Ex-Post Full Surplus Extraction, Straightforwardly*

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Abstract

Consider an estimate of the common value of an auctioned asset that is symmetric in the bidders' types. Such an estimate can be represented solely in terms of the order statistics of those types. This representation forms the basis for a pricing rule yielding truthful bidding as an equilibrium, whether bidders' types are affiliated or independent. We highlight the link between the estimator and full surplus extraction, providing a *necessary and sufficient* condition for *ex-post* full surplus extraction, including the possibility of independent types. The results offer sharp insights into the strengths and limits of simple auctions by identifying the source of informational rents in such environments.

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1 An Illuminating Example

Insights into the strength of auctions as a selling mechanism can be gleaned from clarity in understanding when auctions can extract full surplus. Consider the following common-value auction. Suppose n risk-neutral bidders compete for an indivisible asset worth V, where V has density $g(v)$ on $[0, M]$. For expositional ease consider the case $n = 2$. Each bidder $i = 1, 2$ privately observes his type X_i; let $X = (X_1, X_2)$. A type X_i may be thought of as a signal representing i's private information about asset value. Assume that the signals X_i are independent conditional on $V = v$ and that the density of $X_i|v$ satisfies

$$f_i(x_i|v) = 1_{[x_i \geq v]} 1_{[x_i \leq M]} h_i(x_i) k_i(v),$$

where h_i and k_i are arbitrary positive functions limiting its support to $[v, M]$. Clearly this restricts the functional form of k_i to

$$k_i(v) = \frac{1}{\int_v^M h_i(s)ds}.$$

In particular, for parameters $\alpha = (\alpha_1, \alpha_2) \in \Re^2_{++}$, let

$$h_i(x) = \alpha_i (M - x)^{\alpha_i - 1}$$

so that

$$F_i(x|v) = \frac{\int_v^x \alpha_i (M - s)^{\alpha_i - 1} ds}{\int_v^M \alpha_i (M - s)^{\alpha_i - 1} ds} = 1 - \frac{(M - x)^{\alpha_i}}{(M - v)^{\alpha_i}}. \tag{1}$$

Denote the fundamentals of the common-value problem by $\mathcal{E}_\alpha = (g, F_1, F_2)$. Define also for each such environment

$$v(x_i, x_j) = E[V | X_i = x_i, X_j = x_j]. \tag{2}$$

Lemma 1 *For F_1, F_2 satisfying (1), the estimate of the common value depends only on the minimum of the private signals, i.e. $v(x_i, x_j) = \widetilde{v}(\min\{x_i, x_j\})$.*

The proof is in the Appendix.

Proposition 2 *(Matthews [1977], Milgrom and Weber [1982]) In the environment $\mathcal{E}_\alpha = (g, F_1, F_2)$, for any $\alpha = (\alpha_1, \alpha_2) \in \Re^2_{++}$, a second-price auction, with no reserve price, admits a pure strategy bidding equilibrium of the form (b^*, b^*), where $b^*(x) = v(x, x)$.*[1]

[1] *For the cases $\alpha_1 = \alpha_2$, this is clearly the unique symmetric equilibrium. This paper uses Bayes-Nash equilibrium*

The combination of these two results yields a situation where the winner of this auction realizes no surplus ex-post, since both the estimated value and the price he is facing depend solely on his opponent's signal.

