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A FINITE ALGORITHM FOR THE LINEAR EXCHANGE MODEL

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1. Introduction and Abstract

It is shown that Lemke's algorithm can be used to compute, in a finite number of steps, an equilibrium or reduction for the pure exchange model with linear utilities.

2. The Model

A linear pure exchange model consists of two nonnegative $m \times n$ matrices

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \qquad U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

A trader $i \in \mu = \{1, \ldots, m\}$ enters the market with an endowment w_i and utility u_i for goods $j \in \nu = \{1, \ldots, n\}$, and leaves the market with a nonnegative allocation n-vector of goods x_i.

An equilibrium of (W,U) is a nonnegative nonzero price n-vector p and allocation vectors $x_1, \ldots, x_m$ such that

(a) x_i maximizes $u_i x_i'$ subject to $px_i' \leq pw_i$ and $x_i' \geqq 0$

(b) $\Sigma x_i \leq \Sigma w_i$.

Given a price p, condition (a) requires that each trader maximize his utility subject to his budget. Condition (b) requires that the exchange be feasible, that is, no more of good j leaves the market than enters it.

For simplicity and without loss of generality we may assume that there is a total of one unit of each good available, that is,

$$\Sigma w_i = eW = e = (1, \ldots, 1)$$

further, we assume that each trader has a nonzero endowment, that is, $w_i \neq 0$.

The model is said to be reducible if some proper subset $\beta \subset \mu$ of traders are endowed with all of the goods in $\delta \subseteq \nu$ and if they desire only the goods of δ, that is,

$$W_{\alpha\delta} = 0 \qquad U_{\beta\gamma} = 0$$

where $\alpha = \mu \sim \beta$ and $\gamma = \nu \sim \delta$. In the presence of reducibility an equilibrium may not exist, see Section 5.

3. Solution of the Model

Let M be the $(m+n+m\times n)$-square matrix where blanks indicate zeroes and I is the $n\times n$ identity.

$$
\left[\begin{array}{c|c|c|c|c|c|c}
 & W & \begin{matrix}-e\\0\\\vdots\end{matrix} & \begin{matrix}0\\-e\\0\\\vdots\end{matrix} & & \begin{matrix}\vdots\\0\\-e\\0\end{matrix} & \begin{matrix}\vdots\\0\\-e\end{matrix} \\
\hline
 & -I & I & I & \cdots & I & I \\
\hline
u_1, 0 \ldots & -I & & & & & \\
\hline
0, u_2, 0 \ldots & -I & & & & & \\
\hline
 & \vdots & & & & \vdots & \\
\hline
\ldots 0, u_{m-1}, 0 & -I & & & \cdots & & \\
\hline
\ldots 0, u_m & -I & & & & &
\end{array}\right]
$$

Let v , y , d , and b be the following $(m+n+m\times n)$-vectors, partitioned as M .

$$\begin{bmatrix} s \\ t \\ r_1 \\ r_2 \\ \vdots \\ r_{m-1} \\ r_m \end{bmatrix} \begin{bmatrix} \lambda \\ p \\ q_1 \\ q_2 \\ \vdots \\ q_{m-1} \\ q_m \end{bmatrix} \begin{bmatrix} -e \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -We \\ e \\ e \\ e \\ \vdots \\ e \\ e \end{bmatrix}$$

Let r and q be the (mxn)-vectors $(r_1, \ldots, r_m)$ and $(q_1, \ldots, q_m)$.

