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RATIONALIZABLE TRADE

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Abstract

We formulate necessary and sufficient conditions for interim rationalizable trade between two players.

1 Introduction

“No trade” theorems provide sufficient conditions for the absence of equilibrium trade between asymmetrically informed players.[1] In this paper, we examine when trade is rationalizable, i.e., consistent with common knowledge of rationality. Since rationalizability is a weaker solution concept than equilibrium, trade is sometimes rationalizable even when there is no equilibrium trade. We provide a necessary and sufficient condition for rationalizable trade between two asymmetrically informed players. Our characterization covers situations where the no trade theorems do not apply, e.g., when there are ex post gains from trade and there is no common prior. Thus our necessary and sufficient condition neither implies nor is implied by the standard sufficient conditions for no equilibrium trade.

We begin by reviewing some known special arguments on rationalizable trade that our analysis will unify and relate to the general case.

An Example of Rationalizable Trade. Each of two players is handed a card out of a shuffled deck, and after seeing the card makes the decision to either deposit \$10 in a pot or walk away from the game. Both players make their decisions simultaneously, without observing the other player’s card or decision. If either player walks away, the game terminates without any gains or losses. If neither player walks away, the cards are revealed; if they are the same color, player one collects the \$20 in the pot, otherwise player two collects the \$20.

The decision to accept the trade in this context is to not walk away from the game. Using a plus or a minus sign to denote a positive ex-post gain from trading for player one or two, respectively, the ex-post gains from trade in this game can be summarized by the following sign pattern (the “cross” pattern), that will play an important role throughout this paper.

	b	r
b	$+$	$-$
r	$-$	$+$

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[1] See Appendix A of Dekel and Gul [6] for a review. For example, a result of Sebenius and Geanakoplos [15] showed that two asymmetrically informed, risk neutral, players with a common prior will not trade in equilibrium an object of common ex-post value, unless they expect zero gains from the trade. Morris [11] provides necessary conditions on players’ priors for such results.

Let $[b]$ be the strategy "accept the trade only if the card is black" and let $[r]$ be the strategy "accept the trade only if the card is red." Then $[b]$ is a best response for player one to strategy $[b]$ of player two; $[b]$ for player two is a best response to player one choosing $[r]$; $[r]$ is in turn a best response for player one to strategy $[r]$ of player two; and $[r]$ for player two is a best response to player one choosing $[b]$. Strategies $[b]$ and $[r]$ are therefore rationalizable for both players.[2]

Examples with No Rationalizable Trade. We consider a finite signal version of the trading envelopes problem studied by Nalebuff [13], Geanakoplos [9], and others. Each of two players randomly selects an envelope out of a box containing indistinguishable sealed envelopes. It is common knowledge between the players that each envelope in the box contains some number of dollar bills not exceeding some known number, say ten. After the players privately open their envelopes, they have the option of offering to exchange their envelope with the other player's envelope. If both players offer to trade, the envelopes are swapped; otherwise the players keep their original envelopes. A simple iterative argument shows that trade is not rationalizable in this setting. An player who receives an envelope containing ten dollar bills will clearly not offer to trade. Knowing that, an player who receives nine dollar bills will also refuse to trade, and so on.

More generally, if trade is zero-sum and each player's gain is a monotonic function of both players' signals, then excluding equilibrium trade (e.g., by assuming common prior beliefs) also excludes rationalizable trade. This is because, by monotonicity in best responses (strategic complementarities), the supremum of all rationalizable strategy profiles is an equilibrium (as in Milgrom and Roberts [10]). Morris [12] gave sufficient conditions for no rationalizable trade exploiting such strategic complementarity arguments.

Necessary and Sufficient Conditions. Our results concern *interim* rationalizability, where different types of the same player may hold different conjectures over the opponent's actions;[3] we also assume that players reject trade when they are indifferent and each player has a finite number of possible signals. In this setting, we show that trade is rationalizable if and only if the "sign pattern" of the game (constructed in analogy to the cross pattern of the above card example) contains no cycles. We show that this condition

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[2] Aumann [1] has argued that rationalizability is prima facie too weak a solution concept, precisely because it allows gains from trade in zero sum settings such as the above example.

