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Abstract

We introduce a tractable model of cheap talk among players located on networks. In our model, a player can send a message to another player if and only if he is linked to him. We derive a sharp equilibrium and welfare characterization which reveals two basic insights. In equilibrium, the willingness of a player to communicate with a neighbor decreases with the number of opponents who communicate with the neighbor. The ex-ante equilibrium welfare of every player increases not only with the number of truthful reports transmitted in the network, but also when truthful reports are more evenly distributed across players. We apply our findings to the analysis of homophily in communities, to organization design, and to the study of endogenous network formation. Communication across communities decreases as communities become larger, and communication may be asymmetric: From large communities to small ones. In our set up, fully decentralized organizations maximize all players' welfare. Further, decentralized networks, where information may flow asymmetrically, endogenously form in equilibrium. Finally, we introduce the possibility of public communication in networks, and identify conditions under which public communication Pareto dominates private communication.

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1 Introduction

This paper provides a tractable model of cheap talk among players located on networks. The network describes how communication takes place among players. Each player can send a message to another player if and only if he is linked to him. The analysis is apparently very intricate. Each player may interact with few others, but when contemplating what to report, he must forecast how his messages will alter his counterparts' decisions, taking into account that they may also receive messages from players who are beyond his circle of contacts. Despite these intricacies, we derive a sharp equilibrium and welfare characterization that uncovers two basic congestion effects. *First*, the willingness of a player to communicate with one of his neighbor declines in the number of players communicating with that neighbor. *Second*, the ex-ante equilibrium welfare of every player increases not only with the number of truthful reports transmitted in the network, but also when truthful reports are more evenly distributed across players.

We demonstrate the wide applicability of our general results in a number of illustrations. First, we provide a new perspective to the study of homophily and segregation in communities, by studying equilibrium information transmission within and across groups with different preferences. Second, we investigate how to organize cheap talk in a minimally connected network, so as to explore the implications of our findings for the study of organization design. Third, we examine equilibrium communication in a model where each player can communicate with any other player at a small cost paid ex-ante. This illustration provides a natural counterpart to the existing literature, which studies endogenous network formation where communication is assumed to be truthful. Finally, we introduce the possibility of public communication in networks and determine under which conditions it Pareto dominates private communication.

Model and General Results. Our model is a natural extension of the celebrated model of cheap talk by Crawford and Sobel (1982). There are n players, and a state of the world θ, which is unknown and uniformly distributed on the interval $[0, 1]$. Each player j simultaneously chooses an action y_j, that influences the utility of all players. If θ were known, player i would like that each player's action were as close as possible to $\theta + b_i$, where b_i can be negative and represents player i's bias relative to the common bliss point θ; specifically, player i's payoff is $-\sum_j (y_j - \theta - b_i)^2$. Each player i is privately informed of a signal s_i, which takes a value of one with probability θ and a value of zero with complementary probability. Before players choose their actions, they may simultaneously report their signals to others. A player can send a message only to the set of players he is linked to– his communication neighbors –. Different network architectures represent, for example, different organization structures, existing social networks, existing diplomatic relations among countries, existing R&D collaborations among firms. We first consider the case in which every player can send a message privately to each player in his neighborhood and we define this possibility *private communication.*

An equilibrium is described by a (directed) network in which each link represents a truthful message, termed *truthful communication network*. We provide necessary and sufficient conditions for a strategy profile to be an equilibrium (Theorem 1): for any truthful link from a player i to a player j, it must be the case that the absolute bias difference $|b_i - b_j|$ is smaller than a simple function inversely related to the number of truthful links having player j as the receiver, i.e., the *in-degree* of j.[1] This result identifies an equilibrium congestion effect: the willingness of a player to communicate with one of his neighbor declines in the number of players communicating with that neighbor. Hence, strategic information transmission does not only depend on the conflict of interest among players. The architecture of the communication network and the allocation of players within the network are also essential to understand strategic communication.

In our setup, a truthful communication network maximizes the *ex-ante* utility of a player if and only if it maximizes the *ex-ante* utility of each one of the other players. We find that each player i's ex-ante payoff induced by a player j's choice is an increasing and concave function of the number of players who truthfully communicate with j. Hence, equilibria can be ranked in the *ex-ante* Pareto sense on the basis of the distribution of in-degree that they generate in their corresponding truthful communication networks (Theorem 2). First, if the in-degree distribution of an equilibrium first order stochastic dominates the in-degree distribution of another equilibrium, the former is more efficient than the latter. Second, if the in-degree distribution of an equilibrium is a mean preserving spread of the in-degree distribution of another equilibrium, then the latter is more efficient than the former. This result identifies our welfare congestion effect: by distributing truthful messages evenly across players it is possible to improve efficiency. We provide an example where an equilibrium with more truthful messages is Pareto dominated by another equilibrium with fewer truthful messages that are distributed more evenly.