Define the two linear complementarity problems $*$ and $**$ by

$$* \qquad \begin{cases} v + My + dz = 0 \\ v \geqq 0 \quad y \geqq 0 \quad z \geqq 0 \quad vy = 0 \end{cases}$$

$$** \qquad \begin{cases} v + My + dz = b \\ v \geqq 0 \quad y \geqq 0 \quad z \geqq 0 \quad vy = 0 \end{cases}$$

<u>Theorem 1</u>: Let (s,t,r,λ,p,q,z) be a solution of $*$ with $p \neq 0$. If $p > 0$, then the price p and allocation $x_1, \ldots, x_m$ where

$$x_{ij} = q_{ij}/p_j$$

form an equilibrium. If $p \not> 0$ then the model is reducible, in particular,

$$W_{\alpha\delta} = 0 \qquad U_{\beta\gamma} = 0$$

where

$$\alpha = \{i \in \mu : \lambda_i = 0\} \qquad \beta = \mu \sim \alpha$$

$$\gamma = \{j \in \nu : p_j = 0\} \qquad \delta = \nu \sim \gamma .$$

Proof: For $p > 0$ or $p \not> 0$ define γ and δ as above. From $t \cdot p = 0$ and $t + \Sigma q_i = p$ we have $t = 0$, $q_{i\gamma} = 0$ for $i \in \mu$, and $p = \Sigma q_i$. If $\lambda_i = 0$, then $r_{i\delta} > 0$, $q_{i\delta} = 0$, $q_i = 0$, and $eq_i \leqq w_i p$. If $\lambda_i > 0$, then $s_i = 0$ and $eq_i \leq w_i p$. But $e(\Sigma q_i) = ep = eWp$, therefore $eq_i = w_i p$ for $i \in \mu$.

Assume $p > 0$, from $p = \Sigma q_i$ we have $1 = \sum_i x_{ij}$ or that the allocation is feasible. From $px_i = eq_i = w_i p > 0$, we have that each trader spends his budget, $q_i \neq 0$, and $\lambda_i > 0$. From $r_i + u_i \lambda_i = p$ we have

$$u_{ij}/p_j \leqq 1/\lambda_i - r_{ij}/p_j \lambda_i \ .$$

Since $q_i r_i = 0$, we see that $x_{ij} > 0$ implies

$$u_{ij}/p_j = 1/\lambda_i \geqq u_{ik}/p_k$$

for all $k \in \nu$. Hence x_i maximizes $x_i' u_i$ subject to $x_i' p \leqq w_i p$ and $x_i' \geqq 0$.

Assume $p \not> 0$. If $\lambda_i > 0$ then $u_{i\gamma} = 0$, hence $U_{\beta\gamma} = 0$. If $\lambda_i = 0$, then $w_i p = eq_i = 0$, $w_{i\delta} = 0$, and $W_{\alpha\delta} = 0$. The proof is complete.

To solve * we apply Lemke's algorithms to the nonhomogeneous system ** . For a statement of this algorithm and its output see Sections 7 and 8 of [1]; nevertheless we briefly redescribe these objects now. The algorithm begins with a basic solution to ** of form $(v,0,z)$ and iterates to new adjacent basic solutions of ** as long as possible. The procedure generates either a solution to ** with $z = 0$, a secondary ray of ** , or both. A secondary ray is defined to be a pair,

$(\bar{v},\bar{y},\bar{z})$ and (v,y,z) , of solutions to ** and * , respectively, for which $\bar{v}y = \bar{y}v = 0$, $(y,z) \neq 0$, and $(\bar{y},y) \neq 0$.

<u>Theorem 2</u>: If Lemke's algorithm is applied to ** then either an equilibrium or reduction is generated.

<u>Proof</u>: First, suppose the algorithm terminates with a solution (s,t,r,λ,p,q,z) to ** with $z = 0$. Then