Proposition 3 *The second-price auction with zero reserve leaves no surplus to the winner in the environment $\mathcal{E}_\alpha$ for $\alpha \in \Re^2_{++}$.*

Proof. Lemma 1 implies that expected profit conditional on winning,

$$v(\max\{X_1, X_2\}, \min\{X_1, X_2\}) - v(\min\{X_1, X_2\}, \min\{X_1, X_2\}) = 0,$$

implying full surplus extraction. ■

Note that in this case the winner is ex-post indifferent between participating in the auction or not, while the loser pays nothing which also yields indifference over participation.[2] This raises the question of identifying the environments $\mathcal{E}$ which allow mechanisms that are, ex-post, both full surplus extracting and individually rational.[3]

The interested reader will recognize the similarities between the model above and that employed by Matthews [1984] in his seminal paper on information acquisition. It considers an environment $\mathcal{E}'_\alpha$ where V has density $g(v)$ on $[0, M]$, as above. However, for $\alpha = (\alpha_1, \alpha_2) \in \Re^2_{++}$, let $F_i(x_i|v) = (x/v)^{\alpha_i}$, with support $[0, v]$. This example exhibits $v(x_i, x_j) = \widetilde{v}(\max\{x_i, x_j\})$. In other words, all of the signals except the highest are uninformative.

Both environments $\mathcal{E}_\alpha$ and $\mathcal{E}'_\alpha$ carry the property that $X^\alpha|v$ is more accurate (in the sense of Lehmann [1988] and Persico [2000]) than $X^\beta|v$ for $\alpha \geq \beta$ (demonstrated in the Appendix).[4] Persico [2000] (following Matthews [1984]) used such a model to argue that players have a positive marginal benefit from acquiring better information (which corresponds to choosing a higher α_i) as

throughout as our solution concept.

[2]For $n > 2$, this Proposition applies to an auction where the highest bidder wins, and pays a price equal to the lowest bid submitted. For $n > 2$ the second price auction extracts full surplus ex-ante but is not ex-post individually rational.

[3]An allocation is ex-post individually rational if each player, upon learning the types of all players, evaluates his outcome as at least weakly preferable to not participating (following Holmstrom and Myerson [1983]). Clearly the realization of v will have an impact on the actual profits.

[4]Lehmann's accuracy order is commonly used in the information acquisition literature (see Jewitt [1997], Persico [2001] and Parreiras [2002]) as a weakening of the Blackwell sufficiency condition. A signal X is more informative than a signal Y about an unobserved variable V if the variable $X|v$ can be obtained from $Y|v$ through a transformation that is increasing in v. Persico [2001] notes that this can be interpreted as X being more correlated than Y with V and uses this fact to establish a single-crossing condition for decision problems. A sufficient condition for X to be more informative than Y in this order is that $F_X^{-1}[F_Y(x|v)|v]$ is increasing in v.

long as their opponents are unaware it.[5] However, Proposition 3 implies directly that no bidder would pay any positive price to openly increase the precision of his signal from α to any $\alpha + \varepsilon$, for any $\varepsilon > 0$. This result points out a fundamental difference between environments $\mathcal{E}_\alpha$ and $\mathcal{E}'_\alpha$, due to the differentiated informational content of various order statistics of the private signals.

The intuition of the result is best captured by the notion that even though a bidder's signal may be more accurate than his rival's signal *ex-ante*, it might be *ex-post uninformative*. Lemma 1 shows that expected asset value does not depend on the highest signal regardless of the choice of (α_1, α_2). Furthermore, conditional on observing all the private information the highest signal is irrelevant for the purposes of estimating the common value. Hence, the winning bidder can be precluded from claiming any informational rent, and attains zero expected profit in equilibrium.

The remainder of the paper determines how general this relation is between uninformativeness and full surplus extraction. We expand and complete study of full surplus extraction for common-value auctions in six ways. First, *necessary and sufficient* conditions are obtained here; the stringency of these straightforward conditions thus places in context the comparatively less accessible sufficient conditions in antecedent papers. Second, sufficient conditions allow for a wide range of statistical relationships across bidders' private information, again unlike the prior literature. Third, the mechanism found is a straightforward auction: bidders submit sealed tenders, the winning bidder pays a price which is a non-stochastic function of the submitted bids, without any reserve price, and losing bidders pay nothing.[6] Fourth, under the conditions enabling full surplus extraction, an oral ascending auction with public, irrevocable exits (the *Japanese variant* of the English auction) implements the auction without a reserve price. Fifth, surplus extraction here satisfies ex-post individual rationality: for all realizations of his type and the type(s) of price-setter(s), upon learning the price he is to pay, the winning bidder is still willing to complete the transaction (he is indifferent).[7] Sixth, symmetry plays a narrower role than in antecedent papers.