While derived in a simple Beta-binomial model, the equilibrium and welfare congestion effects are based on a general feature of statistical Bayesian models. The effect of a signal on the posterior update decreases with the precision of prior. In a multi-player communication model, this implies that the marginal equilibrium effect and welfare value of a players truthfully reported signal decreases with the number of truthful messages received from other players. As we shall later explain in details, this fact drives the congestion effects we uncovered.

Illustrations. We demonstrate the relevance of our general results on equilibrium and welfare in three illustrations. The first illustration studies strategic communication between two commu-

[1] This result generalizes the characterization of Proposition 2 by Morgan and Stocken (2008) who study communication from many players to a single receiver. We discuss the relation of our paper with Morgan and Stocken (2008) at the end of this section.

nities of players. Preferences are the same within groups, but differ across groups. In light of our general results, equilibria with complete truthful communication within groups maximize all players' utilities. In this class of equilibria, we show that there is less communication across groups, as communities grow larger. Further, information transmission across groups may be asymmetric: from large community members to small community members and not viceversa. These results may offer a new perspective on a phenomenon that Lazersfeld and Merton (1954) termed *homophily*: the tendency of individuals to associate with others who are similar to themselves. Homophily has been documented across a wide range of characteristics, such as age, race, gender, religion and occupations, e.g., Fong and Isajiw (2000), Baerveldrs et al. (2004), Moody (2001) and McPherson et al. (2001), whereas Currarini et al. (2009) provides a strategic foundation for these empirical patterns. While the study of homophily has so far focused on symmetric relations such as association and friendship, we consider the asymmetric relation of information transmission. Our results predict that there is less truthful exchange of information across individuals with different characteristics in large-population environments, such as metropolitan areas, than in small-population environments, such as rural towns. Further, we predict that large groups of individuals influence the decisions of small groups by credibly reporting information, while there is less truthful communication from small communities to large communities.

Our second illustration is related to the literature on organization design. It is common in that literature to represent the firm internal organization as a minimally connected communication network, see, e.g., Bolton and Dewatripont (1994), Hart and Moore (2005), Sah and Stiglitz (1986) and Radner (1992) for a survey. In our model, we show that the *ordered line* (i.e. the line where communication links are only built between players with adjacent biases) maximizes the ex-ante utilities of all players, within the class of minimally connected networks. Our model assumes that each player's decision influences every other player's utility. Hence, it describes organizations where decision rights are fully decentralized, such as, for example, cooperatives. In such organizations, our result implies that fully decentralized communication is optimal.

This insight complements the findings of the existing literature, which studies the optimal allocation of decision rights, as well as the optimal communication structure within organizations. Most of the literature investigates the optimal communication structure in environments where communication is assumed to be truthful. Two exceptions are Alonso et al. (2008) and Rantakari (2008). They consider cheap talk in a structure with one central head quarter and two peripheral divisions and find that it can be optimal to decentralize decision rights to the divisions.[2] Beyond the simple structure studied in these two papers, the question of optimal allocation of decision rights within general organization structures remains unanswered. This question may be addressed in a simple

[2] On related topics see Dessein (2002), Harris and Raviv (2005).

extension of our model, where we let the set of decision makers be a possibly proper subset of the set of players.

The last illustration is related to the growing literature on strategic network formation, which originated with the seminal papers of Bala and Goyal (2000) and Jackson and Wolinsky (1996), and was extended in several other articles.[3] In these models, players choose to form costly links with others to access their information. Once a link between two players is established, communication is assumed to be truthful. A robust finding of this literature is that equilibrium networks and efficient networks are very centralized: they generate star-type networks where few players have many connections, while the majority of players have only a few. We show that strategic communication may lead to communication networks with properties that are in stark contrast with the findings of the existing literature. We consider a slight modification of our model in which each player can communicate with an other player at a small cost paid *ex-ante* and where the bias difference across adjacent bias players is constant. We show that a class of equilibrium networks that generalizes the ordered line maximizes all players' utilities. In these equilibria, strategic networks are highly decentralized: all moderately bias players have the same in-degree, while the in-degree declines slowly as the bias becomes more extreme. Further, links may not be reciprocal. In that case, moderate bias players influence the decision of extreme bias players through truthful communication, while extremists do not influence the decision of moderates.

