag{123}$$

We have, similar to (118),

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \frac{l^\vartheta r^\vartheta}{\gamma_l^2 \gamma_r^2 \gamma_q} \leq \begin{cases} Cn^{H-1}\left(\log n\right)^2 & \text{if } 2H-\vartheta \geq 1 \\ Cn^{1-3H+2\vartheta} & \text{if } 2H-\vartheta < 1. \end{cases}$$

Essentially the same holds for all other terms in (123) except for

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_l \gamma_r^3 \gamma_q} = \left(\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l \gamma_q}\right)\left(\sum_{r=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_r^3}\right),$$

which is of the form of the bound in (119) because $\sum_{r=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_r^3} < \infty$ in view of $3H - 2\vartheta > 1$ (see (108)). Thus the bound in (119) holds for (120).

We next consider (121). The ideas involved are the same as those used for (120). First,

$$\begin{aligned}
& \left(2\pi\right)^4 \left|E\left[f\left(S_l\right) f\left(S_{l+r}\right) f\left(S_{l+r+q}\right) f\left(S_{l+r+q+s}\right)\right]\right| \\
\leq\ & \frac{C}{\gamma_l \gamma_r \gamma_q \gamma_s} \int \prod_{j_1=[l/2]}^{l-1} \left|\psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1, r)}{\gamma_r} + \frac{\lambda_3 g(j_1+r, q)}{\gamma_q} + \frac{\lambda_4 g(j_1+r+q, s)}{\gamma_s}\right)\right| \\
& \times \prod_{j_2=[r/2]}^{r-1} \left|\psi\left(\frac{\lambda_2 g\left(j_2\right)}{\gamma_r} + \frac{\lambda_3 g\left(j_2, q\right)}{\gamma_q} + \frac{\lambda_4 g\left(j_2+q, s\right)}{\gamma_s}\right)\right| \\
& \times \prod_{j_3=[q/2]}^{q-1} \left|\psi\left(\frac{\lambda_3 g\left(j_3\right)}{\gamma_q} + \frac{\lambda_4 g\left(j_3, s\right)}{\gamma_s}\right)\right| \prod_{j_4=[s/2]}^{s-1} \left|\psi\left(\frac{\lambda_4 g\left(j_4\right)}{\gamma_s}\right)\right| \\
& \times \left|\widehat{f}\left(\frac{\lambda_1}{\gamma_l} - \frac{\lambda_2}{\gamma_r}\right) \widehat{f}\left(\frac{\lambda_2}{\gamma_r} - \frac{\lambda_3}{\gamma_q}\right) \widehat{f}\left(\frac{\lambda_3}{\gamma_q} - \frac{\lambda_4}{\gamma_s}\right) \widehat{f}\left(\frac{\lambda_4}{\gamma_s}\right)\right| d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4.
\end{aligned} \tag{124}$$

This is obtained using the same arguments used in obtaining the bound (122). In exactly the same way as in (122), we first make suitable transformations and then see that, using $\left|\widehat{f}\left(\lambda\right)\right| \leq C\left|\lambda\right|$, (124) is bounded by

$$\frac{C}{\gamma_l \gamma_r \gamma_q \gamma_s} \left(\frac{l^\vartheta + l^\vartheta r^\vartheta + l^\vartheta r^\vartheta q^\vartheta}{\gamma_l} + \frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r} + \frac{q^\vartheta}{\gamma_q}\right)\left(\frac{q^\vartheta}{\gamma_q} + \frac{1}{\gamma_s}\right)\frac{1}{\gamma_s}.$$

In the same way as in (123) it is easy to show, using (108), that the sum

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \sum_{s=1}^{n_{mk}} \frac{1}{\gamma_l \gamma_r \gamma_q \gamma_s} \left(\frac{l^\vartheta + l^\vartheta r^\vartheta + l^\vartheta r^\vartheta q^\vartheta}{\gamma_l} + \frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r}\right)$$
$$\times \left(\frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r} + \frac{q^\vartheta}{\gamma_q}\right) \left(\frac{q^\vartheta}{\gamma_q} + \frac{1}{\gamma_s}\right) \frac{1}{\gamma_s}$$

is bounded by the right hand side of (119). This completes the verification of (R3). ■