[5] Persico [2000] argues in a model containing $\mathcal{E}_\alpha$ as a special case, that information acquisition has a positive marginal revenue. A player has an incentive to move from α to $\alpha + \varepsilon$ in his precision as long as his opponent continues to bid as if the player's information precision is constant. In other words if information acquisition happens in secret the marginal benefit of extra-precision is positive.

[6] Lopomo [2000] points to these conditions as important constraints within which to ask how much surplus a seller can extract. All prior work on full surplus extraction falls outside his constraints.

[7] Robert [1991] shows that mechanisms introduced by Crémer and McLean [1985, 1988], McAfee, McMillan and Reny [1989], and McAfee and Reny [1992] are no longer full-surplus-extracting when either a finite upper bound is placed on the size of a transfer from a bidder to the auctioneer (i.e., limited liability in the weakest sense imaginable), or bidders are made infinitesimally risk-averse. Our results do not suffer these particular forms of robustness problems.

2 A Standard Model

The environment $\mathcal{E}$ generalizes the example above as follows. The joint distribution $F(V, X)$ has the property that (V, X_i) are affiliated, for each $i = 1, \ldots, n$.[8] Affiliation, commonly used in auction literature, is a strong way to characterize statistical dependence implying that higher values of X_i are "good news" about the value of V. Denote by $S_i \subset \mathbb{R}$ the support of X_i. No symmetry assumption on the distribution underlying the X_i's is made directly, however,

$$v(x_1, ..., x_n) := E[V|X_1 = x_1, ..., X_n = x_n]$$

is required to be **symmetric** in all arguments (by affiliation of (V, X_i), it is non-decreasing).

Let $\omega_i : R^n \mapsto R$,for $i = 1, ..., n$ be the set of component-ordering functions. Thus,

$$\omega_1(x_1, ..., x_n) = \max(x_1, ..., x_n), .., \omega_n(x_1, ..., x_n) = \min(x_1, ..., x_n),$$

and $\omega_2(X)$ is the second-highest component of the random vector X. Denote $\omega = (\omega_1, ..., \omega_n)$. We will consider direct revelation mechanisms for the allocation of the asset. Player i's strategy is a function $b_i : S_i \to S_i$ which identifies a message $b_i(x)$ when his type is $X_i = x$. Let $B = (B_1, ...B_n)$ be the resulting vector of messages. We will focus in particular on mechanisms that allocate the asset to the player with the highest message, and charge him according to a function p that **depends on the reports only through their order statistics**, and thus can be expressed as $p(B) = P(\omega(B))$ for some function P. It is convenient to use notation $\omega_{-1} = (\omega_2, ..., \omega_n)$ so that $\omega = (\omega_1, \omega_{-1})$, with corresponding conventions for x_{-1} and B_{-1}. We also abuse notation slightly by using dx_{-1} in shorthand for certain integrations. The interested reader will recognize the importance of the affiliation assumption which allows us to infer the fact that if the individual signals X_i are informative about V so are their order statistics.

[8] Affiliation follows the standard definition of Milgrom and Weber [1982]. Mathematically, signals X and Y are affiliated if their joint density exhibits total positivity of order 2 or, equivalently, if $\frac{\partial^2}{\partial x \partial y} \log f(x, y) \geq 0$ whenever smoothness conditions are satisfied. Affiliation is a strengthening of correlation since it implies, among other things, that conditional on higher values of one variable the other variable increases in the sense of first order stochastic dominance. It is obvious from this definition that statistical independence is a limit case of affiliation.