[3] Alternative versions of rationalizability are discussed in section 3.

is in fact very close to each of the three conditions we discussed above. If exactly one player always gains ex post, there is no rationalizable trade if and only if any one of the following equivalent conditions holds:

1. No two-by-two subset of the game's sign pattern has the *cross pattern* of the card example.
2. The game's sign pattern is that of some *trading envelopes problem*.
3. The game's sign pattern is that of some *monotone zero sum trade*; that is, under some ordering of signal realizations, player 1's gains are increasing in both signals and player 2's gains are decreasing.

The card and trading envelopes examples discussed above are therefore canonical examples of the possibility and impossibility, respectively, of rationalizable trade. This characterization is independent of the players' priors on signals.

The paper's characterization is related to results on rational expectations equilibria (REE) by DeMarzo and Skiadas [7], [8]. For example, Proposition 8 of [7] shows that in finite two-player asymmetric information economies with one risky asset whose payoffs are contingent on the pooled signals, and one risk-free asset, the absence of the cross pattern, properly interpreted, implies the non-existence of partially informative REE, and the uniqueness of a fully informative REE.

2 The Result

We consider a game between two players, labeled 1 and 2. The players' prior beliefs are represented by the probabilities P_1 and P_2, respectively, defined on some measurable space $(\Omega, \mathcal{F})$. Player i observes the realization of a signal $\tau_i : \Omega \to T_i$, where T_i is a finite set, and $P_i[\tau_i = t_i] > 0$ for all $t_i \in T_i$ and $i \in \{1, 2\}$. We use the notation $\tau = (\tau_1, \tau_2)$ and $T = T_1 \times T_2$ throughout, while the expectation operator with respect to P_i is denoted E_i.

Based on their information, players make a decision to accept or reject a trade. If both players accept the trade, player i receives a state-contingent payoff $V_i : \Omega \to \mathbf{R}$, a P_i-integrable random variable. If either player rejects the trade, the payoff to both players is zero. In contexts in which players are not risk neutral, V_i should be thought of as the ex post utility of player i if trade occurs minus the ex post utility of player i if no trade occurs.

A *strategy* for player i is any function of the form $d_i : T_i \rightarrow [0,1]$. We interpret $d_i(t_i)$ as the probability that player i who observes a signal value t_i will offer to trade. The space of all strategies for player i is denoted D_i, and the no trade strategy (identically equal to zero) is denoted 0. Given any set of strategies $S \subseteq D_i$, we let $\text{conv}(S)$ denote the convex hull of S.

Given player two strategy $d_2 \in D_2$, player one's *best response* is the strategy $B_1(d_2) \in D_1$ defined by

$$B_1(d_2)(t_1) = \begin{cases} 1 & \text{if } E_1\left[V_1 d_2(\tau_2) \mid \tau_1 = t_1\right] > 0; \\ 0 & \text{otherwise.} \end{cases}$$

The second player's best response function, B_2, is defined symmetrically. Implicit in these definitions is the assumption that a trade is rejected if there is zero expected gain from trade. This assumption is discussed and justified in section 3.

A strategy profile $(d_1, d_2) \in D_1 \times D_2$ is an *equilibrium* if $d_1 = B_1(d_2)$ and $d_2 = B_2(d_1)$. No trade, $(0,0)$, is always an equilibrium of this game. The no equilibrium trade result gives sufficient conditions for this to be the unique equilibrium:

Proposition 1 *If $P_1 = P_2$ and $V_1 + V_2 \leq 0$, then the no trade equilibrium, $(0,0)$, is the unique equilibrium.*

Proof. Suppose that $(d_1, d_2) \neq (0,0)$ is an equilibrium. Then $d_1 = B_1(d_2)$ implies $E_1[V_1 d_1(\tau_1) d_2(\tau_2)] > 0$, and $d_2 = B_2(d_1)$ implies $E_2[V_2 d_1(\tau_1) d_2(\tau_2)] > 0$. The two inequalities are clearly inconsistent with the proposition's assumptions. ■

We consider instead the set of strategies that are rationalizable. The set of rationalizable strategies is identified by iteratively deleting strategies that are not best responses to remaining strategies of the opponent. So starting with $I_i^0 = D_i$, we recursively define:

$$\begin{aligned} I_1^{k+1} &= \left\{ d_1 \in I_1^k \mid \forall t_1 \in T_1 : \exists d_2 \in \text{conv}\left(I_2^k\right) : d_1(t_1) = B_1(d_2)(t_1) \right\}, \\ I_2^{k+1} &= \left\{ d_2 \in I_2^k \mid \forall t_2 \in T_2 : \exists d_1 \in \text{conv}\left(I_1^k\right) : d_2(t_2) = B_2(d_1)(t_2) \right\}. \end{aligned}$$

The set of *interim rationalizable strategies* for player i is $I_i^\infty = \bigcap_{k \geq 0} I_i^k$. This is the *interim* version of rationalizability because different types of each player are allowed to hold different conjectures about the opponent's actions.[4] The alternative ex ante version of rationalizability is discussed in section 3.

