Subscription Equilibrium with Production: Neutrality and Constrained Suboptimality of Equilibria*

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Abstract

We revisit the analysis of subscription equilibria in a full fledged general equilibrium model with public goods. We study the case of a non-profit, or public, firm that produces the public good using private goods as inputs, which are to be financed by voluntary contributions of households. We analyze policy interventions that will lead to an increase of the public good level at subscription equilibria, and show that most of the standard neutrality results do not survive in our general equilibrium model with many private goods and relative price effects allowed. We also take a direct approach to welfare analysis and study interventions that has the goal of Pareto improving upon subscription equilibrium outcomes. We delineate conditions under which, for a generic set of economies, well chosen interventions will Pareto improve upon a given subscription equilibrium outcome. In particular, we show that a general non-neutrality result in terms of utilities holds even if all households are contributors.

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1 Introduction

In a series of path-breaking articles that have set forth a definitive theory of public goods, Samuelson (1954, 1955, 1958) presented the first modern analyses of public goods within a general equilibrium context. His main concern being the normative one, Samuelson provided a characterization of welfare optima in public good economies, but he did not elaborate on the process through which the level of a public good is to be determined. For a complete theory of equilibrium, one has to specify how the level of a public good is to be determined, and, owing to the distinctive nature of public goods, this is typically going to be a collective (political) decision-making process that goes beyond the standard pure market equilibrium notion. Starting with Foley (1967), there have been various attempts to provide theories of politico-economic equilibrium with public goods in a general equilibrium context.[1] The problem from the view point of economic theory, however, is the fact that one has to provide precise institutional details of how such collective decisions are to be made, an area of inquiry that perhaps intersects more with political science than standard economic theory.

To provide an analysis of the public good problem from pure economic theory point of view, as well as to serve as a benchmark extension of an analysis of completely decentralized private good economies to public good economies, a useful starting point is to study which equilibria will be established in the absence of a central authority or mutual agreement among the agents. Towards this end, Malinvaud (1972, p. 213) proposed to study the system whereby the public good is financed by *subscription*, with each household making a contribution to increase the production of public good. The contributions are to be voluntary and contribution decisions are to be made by each household independently of other households, the complete autonomy of households thus being fully respected. Thus, Malinvaud (1972)'s *subscription equilibrium* is the non-cooperative (Nash) equilibrium of the game correponding to the economy under consideration, with contribution level as the action taken by the agents and with their relevant payoff functions appropriately defined.

In this paper we revisit the analysis of subscription equilibria in a full fledged general equilibrium model with public goods. We proved existence and regularity of subscription equilibria for a generic set of economies in Villanacci and Zenginobuz (2006). Observing that, as the outcome of a non-cooperative game, the public good provided at subscription equilibria will typically be suboptimal, we here analyze policy interventions that will lead to an increase of the public good level at subscription equilibria (the neutrality results - see discussion below). We also take a direct approach to welfare analysis and study interventions that have the goal of Pareto improving upon subscription equilibrium outcomes.

The subscription equilibrium notion of Malinvaud (1972) is in fact the private (voluntary) contribution equilibrium notion for charitable contributions which has since come to be much studied in the public economics literature. A vast number of studies have also applied the same notion in other relevant contexts, such as contributions to election campaigns of political parties, contributions to activities of special interest groups, behavior of the family members in the economic activities of a family, contributions to multinational foreign aid packages (e.g. famine relief effort in Somalia).[2] However, most of these studies adopt what is essentially a partial equilibrium framework. Moreover, in cases where the model used has general equilibrium features, it is typically cast with assumptions that are very restrictive from a genuine general equilibrium analysis point of view.

A brief review of the standard model used to study voluntary contribution equilibria will help clarify the restrictivenss of its assumptions for a genuine general equilibrium treatment of the problem.

Using a partial equilibrium model, Warr (1982) showed that the level of public good provided at the equilibrium of a voluntary contribution game is invariant to (small) redistribution of initial endowments among an unchanged set of contributors. This invariance property, which has come to be termed as "neutrality" property, has an important implication, namely that an exogenous,

[1] See Milleron (1972) for a survey of general equilibrium models with public goods.

[2] A vast number of contributions on these issues start with the initial contribution by Olson (1965), and include, among others, papers by McGuire (1974), Chamberlain (1976), Becker (1981), Young (1982), Warr (1982, 1983), Brennan and Pincus (1983), Kemp (1984), Roberts (1984), Bergstrom, Blume, and Varian (1986), Bernheim (1986), Cornes and Sandler (1986), and Andreoni (1988).

tax-financed increase in government spending on the public good will reduce voluntary private spending on the public good by an equal amount, thus perfectly "crowding out" voluntary private contributions. Bergstrom, Blume, and Varian (1986) revisited the neutrality issue using a simple general equilibrium model with a single private good and a single public good that is produced through a linear production technology using the private good. One of their results demonstrates the importance of the assumption that redistribution of income does not change the set of contributing agents in order to prove the neutrality result in their framework. If the redistribution of income does change the set of contributing agents and/or alters the total wealth of the current set of contributing agents, the government provision of a public good will no longer necessarily "crowd out" private contributions.

Even though it is cast as a general equilibrium model, the specific assumptions employed by Bergstrom, Blume, and Varian (1983) in fact render it a partial equilibrium one. In their model, which is now the canonical model for studying voluntary contribution equilibria, the presence of only one private good, *and* the linear production technology for the only public good together imply that there are no relative prices to be determined in equilibrium (hence its partial equilibrium nature). The linear coefficient of conversion between the private and the public good is the only possible equilibrium price in such a setting, a fact which in turn allows normalization of the prices of both the private and the public good to one without loss of generality. That feature of their model prevents the possibility of using a powerful channel for intervention, namely the changes in relative prices.

When more than one private good and a non-linear production technology are allowed, modeling of how and by whom the public good is produced becomes a crucial preliminary issue to be resolved, both from the production technology and the market institutional viewpoints.

Regarding the production technology, the linear production technology assumption also allows taking profits of firms equal to zero, with the implication that the presence of firms plays no basic role in the model. That is to say, given zero profits of the linear technology case, the households can simply be thought to produce the public good themselves with a constant conversion rate between the private and the public good. Observe also that with many inputs linearity of a production function implies constant returns to scale, but the converse will not hold, except for the single input case. When the production function is linear, and firms produce a strictly positive quantity of output, prices are completely determined by the production coefficients. Therefore, equilibrium prices can be said to be "fixed by the technology". That is, they change only if technology changes, and they are not affected by changes in endowments or preferences. Outside the case of linear production function, equilibrium prices are not fixed by the production technology either in the case of (generic) constant returns to scale, where firms' profits will be zero, or for strictly concave production technologies with non-zero equilibrium profits.

Regarding the market institutional aspect, if a profit-maximizing (private) firm is assumed to produce the public good under non-constant returns to scale technology, then how the (non-zero) profits of the firm are apportioned among its shareholders will have an impact on equilibrium outcomes. An alternative is to consider the production of the public good as being carried out by a non-profit, or a public firm subject to a balanced budget constraint. In that case the amount of public good to be produced by the non-profit firm can be taken as the maximum amount that can be produced with the amount of contributions collected from consumers.

In this paper, we study the case of a non-profit, or public, firm that produces the public good using private goods as inputs, which are to be financed by voluntary contributions of households. Thus, by definition, the firm producing the public good has no profits, and profit aspect of the production side of the economy becomes exactly the same as in the standard one private good and linear production technology for the public good case. We adopt a non-constant returns to scale production technology to allow for genuine relative price effects.