3 A Simple Pricing Rule

Consider the problem of estimating asset value in the hypothetical situation in which all n players' types are observed. Since the expected value of V is a symmetric function of the private signals, it depends on the types only through their order statistics; hence let $H(\omega(X))$ denote an estimator of asset value.[9] That is,

$$H(\omega(x_1, ..., x_n)) = v(x_1, ..., x_n).$$

Construct an auction that allocates the good to the highest bidder and uses a pricing rule

$$P^T(B) = H((\omega_2, \omega_2, \omega_3, ..., \omega_n)(B_1, B_2, ..., B_n)) = H(\omega_2, \omega_{-1}(B)),$$

which does not depend on the winner's bid and imposes no charges on losing bidders. Thus, this price would be the estimate of the asset value had the highest bidder's type matched the second-highest bid, and had each other bidder bid his type. Finally, we remark that the auction described above is a *direct mechanism* which implements the allocation of the *Japanese variant* of the English auction, as described in Milgrom and Weber [1982].

Lemma 4 *The pricing rule $P^T(B) = H(\omega_2, \omega_{-1}(B))$ generates a direct revelation mechanism, i.e. in this mechanism, it is an equilibrium for each player i to adopt a strategy $b_i(x) = x$, that is, to truthfully reveal his type.*

Proof. Define for each y the set $A^1(y) = \{x_{-1} | \max(x_{-1}) \leq y\}$.

Let the other $n-1$ players adopt $b(x) = x$, and consider (without loss of generality) the problem facing player 1 with type $X_1 = x_1$:

$$B \in \arg\max_b \int_{A^1(b)} (H(\omega(x_1, x_{-1})) - H(\omega_2, \omega_{-1}(b, x_{-1}))\, f(x_{-1}|x_1)\, dx_{-1}.$$

The integrand is positive on $A^1(x)$ and non-positive on $A^1(b) \setminus A^1(x)$ for $b \geq x_1$, and hence expected profit is maximized by setting $b = x$. ■

It is important to note that the distribution of the X_i's affects this auction only through its effect on the estimator H. In particular, truth-telling is an equilibrium regardless of whether the types are independent or strictly affiliated. The intuition for this result is relatively straightforward: if the

[9] Note that H is an increasing function in all its components given the affiliation assumption.

highest private signal carries no information about the true value of the asset then our mechanism can elicit truthful revelation of information without "leaving money on the table" in terms of informational rents.

The "No Residual Uncertainty Class": Consider the environments for which $F(V, X)$ degenerates to a mass point given realizations $X = (x_1, ..., x_n)$.[10] An immediate implication of this special case is that the absence of residual uncertainty allows for the introduction of differentiated risk attitudes. Let each player i seek to maximize the expectation of an increasing utility function u_i of profit, normalized via $u_i(0) = 0$.

Corollary 1 *In the No Residual Uncertainty class, for the pricing rule $P_T(B) = H(\omega_2, \omega_{-1})(B)$, it remains an equilibrium for each bidder i to adopt $b_i(x) = x$ even if bidders have differentiated attitudes towards risk.*

Proof. The proof above is simply modified to use the integral:

$$x \in \arg\max_b \int_{A_1(b)} u_i(H(\omega(x_1, x_{-1})) - H(\omega(b, x_{-1}))f(x_{-1}|x_1)dx_{-1}$$

for which the integrand's sign is unaffected by the introduction of u_i. ■

For the traditional model in which types are independent conditional on V, and stochastically ordered by V (Wilson [1977] and following papers), the Theorem above holds for risk-neutral bidders. The presence of residual uncertainty prevents its application to risk-averse bidders.