Write L_i for the set of signal profiles where player i expects a strict gain from trade, i.e.,

$$L_i = \left\{ t \in T : E_i \left[V_i 1_{\{\tau = t\}} \right] > 0 \right\}, \qquad i \in \{1, 2\}, \tag{1}$$

where $1_{\{\tau=t\}}$ denotes the random variable that is equal to one on the event $\{\tau = t\}$ and zero otherwise. A *n-cycle* of (L_1, L_2) is any set of the form $\{t^1, t^2, \ldots, t^n\} \subseteq L_1$ such that $\left(t_1^{k+1}, t_2^k\right) \in L_2$ for all $k \in \{1, \ldots, n-1\}$, and $(t_1^1, t_2^n) \in L_2$. Thus writing a $+$ for elements of L_1 and a $-$ for elements of L_2, the following are examples of a 1-cycle, a 2-cycle, and a 4-cycle, respectively:

	t_2
t_1	$\pm$

	t_2	t_2'
t_1	$+$	$-$
t_1'	$-$	$+$

	t_2^1	t_2^2	t_2^3	t_2^4
t_1^1	$+$			$-$
t_1^2	$-$	$+$		
t_1^3		$-$	$+$	
t_1^4			$-$	$+$

(L_1, L_2) is *acyclic* if there is no n-cycle for any n.

Proposition 2 *There is no interim rationalizable trade* ($I_1^\infty = I_2^\infty = \{0\}$) *if and only if* (L_1, L_2) *is acyclic.*

Proof. Starting with $T_i^0 = T_i$, we recursively define the sets:

$$\begin{aligned} T_1^{k+1} &= \left\{ t_1 \in T_1^k : (t_1, t_2) \in L_1 \text{ for some } t_2 \in T_2^k \right\}, \\ T_2^{k+1} &= \left\{ t_2 \in T_2^k : (t_1, t_2) \in L_2 \text{ for some } t_1 \in T_1^k \right\}, \end{aligned}$$

and we let $T_i^\infty = \bigcap_{k \geq 0} T_i^k$. Given any set $T_i' \subseteq T_i$, we let $D_i(T_i')$ denote the set of all $d_i \in D_i$ such that $d_i(t_i) = 0$ if $t_i \notin T_i'$. Now we may verify by induction that $I_i^k = D_i\left(T_i^k\right)$ for all k; this is true by definition for $k = 0$. Suppose that it is true for k. Now if $t_1 \notin T_1^{k+1}$, then $(t_1, t_2) \notin L_1$ for all $t_2 \in T_2^k$. Thus $E_1[V_1 d_2 | \tau_1 = t_1] \leq 0$ for all $d_2 \in I_2^k$ and not offering to

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[4] Bernheim [3] and Pearce [14] defined rationalizability for complete information games. Incomplete information can always be represented by a move by nature at the beginning of the game. Applying Pearce's notion of *extensive form rationalizability* to that game gives interim rationalizability.

trade is the best response for player 1 given signal t_1. On the other hand, if $t_1 \in T_1^{k+1}$, then $(t_1, t_2) \in L_1$ for some $t_2 \in T_2^k$. The strategy of accepting only given signal t_2 $\left(d_2(t_2') = \begin{cases} 1, \text{ if } t_2' = t_2 \\ 0, \text{ otherwise} \end{cases} \right)$ is in T_2^k. So offering to trade is a best response for player 1 given signal t_1. But the strategy of never offering to trade is also in T_2^k. So not offering to trade is also a best response for player 1 given signal t_1. Thus $I_1^{k+1} = D_1\left(T_1^{k+1}\right)$. A symmetric argument implies $I_2^{k+1} = D_2\left(T_2^{k+1}\right)$. So $I_i^\infty = D_i(T_i^\infty)$, $i \in \{1, 2\}$. Thus there is interim rationalizable trade if and only if $T_i^\infty \neq \emptyset$.

Now if (L_1, L_2) has a cycle, that cycle clearly survives all steps in the above elimination, and therefore trade is interim rationalizable.