In our more general framework, the relative price effects, which are absent with a single private good and under constant returns to scale technology, come to play an important role. Relative price effects provide a powerful channel through which government interventions can bring about redistributive wealth effects, which, in turn, change equilibrium outcomes. We show that most of the standard neutrality results do not survive when more than one private good and genuine relative price effects are allowed in a full fledged general equilibrium model. In particular, regarding the neutrality results of Bergstrom, Blume, and Varian (1983) mentioned above, we show that (a) there are redistributions of a numeraire good that does not affect contributors' total wealth but nevertheless increase the level of public good provided at private provision equilibria; and (b) a redistribution in favor of contributors is neither a necessary nor a sufficient condition to increase the level of public good at a private provision equilibria

In our analyses of interventions that have the goal of Pareto improving upon the market outcome, we study several types of policies, ranging from imposing taxes on firms and on households, to directly intervening in the production decisions of the non-profit (public) firm. In fact, it would be possible to apply our approach to other forms of intervention to allow a policy maker choose the one more suitable for the institutional and political environment under consideration. Note that the type of interventions sought here will coexist along with private provision of public goods. We delineate conditions under which, for a generic set of economies, a given type of intervention will Pareto improve upon a given subscription equilibrium outcome. In particular, we show that a general non-neutrality result in terms of utilities holds even if all households are contributors, which is the case where the existing neutrality results on the amount of public good produced apply with full force.

The approach we use to prove our results is based on differential techniques, which amount to computing the derivative of the *equilibrium* values of the "goal function" - the household welfare levels in this case- with respect to some policy tools - taxes and/or government's direct provision of the public good.[3]

The plan of our paper is as follows. In section 2, we present the set up of our model and the existence and regularity results proved in Villanacci and Zenginobuz (2006). In Section 3, we briefly discuss non-optimality of equilibria. In Section 4, we present the general strategy used to prove our main results. In Section 5, we prove results on the possibility of a government intervention to influence the total amount of public good through different types of intervention. In Section 6, we describe how to increase households' welfare using two different types of interventions: the first one requires the planner to use the available production technology; the second one consists in taxing prices faced by the firm. In both interventions, taxes are imposed also on households (in fact, one contributor[4]). This further intervention can be avoided; we decided to describe it in details because it lightens a requirement imposed on the number of households and because taxing households seems quite a natural kind of intervention.[5]

2 Set-up of the Model and Preliminary Results

We consider a general equilibrium model with private provision of a public good. There are C, $C \geq 1$, private commodities, labelled by $c = 1, 2, ..., C$. There are H households, $H > 1$, labelled by $h = 1, 2, ..., H$. Let $\mathcal{H} = \{1, ..., H\}$ denote the set of households. Let x_h^c denote consumption of private commodity c by household h; e_h^c embodies similar notation for the endowment in private goods.

The following standard notation is also used:

- $x_h \equiv (x_h^c)_{c=1}^C$, $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- $e_h \equiv (e_h^c)_{c=1}^C$, $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- p^c is the price of private good c. Prices are expressed in units of the numeraire good C, whose price is therefore normalized to 1. Define , $p^{\setminus} \equiv (p^c)_{c=1}^{C-1}$ and $p \equiv \left(p^{\setminus}, 1\right)$.

[3] Therefore, all our arguments are "local" in their nature. We also note that, since price effects may in principle go in any direction, all our non-neutrality results hold only *typically* in the relevant space of economies.

[4] Recall that, given our assumptions, in each equilibrium at least one consumer is a contributor, and in fact *each* consumer may be a contributor.

[5] A more detailed version of the paper, containing even the most elementary proofs, is available upon request from the authors.

- $g_h \in \mathbb{R}_+$ is the amount of resources (measured in units of the numeraire good) that consumer h provides. Let $g \equiv (g_h)_{h=1}^H$, $G \equiv \sum_{h=1}^H g_h$, and $G_{\setminus h} \equiv G - g_h$.
- y^g is the amount of public good produced in the economy.

The preferences over the private goods and the public good of household h are represented by a utility function

$$u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++} \to \mathbb{R}, \qquad u_h : (x_h, y^g) \mapsto u_h(x_h, y^g)$$

Assumption 1 *$u_h(x_h, y^g)$ is a smooth, differentiably strictly increasing (i.e., for every $(x_h, y^g) \in \mathbb{R}_{++}^{C+1}$, $Du_h(x_h, y^g) \gg 0$)[6], differentiably strictly quasi-concave function (i.e.,$\forall (x_h, y^g) \in \mathbb{R}_{++}^{C+1}$,$\forall v \in \mathbb{R}^{C+1} \setminus \{0\}$, if $Du_h(x_h, y^g) v = 0$, then $vD^2 u_h(x_h, y^g) v < 0$) and for each $\underline{u} \in \mathbb{R}$ the closure (in the standard topology of $\mathbb{R}^{C+1}$) of the set $\left\{(x_h, y^g) \in \mathbb{R}_{++}^{C+1} : u_h(x_h, y^g) \geq \underline{u}\right\}$ is contained in $\mathbb{R}_{++}^{C+1}$.*

Let $\mathcal{U}$ be the set of utility functions u_h satisfying Assumption 1.

The production technology available to produce the public good is described by the following production function.

$$f : \mathbb{R}_{++}^C \to \mathbb{R}_{++}, \qquad f : y \longmapsto f(y)$$

Assumption 2 f is C^2, differentiably strictly increasing, differentiably strictly concave (i.e., $\forall y \in \mathbb{R}_{++}^C$, $D^2 f$ is negative definite), and $\forall \underline{f} \in \mathbb{R}_{++}$, $cl_{\mathbb{R}^C}\{y \in \mathbb{R}_{++}^C : f(y) \geq \underline{f}\} \subseteq \mathbb{R}_{++}^C$.

Let $\mathcal{F}$ be the set of production functions f satisfying Assumption 2.

The government collects resources from the contributors, and maximizes the production of public goods, given the constraint to balance the budget, i.e., it solves the following problem. For given $p^{\setminus} \in \mathbb{R}_{++}^{C-1}$ and $G \in \mathbb{R}_{++}$,

$$\max_{y \in \mathbb{R}_{++}^C} f(y) \quad s.t \quad -py + G = 0 \qquad (\alpha) \tag{1}$$

with $G = \sum_h g_h$ and where we follow the convention of writing associated Lagrange or Kuhn-Tucker multipliers next to the constraint.

For given $p^{\setminus} \in \mathbb{R}_{++}^{C-1}$ and $G \in \mathbb{R}_{++}$ a solution to problem (1) is characterized by Lagrange conditions.

Define

$$\widehat{f} : \mathbb{R}_{++}^{C-1} \times \mathbb{R}_{++} \to \mathbb{R}_{++}, \quad \left(p^{\setminus}, G\right) \mapsto \max\ (1)$$

Remark 1 As an application of the envelope and the implicit function theorems, we have that $\forall \left(p^{\setminus}, G\right) \in \mathbb{R}_{++}^C$, $D_G \widehat{f}\left(p^{\setminus}, G\right) = \alpha > 0$ and $D_{GG} \widehat{f}\left(p^{\setminus}, G\right) = \left(p\left(D^2 f\left(y\left(p^{\setminus}, G\right)\right)\right)^{-1} p\right)^{-1} < 0$, where $y\left(p^{\setminus}, G\right) = \arg\max(1)$.