4 A Sufficient Condition for Full Extraction

The success of the motivating example depended on the fact that conditional on observing all the private information the highest signal is irrelevant for the purposes of estimating the common value. In our notation this stipulation that the maximum of the n types is uninformative means that $\partial_1 H = 0$. When this high type is uninformative, removing it from the pricing function does not bias the price as an estimator of asset value. Hence, we have

Theorem 1 *Uninformativeness of the maximum type is sufficient for the sealed-bid auction with pricing rule P^T (no reserve price, and no payments from losing players) to be an ex-post full-surplus-extracting mechanism in the environment $\mathcal{E}$.*

[10] In other words, V is a nonstochastic function of the types. Several papers have studied "average-value" where $V = \frac{1}{N}(\sum_{i=1}^{N} X_i)$. In that case $P^T(B) = \frac{1}{N}(2\omega_2 + \sum_{i=3}^{N} \omega_i)(B)$.

Proof. By Lemma 4, in the truth-telling equilibrium, the winning player's expected profit is $H(\omega(X)) - H(\omega_2, \omega_{-1}(X))$, which is 0, by stipulation, as is that of every losing player. ■

Corollary 2 *In the No Residual Uncertainty class, if the highest type is uninformative, the pricing rule* $P(B) = H(\omega_2, \omega_{-1})(B)$*, extracts full surplus even in the presence of differentiated risk attitudes.*

Corollary 2 is a simple consequence of Corollary 1.

Multi-Unit Auctions: Briefly, consider an extension in which n players with unit demands compete for $k \leq n$ homogeneous assets which have a common-value of

$$V = V(X) = H(\omega_1, .., \omega_k, \omega_{k+1}, .., \omega_n(X)).$$

A simple extension of Lemma 4 finds truth-telling to be equilibrium bidding when the k highest players obtain one asset each at a common price

$$H(\omega_{k+1}, .., \omega_{k+1}, \omega_{k+1}, .., \omega_n(B)).$$

If the first k order statistics are uninformative, an ex-post full-surplus-extracting mechanism is attained.

This method of approaching full surplus extraction is simple yet powerful. It relies upon a pricing rule that yields truthful revelation in equilibrium without regard for the underlying stochastic relationships across types. By construction it is also ex-post individually rational. The key to its performance is the exploitation of the fact that the winner's information is irrelevant in the estimation of the common value.

5 Necessity of Uninformativeness

Full surplus extraction is neither the usual nor a predictable occurrence. So it is not surprising that a sufficient condition appears restrictive. Our interest in this condition is bolstered by two characterizations regarding its necessity. In other words, is our sufficient condition of uninformativeness too restrictive? The next theorem shows that as long as the goal is ex-post full surplus extraction, the answer to the question is no.

Theorem 2 *In environment* $\mathcal{E}$ *ex-post full surplus extracting mechanisms exist only if* $\partial_1 H = 0$.

The proof of this result can be found in the appendix. Simply stated, the result indicates that in these simple environments informativeness and ex-post rents for bidders are equivalent.

Theorem 3 *If $\partial_1 H > 0$ and types are independent, no interim full-surplus-extracting mechanism exists in environment $\mathcal{E}$.*

(Proof in Appendix.)[11]

The intuition for Theorem 3 follows from its simple proof. Full-surplus extraction requires monotonicity of the allocation probability in types. If the highest signal is informative, then the seller has to reward the holder of that signal for revealing it via an information rent. The classical way of reducing this information rent is the introduction of reserve prices which commit the auctioneer to not selling and therefore prevent full-surplus extraction.[12] One should also note that the monotonicity mentioned above implies a special role for the informational content of the highest signal. Matthews' example where $\partial_1 H > 0$, and $\partial_k H = 0$ for $k > 1$ does not allow us to exploit the uninformativeness of the k^{th}-highest signal in a manner similar to the mechanism described in Section 3. The key to the results above is the fact that the winner's signal needs to be uninformative. The interested reader will find it easy to show that increasing the allocation probability for the k^{th}-highest bidder destroys the incentive compatibility of the mechanism in general.

