A Practical Competitive Market Model for Indivisible Commodities

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Abstract

A general and practical competitive market model for trading indivisible goods is introduced. There are a group of buyers and a group of sellers, and several indivisible goods. Each buyer is initially endowed with a sufficient amount of money and each seller is endowed with several units of each indivisible good. Each buyer has reservation values over bundles of indivisible goods above which he will not buy and each seller has reservation values over bundles of his own indivisible goods below which he will not sell. Buyers and sellers' preferences depend on the bundle of indivisible goods and the quantity of money they consume. All preferences are assumed to be quasi-linear in money and money is treated as a perfectly divisible good. It is shown in an extremely simple manner that the market has a Walrasian equilibrium if and only if an associated linear program problem has an optimal solution with its value equal to the potential market value. In addition, it is shown that the equilibrium prices of the goods and the profits of the agents are the optimal solutions of the linear program problem.

Keywords: Market, indivisibility, Walrasian equilibrium, linear program, potential market value

JEL Classification: C6, C62, C68, D4, D41, D46, D5, D50, D51

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1 Introduction

In recent years there have been renewed efforts to investigate the equilibrium existence problem with indivisibilities; see e.g., Bikhchandani and Mamer (1997), Laan, Talman and Yang (1997), Bevia, Quinzii and Silva (1999), Gul and Stacchett (1999). These new studies have generalized in one way or another the equilibrium models of Koopmans and Beckman (1957), Shapley and Shubik (1972), Shapley and Scarf (1974), Gale (1984), Quinzii (1984), Kaneko and Yamamoto (1986) to deal with more realistic equilibrium problems of indivisibilities. In the earlier studies it is typically assumed that each agent possesses one indivisible object and consumes no more than one indivisible object but has preferences over different indivisible objects, whereas in the recent studies this restrictive assumption has been relaxed so that agents can consume as many indivisible goods as they wish.

Most of the recent studies have focused on the particular but important case where agents' preferences are quasi-linear in money. In the present study we will also confine our attention to this case but deal with a more general and more practical market model. It is worth noting that our analysis seems to be extremely simple and elementary compared with the existing analyses, and furthermore it provides an easy way of computing equilibrium prices and benefits of trading.

The rest of the article is organized as follows. In Section 2 we introduce the buyer-seller model and several concepts. In Section 3 we establish a necessary and sufficient condition for the existence of an equilibrium in the buyer-seller model and discuss its implications. Finally in Section 4 we present a symmetric trading model and give an equilibrium existence theorem.

2 The Market Model

First, we introduce some notation. The set I_k denotes the set of the first k positive integers. The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$. The vector $\mathbf{0}$ denotes the vector of zeros. Furthermore, $x \cdot y$ means the inner product of vectors x and y.

Now we will start to introduce the market model for trading indivisible goods. In the market there are two disjoint finite groups, a group B of buyers and a group S of sellers, and n indivisible commodities. Each buyer i is initially endowed with a sufficient amount m_i of money and each seller j endowed with a bundle $W^j \in \mathbb{Z}^n_+$ of indivisible goods. These goods can be houses, cars, trucks, aircrafts, computers, machines, and so on. Let W denote the total indivisible goods in the market, i.e., $W = \sum_{j \in S} W^j$. Thus, for each indivisible good $h = 1, \cdots, n$, there are W_h units available in the market. It is understood that $W_h > 0$ for every $h = 1, \cdots, n$. Each buyer i has reservation values V_i over bundles of indivisible goods above which he will not buy. That is, V_i is a mapping from $\mathbb{Z}^n_+$ to the set $\mathbb{R}_+$ of nonnegative real numbers with $V_i(\mathbf{0}) = 0$. Each seller j has reservation values V_j over bundles of his own indivisible goods below which he will not sell. Namely, V_j is also a mapping from $\mathbb{Z}^n_+$ to the set $\mathbb{R}_+$ with $V_j(\mathbf{0}) = 0$. Buyers and sellers' preferences $u_h(x, m)$ depend on the bundle x of indivisible goods and the quantity m of money they consume. It is assumed that all preferences are quasi-linear in money (i.e., $u_h(x, m) = V_h(x) + m$) and that money is a perfectly divisible good. It is natural to assume that the free disposal condition holds for every seller, i.e., $V_j(x) \leq V_j(y)$ for all $j \in S$ and all $x \leq y \leq W^j$. Furthermore, we will assume that every buyer i has a sufficient amount m_i of money, i.e., $m_i \geq \max\{ V_i(x) \mid x \leq W,\ x \in \mathbb{Z}^n_+\}$. This market model will be represented by $\mathcal{M} = ((V_i, m_i, i \in B); (V_j, W^j, j \in S))$. Two remarks are in order.