Public Communication. In the last part of the paper we move beyond private communication to study a wide spectrum of different communication modes. In private communication, each player can send a different message to each one of his neighbors. The opposite extreme communication mode is *public communication*, where each player can only send the same message to all his neighbors. In general, a communication mode of a player is described by an exogenous partition of his communication neighborhood, together with the restriction that the player sends the same message to all players belonging to the same element of the partition. We first extend the conditions for equilibrium developed for the private communication case to any arbitrary communication mode (Theorem 3). We then turn to study a key question in the literature of multi-player communication: Whether public or private communication is more efficient in aggregating private information. We answer this question in two setups.

Building on the work of Farrell and Gibbons [1989], our first model considers a scenario where a single sender communicates with a set of decision makers. In the private communication mode, the sender

[3] Extensions have covered, among others, the case of players' heterogeneity (Galeotti at al. (2006), Galeotti (2006), Hojman and Szeidl (2008), Jackson and Rogers (2005)), endogenous information acquisition (Galeotti and Goyal (2008)), investment in links' reliability (Bloch and Dutta (2007) and Rogers (2008)), and investment in the quality of pairwise costly communication (Calvo-Armengol, de Marti and Prat (2009)). For a survey of the literature see Goyal (2007) and Jackson (2008).

only communicates with players of similar biases. In the case of public communication, a sender with a bias similar to the average bias of the receivers has higher incentives to communicate truthfully. Hence, we conclude that private communication Pareto dominates public communication if and only if the sender's bias is extreme relative to the average bias of the receivers. This finding reiterates the observation that certain allocations of players in a communication network help improving the possibility of truthful communication.

The second model characterizes Pareto dominant equilibrium under public communication when each player is both a sender and a receiver of information and the bias difference across adjacent players is constant. We show that as the bias difference increases, the most extreme bias players stop sending a truthful message and for large bias differences only the median bias player truthfully communicates. So, in contrast with the case of private communication, under public communication a centralized communication network may arise, when the level of conflicts between players is sufficiently large. Finally, using the equilibrium characterization of top Pareto equilibria under private and public communication, we numerically calculate the welfare that they generate. We find that, in most cases, whenever public communication is feasible, it Pareto dominates private communication. This result suggests that systems of communication based on public communication technologies lead to more efficient outcomes. For example, this is consistent with the hypothesis that web-applications, such as consumer ratings in eBay (which are public in nature) are more efficient in aggregating information than word of mouth communication.

Related Literature on Communication Games. We have already discussed the relation between our article and the literature on homophily, organization design, and strategic network formation. We now discuss how our paper relates to the literature on information transmission with multiple senders and receivers, which builds on the classical model of cheap talk by Crawford and Sobel [1982]. One strand of the literature introduces multiple senders who, unlike our paper, are perfectly informed about the state of the world (Gilligan and Krehbiel (1987, 1989), Krishna and Morgan (2001a, 2001b), Battaglini (2002), and Ambrus and Takahashi (2008)). The main result of that literature is that full-revealing information transmission can be achieved, under some regularity conditions.

The second strand of the literature considers the case with one receiver and multiple senders who are imperfectly informed about the state of the world. For example, Austen Smith (1993) shows that sequential communication by two senders induces more information transmission than simultaneous communication. Wolinsky (2002) shows that communication becomes more informative when senders communicate among each other before reporting to the decision maker. Battaglini (2004) shows that fully revealing communication can be approximated when either the sender's information becomes perfectly precise, or the number of senders diverges to infinity.

The paper in the multiple senders–one receiver literature which is most closely related to our study is Morgan and Stocken (2008). In fact, we use the same statistical structure, and extend their equilibrium characterization to the case where players are simultaneously senders and receivers according to a communication network structure. Beyond this related starting point, however, the content of the two papers is completely different. Morgan and Stocken (2008) study the statistical properties of private communication from multiple senders to a single receiver applied to the problem of polling, and compare polling with elections. Instead, our paper considers any communication network structure, illustrates the equilibrium implications in the network and organization design literatures, and studies public as well as private communication.