Verification of (R*3). We start with the **remark** that when we verified (R3) for $f(S_l)$, it was clear that the same verification will hold for $f(S_{l,\nu})$ also for any $\nu \geq 1$, where $S_{l,\nu}$ is as in (65), because S_l and $S_{l,\nu}$ have the same structural form. Recall that $\zeta^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l)\,\omega_l$, where ω_l is a sum of ν terms (see (7)), $\nu \geq 1$. It is convenient to prove the claim by induction on ν. Therefore we shall use the notation

$$\omega_{l,\nu} = \sum_{j=l-\nu+1}^{l} d_{l-j}\eta_j.$$

For $\nu = 1$, we have $\omega_{l,1} = \eta_l$, and

$$\sum f(S_l)\,\eta_l = \sum \left(f(S_l)\,\eta_l - f_*(S_{l,1})\right) + \sum f_*(S_{l,1}), \tag{125}$$

where

$$f_*(S_{l,1}) = E_{l-1}\left[f(S_l)\,\eta_l\right]$$

is as in (67), corresponding to $\nu = 1$. (Here and in the rest of the proof the sum $\sum$ stands for $\sum_{l=\left[n\frac{k}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]}$.) According to Lemma 7, $f_*(x)$ satisfies the conditions of $f(x)$ of Theorem 1. Therefore, in view of the remark made above, we are implicitly assuming that (R3) is verified for $f_*(S_{l,1})$.

For the remaining term in (125) note that $f(S_l)\,\eta_l - f_*(S_{l,1})$ form martingale differences, and hence (see Hall and Heyde (1980, Theorem 2.11))

$$\begin{aligned} & E\left[\left(\sum \left(f(S_l)\,\eta_l - f_*(S_{l,1})\right)\right)^4\right] \\ \leq\ & CE\left[\left(\sum E_{l-1}\left[\left(f(S_l)\,\eta_l\right)^2\right]\right)^2\right] + C\sum E\left[\left(f(S_l)\,\eta_l\right)^4\right]. \end{aligned} \tag{126}$$

We have $E_{l-1}\left[\left(f(S_l)\,\eta_l\right)^2\right] = f_*^{(2)}(S_{l,1})$ where $f_*^{(2)}(x) = E\left[f^2(x+\xi_l)\,\eta_l^2\right]$. That is,

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum E_{l-1}\left[\left(f(S_l)\,\eta_l\right)^2\right]\right)^2\right] = \left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum f_*^{(2)}(S_{l,1})\right)^2\right].$$

In addition $\int \left(f_*^{(2)}(x) + \left| f_*^{(2)}(x) \right|^2 \right) dx < \infty$, which will imply (see (114) or the Remark at the end of Jeganathan (2004a)) that

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f_*^{(2)}(S_{l,1})\right)^2\right] \leq C\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \frac{1}{\gamma_l}\left(1+\sum_{r=1}^{n_{mk}}\frac{1}{\gamma_r}\right) \tag{127}$$

and hence, because $\max_{1\leq k\leq m} \frac{\gamma_n}{n}\sum_{r=1}^{n_{mk}} \frac{1}{\gamma_r} \sim C\left(\frac{1}{m}\right)^{1-H}$ and $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \leq C$,

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{k=1}^{m} E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f_*^{(2)}(S_{l,1})\right)^2\right] \to 0$$

as $n \to \infty$ first and then $m \to \infty$.

Similarly $\frac{\gamma_n}{n}\sum_{l=1}^{n} E\left[\left(f(S_l)\eta_l\right)^4\right] \leq C$, that is, $\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=1}^{n} E\left[\left(f(S_l)\eta_l\right)^4\right] \leq C\frac{\gamma_n}{n} \to 0$.

Thus (R*3) holds for the case $\nu = 1$. We remark that the same arguments show that (R*3) holds also for $f(S_{l,1})\eta_{l-1} = f(S_{l,1})\omega_{l-1,1}$.