Remark 1: If there is only one unit of each indivisible good (i.e., $W_h = 1$ for all $h \in I_n$), and furthermore the reservation values of every seller over his own goods are all zero (i.e., $V_j(L) = 0$ for all $j \in S$ and all $L \in \mathbb{Z}^n_+$ with $L \leq W^j$), then the model here is reduced to those of Bikhchandani and Mamer (1997), Bevia et al. (1999), Gul and Stacchett (1999). Our model imposes the free disposal condition on sellers, whereas their models impose that condition on buyers.

Remark 2: The model here is less general than that of Laan et al. (1997) in which the quasi-linearity in money is not imposed but some sort of separability over indivisible goods is needed.

A vector x in $\mathbb{Z}^n_+$ is called a *bundle*; and a *feasible bundle* or *feasible resource* if it is less than or equal to W. Let $F(W)$ denote the set of all feasible resources. A family $x(B) = (x^i, i \in B)$ of feasible bundles is called an *allocation* if $\sum_{i \in B} x^i \leq W$. For each $j \in S$, define

$$F(W^j) = \{x \in \mathbb{Z}^n_+ \mid x \leq W^j\}.$$

A nonempty subset of B is called a coalition. A price vector $p \in \mathbb{R}^n_+$ indicates a price for each indivisible good. Given a price vector $p \in \mathbb{R}^n_+$, the demand correspondence $D_i(p)$ of buyer i is defined by

$$D_i(p) = \{x \mid (V_i(x) - p \cdot x) = \max\{V_i(y) - p \cdot y \mid p \cdot y \le m_i,\ y \le W,\ y \in \mathbb{RZ}^n_+\}\}.$$

Note that $m_i \ge V_i(y)$ for every $y \in F(W)$. This implies that the budget constraint $p \cdot y \le m_i$ is redundant. Thus, the set $D_i(p)$ can be simplified as

$$D_i(p) = \{x \mid (V_i(x) - p \cdot x) = \max\{V_i(y) - p \cdot y \mid y \le W,\ y \in \mathbb{Z}^n_+\}\}.$$

In the definition of $D_i(p)$, buyer i always chooses a feasible bundle, i.e., $y \le W$. This constraint is not essential and can be deleted if buyer i's reservation function V_i is bounded on $\mathbb{Z}^n_+$ and his initial endowment m_i is no less than $\max\{V_i(x) \mid x \in \mathbb{Z}^n_+\}$. Our analysis also applies to this more general case.

The supply correspondence $S_j(p)$ of seller j is similarly defined by

$$S_j(p) = \{x \mid (p \cdot x - V_j(x)) = \max\{p \cdot y - V_j(y) \mid y \le W^j,\ y \in \mathbb{Z}^n_+\}\}.$$

A tuple $((x^i, i \in B), p)$ is a Walrasian equilibrium if p is a vector in $\mathbb{R}^n_+$; and if $x^i \in D_i(p)$ for every $i \in B$; and if $W^j \in S_j(p)$ for every $j \in S$; and if $\sum_{i \in B} x^i = W$. At such an equilibrium, buyer i's profit is given by $V_i(x^i) - p \cdot x^i$, whereas seller j's profit is given by $p \cdot W^j - V_j(W^j)$.

Without further assumptions imposed on the model, there will be no Walrasian equilibrium. We will give a simple example to illustrate this point. There are two buyers 1 and 2 and one seller 3 in the market. Buyer 1 initially has two dollars, buyer 2 has also two dollars and the seller has two indivisible goods A and B. Their reservation values are given by $V_1(\emptyset) = V_2(\emptyset) = V_3(\emptyset) = 0$, $V_1(A) = 7/6$, $V_1(B) = 1$, $V_1(AB) = 4/3$, $V_2(A) = 1/3$, $V_2(B) = 1/4$, $V_2(AB) = 5/6$, $V_3(A) = 1/4$, $V_3(B) = 1/4$, $V_3(AB) = 1/3$. Although this market satisfies all assumptions specified above, there exists no equilibrium. First of all, a price vector p could be an equilibrium price only if $p_A + p_B \ge 1/3$ with $p_A \ge 1/12$ and $p_B \ge 1/12$. Secondly, an equilibrium must be Pareto optimal. In fact, the allocation in which buyer 1 gets A and buyer 2 gets B is the unique Pareto optimal allocation. We will show that this allocation cannot be an equilibrium. Suppose to the contrary that this is an equilibrium. Then the following system of linear inequalities must have a solution:

$$\begin{array}{rcl} 7/6 - p_A & \ge & 1 - p_B \\ 7/6 - p_A & \ge & 4/3 - p_A - p_B \\ 7/6 - p_A & \ge & 0 \\ 1/4 - p_B & \ge & 1/3 - p_A \\ 1/4 - p_B & \ge & 5/6 - p_A - p_B \\ 1/4 - p_B & \ge & 0. \end{array}$$

It follows that $7/12 \leq p_A \leq 6/7$, $1/6 \leq p_B \leq 1/4$, and $1/12 \leq p_A - p_B \leq 1/6$. Obviously, these inequalities have no solutions p_A and p_B. Therefore, the market has no equilibrium.