Household's problem is the following one. For given $p^{\setminus} \in \mathbb{R}_{++}^{C-1}, G_{\setminus h} \in \mathbb{R}_+, e_h \in \mathbb{R}_{++}^C$,

$$\max_{(x_h, g_h) \in \mathbb{R}_{++}^C \times \mathbb{R}} u_h\left(x_h, \widehat{f}\left(p, g_h + G_{\setminus h}\right)\right) \quad s.t. \quad \begin{array}{l} -px_h + pe_h - g_h = 0 \\ g_h \geq 0 \end{array}$$

Equivalently, we can write the household's problem as follows. For given $p^{\setminus} \in \mathbb{R}_{++}^{C-1}, G_{\setminus h} \in \mathbb{R}_+, e_h \in \mathbb{R}_{++}^C$,

$$\max_{\left(x_h, g_h, y_h^g\right) \in \mathbb{R}_{++}^C \times \mathbb{R}_{++}} u_h\left(x_h, y_h^g\right) \quad s.t. \quad \begin{array}{ll} -px_h + pe_h - g_h \geq 0 & \lambda_h \\ g_h \geq 0 & \mu_h \\ -y_h^g + \widehat{f}\left(p, g_h + G_{\setminus h}\right) \geq 0 & \eta_h \end{array} \tag{2}$$

Observe that in the latter formulation, in equilibrium it must be the case that for every h, $y_h^g = y^g$.

[6] For vectors y, z, $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the correponding element of z; $y > z$ means that $y \geq z$ but $y \neq z$.

Remark 2 For given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}, G_{\setminus h} \in \mathbb{R}_+, e_h \in \mathbb{R}^C_{++}$, a solution to problem (2) is characterized by Kuhn-Tucker conditions.

Definition 3 *An economy is an element* $\pi \equiv (e, u, f)$ *in* $\Pi \equiv \mathbb{R}^{CH}_{++} \times \mathcal{U}^H \times \mathcal{F}$.

Definition 4 *A vector* $(y, x, y^g, g, p^{\setminus})$ *is an equilibrium for an economy* $\pi \in \Pi$ *if:*

1. *the public firm maximizes, i.e., it solves problem (1) at* $\left(p, \sum_{h=1}^H g_h\right)$*;*
2. *households maximize, i.e., for each* h *,* (x_h, y_h^g, g_h) *solves problem (2) at* $p^{\setminus} \in \mathbb{R}^{C-1}_{++}, \sum_{h' \neq h} g_h \in \mathbb{R}_+, e_h \in \mathbb{R}^C_{++}$*; and*
3. *markets clear , i.e.,* (x, y) *solves*

$$-\sum_{h=1}^H x_h^{\setminus} - y^{\setminus} + \sum_{h=1}^H e_h^{\setminus} \quad = 0$$

where for each h*,* $x_h^{\setminus} \equiv (x_h^c)_{c \neq C}$ *,* $e_h^{\setminus} \equiv (e_h^c)_{c \neq C} \in \mathbb{R}^{C-1}_{++}$ *and* $y^{\setminus} \equiv (y^c)_{c \neq C} \in \mathbb{R}^{C-1}$.[7]

In the remainder of the paper we are going to use two equivalent equilibrium systems. System (3) below simply lists Kuhn-Tucker conditions of the agent's maximization problems and market clearing conditions. System (6) is used to show generic regularity and the result on the effectiveness of policy interventions.

Define

$$\Xi' \equiv \mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \left(\mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}\right)^H \times \{g \in \mathbb{R}^H : \Sigma_{h=1}^H g_h > 0\} \times \mathbb{R}^{C-1}_{++}$$

$$\xi' \equiv \left(y, \alpha, y^g, (x_h, y_h^g, \lambda_h, \eta_h, \mu_h)_{h=1}^H, g, p^{\setminus}\right)$$

and

$$F_1 : \Xi' \times \mathbb{R}^{CH}_{++} \to \mathbb{R}^{\dim \Xi'}, \quad F_1 : (\xi', \pi) \mapsto \text{left hand side of (3) below}$$

$$\begin{array}{lll}
(1) & Df(y) - \alpha p & = 0 \\
(2) & -py + \sum_h g_h & = 0 \\
(3) & y^g - f(y) & = 0 \\
(h.1) & D_{x_h} u_h (x_h, y_h^g) - \lambda_h p & = 0 \\
(h.2) & D_{y^g} u_h (x_h, y_h^g) - \eta_h & = 0 \\
(h.3) & -\lambda_h + \mu_h + \eta_h D_G \widehat{f}\left(p, g_h + G_{\setminus h}\right) & = 0 \\
(h.4) & -px_h + pe_h - g_h & = 0 \\
(h.5) & -y_h^g + \widehat{f}\left(p, g_h + G_{\setminus h}\right) & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^{\setminus} - y^{\setminus} + \sum_{h=1}^H e_h^{\setminus} & = 0
\end{array} \tag{3}$$

Note that in the above system we in fact have

$$y^g = f(y) = \widehat{f}\left(p, g_h + G_{\setminus h}\right) = y_h^g \qquad \text{for all } h \tag{4}$$

where we used the definition of $\widehat{f}$ and equations (1) and (2) in the above system.

Observe that $(y, x, y^g, g, p^{\setminus})$ is an equilibrium associated with an economy π if and only if there exists $(\alpha, \lambda, \eta, \mu)$ such that $F_1\left(y, \alpha, y^g, (x_h, y_h^g, \lambda_h, \eta_h, \mu_h)_{h=1}^H, g, p^{\setminus}, \pi\right) = 0$. With innocuous abuse of terminology, we will call ξ'an equilibrium.

Using an homotopy argument, in Villanacci and Zenginobuz (2006), we show the following result.

[7] Clearly, the Walras' law applies in this model.

Theorem 5 *For every economy $\pi \in \Pi$, an equilibrium exists.*

We now introduce the equilibrium system we use to show the results of the present paper.

Lemma 6 *For every $e \in \mathbb{R}^{CH}_{++}$, ξ' is a solution to system (3) if and only if it is a solution to the following system*

$$\begin{array}{lll}
(1) & Df(y) - \alpha p & = 0 \\
(2) & -py + \sum_h g_h & = 0 \\
(3) & y^g - f(y) & = 0 \\
(h.1) & D_{x_h} u_h(x_h, y^g) - \lambda_h p & = 0 \\
(h.3') & \alpha D_{y^g} u_h(x_h, y^g) - \lambda_h + \mu_h & = 0 \\
(h.4) & -px_h + pe_h - g_h & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^\backslash - y^\backslash + \sum_{h=1}^H e_h^\backslash & = 0 \\
(h.5') & -y_h^g + y^g & = 0 \\
(h.6') & \alpha\eta_h - \lambda_h + \mu_h & = 0
\end{array} \tag{5}$$

Proof. The proof follows from the comparison of systems (3) and (5), and from Remark 1. ■

Since η_h appears only in equation $(h.6')$ and it is uniquely determined by that equation, and y_h^g appears only in equation $(h.5')$ and it is uniquely determined by that equation,we can erase those variables and equations and get the following basically equivalent system.

$$\begin{array}{lll}
(1) & Df(y) - \alpha p & = 0 \\
(2) & -py + \sum_h g_h & = 0 \\
(3) & y^g - f(y) & = 0 \\
(h.1) & D_{x_h} u_h(x_h, y^g) - \lambda_h p & = 0 \\
(h.2) & \alpha D_{y^g} u_h(x_h, y^g) - \lambda_h + \mu_h & = 0 \\
(h.3) & -px_h + pe_h - g_h & = 0 \\
(h.4) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^\backslash - y^\backslash + \sum_{h=1}^H e_h^\backslash & = 0
\end{array} \tag{6}$$

Define

$$\widetilde{\Xi} \equiv \mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \left(\mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}\right)^H \times \{g \in \mathbb{R}^H : \sum_{h=1}^H g_h > 0\} \times \mathbb{R}^{C-1}_{++}$$

$$\widetilde{\xi} \equiv \left(y, \alpha, y^g, (x_h, \lambda_h, \mu_h)_{h=1}^H, g, p^\backslash\right)$$

and

$$F_2 : \widetilde{\Xi} \times \mathbb{R}^{CH}_{++} \to \mathbb{R}^{\dim \widetilde{\Xi}}, \quad F_2 : \left(\widetilde{\xi}, e\right) \mapsto \text{ left hand side of system (6)}$$

We can now prove that there is a large set of the endowments (the so-called regular economies) for which associated equilibria are finite in number, and that equilibria change smoothly with respect to endowments - see Theorem 8 below. To do this, we need to restrict the set of utility functions adding the following assumptions

Assumption 3. $\forall h$, u_h is differentiably strictly concave, i.e., $\forall (x_h, G) \in \mathbb{R}^{C+1}_{++}$, $D^2 u_h(x_h)$ is negative definite.