Now suppose that $\nu = i$, $i \geq 2$, and that (R*3) holds for $\nu = i-1$. Taking into account the preceding remark, this means we can assume that (R*3) holds for $f(S_{l,1})\omega^*_{l,i-1}$, where

$$\omega^*_{l,i-1} = \omega_{l,i} - \eta_l = \sum_{j=l-i+1}^{l-1} d_{l-j}\eta_j.$$

We have

$$\sum f(S_l)\omega_{l,i} = \sum \left(f(S_l)\omega_{l,i} - E_{l-1}\left[f(S_l)\omega_{l,i}\right]\right) + \sum E_{l-1}\left[f(S_l)\omega_{l,i}\right],$$

where

$$\sum E_{l-1}\left[f(S_l)\omega_{l,i}\right] = \sum E_{l-1}\left[f(S_l)(\omega_{l,i}-\eta_l)\right] + \sum E_{l-1}\left[f(S_l)\eta_l\right].$$

Here $E_{l-1}\left[f(S_l)\eta_l\right] = f_*(S_{l,1})$ is as before, for which as noted earlier the verification of (R3) will be the same as that for S_l. Also,

$$E_{l-1}\left[f(S_l)(\omega_{l,i}-\eta_l)\right] = E_{l-1}\left[f(S_l)\omega^*_{l,i-1}\right] = \omega^*_{l,i-1}E_{l-1}\left[f(S_l)\right] = g(S_{l,1})\omega^*_{l,i-1},$$

where $g(x) = E\left[f(x+\xi_l)\right]$. This form is the same as that of $f(S_{l,1})\omega^*_{l,i-1}$, for which we have assumed the induction hypothesis that (R*3) holds.

Regarding the remaining term $\sum\left(f(S_l)\omega_{l,i} - E_{l-1}\left[f(S_l)\omega_{l,i}\right]\right)$, which is a sum of martingale differences, we have the bound analogous to (126), in which the second term is

treated in the same way as the second term in (126). The first term is $E\left[\left(\sum E_{l-1}\left[\left(f\left(S_l\right)\omega_{l,i}\right)^2\right]\right)^2\right]$, where (recall $\omega_{l,i}=\omega^*_{l,i-1}+\eta_l$)

$$E_{l-1}\left[\left(f\left(S_l\right)\omega_{l,i}\right)^2\right]\leq 2\left|\omega^*_{l,i-1}\right|^2 E_{l-1}\left[f^2\left(S_l\right)\right]+2E_{l-1}\left[f^2\left(S_l\right)\eta_l^2\right].$$

Letting $g\left(S_{l,1}\right)=E_{l-1}\left[f^2\left(S_l\right)\right]$, it is implicit in the arguments of the verification of (R*2) that the bound in (127) holds for

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\left|\omega^*_{l,i-1}\right|^2 g\left(S_{l,1}\right)\right)^2\right]$$

also. Also the term $E_{l-1}\left[f^2\left(S_l\right)\eta_l^2\right]=f^{(2)}_*\left(S_{l,1}\right)$ has already been treated. It thus follows that (R*3) holds for $\sum f\left(S_l\right)\omega_l$ when $\nu=i$. This completes the verification of (R*3). ■

Proof of Theorem 3. (This proof can be read immediately after (109) and (124).) We consider the situation of the statement (I) of the theorem. First consider the case $r=1$. Then, in view of Proposition 13, it is enough to show that

$$E\left[\left(\frac{\gamma_n}{n}\sum_{j=1}^{n} f\left(S_j,S_{j+1}\right)-\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+1}\right)\right]\right)^2\right]\to 0. \tag{128}$$

The proof of this is the same as that of (113) but now the inequalities (109) and (124) will be used. To see this note that the inequality (124) holds with $f\left(S_l,S_{l+r}\right)f\left(S_{l+r+q},S_{l+r+q+s}\right)$ in place of $f\left(S_l\right)f\left(S_{l+r}\right)f\left(S_{l+r+q}\right)f\left(S_{l+r+q+s}\right)$, and hence in particular (taking $r=1,q=i-1,s=1$)

$$\left|E\left[f\left(S_l,S_{l+1}\right)f\left(S_{l+i},S_{l+i+1}\right)\right]\right|\leq\frac{C}{\gamma_l\gamma_i}.$$

Similarly (109) gives $\left|E\left[f^2\left(S_l,S_{l+1}\right)\right]\right|\leq\frac{C}{\gamma_l}$. Hence the proof of (128) is the same as that of (113). This is also the case for the situation of the statement (II). The proof of the general case $r\geq 2$ is similar; using the statement of Lemma 14 for the general case $r\geq 2$. We omit the details. ■

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