3 The Equilibrium Existence Theorem

In this section we will establish an existence theorem for the market model. Let Π_W^B denote the family of all allocations $x(B) = (x^i, i \in B)$ satisfying $\sum_{i\in B} x^i = W$. Given a coalition H and a feasible resource K, we define

$$\Pi_K^H = \{\bar{x}(H) \mid \bar{x}(H) = (x^i,\ i \in H) \text{ for some } x(B) \in \Pi_W^B \text{ with } \sum_{i\in H} x^i = K\}.$$

Extending Yang (1990), we define the potential value of the coalition H and the feasible resource K as

$$R(H,K) = \max_{x(H)\in\Pi_K^H} \{\sum_{i\in H} V_i(x^i)\}.$$

The value of $R(B,W)$ will be called the *potential market value*. Now we introduce the following linear program problem associated with the market model.

$$\begin{array}{c} \min \ \sum_{i\in B} x_i + p\cdot W \\ \text{s.t. } \sum_{i\in H} x_i + p\cdot K \geq R(H,K),\ \ \forall\ \emptyset \neq H \subset B,\ \forall\ K \leq W,\ \ K \in \mathbb{Z}_+^n \\ p\cdot W^j - V_j(W^j) \geq p\cdot L - V_j(L),\ \forall\ j \in S,\ \forall\ L \leq W^j,\ L \in \mathbb{Z}_+^n. \end{array} \quad (3.1)$$

We are now ready to establish the main existence theorem which states a necessary and sufficient condition for the existence of a Walrasian equilibrium.

Theorem 3.1 *Given a market model $\mathcal{M} = ((V_i, m_i, i \in B); (V_j, W^j, j \in S))$, there exists a Walrasian equilibrium if and only if the linear program problem (3.1) has an optimal solution with its value equal to the potential market value $R(B,W)$.*

Proof: Now suppose that $((x^i, i \in B), p)$ is a Walrasian equilibrium. Then for all $i \in B$ and all $K \in F(W)$ it holds

$$V_i(x^i) - p\cdot x^i \geq V_i(K) - p\cdot K,$$

and for all $j \in S$ and all $L \in F(W^j)$, it holds

$$p\cdot W^j - V_j(W^j) \geq p\cdot L - V_j(L).$$

Thus, the second inequalities of the linear program are satisfied. Now let $x_i = V_i(x^i) - p \cdot x^i$ for every $i \in B$. Let $y(B)$ be an arbitrary element in Π_W^B. Then it holds that

$$V_i(x^i) - p \cdot x^i \geq V_i(y^i) - p \cdot y^i$$

for all $i \in B$. It follows that $\sum_{i \in B} V_i(x^i) \geq \sum_{i \in B} V_i(y^i)$. This implies that

$$\sum_{i \in B} x_i + \sum_{i \in B} p \cdot x^i = \sum_{i \in B} V_i(x^i) = R(B, W).$$

We still have to show that (x, p) also satisfies the first constraints of the linear program. Suppose to the contrary that some constraint is violated. Then there would be a nonempty subset H of B and a feasible resource $K \in F(W)$ so that

$$\sum_{i \in H} x_i + p \cdot K < R(H, K).$$

This implies that there are some $i \in H$ and some feasible resource $D \leq K$ satisfying

$$x_i + p \cdot D < V_i(D).$$

But this is impossible, since we have

$$x_h \geq V_h(T) - p \cdot T$$

for all $h \in B$ and all $T \in F(W)$.