Assumption 4. For all h and $(x_h, y^g) \in \mathbb{R}^{C+1}_{++}$

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, y^g) & [D_{x_h} u_h(x_h, y^g)]^T \\ D_{y^g x_h} u_h(x_h, y^g) & D_{y^g} u_h(x_h, y^g) \end{bmatrix} \neq 0$$

Remark 7 In the case of household h being a contributor (and therefore μ_h being equal to zero) the above assumption implies that

$$\begin{bmatrix} D_{x_h x_h} & -p^T \\ \alpha D_{y^g x_h} & -1 \end{bmatrix}$$

which is what we need in the proof of some of our result.

Assumption 4 has an easy and appealing economic interpretation. It is easy to see that it is implied by the public good being a normal good, providing the household is a contributor.

Call $\widetilde{\mathcal{U}}$ the subset of $\mathcal{U}$ whose elements satisfy Assumptions 3 and 4. Define

$$pr : F_2^{-1}(0) \rightarrow \mathbb{R}^{CH}_{++}, \quad pr : (\xi, e) \mapsto e$$

We can state the needed generic regularity result.

Theorem 8 *For each $(u, f) \in \widetilde{\mathcal{U}} \times \mathcal{F}$, there exists an open and full measure subset $\mathcal{R}$ of $\mathbb{R}^{CH}_{++}$ such that*

1. *there exists $r \in \mathbb{N}$ such that $F_{2,e}^{-1}(0) = \left\{\widetilde{\xi}^i\right\}_{i=1}^{r}$;*
2. *$\forall h \in \mathcal{H}$, either $g_h > 0$ or $\mu_h > 0$*
3. *there exist an open neighborhood Y of e in $\mathbb{R}^{CH}_{++}$, and for each i an open neighborhood U_i of $\left(\widetilde{\xi}^i, e\right)$ in $F_2^{-1}(0)$ such that $U_j \cap U_k = \emptyset$ if $j \neq k$, $pr^{-1}(Y) = \cup_{i=1}^{r} U_i$ and $pr_{|U_i} : U_i \rightarrow Y$ is a diffeomorphism.*

3 Non-optimality of Subscription Equilibria

It is well known that typically subscription equilibria are not Pareto optimal. That result is also a corollary of the theorems in Section 5.

A large part of the literature studied one private good, linear technology models and tried to propose policy interventions aimed at increasing the equilibrium level of G, whose underprovision was implicitly considered the main reason of inefficiency.

In Section 5, we address that problem and in our more general framework, we confirm some existing results and we show that some others do not generalize. All policy interventions that have been studied require a reduction in some household's wealth. That negative effect on her equilibrium utility level is to be balanced with the positive effect due to an increase in G, the total effect being unclear. In fact, we show the total effect can lead to a decrease in that household utility level.

Building up on that simple observation, we take a direct approach to welfare analysis and describe interventions that are able to Pareto improve upon subscription equilibrium outcomes.

In the remainder of the paper, we first lay out a general strategy to deal with some policy questions in a general equilibrium model. Then, we apply that strategy to the policy interventions referred to above.

We prove in some detail the result for the technically easiest case - a redistribution from non-contributors to contributors in order to increase G. Other proofs are the same in spirit and involve very similar arguments.

4 A General Methodology

Our starting point is the equilibrium function F_2 defined using system (6). We then proceed in four steps.[8]

[8] We apply the general approach introduced by Geneakoplos and Polemarchakis (1986), using the strategy laid out by Cass and Citanna (1998) and Citanna, Kajii and Villanacci (1998).

Step (i):

We first define a new equilibrium function

$$F_3 : \Xi \times \mathbb{R}^T \times \widetilde{\Pi} \rightarrow \mathbb{R}^{\dim \Xi}, \qquad : (\xi, \theta, \pi) \mapsto F_3(\xi, \theta, \pi)$$

taking into account the planner's intervention effects on agents behaviors via the policy tools $\theta \in \mathbb{R}^T$.

We then define a function F_4, describing the constraints on the planner intervention

$$F_4 : \Xi \times \mathbb{R}^T \times \widetilde{\Pi} \rightarrow \mathbb{R}^k, \qquad : (\xi, \theta, \pi) \mapsto F_4(\xi, \theta, \pi)$$

and then we consider the function

$$\widetilde{F} \equiv (F_3, F_4)$$

whose zeros can be naturally interpreted as equilibria with planner's intervention. We can partition the vector θ of tools into two subvectors $\theta_1 \in \mathbb{R}^{T_1}$ and $\theta_1 \in \mathbb{R}^{T_2}$. The former can be seen as the vector of independent tools and the latter as the vector of dependent tools: once the value of the first vector is chosen, the value of the second one is uniquely determined.[9] We find a value $\overline{\theta}_1$ (and associated $\overline{\theta}_2$) at which equilibria with and without planner's intervention coincide (that value being simply zero in all cases studied below). Define $\overline{\theta} \equiv \left(\overline{\theta}_1, \overline{\theta}_2\right)$

We finally introduce a goal function $\mathcal{G}$ defined as

$$\mathcal{G} : \Xi \times \mathbb{R}^T \times \widetilde{\Pi} \rightarrow \mathbb{R}^J, \qquad : (\xi, \theta, \pi) \mapsto \mathcal{G}(\xi, \theta, \pi)$$

The object of our analysis is to study the local effect of a change in the values of independent tools θ_1, around the no-intervention value $\overline{\theta}_1$, on $\mathcal{G}$ when its arguments assume their equilibrium (with planner intervention) values.

Step (ii):

We construct the function linking (independent) tools to goals. An important step towards that construction is provided verifying the following condition.

Condition 9 *For each $(u, f) \in \widetilde{\mathcal{U}} \times \mathcal{F}$, there exists an open and full measure subset $\mathcal{R}$ of $\mathbb{R}^{CH}_{++}$ such that for every $e \in \mathcal{R}$ and for every ξ such that $\widetilde{F}\left(\xi, \overline{\theta}, e, u, f\right) = 0$,*

$$D_{(\xi, \theta_2)} \widetilde{F}\left(\xi, \overline{\theta}, \pi\right) \text{ has full row rank } \dim \Xi + T_2 \tag{7}$$

If the above condition is satisfied, there exists an open and dense subset Π^* of $\widetilde{\Pi}$ such that for each $\pi \in \Pi^*$, condition (7) holds. Then as a consequence of the Implicit Function Theorem, $\forall \pi \in \Pi^*$ and $\forall \xi$ such that $F_2(\xi, \pi) = 0$, there exist an open set $V \subseteq \mathbb{R}^{T_1}$ containing $\overline{\theta}_1$ and a unique C^1 function $h_{(\xi,\pi)} : V \rightarrow \mathbb{R}^{\dim \Xi + T_2}$ such that $h_{(\xi,\pi)}\left(\overline{\theta}_1\right) = \left(\xi, \overline{\theta}_2\right)$, and

$$\text{for every } \theta_1 \in V, \quad \widetilde{F}\left(h_{(\xi,\pi)}(\theta_1), \theta_1, \pi\right) = 0$$

In words, the function $h_{(\xi,\pi)}$ describes the effects of local changes of θ_1 around $\overline{\theta}_1$ on the *equilibrium values* of ξ and θ_2.