The third strand of literature on cheap talk related to our paper introduces multiple receivers with a single sender. Farrell and Gibbons (1989) compare private and public communication by a sender to two receivers, and identify different effects that may favor the former or the latter. Their results have been generalized by Goltsmann and Pavlov (2008), whereas Koessler and Martimort (2008) extend the analysis to the case of a multi-dimensional state of the world. Caillaud and Tirole (2007) study a model where a sender communicates to committee members, who may collect independent information. Our comparison of public and private communication contributes to this strand of the literature by showing that private communication Pareto dominates public communication if and only if the sender's bias is extreme relative to the receivers' biases.

The study of strategic information transmission when all players are simultaneously senders and receivers is largely unexplored.[4] An exception is a nice, recent paper by Hagenbach and Koessler (2009). The main difference in the analysis is that our equilibrium and welfare congestion effects are absent in their paper. This contrast is substantial because these congestion effects are our main general insights, and drive all our results in illustrations. This difference arises because they consider the following statistical model: With probability one, the state of the world θ equals the sum of each player i's individual binary signal s_i, which are independent across players. As a result, the marginal effect of one truthful message on the action chosen by a receiver is constant in the number of truthful messages received. Hence, their condition for a player to truthfully transmit his information to an opponent does not depend on the communication strategies used by other players. In our standard Bayesian statistical model, instead, the marginal effect of one truthful message on the action chosen by a receiver decreases in the number of truthful messages received. This fact implies the equilibrium and welfare congestion effects that are key to our analysis.

The rest of the paper is organized as follows. Section 2 develops the model under private communication and section 3 develops the results on equilibrium and welfare. Section 4 contains our three

[4] A few papers consider the possibility that players are simultaneously senders and receivers, but focus on information aggregation in committees, e.g., Ottaviani and Sorensen (2001), Austen-Smith and Feddersen (2006) and Visser and Swank (2007).

illustrations. In section 5 we move beyond the case of private communication and compare welfare under private and public communication. Section 6 concludes. All proofs are in an Appendix.

2 Model

The set of players is $N = \{1, 2, ..., n\}$, player i's individual bias is b_i and $b_1 \leq b_2 \leq ... \leq b_n$. The vector of biases $\mathbf{b} = \{b_1, ..., b_n\}$ is common knowledge. The state of the world θ is uniformly distributed on $[0, 1]$. Every player i receives a private signal $s_i \in \{0, 1\}$ on the realization of the state of the world, where $s_i = 1$, with probability θ.

A communication network $\mathbf{g} \in \{0, 1\}^{n \times n}$ is a (possibly directed) graph: i can send his own signal to j whenever $g_{ij} = 1$. We assume that $g_{ii} = 0$ for all $i \in N$. The communication neighborhood of i is the set of players to whom i can send his signal and it is denoted by $N_i(\mathbf{g}) = \{j \in N : g_{ij} = 1\}$. A communication strategy of a player i specifies, for every $s_i \in \{0, 1\}$, a vector $\mathbf{m}_i(s_i) = \{m_{ij}(s_i)\}_{j \in N_i(\mathbf{g})}$. We denote a communication strategy of i as $\mathbf{m}_i : \{0, 1\} \to \{0, 1\}^{|N_i(\mathbf{g})|}$; $\mathbf{m} = \{\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n\}$ denotes a communication strategy profile. The mixed strategy extension of strategy $\mathbf{m}_i$ is μ_i. We let $\hat{\mathbf{m}}_i$ be the messages sent by agent i to his communication neighborhood, and $\hat{\mathbf{m}} = (\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2, ..., \hat{\mathbf{m}}_n)$.[5]

After communication occurs, each player i chooses an action $\hat{y}_i \in \Re$. Let $N_i^{-1}(\mathbf{g}) = \{j \in N : g_{ji} = 1\}$ be the set of players communicating with agent i. Then, agent i's action strategy is $y_i : \{0, 1\}^{|N_i^{-1}(\mathbf{g})|} \times \{0, 1\} \to \Re$; $\mathbf{y} = \{y_1, ..., y_n\}$ denotes an action strategy profile. Given state of the world θ, the payoffs of i facing a profile of actions $\hat{\mathbf{y}}$ is:

$$u_i(\hat{\mathbf{y}}|\theta) = -\sum_{j \in N} (\hat{y}_j - \theta - b_i)^2 .$$

In words, agent i's payoffs depend on how his own action y_i and the action taken by other players is close to his ideal action $b_i + \theta$.[6]