To prove the converse, observe that

$$\begin{aligned} t_1 &\in T_1^\infty \quad \Rightarrow \quad (t_1, t_2) \in L_1 \text{ for some } t_2 \in T_2^\infty, \\ t_2 &\in T_2^\infty \quad \Rightarrow \quad (t_1, t_2) \in L_2 \text{ for some } t_1 \in T_1^\infty. \end{aligned}$$

Suppose now that $t_1^0 \in T_1^\infty$. Then there exists $t_2^0 \in T_2^\infty$ such that $(t_1^0, t_2^0) \in L_1$. Similarly, there exists $t_1^1 \in T_1^\infty$ such that $(t_1^1, t_2^0) \in L_2$. Repeating this argument, we generate a sequence $\left\{t^k : k = 0, 1, \ldots\right\}$ such that $\left(t_1^k, t_2^k\right) \in L_1$ and $\left(t_1^{k+1}, t_2^k\right) \in L_2$ for all k. If (L_1, L_2) is acyclic, then this sequence must take an infinite number of values, contradicting our assumption that T is finite. ■

3 Discussion

PRIORS AND THE SIZE OF EX POST GAINS FROM TRADE DO NOT MATTER. Players' prior beliefs over the signal space T are irrelevant to the existence of interim rationalizable trade; all that matters is which elements of T each player thinks possible (i.e., assigns positive probability to). Similarly, the size of the ex post gain from trade $E_i\left[\left. V_i \right| \tau = t\right]$ is irrelevant; all that matters is whether it is strictly positive. This implies that attitudes to risk and endowments are irrelevant to the existence of interim rationalizable trade as long as trades depend only on the realized signal profile. Specifically, suppose that player i has a strictly increasing von Neumann-Morgenstern utility function u_i, and an endowment given by the random variable e_i. The proposed dollar trade is given by the function $x : T \to \mathbf{R}$, where $x(t)$ represents the amount that player two will pay player one on the event $\{\tau = t\}$. The implied net utility gains are given by functions $V_1 = u_1(e_1 + x(\tau)) - u_1(e_1)$ and $V_2 = u_2(e_2 - x(\tau)) - u_2(e_2)$. But calculating the L_1 and L_2 sets for V_1 and V_2 gives $L_1 = \{t \in T : P_1[\tau = t] > 0 \text{ and } x(t) > 0\}$ and $L_2 = \{t \in T : P_2[\tau = t] > 0 \text{ and } x(t) < 0\}$. Thus utility functions and endowments are irrelevant for the existence of interim rationalizable trade.

INFINITE SIGNAL SPACES. The proof of Proposition 2 remains valid if T_1 is finite and T_2 is countably infinite, but Proposition 2 is not valid if both T_1 and T_2 are countably infinite. For example, consider a countably infinite version of the trading envelopes game, where $T_1 = T_2 = \{1, 2, 3, ..\}$, $V = \tau_1 - \tau_2$, and $P_i(\tau_j > \tau_i | \tau_i = t_i) > 0$ for all $t_i \in T_i$, $i \in \{1, 2\}$, $j \neq i$. In this case, (L_1, L_2) is acyclic, but all strategies are interim rationalizable.[5]

ALTERNATIVE CHARACTERIZATIONS OF ACYCLICITY. In the appendix, we report a number of alternative chacterizations of acyclicity. They imply the three characterizations discussed in the introduction.

The Trading Envelopes Problem. We noted that an easy sufficient condition for no interim rationalizable trade is that the trade has the structure of a trading envelopes problem, i.e., there exist functions $f_i : T_i \to \{1, 2, ..\}$ with the interpretation that $f_i(t_i)$ is the number of dollars in the envelope of type t_i of player i; and so $V_1(\omega) = -V_2(\omega) = f_2(t_2(\omega)) - f_1(t_1(\omega))$. In fact, (L_1, L_2) is acyclic if and only if it could have been derived from a trading envelopes problem:

Lemma 3 *(L_1, L_2) is acyclic if and only if there exist functions $f_i : T_i \to \{1, 2, ...\}$ such that $L_1 \subseteq \{t \in T : f_2(t_2) > f_1(t_1)\}$ and $L_2 \subseteq \{t \in T : f_1(t_1) > f_2(t_2)\}$.*

Monotonicity. Another sufficient condition for no interim rationalizable trade was that the trade was zero sum and monotonic in both players' signals, i.e., $V_1(\omega) = -V_2(\omega) = x(t_1(\omega), t_2(\omega))$ where x is weakly increasing