For every economy $\pi \in \Pi^*$, and every $\xi \in F_\pi^{-1}(0)$, we can then define, as desired,

$$g_{(\xi,\pi)} : V \rightarrow \mathbb{R}^J, \;\; g_{(\xi,\pi)} : \theta_1 \mapsto \mathcal{G}\left(h_{(\xi,\pi)}(\theta_1), \theta_1, \pi\right)$$

In what follows, unless explicitly needed, we will omit the subscript (ξ, π) of the function g.

Step (iii):

Using the function g above, we give a sufficient condition which guarantees that changes in the values of policy tools have a non-trivial effect on the values of the goals.

Technically, this amounts to showing that there exists an open and dense subset $\Pi^{**} \subseteq \widetilde{\Pi}$ such that for each $\pi \in \Pi^{**}$ and for each associated equilibrium ξ, the planner can "move" the equilibrium

[9] In our proposed different types of intervention, we will have $k = T_2$, i.e., the number of constraints imposed on the planner's behavior is equal to the number of dependent tools.

value of the goal function in any direction locally around $g\left(\overline{\theta}_1\right)$, the value of the goal function in the case of no intervention. More formally, we need to show that g is *essentially surjective at* $\overline{\theta}^*$, i.e., the image of each open neighborhood of $\overline{\theta}_1$ in $\mathbb{R}^{T_1}$ contains an open neighborhood of $g\left(\overline{\theta}_1\right)$ in $\mathbb{R}^k$. A sufficient condition[10] for that property is

$$\text{rank}\ \left[Dg(\overline{\theta}^*)\right]_{T_1\times J} = J \tag{8}$$

Therefore, recalling the distinction between dependent and independent tools above, we must have

$$J = \#\ \text{goals} \quad \leq \quad \#\ \text{independent tools} = T_1$$

Step (iv):

We want to show that the statement (8) holds in an open and dense subset Π^{**} of $\widetilde{\Pi}$. Following Cass (1992), a sufficient condition for that is to show that for each $\pi \in \Pi^{**}$ the following system has no solutions $(\xi, c) \in \Xi \times \mathbb{R}^{\dim \Xi + T_2 + k}$

$$\begin{cases} F_2\left(\xi,\pi\right) & = & 0 & (1) \\ c\left[D_{\xi,\theta_2}\left(\widetilde{F},\mathcal{G}\right)\left(\xi,\overline{\theta},\pi\right)\right] & = & 0 & (2) \\ cc-1 & = & 0 & (3) \end{cases} \tag{9}$$

Openness of Π^{**}. It follows from the properness of the projection function from the equilibrium set (in fact, manifold) to the parameter space.

Density of Π^{**}. Define the function

$$F^* : \Xi \times \mathbb{R}^{\dim \Xi + T_2 + k} \times \widetilde{\Pi} \to \mathbb{R}^{\dim \Xi} \times \mathbb{R}^{\dim \Xi + + T_2 + k} \times \mathbb{R}$$
$$F^* : (\xi, c, \pi) \mapsto \text{left hand side of system (9)}$$

As an application of a finite dimensional version of Parametric Transversality Theorem, the denseness result is established by showing that 0 is a regular value for F^*. More precisely, since π is an element of the infinite dimensional set $\widetilde{\Pi}$, we choose to look at a finite dimensional subset (submanifold) of that set parametrized by a vector a, taking advantage of the generic regularity of equilibria. The construction of the parametrization used is as follows.

We use a finite local parameterization of both the utility and the transformation functions.[11] For the former, we are going to use the following form:

$$\overline{u}_h\left(x_h,g_h\right) = u_h\left(x_h,g_h\right) + \left((x_h,g_h)-(x_h^*,g_h^*)\right)^T A_h\left((x_h,g_h)-(x_h^*,g_h^*)\right)$$

with

$$A_h \equiv \begin{bmatrix} A_{xx,h} & 0 \\ 0 & a_{gg,h} \end{bmatrix}$$

where $u_h \in \widetilde{\mathcal{U}}_h$, (x_h^*, g_h^*) are equilibrium values, $A_{xx,h}$ is a symmetric negative definite matrix, and $a_{gg,h}$ is a strictly negative number. Same second order local parameterization is used for the production function, using a symmetric negative definite matrix A_f.

We can then define $a \equiv \left((a_h, a_{gg,h})_{h=1}^H, \widehat{a}_f\right)$, where $(a_h, a_{gg,h})$ and $\widehat{a}_f$ are the vectors of distinct elements of the symmetric matrices A_h, for $h = 1, ..., H$, and $\widehat{A}_f$, .

We then redefine the functions F_2, $\widetilde{F}$, $\mathcal{G}$, and F^* by replacing $\widetilde{\mathcal{U}} \times \mathcal{F}$ in their domain with a open ball $\widehat{\mathcal{A}}$ in a finite Euclidean space with generic element a. Call F_A, $\widetilde{F}_A$, $\mathcal{G}_A$, and F_A^* the functions so obtained. We can then rewrite (9) as $F_A^*(\xi, c, e, a) = 0$, i.e.,

$$\begin{cases} F\left(\xi,e,a\right) & = & 0 & (1) \\ c\left[D_{(\xi,\theta_2)}\left(\widetilde{F}_A,\mathcal{G}_A\right)\left(\xi,\overline{\theta},e,a\right)\right] & = & 0 & (2) \\ cc-1 & = & 0 & (3) \end{cases} \tag{10}$$

[10] See, for example, Chapter 1 in Golubitsky and Guillemin (1973).

[11] For further details on the content of this appendix, see Cass and Citanna (1998) and Citanna, Kaji and Villanacci (1998).

We are then left with showing that 0 is a regular value for F_A^*, i.e., either

$$\begin{array}{l} F_A^*(\xi, e, a) = 0 \text{ has no solutions } (\xi, c) \\ \text{for all values of } (e, a) \text{ in an open and dense subset of } \mathbb{R}_{++}^{CH} \times \widehat{\mathcal{A}} \end{array} \tag{11}$$

or, using generic regularity, that

$$\begin{array}{l} \text{for each } (\xi, c, e, a) \in (F_A^*)^{-1}(0), \\ \left[\begin{array}{cc} \left[D_{(\xi,\theta_2)}\left(\widetilde{F}_A, G_A\right)(...)\right]^T & N_a(c) \\ c & \end{array} \right] \\ \text{has full rank} \end{array} \tag{12}$$

where $N_a(c)$ is the partial Jacobian of the left hand side of equations (2) and (3) in system (10) with respect to a.

In what follows, we apply the strategy described in the previous section. We first describe in words the type of intervention and then we indicate the specific form of the functions F_3, F_4 and $\mathcal{G}$ consistent with that intervention. We finally state the theorem on the essential surjectivity of the corresponding function g. To keep notation as light as possible, we use the same notation about the above functions and related sets in each section.

5 Government Intervention on the Public Good Level

5.1 Redistributing among contributors

The following theorem is a restatement of a theorem by BBV for the case of many private goods, and its proof is a straight forward adaptation of their proof.

Theorem 10 *Consider an equilibrium associated with an arbitrary economy and a redistribution of the private numeraire good among contributing households such that no household loses more wealth than her original contribution. All the equilibria after the redistribution are such that the consumption of private goods and the total amount of consumed public good are the same as before the redistribution.*

As a simple Corollary to Theorem 10, we get the following:

Proposition 11 *The set of equilibria after a local redistribution from an arbitrary set of non-contributors to one contributor is equal to the set of equilibria after a local redistribution from that same arbitrary set of non-contributors to an arbitrary set of contributors.*

This result follows from the fact that each equilibrium with only 1 contributor being subsidized can be obtained from each equilibrium with more than one contributor being subsidized using appropriate redistributions among contributors. Making use of this result, we consider taxes or subsidies on only one contributor in all of the different types of planner interventions we study below.