It is also instructive to relate the examples in section 1 to the issue of alignment of virtual valuations with types, which is the theoretical linchpin of several classic auction papers. The virtual valuation of player i is defined as

$$V_i = v(x_i, x_{-i}) - \frac{1 - F(x_i|x_{-i})}{f(x_i|x_{-i})} \partial_i v(x_i, x_{-i}).$$

It is a dictum of auction theory that expected revenue is increasing in the probability that the asset is sold to the player with the highest virtual valuation.[13] When $\partial_1 H = 0$, as in the original example,

[11] It is noteworthy that statistical independence of players' types and ex-post individual rationality of the mechanism play the unexpected role of substitutes as elements of necessary conditions.

[12] See Myerson [1981], Bulow and Klemperer (1996). The sort of lotteries employed in Crémer and McLean (1988) are unavailable in the independent case.

[13] Myerson [1981] first defines virtual valuations, for the independent-types case, and establishes this result when dependence of valuations on rivals' types is limited to additive "revision effects." Bulow and Klemperer [1996] define virtual valuations, calling them "marginal revenues," for general auction forms. They assume what they call regularity, by which they mean that higher types have higher virtual valuations. Campbell and Levin [2005] cite the dictum without claiming credit for it, apparently treating the demonstration in Bulow and Klemperer [1996] as not depending on the regularity assumption made there. Campbell and Levin discuss methods of decreasing the probability that the highest type attains the asset, for the case when the highest type does not have the highest virtual valuation.

no rival has a higher virtual valuation than the bidder with the highest type. If, further, $\partial_i H > 0$, $\forall i \neq 1$, then the highest type is uniquely the type with the highest virtual valuation. These are the cases where auctions work well, and the simple auction presented here exceptionally well. In the Matthews example, the highest type has the *lowest* virtual valuation. Hence, instruments that are working to reduce the probability that the highest type wins, can potentially increase the expected revenue of the seller.

An interesting example that follows from this theorem is that if the private signals are independent and the common value is the average, full surplus cannot be extracted. It is striking that the seemingly small change to having the common value equal the median allows any standard auction format to extract full surplus.

6 Conclusions

This paper provides a simple yet sharp insight into the optimality of auction mechanisms. For a subset of allocation problems, identified via the condition $\partial_1 H = 0$ above, simple auctions can extract surplus fully and robustly. In particular, they are robust to the stochastic structure of bidders' private information (affiliation and uninformativeness of the highest order statistic are all that is required), and for a further subset, robust to changes in bidders' risk attitudes. The uninformativeness condition allows the auctioneer to achieve the desired result even in environments of independent private information. The mechanisms which achieve this are ex-post individually rational and implementable by simple auctions.

Of course, the set of problems for which estimators based only on the $(n-1)$-lowest order statistics can be constructed is small. This is a direct consequence of the stringency of the ex-post zero surplus requirement. However, any environment with $\partial_1 H > 0$ imposes a cost on the auction designer: either a mechanism [i] must depend in a much more complex way upon detailed information about the distribution of types, or it [ii] must move away from selling to the highest type with probability one. Choice [i] is reflected in the antecedent full-surplus-extracting mechanisms of Crémer and McLean [1985, 1988], McAfee, McMillan and Reny [1989], and McAfee and Reny [1992]; all approach the problem from an interim perspective. Their mechanisms differ from ours in complexity and in dependence on risk neutrality, unlimited liability, and particular details of the information structure.

Choice [ii] is characteristic of the literature on optimal auctions when some surplus remains with the bidders. Myerson [1981] sells to the bidder with the highest virtual valuation, and then only if a reserve price is met (Riley and Samuelson [1981] and Harris and Raviv [1981] find corresponding, less general characterizations). A host of papers depend upon a reserve price, with perhaps the most accessible being Bulow and Roberts [1989]. Harstad and Bordley [1996] sell a single asset to a randomly chosen bidder among the k highest. Bulow and Klemperer [1996] use the $(n-1)$ losing bids to formulate a take-it-or-leave-it-offer (which differs from a reserve price only in how late in the auction it is announced) to the high bidder. Campbell and Levin [2005] consider a specialization of the Matthews example with independent types, and find that posted prices may generate more revenue than a simple auction. For situations so far from the condition found above to be critical, no auction will be a very satisfactory mechanism.