The equilibrium concept is Perfect Bayesian Equilibrium. To avoid dealing with payoff equivalent equilibria and off-path beliefs, we focus on equilibria where each agent i's communication strategy μ_{ij} with an agent $j \in N_i(\mathbf{g})$, may take only two forms: the truthful one, $m_{ij}(s_i) = s_i$ for all s_i, and the babbling one, $\mu(\hat{m}_{ij}|s_i) = 1/2$ for all $\hat{m}_{ij}$ and s_i. Note that in these equilibria, all messages are on

[5] Note that we allow each player to communicate privately with each of his neighbors. In some situations the technology of communication might force players to communicate the same signal to a subset or all their neighbors. Section 5 investigates these possibilities.

[6] Depending on the particular context, a model where only a subset of players takes an action and/or some players are affected only by the actions taken by a subset of the population may be more plausible. Our method of analysis and our results can be simply extended to these settings.

path. With some slight abuse of notation we shall therefore use $\mathbf{m}$ to indicate such communication strategies.

Given the received messages $\hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}$, by sequential rationality, agent i chooses y_i to maximize his expected payoff. Therefore, agent i's optimization reads

$$\begin{aligned} & \max_{y_i} \left\{ -E\left[\sum_{j \in N} (y_j - \theta - b_i)^2 \middle| s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i} \right] \right\} \\ = & \max_{y_i} \left\{ -E\left[(y_i - \theta - b_i)^2 \middle| s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i} \right] \right\}. \end{aligned}$$

Hence, agent i chooses

$$y_i\left(s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}\right) = b_i + E\left[\theta | \, s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}\right], \tag{1}$$

where the expectation is based on equilibrium beliefs: All the messages $\hat{m}_{ji}$ received by an agent j who adopts a babbling strategy are disregarded as uninformative, and all $\hat{m}_{ji}$ received by an agent j who adopts a truthful strategy are taken as equal to s_j. Hereafter, whenever we refer to a strategy profile $(\mathbf{m}, \mathbf{y})$, each element of $\mathbf{y}$ is assumed to satisfy condition 1.

We further note that the agents' updating is based on the standard Beta-binomial model. So, suppose that an agent i holds k signals, i.e. he holds the signal s_i and $k-1$ neighbors truthfully reveal their signal to him. If l out of such k signals equal 1, then the conditional pdf is:

$$f\left(l|\theta, k\right) = \frac{k!}{l!\left(k-l\right)!}\theta^l\left(1-\theta\right)^{(k-l)},$$

and his posterior is:

$$f\left(\theta|l, k\right) = \frac{(k+1)!}{l!\left(k-l\right)!}\theta^l\left(1-\theta\right)^{(k-l)}.$$

Consequently, $f(l|\theta, k) = f(\theta|l, k)/(k+1)$ and $E\left[\theta|l, k\right] = \left(l+1\right)/\left(k+2\right).$

In the first stage of the game, in equilibrium, each agent i adopts either truthful communication or babbling communication with each agent $j \in N_i(\mathbf{g})$, correctly formulating the expectation on the action chosen by agent j as a function of his message $\hat{m}_{ij}$ and with the knowledge of the equilibrium strategies $\mathbf{m}_{-i}$ of the opponents.

3 Results

We first provide a characterization of equilibrium. We then investigate the relationship between equilibrium communication and Pareto efficiency.

3.1 Equilibrium Truthful Networks

Our first result provides necessary and sufficient conditions for equilibrium. A communication network $\mathbf{g}$ together with a strategy profile $(\mathbf{m}, \mathbf{y})$ induces as subgraph of $\mathbf{g}$, in which each link involves truthful communication. We refer to this network as the *truthful network* and denote it by $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$. Formally, $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ is a binary directed graph where $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1$ if and only if $g_{ij} = 1$ and $m_{ij}(s) = s$, for every $s = \{0, 1\}$. Note that $\mathbf{g} = \mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ if and only if $m_{ij} = 1$ for all $i \in N$, $j \in N_i(\mathbf{g})$. Given a truthful network $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$, let $k_i(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ be the number of agents who send a truthful message to i. We term $k_i(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ the in-degree of player i.