We now look at the case in which the planner redistributes endowments of one private good between a (strictly) contributing household, say $h = 1$, and one or two (strictly) non-contributing household, say $h = 2, 4$.[12]

5.2 Redistributing between a contributor and a non-contributor

The planner redistributes resources between a contributor and a non-contributor in order to increase the total production of public good. Therefore, she taxes household 1 and 2 by an amount ρ_1 and ρ_2 respectively. Household $h \in \{1, 2\}$'s budget constraint becomes:

$$-p(x_h - e_h) - g_h - \rho_h = 0 \tag{13}$$

[12] It can be easily shown that the set of economies for which there exists at least one or two non-contributors is open (and non-empty).

The balanced budget constraint requires

$$\rho_1 + \rho_2 = 0$$

Note that # goals =1, # constraints = 1, tools are ρ_1, ρ_2 and thus # tools = 2.
Therefore

$$\begin{array}{ll} F_3 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}^{\dim \Xi}, & (\xi, \rho, \pi) \mapsto \text{ LHS of (6) with eqn. } (h.3) \text{ replaced by (13), } h \in \{1, 2\} \\ F_2 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \rho_1 + \rho_2 \\ \mathcal{G} : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \sum_{h=1}^{H} g_h \end{array}$$

Theorem 12 *There exists an open and dense subset S^* of the set of the economies for which there exists at least one non-contributor, such that $\forall \pi \in S_1^*$ and $\forall \xi' \in F_\pi^{-1}(0)$ the function g is essentially surjective at 0, i.e.,there exists a redistribution of the endowments of private good C between one contributor and one non-contributor which increases (or decreases) the level of provided public good.*[13]

5.3 Redistributing between non-contributors

The planner redistributes resources between a contributor and a non-contributor in order to increase the total production of public good.

Theorem 13 *If $C \geq 2$, for an open and dense subset S^* of the set of (the economies for which there exist at least two non-contributors, at any equilibrium ξ', the function g is essentially surjective at 0, i.e., there exist taxes on two non-contributors which increases (or decreases) the level of provided public good.*

The requirement $C \geq 2$ in the theorem brings out the importance of having more than one private good in obtaining non-neutrality results in our analysis. To see why having more than one private good is essential to affect relative price changes, consider the case of one public and one private good. Redistributing the private good among non-contributors will not change the demand of the public good because contributors are not affected by this intervention and non-contributors do not become contributors (because, generically, we are not on the border line cases and taxes are small). Therefore, there will also be no change in the overall the demand for the single private good. With no other private good available, the overall effect is just a reallocation of the demand for the private good from a non-contributor to another.

5.4 Taxing one contributor and two non-contributors

In this subsection,we show that we can take away some amount of the numeraire good from the contributor (tax her positively) and still increase the amount of G.

We consider the case in which the planner redistributes endowments of one private good among three households, say $h = 1, 2$ and 4, where household 1 is a contributor and the other two are non-contributors.

Therefore

$$\begin{array}{ll} F_3 : \Xi \times \mathbb{R}^3 \times \Pi \to \mathbb{R}^{\dim \Xi}, & (\xi, \rho, \pi) \mapsto \text{LHS of (6) with eqn. } (h.3) \text{ replaced by (13), } h \in \{1, 2, 4\} \\ F_4 : \Xi \times \mathbb{R}^3 \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \sum_{h=1,2,4} \rho_h \\ \mathcal{G} : \Xi \times \mathbb{R}^3 \times \Pi \to \mathbb{R}^2, & (\xi, \rho, \pi) \mapsto \left(\rho_1, \sum_{h=1}^{H} g_h\right) \end{array}$$

Note that # goals =2, # constraints = 1, tools are ρ_1, ρ_2, ρ_4 and thus # tools = 3.

[13] This result is in fact a Corollary ot Theorem 14 below.

Theorem 14 *If $C \geq 2$, for an open and dense subset S^* of the set Π of the economies for which there exists at least two non-contributors, at any equilibrium ξ', the function $\widehat{g}_a$ is locally onto around* 0.

In more intuitive terms, the theorem says that, typically in the relevant set of economies, there exists a redistribution of the endowments of private good C among one contributor and two non-contributors such that

the contributor may be taxed, or subsidized or neither of the two, and, *in a completely unrelated manner*,

the level of provided public good may be increased or decreased or left constant.

In other words, the planner can choose arbitrarily the signs of changes (negative, positive or zero) of both ρ_1 and G and then find a value of (ρ_2, ρ_4) in an arbitrarily small neighborhood of zero which induces those desired sign changes.

5.5 Dealing with wealth of all contributors: Taxing one contributor and two non-contributors

In this subsection, we want to increase the equilibrium level of G, even penalizing contributors, this time not in term of a negative tax, but in terms of a negative change in their total wealth. Therefore, we have

$$\begin{array}{ll} F_3 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}^{\dim \Xi}, & (\xi, \rho, \pi) \mapsto \text{LHS of (6) with eqn. } (h.3) \text{ replaced by (13)}, \ h \in \{1,2,4\} \\ F_4 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \sum_{h=1,2,4} \rho_h \\ \mathcal{G} : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \left(\sum_{h \in \mathcal{H}^+} pe_h + \rho_1, \sum_{h=1}^{H} g_h \right) \end{array}$$

Theorem 15 *For an open and dense subset S^* of the set Π of the economies for which there exists at least two non-contributors, at any equilibrium ξ', the function $\widehat{g}_a$ is locally onto around* 0.

The above result is similar to the result in Theorem 14, the other goal of the intervention beside G, being *total wealth of contributors* instead of the level of taxes on a contributor. In other words, the planner can choose arbitrarily the signs of changes (negative, positive or zero) of both total wealth of contributors and G and then find a value of (ρ_1, ρ_2, ρ_4) in an arbitrarily small neighborhood of zero which induces those desired sign changes.

5.6 Increasing G and Pareto improving

It is easy to prove that an increase in the total equilibrium level of G does not imply a Pareto improvement. To do so, consider as F_3 and F_4 the same functions as in the previous subsection and $\mathcal{G}$ as follows

$$\mathcal{G} : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, \quad (\xi, \rho, \pi) \mapsto \left(u_2(x), \sum_{h=1}^{H} g_h \right)$$

Theorem 16 *For an open and dense subset S^* of the set Π of the economies for which there exists at least two non-contributors, at any equilibrium ξ', the function g is locally onto around* 0*, i.e., the total level of the public good may increase without leading to a Pareto superior equilibrium outcome.*

6 Government Interventions on Welfare

6.1 Intervening in firm production

The planner taxes one contributor and changes the choice of inputs and output of the production of the public good. The idea is that the manager of the public firm chooses y to solve problem (1) and then the ministry of Public Economics, i.e., the planner, decides to use some extra inputs θ^y to produce extra public good financing it with taxes ρ_1 on household 1 who is a contributor.

The constraint on planner intervention is simply

$$\rho_1 - p\theta^y \quad = 0$$

In this case, # goals $= H$, # constraints $= 1$, tools: ρ_1, θ^y whose number is $1 + C$. Therefore, we must have $H \leq C$. In fact, for technical reasons, we need to impose $H \leq C - 1$.