7 Appendix: Proofs

Proof of Lemma 1: By Bayes' rule,

$$v(x_i, x_j) = \frac{\int_0^M vg(v) \prod_{i=1,2} 1_{[x_i \geq v]} 1_{[x_i \leq M]} h_i(x_i) g_i(v) dv}{\int_0^M g(v) \prod_{i=1,2} 1_{[x_i \geq v]} 1_{[x_i \leq M]} h_i(x_i) g_i(v) dv}.$$

Note that $\prod_{i=1,2} 1_{[x_i \geq v]} = 1_{[\min(x_j, x_i) \geq v]}$, so cancelling all common terms which depend only on x_i and $x_j$in numerator and denominator yields

$$v(x_i, x_j) = \frac{\int_0^M vg(v) 1_{[\min(x_j, x_i) \geq v]} \prod_{i=1,2} g_i(v) dv}{\int_0^M g(v) 1_{[\min(x_j, x_i) \geq v]} \prod_{i=1,2} g_i(v) dv}.$$

That both numerator and denominator depend only on $\min\{x_i, x_j\}$ yields the conclusion.

Claim 1 *In the environment $\mathcal{E}_\alpha$, $X^\alpha | v$ is more accurate (in the sense of Lehmann [1988] and Persico [2000]) than $X^\beta | v$ for $\alpha > \beta$.*

Proof. It suffices to show that

$$F^{\alpha^{-1}}[F^\beta[x|v]|v] \text{ is increasing in } v, \text{ for } x \in (v, M).$$

Note that

$$F^{\alpha^{-1}}[x|v] = M - (M - v)(1 - x)^{\frac{1}{\alpha}},$$

so that

$$F^{\alpha^{-1}}[F^{\beta}[x|v]|v] = M - (M - v)\frac{(x - v)^{\frac{\beta}{\alpha}}}{(M - v)^{\frac{\beta}{\alpha}}}.$$

Let $\delta = \frac{\beta}{\alpha} < 1$, then

$$\frac{d}{dv}F^{\alpha^{-1}}[F^{\beta}[x|v]|v] = \frac{d}{dv}\left(-\frac{(M - x)^{\delta}}{(M - v)^{\delta - 1}}\right) > 0,$$

by direct calculation. ■

Proof of Theorem 2: Consider, as in Myerson [1981], the class of mechanisms consisting of a pair of functions $(\pi, y) = \{(\pi_i, y_i), i = 1, ..., n\}$ that define, respectively, the probability of player i receiving the asset and i's payment, as a function of the profile of reported types. The ex-post individual rationality condition (IR) for a mechanism (π, y) to extract full-surplus is

$$\pi_i(x_i, x_{-i})v(x_i, x_{-i}) - y_i(x_i, x_{-i}) = 0.$$

for all (x_i, x_{-i}).

The incentive compatibility condition (IC) is

$$\int_{X_{-i}} (\pi_i(\overline{x}_i, x_{-i})v(x_i, x_{-i}) - y_i(\overline{x}_i, x_{-i}))f(x_{-i}|x_i)dx_{-i} \leq 0$$

for all $\overline{x}_i$ and x_i. The (IR) condition for $\overline{x}_i$ can be written

$$\pi_i(\overline{x}_i, x_{-i})v(\overline{x}_i, x_{-i}) = y_i(\overline{x}_i, x_{-i}),$$