On the other hand, let (x, p) be an optimal solution of the linear program with its value equal to $R(B, W)$. Then there must exist an allocation $(x^i, i \in B)$ satisfying $\sum_{i \in B} x^i = W$ so that

$$\sum_{i \in B} x_i + p \cdot W = \sum_{i \in B} V_i(x^i) = R(B, W).$$

Furthermore, it follows from the linear program that for all $i \in B$ and all $K \in F(W)$

$$x_i \geq V_i(K) - p \cdot K.$$

Note that

$$x_i \geq V_i(x^i) - p \cdot x^i$$

for all $i \in B$. Furthermore, it holds that

$$x_i = V_i(x^i) - p \cdot x^i$$

for all $i \in B$. Therefore we have

$$V_i(x^i) - p \cdot x^i \geq V_i(K) - p \cdot K$$

for all $i \in B$ and all $K \in F(W)$. Thus, $x^i \in D_i(p)$ for all $i \in B$. Furthermore, note that (x, p) also satisfies the second constraints of the linear program, i.e., $W^j \in S_j(p)$ for all $j \in S$. Since the free disposal holds for sellers, it is easy to see that p is nonnegative. By definition $((x^i, i \in B), p)$ is a Walrasian equilibrium. This demonstrates the theorem. □

This theorem tells us that the equilibrium problem with indivisibilities studied here is equivalent to a linear program problem and its equilibrium price vector and the profit vector of buyers are the optimal solution (p, x) of the linear program problem, and the profits of sellers are given by its second constraints. Thus, one can easily compute the equilibrium prices and the benefits of trading.

Now we will return to the previous example. With respect to that example to see whether there is an equilibrium or not we only need to check the following linear program problem:

$$\begin{array}{ll} \min & x_1 + x_2 + p_A + p_B \\ s.t. & x_1 + p_A \geq 7/6 \\ & x_1 + p_B \geq 1 \\ & x_1 + p_A + p_B \geq 4/3 \\ & x_2 + p_A \geq 1/3 \\ & x_2 + p_B \geq 1/4 \\ & x_2 + p_A + p_B \geq 5/6 \\ & x_1 + x_2 + p_A \geq 7/6 \\ & x_1 + x_2 + p_B \geq 1 \\ & x_1 + x_2 + p_A + p_B \geq 17/12 \\ & p_A + p_B \geq 1/3 \\ & x_1 \geq 0,\ x_2 \geq 0,\ p_A \geq 1/12,\ p_B \geq 1/12. \end{array}$$

The potential market value is 17/12, but the linear program problem has an optimal solution $(x_1, x_2, p_A, p_B) = (4/6, 0, 3/6, 2/6)$ with its value 18/12. According to Theorem 3.1, the market has no equilibrium.

4 A Further Discussion

In the previous sections we differentiate the rule of buyers and sellers. In other words, the model is not symmetric. In this section we will show the previous analysis can be easily adapted to deal with the symmetric model. Now we will introduce a symmetric market model for trading indivisible goods. In the market there are a finite set A of traders (buyers, or sellers, or both), and n indivisible commodities. Each trader i is initially endowed with some amount m_i of money and a bundle $W^i \in \mathbb{Z}^n_+$ of indivisible goods. Let W denote the total indivisible goods in the market, i.e., $W = \sum_{i\in A} W^i$. Thus, for each indivisible good $h = 1, \cdots, n$, there are W_h units available in the market. It is understood that $W_h > 0$ for every $h = 1, \cdots, n$. Each trader i has reservation values V_i over bundles of indivisible goods. It is assumed that V_i is a mapping from $\mathbb{Z}^n_+$ to the set $\mathbb{R}_+$ of nonnegative real numbers with $V_i(\mathbf{0}) = 0$. Traders' preferences $u_h(x, m)$ depend on the bundle x of indivisible goods and the quantity m of money they consume, and are quasi-linear in money (i.e., $u_h(x, m) = V_h(x) + m$). Money is treated as a perfectly divisible good. We impose a mild assumption that free disposal condition holds for every trader, i.e., $V_i(x) \leq V_i(y)$ for all $i \in A$ and all $x \leq y \leq W$, and that every trader i has a sufficient amount m_i of money, i.e., $m_i \geq V_i(W) - V_i(W^i)$. This market model will be represented by $\mathcal{M} = (V_i, m_i, i \in A)$.

Now we will associate this model and its equilibrium existence with the following linear program problem.

$$\begin{array}{c} \min \ \sum_{i\in A} x_i + p \cdot W \\ s.t. \ \sum_{i\in H} x_i + p \cdot K \geq R(H, K), \ \forall \ \emptyset \neq H \subset A, \ \forall \ K \leq W, \ K \in \mathbb{Z}^n_+ \end{array} \tag{4.2}$$

Applying the argument of Theorem 3.1, one can demonstrate the following theorem which states a necessary and sufficient condition for the existence of a Walrasian equilibrium.

Theorem 4.1 *Given a symmetric market model $\mathcal{M} = (V_i, m_i, i \in A)$, there exists a Walrasian equilibrium if and only if the linear program problem (4.2) has an optimal solution with its value equal to the potential market value $R(A, W)$.*

We remark that in equilibrium the profit of trader i from trading is equal to $x_i + p \cdot W^i - V_i(W^i)$, where (x, p) is an optimal solution of problem (4.2).

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