Theorem 1 *Consider a communication network* $\mathbf{g}$*. A strategy profile* $(\mathbf{m}, \mathbf{y})$ *is equilibrium if and only if for every* (i, j) *with* $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1$ *the following condition holds:*

$$|b_i - b_j| \leq \frac{1}{2\left[k_j(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) + 3\right]}. \tag{2}$$

Theorem 1 tells us that the ability of a player i to credibly communicate with a player j depends on the difference in their biases and on how many other players credibly communicate with j, i.e. j's in-degree. In particular, it shows that a high player j's in-degree prevents player i from sending a truthful message to j. The intuition for this result stems from the single-peakedness of players' payoff functions. If i babbles, his message is ignored. If i is truthful to j, then j raises his action when i reports $\hat{m}_{ij} = 1$ and lowers it when $\hat{m}_{ij} = 0$. The key observation is that the effect of i's communication on j's action depends on how well informed j is in equilibrium. If j is well informed, i's message affects j's action only slightly, but if j is poorly informed, i's message moves j's choice significantly.

To see that the previous property provides incentives to truthfully communicate to low in-degree players, suppose that $b_i > b_j$ so that i has an incentive to bias j's action upwards. When many opponents truthfully communicate with j, this player is well informed. Hence, a small increase in j's action is always beneficial in expectation to i, as it brings j's action closer to i's expected bliss point. As a result, player i will not be able to truthfully communicate the signal $s_i = 0$. In contrast, when j has a low in-degree, then i's report $\hat{m}_{ij} = 1$ moves j's action upwards significantly, possibly over i's bliss point. In this case, biasing upwards j's action may result in a loss for player i and i will not be willing to deviate from the equilibrium truthful communication strategy. Of course, the impact of j's in-degree on the ability of player i to communicate truthfully with j will depend also on their relative bias difference as illustrated in condition (2).

Building upon Theorem 1, the following remark provides a simple algorithm to determine the conditions under which a given communication strategy is part of equilibrium.

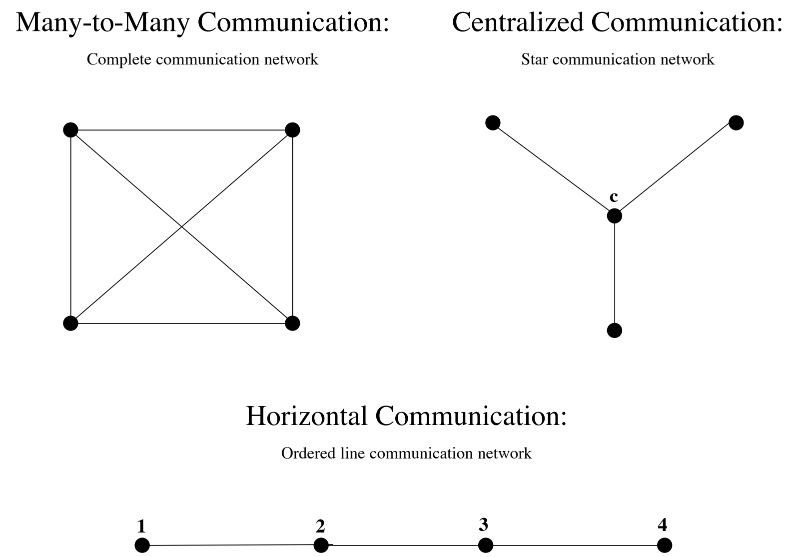

Figure 1: Examples, $n = 4$.

Remark 1 *Consider a communication network* $\mathbf{g}$. *A strategy* $(\mathbf{m}, \mathbf{y})$ *is equilibrium if and only if* $v(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) \leq 1/2$, *where* $v(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))) = \max_{(i,j)\in V(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))} |b_i - b_j|(k_j(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}) + 3)$ *and* $V(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) = \{(i, j) : c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1\}$.

We now provide three examples to illustrate how the architecture of the communication network and the allocation of players (based on their biases) in the network is essential to understand the strategic exchange of information. Figure 1 depicts the communication networks used in these examples; there, an edge connecting i and j means that i can send a message to j and *vice-versa*.