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[5] The importance of allowing an infinite number of signals is well known from discussions of the trading envelopes problem (Nalebuff [13] and Geanakoplos [9]). Bhattacharya and Lipman [4] showed that trade is possible even in (interim) *equilibrium* with risk neutral agents, a common prior and zero sum trades. These assumptions are sufficient to rule out trade if there are only a finite number of signals, since the assumptions imply no ex ante gains from trade. But if there are an infinite number of possible signals and unbounded interim utilities from the trade, ex ante utilities may not be well-defined and the usual equilibrium no trade argument cannot be applied. They use a version of the trading envelopes problem to make this point.

in both signals. Again, (L_1, L_2) is acyclic if and only if it could have been derived from some monotone zero sum trade:

Lemma 4 *(L_1, L_2) is acyclic if and only if there exists an ordering of signals and (under that ordering) a weakly increasing function $x : T \to \mathbf{R}$ such that $L_1 \subseteq \{t \in T : x(t) > 0\}$ and $L_2 \subseteq \{t \in T : x(t) < 0\}$.*

The Cross Pattern. A sufficient condition for rationalizable trade was the simple cross pattern. But the cross pattern is in fact necessary for rationalizable trade, in the case where exactly one player strictly gains contingent on each signal realization:

Lemma 5 *If (L_1, L_2) is a partition of T, then (L_1, L_2) is acyclic if and only if (L_1, L_2) contains no 2-cycles.*

Notice that (L_1, L_2) will partition T whenever trades are zero sum ex post, they depend only on the vectors of signals observed by the players and all profiles of signal realizations are possible, i.e., if there exists $x : T \to \mathbf{R}$ such that, for all $t \in T$, $E[V_1 \mid \tau = t] = -E[V_2 \mid \tau = t] = x(t) \neq 0$ and $P_i[\tau = t] \neq 0$ for each i.

INTERIM VERSUS EX ANTE RATIONALIZABILITY. In applying rationalizability in an incomplete information setting, we face a modelling choice: should we think of players choosing their actions at the interim stage - after observing their private signals - in which case it is natural to allow different types of the same player to have different conjectures over their opponent's strategy; or should we think of players choosing their strategies at the ex ante stage - before observing their private signals - in which case it is natural to require different types of the same player to have the same conjecture over their opponent's strategy? The former leads to the interim definition of rationalizability that we described earlier. The latter gives the following definition of ex ante rationalizability. Starting with $X_i^0 = D_i$, we recursively define the strategy sets:

$$\begin{aligned} X_1^{k+1} &= \left\{ d_1 \in X_1^k \mid \exists d_2 \in \text{conv}\left(X_2^k\right) : \forall t_1 \in T_1 : d_1(t_1) = B_1(d_2)(t_1) \right\}, \\ X_2^{k+1} &= \left\{ d_2 \in X_2^k \mid \exists d_1 \in \text{conv}\left(X_1^k\right) : \forall t_2 \in T_2 : d_2(t_2) = B_2(d_1)(t_2) \right\}. \end{aligned}$$

The set of *ex ante rationalizable strategies* for player i is $X_i^\infty = \bigcap_{k \geq 0} X_i^k$.[6] The difference between the above recursion and that in the definition of interim rationalizability is the transposition of the existential and universal quantifiers. Clearly, $X_i^k \subseteq I_i^k$ for all k, and $X_i^\infty \subseteq I_i^\infty$. So, if (L_1, L_2) is acyclic, there is also no ex ante rationalizable trade. On the other hand, the following example demonstrates that acyclicity of (L_1, L_2) is not necessary for the impossibility of ex ante rationalizable trade.

Example 6 *Suppose that, for* $i \in \{1,2\}$, $T_i = \{t_i^1, t_i^2, t_i^3\}$, $P_i[\tau = t] = 1/9$ *for all* $t \in T$, *and* $V_1 = -V_2 = V$, *where* V *is given as a function of types by the following table:*

	t_2^1	t_2^2	t_2^3
t_1^1	0	1	1
t_1^2	-1	$-\varepsilon$	ε
t_1^3	-1	ε	$-\varepsilon$

.