$$es + eWp - e(\Sigma q_i) = -eWe$$

$$et - ep + e(\Sigma q_i) = ee$$

or $-es = et$, and hence, $s = t = 0$. Hence $(0,0,r,\lambda,p+\epsilon,q,0)$ is a solution to * . Now apply Theorem 1. Otherwise, the algorithm generates solutions (s,t,r,λ,p,q,z) and $(\bar{s},\bar{t},\bar{r},\bar{\lambda},\bar{p},\bar{q},\bar{z})$ to * and ** with $\bar{\lambda}s = \bar{s}\lambda = 0$, $(\lambda,p,q,z) \neq 0$, and $(\bar{\lambda},\bar{p},\bar{q},\lambda,p,q) \neq 0$. Suppose $p = 0$, then $r = 0$, $\lambda = 0$, $t = 0$, $q = 0$, $z > 0$, $s > 0$, $\bar{\lambda} = 0$, $\bar{r} > 0$, $\bar{q} = 0$, $-\bar{p}\,\bar{p} = \bar{p}\,e$, and $\bar{p} = 0$, but this contradicts $(\bar{y},y) \neq 0$, so $p \neq 0$. Hence $p \neq 0$ and we apply Theorem 1 and the proof is complete. If $p > 0$, then $\lambda > 0$ and $z = 0$ from Theorem 1. Consequently, $\bar{s} = 0$, $\bar{t} = 0$, and $\bar{z} = 0$.

4. An Example

Given the model

$$W = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} \qquad U = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

we have

$$M = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 1/4 & 1/2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 3/4 & 1/2 & 0 & 0 & -1 & -1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ \hline 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

For this model Lemke's algorithm passes through the following sequence of solutions to ** .

Iteration	λ_1	λ_2	p_1	p_2	q_{11}	q_{12}	q_{21}	q_{22}	z
0	0	0	0	0	0	0	0	0	∞
1	0	0	0	0	0	0	0	0	5/4
2	0	1/4	0	0	0	0	0	0	5/4
3	0	1/4	0	0	0	0	0	1/2	3/4
4	1/3	1/4	0	0	0	0	0	1/2	3/4
5	1/3	1/4	0	0	1/2	0	0	1	1/4
6	1/3	3/8	0	1/2	1	0	0	3/2	0

The algorithms terminated with a solution to ** with $z = 0$. Using Theorems 1 and 2 we see that $p = (1, 3/2)$, $x_1 = (1,0)$, $x_2 = (0,1)$ is an equilibrium.

5. Acknowledgment

The linear pure exchange model was evidently first studied as such and in detail in Gale [2]. The concept of reducibility was introduced there, and assuming irreducibility, it was shown using the Kakutani fixed point theorem that an equilibrium exists. In the presence of reducibility it was demonstrated that there may be no equilibrium, for example,

$$W = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$$

However, if the model splits, that is

$$W_{\alpha\delta} = U_{\alpha\delta} = 0$$

$$W_{\beta\gamma} = U_{\beta\gamma} = 0$$

to irreducible submodels $(W_{\alpha\gamma}, U_{\alpha\gamma})$ and $(W_{\beta\delta}, U_{\beta\delta})$, it was observed that an equilibrium is obtained by combining the submodel equilibria. Finally, it was shown that distinct equilibrium prices lead to the same payoffs $x_i u_i$.

Stymied in an effort to compute an equilibrium of the linear pure exchange model using Lemke's algorithm, the author approached David Gale with the following question. If W and U are rational, does there exist a rational equilibrium? The success of the present paper rests upon the argument given in Gale [3] which supplied an affirmative answer.

6. Comments

By partitioning the rows of ** into one "market system" and m "trader systems" with price the only common variables, the number of operations required for each iteration of Lemke's algorithm can be reduced from order m^2n^2 to $m^2n + n^2m$; this savings does not alter the number of iterations required for termination. That the algorithm can be interpreted as a "global market adjustment mechanism" will be interesting to explore. Also under study are extensions of the overall method to include piecewise linear utilities, production, etc., if successful this avenue could prove important in real economic modeling.

7. References

[1] Eaves, B. C., "The Linear Complementarity Problem," Management Science, Vol. 17, No. 9, May, 1971.

[2] Gale, D., "Price Equilibrium for Linear Models of Exchange," Technical Report P-1156, The Rand Corporation, August 15, 1957.

[3] Gale, D., "Exchange Models with Linear Utilities Functions Have Rational Equilibria," private communication, June 17, 1974.