for all x_{-i}. The (IC) condition therefore becomes

$$\int_{X_{-i}} \pi_i(\overline{x}_i, x_{-i})(v(x_i, x_{-i}) - v(\overline{x}_i, x_{-i}))f(x_{-i}|x_i)dx_{-i} \leq 0$$

for all $\overline{x}_i$ and x_i. So whenever $\overline{x}_i \leq x_i$, the monotonicity of v rules out the possibility of a strictly negative integrand:

$$\pi_i(\overline{x}_i, x_{-i})(v(x_i, x_{-i}) - v(\overline{x}_i, x_{-i})) = 0,$$

for all x_{-i}. Note that the monotonicity of $v(.,.)$ also implies, for all x_{-i}:

$$\pi_i(x_i, x_{-i})(v(x_i, x_{-i}) - v(\overline{x}_i, x_{-i})) \geq 0.$$

Combining the last two inequalities:

$$(\pi_i(x_i, x_{-i}) - \pi_i(\overline{x}_i, x_{-i}))(v(x_i, x_{-i}) - v(\overline{x}_i, x_{-i})) \geq 0,$$

for all $\overline{x}_i \leq x_i$. That is, the mechanism's allocation probability function π must satisfy monotonicity. This relationship holds for all i. Full surplus extraction and symmetry guarantee that fixing an arbitrary x_{-i}, there must exist a value of $\overline{x}_i$ such that $\pi_i(\overline{x}_i, x_{-i}) > 0$. Monotonicity of π then implies that $\pi_i(x_i, x_{-i}) > 0$, for all $x_i \geq \overline{x}_i$. In particular, choose $\widetilde{x}_i = \max\{\overline{x}_i, x_{-i}\}$, which ensures that $\pi_i(\widetilde{x}_i, x_{-i}) > 0$. Then $v(x_i, x_{-i}) = v(\widetilde{x}_i, x_{-i})$ for all $x_i \geq \widetilde{x}_i$.

As x_{-i} was arbitrary, this establishes that a necessary condition for ex-post full surplus extraction is $\partial_1 H = 0$.

Proof of Theorem 3: Proof is by contradiction. Using the same notation as above, let $Z(\widehat{x}_i, x_i)$ represent player i's expected payoff under a mechanism (π, y) if he reports $\widehat{x}_i$ and his type is x_i:

$$Z(\widehat{x}_i, x_i) = \int (\pi_i(\widehat{x}_i, x_{-i})v(x_i, x_{-i}) - y_i(\widehat{x}_i, x_{-i}))f(x_{-i})dx_{-i}.$$

A mechanism must satisfy the classic conditions

$$(IR) \quad Z(x_i, x_i) \geq 0, \text{ for all } x_i, \text{ and}$$

$$(IC) \quad Z(x_i, x_i) \geq Z(\widehat{x}_i, x_i), \text{ for all } \widehat{x}_i, x_i.$$

Full surplus extraction (FS) requires that the (IR) constraint be satisfied with equality, and the asset to be sold with probability 1 whenever its value is positive. Since the asset must be sold, it must be possible to fix an i, some (x_i, x_{-i}) so that $\pi_i(x_i, x_{-i}) > 0$ over some set of positive measure in the x_{-i} space, and consider any $\widehat{x}_i > x_i$. Using (FS) and (IC):

$$0 = Z(\widehat{x}_i, \widehat{x}_i) = Z(x_i, x_i) \geq Z(x_i, \widehat{x}_i).$$

This inequality can be rewritten as

$$\int \pi_i(x_i, x_{-i})(v(\widehat{x}_i, x_{-i}) - v(x_i, x_{-i}))f(x_{-i})dt_{-i} \leq 0.$$

A contradiction to $\partial_1 H > 0$ has been reached: $\partial_1 H > 0$ implies that the difference above, $v(\widehat{x}_i, x_{-i}) - v(x_i, x_{-i})$, is everywhere positive, and by construction is being multiplied by a positive probability over some positive measure set.

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