Define $\rho \equiv (\rho_1, \theta^y) \in \mathbb{R}^{C=1}$ and

$$\begin{array}{ll} F_3 : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}^{\dim \Xi}, & (\xi, \rho, \pi) \mapsto \text{LHS of (6) with eqn.} (h.3) \text{ replaced by (13)}, h \in \{1,2,4\} \\ F_4 : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto \rho_1 - p\theta^y \\ \mathcal{G} : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}, & (\xi, \rho, \pi) \mapsto (u_h(x_h))_{h=1}^H \end{array}$$

Theorem 17 *Assume that $H \leq C-1$. For an open and dense subset S_1^* of the set of the economies, at any equilibrium ξ', the function g is essentially surjective at 0, i.e., there exists a tax on a contributor and a choice of change in inputs θ^y which Pareto improves or impairs upon the equilibrium ξ'.*

6.2 Communicating non-market prices to the manager

The planner tells the manager of the public firm to maximize the amount of produced public good at prices $(1-\sigma^c)\, p^c$under the constraint of not spending more than $\sum_h g_h$. Of course if, just to fix ideas, $\sigma^c > 0$ for each c, the manager is not spending $\sum_{c=1}^C (1-\sigma^c)\, p^c y^c$, but the higher amount $\sum_{c=1}^C p^c y^c$. The difference $\sum_{c=1}^C \sigma^c p^c y^c$ has to be financed by taxes ρ_1 .

The basic idea of all the intervention is of course that market prices are not the "right ones". The intervention is in sense the most direct one: change the prices at which public good is produced.

1. Public firm solves

$$\max_{y \in \mathbb{R}^C_{++}} f(y) \quad s.t. \quad -\sum_{c=1}^C (1-\sigma^c)\, p^c y^c + G = 0 \tag{14}$$

2. Household 1 has to pay for the "discounts" that the public firm obtained in buying inputs:

$$\rho_1 - \sum_{c=1}^C \sigma^c p^c y_1^c = 0 \tag{15}$$

3. Household 1's budget constraint becomes:

$$-p(x_1 - e_1) - g_1 - \rho_1 = 0$$

Note that # goals $=H$, # constraints $= 1$, tools are $\rho_1, \tau \equiv \left(\tau_f^c\right)_{c=1}^C$ and thus # tools $= C+1$.

Therefore, we must have $H \leq C$. As in the case of the previous subsection, we have in fact to impose that $H \leq C - 1$

Then the system with planner intervention is

$$\begin{array}{lll} (1) & Df(y) - \alpha\left((\mathbf{1}-\sigma^c)\, p^c\right)_{c=1}^C & = 0 \\ (2) & -\left((\mathbf{1}-\sigma^c)\, p^c\right)_{c=1}^C \cdot y + \sum_h g_h & = 0 \\ (h.1) & D_{x_h} u_h(x_h, y^g) - \lambda_h p & = 0 \\ (h.2) & \alpha D_{y^g} u_h(x_h, y^g) - \lambda_h + \mu_h & = 0 \\ (h.3) & -px_h + pe_h - g_h - \rho_h & = 0 \\ (h.4) & \mu_h & = 0 \\ (M) & -\sum_{h=1}^H x_h^\backslash - y^\backslash + \sum_{h=1}^H e_h^\backslash & = 0 \\ (P1) & y^g - f(y) & = 0 \end{array} \tag{16}$$

with $\rho_h = 0$ if and only if $h \neq 1$.

Define and

$$F_3 : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}^{\dim \Xi}, \qquad (\xi, \rho_1, \sigma, \pi) \mapsto \text{ LHS of (16)}$$
$$F_4 : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}, \qquad (\xi, \rho_1, \sigma, \pi) \mapsto \rho_1 - p\theta^y$$
$$\mathcal{G} : \Xi \times \mathbb{R}^{C+1} \times \Pi \to \mathbb{R}, \qquad (\xi, \rho_1, \sigma, \pi) \mapsto (u_h (x_h))_{h=1}^H$$

Theorem 18 *Assume that $H \leq C-1$. For an open and dense subset S^* of the set of the economies, at any equilibrium ξ', the function g is essentially surjective at 0, i.e., there exists a tax on a contributor and a choice of change in input prices σ which Pareto improves or impairs upon the equilibrium ξ'.*

7 Appendix. The proof of Theorem 12

Condition 9 in Subsection 4 can be easily verified, exploiting generic regularity.

Since equations (2) and (3) in system (9) simply say that $\left[D_{(\xi,\tau)}\left(\widetilde{F}, G\right)\left(\xi, \overline{\theta}, e, a\right)\right]$ has full row rank, showing that system (9) has no solutions is equivalent to showing that the following system has no solutions

$$\begin{cases} F(\xi, e, a) & = & 0 & (1) \\ c \cdot \Gamma\left(\xi, \overline{\theta}, e, a\right) & = & 0 & (2) \\ cc - 1 & = & 0. & (3) \end{cases}$$

where $\Gamma\left(\xi, \overline{\theta}, e, a\right)$ is a matrix obtained from $\left[D_{(\xi,\theta_2)}\left(\widetilde{F}_A, \mathcal{G}_A\right)\left(\xi, \overline{\theta}, e, a\right)\right]$ using elementary row and column operations (which are rank preserving). Then both condition (11) and (12) do hold if $\left[D_{(\xi,\theta_2)}\left(\widetilde{F}_A, \mathcal{G}_A\right)\left(\xi, \overline{\theta}, e, a\right)\right]$ is substituted by $\Gamma\left(\xi, \overline{\theta}, e, a\right)$. We are therefore left with showing that form of those conditions.

Below we compute $D_{(\xi,\rho_1,\sigma)}\left(\widetilde{F}_A, \mathcal{G}_A\right)$. The components of $\left(\widetilde{F}_A, \mathcal{G}_A\right)$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the corresponding partial Jacobian is displayed.

	y	α	x_1	g_1	λ_1	μ_1	x_2	g_2	λ_2	μ_2	$p^{\backslash}$	y^g	ρ_1	ρ_2
(1) $Df(y)$ $-\alpha p$	$D^2 f$	$-p^T$									αI 0			
(2) $-py+$ $\Sigma_h g_h$	$-p$			1				1			$-y^{\backslash}$			
(1.1) $D_{x_1} u_1+$ $-\lambda_1 p$			$D^1_{x_1 x_1}$		$-p^T$						$-\lambda_1 I$ 0	$D^1_{x_1 y^g}$		
(1.2) $\alpha D_{y^g} u_1$ $-\lambda_1+\mu_1$		$D_{y^g} u_1$	$\alpha D^1_{y^g x_1}$		-1	1						$\alpha D^1_{y^g y^g}$		
(1.3) $-pz_1+$ $-g_1-\rho_1$			$-p$	-1							$-z_1^{\backslash}$		-1	
(1.4) μ_1						1								
(2.1) $D_{x_2} u+$ $-\lambda_2 p$							$D^2_{x_2 x_2}$		$-p^T$		$-\lambda_2 I$ 0	$D^2_{x_2 y^g}$		
(2.2) $\alpha D_{y^g} u_2$ $-\lambda_2+\mu_2$		$D_{y^g} u_2$					$\alpha D^2_{y^g x_2}$		-1	1		$\alpha D^2_{y^g y^g}$		
(2.3) $-pz_2+$ $-g_2-\rho_2$							$-p$	-1			$-z_2^{\backslash}$			-1
(2.4) g_2								1						
(M) $-x^{\backslash}-y^{\backslash}$ $+e$	$-I0$		$-I0$				$-I0$							
$(P1)$ $-y^g+$ $f(y)$	Df											-1		
$(P2)$ $\rho_1+\rho_2$													1	1
$(P3)$ $\Sigma_h g_h$				1				1						

Performing some elementary row and column operations and erasing some irrelevant rows and columns, we get the following matrix $\Gamma\left(\xi, \overline{\theta}, e, a\right)$.