If $\varepsilon \in (0, 1/3)$, *then ex ante rationalizable trade is impossible, even though* (L_1, L_2) *has a* 2*-cycle .*

However, a partial converse is possible. Suppose that (L_1, L_2) has a cycle, $\{t^1, \dots, t^n\}$. Modifying P_i so that $P_i[\tau = t] = 1/2n$ for all t associated with that cycle (that is, any t of the form $\left(t_1^k, t_2^k\right)$ or $\left(t_1^{k+1}, t_2^k\right)$ or (t_1^1, t_2^n)) while the conditional probabilities $P_i[\cdot \mid \tau = t]$ remain the same,[7] it follows easily that all strategies of the form $1_{\left\{\tau_i = t^k\right\}}$ are ex-ante rationalizable under the new priors.

REFINEMENTS. We assumed that trade is rejected if a player is indifferent between accepting and rejecting. We have in mind that there is a small cost associated with accepting trade (whether or not the trade is implemented). All the analysis of this paper would remain the same in the

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[6] Both ex ante and interim rationalizability assume that the players' priors over the state space are common knowledge. Battigalli [2] introduced a notion of incomplete information rationalizability that does not depend on agents' prior beliefs about types, and is in general weaker than interim rationalizability (in static games). But since our characterization of interim rationalizable trade was independent of the prior beliefs, our interim rationalizability characterization would hold unchaged for Battigalli's notion. An early version of Battigalli [2] noted the impossibility of rationalizable trade (in his sense) in the finite signal trading envelopes example.

[7] We could also assume that $P_i[\tau = t] > 0$ for all $t \in T$ by letting $P_i[\tau = t] = 1/2n - \varepsilon > 0$ for all $t \in \left\{t^1, \dots, t^n\right\}$, and sufficiently small $\varepsilon > 0$.

presence of a sufficiently small such cost, which was therefore omitted from our formal setup. This assumption ensures that a player's best response to no trade is no trade. If we did not impose this refinement, anything would be a best response to no trade, and therefore any behavior would be interim rationalizable. Thus some form of refinement is required to obtain any interesting results.

Our particular choice of refinement implies that no trade is always interim rationalizable for every type. It is *this* property that ensures that the existence of interim rationalizable trade is independent of the players' prior beliefs about signals. Suppose instead we deleted one round of weak dominated strategies before iterated deletion of strictly dominated strategies (Dekel and Fudenberg [5]). Then weak dominance considerations would sometimes require that accepting trade was the only possible best response by the opponent. This could make a significant difference, as in the following example.[8]

Example 7 *$T_1 = \{t_1\}$, $T_2 = \{t_2^1, t_2^2\}$, $P_i[\tau = t] = \frac{1}{2}$ for each $i \in \{1,2\}$ and $t \in T$, and (V_1, V_2) is given as a function of types by the following table:*

	t_2^1	t_2^2
t_1	$-2, 1$	$1, 1$

Trade is interim rationalizable at (t_1, t_2^2), since (t_1, t_2^2) is a 1-cycle. However, deleting weak dominated strategies would imply that both types of player 2 must trade. But now accepting trade cannot be a best response for the unique type of player 1.

Thus while we believe that our refinement is very natural for this trading game, it is important to note the subtle role that it plays.

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APPENDIX

In this appendix, we provide a number of alternative characterizations of acyclicity.

Condition A (L_1, L_2) is acyclic.

Condition B There exists a partition, (L_+, L_-), of T without cycles such that $L_1 \subseteq L_+$ and $L_2 \subseteq L_-$.

Condition C There exists a partition, (L_+, L_-), of T without 2-cycles such that $L_1 \subseteq L_+$ and $L_2 \subseteq L_-$.

Condition D There exist functions $f_i : T_i \to \{1, 2, \ldots\}$ such that $L_1 \subseteq \{t \in T : f_2(t_2) > f_1(t_1)\}$ and $L_2 \subseteq \{t \in T : f_1(t_1) > f_2(t_2)\}$.

Condition E There exist bijections $\phi_1 : T_1 \to \{1, 2, .., n_1\}$ and $\phi_2 : T_2 \to \{1, 2, .., n_2\}$ and an increasing function, $x : \{1, .., n_1\} \times \{1, ..., n_2\} \to \mathbf{R}$, such that $L_1 \subseteq \{t \in T : x(\phi_1(t_1), \phi_2(t_2)) > 0\}$ and $L_2 \subseteq \{t \in T : x(\phi_1(t_1), \phi_2(t_2)) < 0\}$.

Lemma 8 *Conditions A through E are all equivalent.*

Now condition C gives lemma 5, condition D gives lemma 3 and condition E gives lemma 4.