	y	α	x_1	g_1	λ_1	x_2	λ_2	$p^{\backslash}$	y^g	ρ_1	ρ_2
(1) $Df(y)$ $-\alpha p$	$D^2 f$	$-p^T$						αI 0			
(2) $-py+$ $\Sigma_h g_h$	$-p$							$-y^{\backslash}$			
(1.1) $D_{x_1} u_1+$ $-\lambda_1 p$			$D^1_{x_1 x_1}$		$-p^T$			$-\lambda_1 I$ 0	$D^1_{x_1 y^g}$		
(1.2) $\alpha D_{y^g} u_1$ $-\lambda_1+\mu_1$		$D_{y^g} u_1$	$\alpha D^1_{y^g x_1}$		-1				$\alpha D^1_{y^g y^g}$		
(1.3) $-pz_1+$ $-g_1-\rho_1$			$-p$	-1				$-z_1^{\backslash}$		-1	
(2.1) $D_{x_2} u+$ $-\lambda_2 p$						$D^2_{x_2 x_2}$	$-p^T$	$-\lambda_2 I$ 0	$D^2_{x_2 y^g}$		
(2..3) $-pz_2+$ $-g_2-\rho_2$						$-p$		$-z_2^{\backslash}$			-1
(M) $-x^{\backslash}-y^{\backslash}$ $+e$	$-I0$		$-I0$			$-I0$					
$(P1)$ $-y^g+f(y)$								$-\alpha y^{\backslash}$	-1		
$(P2)$ $\rho_1+\rho_2$										1	1
$(P3)$ Σg_h				1							

Then to check that condition 11 (in terms of Γ) holds, it is enough to check that the following system has no solutions.

$$\left\{\begin{array}{lll}
(1) & D^2 f\ c_y - p^T c_\alpha + \begin{bmatrix} -I \\ 0 \end{bmatrix} c_{p^{\backslash}} & = 0 \\
(2) & -pc_y + D_{y^g} u_1 \cdot c_{g_1} & = 0 \\
(1.1) & D_{x_1 x_1} u_1 \ \cdot c_{x_1} + \alpha D_{y^g x_1 u_1} \cdot c_{g_1} - p^T c_{\lambda_1} + \begin{bmatrix} -I \\ 0 \end{bmatrix} c_{p^{\backslash}} & = 0 \\
(1.2) & -c_{\lambda_1} + c_G & = 0 \\
(1.3) & -pc_{x_1} - c_{g_1} & = 0 \\
(2.1) & D_{x_2 x_2} u_2 \cdot c_{x_2} - p^T c_{\lambda_2} + \begin{bmatrix} -I \\ 0 \end{bmatrix} c_{p^{\backslash}} & = 0 \\
(2.3) & -pc_{x_2} & = 0 \\
(M) & [\alpha I0]\, c_y + -y^{\backslash} c_\alpha + \sum_h \left([-\lambda_h I0]\, c_{x_h} - z_h^{\backslash} c_{\lambda_h} \right) - \alpha y^{\backslash} c_{y^g} & = 0 \\
(P1) & D_{x_1 y^g} u_1 \cdot c_{x_1} + \alpha D_{y^g y g} u_1 \cdot c_{g_1} + .. + D_{x_2 y^g} u_2 \cdot c_{x_2} - c_{y^g} & = 0 \\
(G1) & -c_{\lambda_1} + c_{\rho_1} & = 0 \\
(G2) & -c_{\lambda_2} + c_{\rho_1} & = 0 \\
(L) & cc - 1 & = 0
\end{array}\right.$$

To show that condition (12) (in terms of Γ) holds we have to check that the following matrix M has full rank.

$D^2 f$	$-p^T$						$-I$ 0				N_{c_y}		
$-p$			$D^1_{y^g}$										
		$D_{x_1 x_1}$	$\alpha D_{y^g x_1}$	$-p^T$			$-I$ 0					$N_{c_{x_1}}$	
				-1						1			
		$-p$	-1										
					$D_{x_2 x_2}$	$-p^T$	$-I$ 0						$N_{c_{x_2}}$
					$-p$								
$\alpha I0$	$-y\backslash$	$-\lambda_1 I0$		$-z_1^{\backslash}$	$-\lambda_2 I0$	$-z_2^{\backslash}$		$-\alpha y\backslash$					
		$D_{x_1 y^g}$	$\alpha D^1_{y^g y g}$		$D_{x_2 y^g}$			-1					
				-1					1				
						-1			1				
c_y	c_α	c_{x_1}	c_{g_1}	c_{λ_1}	c_{x_2}	c_{λ_2}	$c_{p\backslash}$	c_{y^g}	c_{ρ_1}	c_G			

which can be done through some cumbersome computations.

References

Andreoni, J. (1988), Privately Provided Public Goods in a Large Economy: The Limits of Altruism, *Journal of Public Economics*, 35, 57-73.

Becker, G., 1981, *A Treatise on the Family,* Cambridge: Harvard University Press.

Bergstrom, T., Blume, L. and H. Varian, (1986), On the Private Provision of Public Goods, *Journal of Public Economics, 29,* 25-49.

Bernheim, D. (1986), On the Voluntary and Involuntary Provision of Public Goods, *American Economic Review*, 76, 789-793.

Brennan, G, and J. Pincus, 1983, "Government expenditure growth and resource allocation", *Oxford Economic Papers*, 35, 351-365.

Cass D. (1992), Two Observations About the Implicit Function Theorem in General Equilibrium Theory, University of Pennsylvania, mimeo.

Cass D, Citanna A. Pareto improving financial innovation in incomplete markets. Economic Theory 1998; 11; 467-494.

Citanna A, Kajii A, Villanacci A. Constrained suboptimality in incomplete markets: a general approach and two applications. Economic Theory 1998; 11; 495-521.

Chamberlain, J., 1976, "A diagrammatic exposition of the logic of collective action", *Public Choice*, 26, 59-74.

Cornes, R., and T. Sandler (2000), Pareto Improving Redistribution in the Pure Public Good Model, *German Economic Review*, 1, 169-186.

Foley, D. (1967), Resource Allocation and the Public Sector, *Yale Economic Essays*, 7, 45-98.

Geanakoplos J, Polemarchakis HM. Existence, regularity, and constrained suboptimality of competitive allocations when assets structure is incomplete. In W. P. Heller, R. M. Starr, and D. A. Starret Eds, Essays in honor of K. J. Arrow, Cambridge University Press, Cambridge Vol. 3; 1986. 65-95.

Golubitsky M., and Guillemin V. (1973), *Stable Mappings and Their Singularities*, Springer–Verlag, New York.

Kemp, M., 1984, "A note on the theory of international transfers", *Economic Letters*, 14 (2-3), 259-262.

McGuire, M., 1974, "Group homogeneity and aggregate provision of a pure public good under cournot behavior", *Public Choice*, 18, 107-126.

Milleron, J. C., 1972, "Theory of value with public goods: A survey article", *Journal of Economic Theory*, 5, 419-477.

Olson, M., 1965, The Logic of Collective Action, Cambridge: Harvard University Press.

Roberts, R., 1984, "A positive model of private charity and wealth transfers", *Journal of Political Economy*, 92, 135-148.

Samuelson, P. A. (1954), The Pure Theory of Public Expenditure, *Review of Economics and Statistics*, 36, 387-9.

Samuelson, P. A. (1955), Diagrammetic Exposition of a Theory of Public Expenditure, *Review of Economics and Statistics*, 37, 350-6.

Samuelson, P. A. (1958), Aspects of Public Expenditure Theories, *Review of Economics and Statistics*, 40, 332-8.

Tinbergen, J., (1956), *Economic policy: Problems and Design*, North-Holland, Amsterdam.

Villanacci, A. and U. Zenginobuz, (2006), Subscription Equilibria with Public Production: Existence and Regularity, *Research in Economics,* 60, 199–215.

Warr, P. (1982), Pareto Optimal Redistribution and Private Charity, *Journal of Public Economics*, 19, 131-138.

Warr, P. (1983), The Private Provision of Public Goods is Independent of the Distribution of Income, *Economics Letters*, 13, 207-211.

Young, D., 1982, "Voluntary purchase of public goods", *Public Choice*, 38, 73-86.