Proof. $(A \Rightarrow B)$ Suppose that (L_1, L_2) has no cycles and $t \notin L_1 \cup L_2$. We claim that either $(L_1 \cup \{t\}, L_2)$ or $(L_1, L_2 \cup \{t\})$ must then have no cycles, as well. Intuitively, suppose that attaching t to either L_1 or L_2 creates a cycle. An illustration of this situation appears below.

			t_2		
	$+$		$-$		
	$-$	$+$			
t_1		$-$	$*$		$+$
			$+$	$-$	
				$+$	$-$

It is then clear that the union of these two incomplete cycles forms a cycle without t, contradicting the assumption that (L_1, L_2) contains no cycles.

More formally, suppose that $(L_1 \cup \{t\}, L_2)$ has the cycle $(t^1, \ldots, t^m)$, and $(L_1, L_2 \cup \{t\})$ has the cycle $(s^1, \ldots, s^n)$, each chosen to be of minimal cardinality. Since (L_1, L_2) has no cycles, both of these cycles contain t, and since they were chosen to be minimal, they only contain t once. Without loss in generality, we assume that $t = (t_1^1, t_2^1) = (s_1^1, s_2^n)$. This implies that $(s_1^1, t_2^m) = (t_1^1, t_2^m) \in L_2$ and $(t_1^2, s_2^n) = (t_1^2, t_2^1) \in L_1$. But then the definition of a cycle implies that $(t^2, \ldots, t^m, s^1, .., s^n)$ is a cycle of (L_1, L_2), a contradiction. Repeating this procedure, all elements of T can be signed without creating cycles, confirming condition B.
$(B \Rightarrow C)$ Immediate.

For the next part of this proof, we use an argument similar to that of Proposition 7 in DeMarzo and Skiadas [7].
$(C \Rightarrow D)$ Let (L_+, L_-) be a partition of T without 2-cycles such that $L_1 \subseteq L_+$ and $L_2 \subseteq L_-$ (i.e., assume condition C holds). Define the sets

$$\begin{aligned} F_1(t_1) &= \{t_2' \in T_2 : (t_1, t_2') \in L_+\}, \text{ and} \\ F_2(t_2) &= \{t_2' \in T_2 : (t_1', t_2') \in L_+ \text{ for some } (t_1', t_2) \in L_-\}; \end{aligned}$$

and let $f_1(t_1) = K - 2|F_1(t_1)|$ and $f_2(t_2) = K - 1 - 2|F_2(t_2)|$, for some large positive integer K (where the notation $|F|$ represents the cardinality of the set F). By definition, $(t_1, t_2) \in L_-$ implies $F_1(t_1) \subseteq F_2(t_2)$; thus $K - f_1(t_1) = 2|F_1(t_1)| \leq 2|F_2(t_2)| = K - 1 - f_2(t_2)$; so $f_1(t_1) - f_2(t_2) \geq 1$. Also, $(t_1, t_2) \in L_+$ implies $F_2(t_2) \subseteq F_1(t_1) \setminus \{t_2\}$ and $t_2 \in F_1(t_1)$; thus $K - 1 - f_2(t_2) = 2|F_2(t_2)| \leq 2(|F_1(t_1)| - 1) = K - f_1(t_1) - 2$; so $f_2(t_2) - f_1(t_1) \geq 1$.
$(D \Rightarrow E)$ Let $f_i : T_i \to \{1, 2, \ldots\}$ satisfy $L_1 \subseteq \{t \in T : f_2(t_2) > f_1(t_1)\}$ and $L_2 \subseteq \{t \in T : f_1(t_1) > f_2(t_2)\}$ (i.e., assume condition D holds). Choose bijections $\phi_1 : T_1 \to \{1, 2, .., n_1\}$ and $\phi_2 : T_2 \to \{1, 2, .., n_2\}$ such that $f_2(t_2) > f_2(t_2') \Rightarrow \phi_2(t_2) > \phi_2(t_2')$ and $f_1(t_1) > f_1(t_1') \Rightarrow \phi_1(t_1) < \phi_1(t_1')$. Let $x(i, j) = f_2\left([\phi_2]^{-1}(j)\right) - f_1\left([\phi_1]^{-1}(i)\right)$. By construction, (ϕ_1, ϕ_2, x) satisfy condition E.
$(E \Rightarrow A)$ Immediate. ■