Example 1: Many to Many Communication. A situation where there are no constrains in communication among players is equivalent to assume that the communication network is complete, i.e., $g_{ij} = 1$ for all $i, j \in N$. An equilibrium where every player communicates truthfully with all other players–a fully revealing equilibrium–exists if and only if

$$(b_n - b_1) \leq \frac{1}{2(n+2)}.$$

For every vector of biases, a fully revealing equilibrium fails to exist as long as the size of the population is large enough. Furthermore, it is easy to check that the condition for a fully revealing equilibrium in the complete network is sufficient for the existence of a fully revealing equilibrium in every arbitrary communication network $\mathbf{g}$. ■

Example 2: Centralized Communication. Suppose that there is a player, the center, who can communicate with all other players, whereas each of the other players can only communicate with the center. Formally, consider the star communication network centered on i, i.e. $g_{ij} = g_{ji} = 1$ for all

$j \in N$ and there are no other links. A fully revealing equilibrium exists if and only:

$$\max[b_i - b_1, b_n - b_i] \leq \frac{1}{2(n+2)}.$$

We first note that, similarly to the "many to many communication" example, a fully revealing equilibrium in the star network fails to exist for sufficiently large groups. Secondly, the decision of who occupies the central position is important to determine whether a fully revealing equilibrium exists. The worse situation is when a player with an extreme bias–either player 1 or player n–is the center. In this case the condition for a fully revealing equilibrium in the star network is analogous to the one in the complete network. In contrast, the condition for fully revealing equilibrium existence is most easily met when the center of the star is a player with a moderate bias, i.e., the player $i^* = \arg\min_{i \in N}\{\max[b_i - b_1, b_n - b_i]\}$. ■

Example 3: Horizontal Communication. We now consider a situation where each player can communication with his bias immediate neighbors. That is, the communication network is an ordered line, i.e., $g_{ii+1} = g_{i+1i} = 1$ for all $i = 1...n-1$. In this case player 1 and player n send a message to and receive a message from only one player, while all other players send a message to and receive a message from other two players. Let i^* be the player such that $b_{i^*+1} - b_{i^*} \geq b_{i+1} - b_i$ for all $i = 1, ..., n-1$. In the ordered line network a fully revealing equilibrium exists if and only if

$$b_{i^*+1} - b_{i^*} \leq \frac{1}{10}.$$

In contrast with the star network and the complete network, in the ordered line network a fully revealing equilibrium exists as long as the bias difference of adjacent bias players is not too large; this condition is independent of the size of the group. In fact, it is easy to see that whenever a fully revealing equilibrium exists in the star network, then a fully revealing equilibrium also exists in the ordered line, while the reverse is not necessarily true. ■

3.2 Welfare

We now consider welfare generated in equilibrium. Because of the quadratic utility formulation, if we let $\sigma_j^2(\mathbf{m}, \mathbf{y})$ be the residual variance of θ that player j expects to have once communication has occurred, we can write player i's expected utility in equilibrium $(\mathbf{m}, \mathbf{y})$ as follows:

$$EU_i(\mathbf{m}, \mathbf{y}) = -\left[\sum_{j \in N}(b_j - b_i)^2 + \sum_{j \in N}\sigma_j^2(\mathbf{m}, \mathbf{y})\right].$$

This is a simple extension of the welfare characterization by Crawford and Sobel [1982] to multiple senders and multiple receivers. A nice feature of our model is that we can express the sum of residual variances of θ as a function of a simple property of the equilibrium truthful network, namely its distribution of in-degrees. That is

$$\sum_{j \in N} \sigma_j^2(\mathbf{m}, \mathbf{y}) = \frac{1}{6} \sum_{k=0}^{n-1} \frac{1}{k+3} P(k|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})),$$

where $P(k|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ is the proportion of players with in-degree k in the equilibrium truthful network and $P(\cdot|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) : \{0, ..., n-1\} \rightarrow [0, 1]$ is its in-degree distribution.

We can now state the following result.

Theorem 2 *Consider communication networks* $\mathbf{g}$ *and* $\mathbf{g}'$*. Suppose that* $(\mathbf{m}, \mathbf{y})$ *and* $(\mathbf{m}', \mathbf{y}')$ *are equilibria in* $\mathbf{g}$ *and* $\mathbf{g}'$*, respectively. Equilibrium* $(\mathbf{m}, \mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}', \mathbf{y}')$ *if and only if*

$$\sum_{k=0}^{n-1} \frac{1}{k+3} P(k|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) < \sum_{k=0}^{n-1} \frac{1}{k+3} P(k|\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g}')). \tag{3}$$

A simple inspection of condition 3 allows us to rank equilibria in the Pareto sense based on stochastic dominance relations between the in-degree distributions of their corresponding truthful networks.

Corollary 1 *Consider communication networks* $\mathbf{g}$ *and* $\mathbf{g}'$*. Suppose that* $(\mathbf{m}, \mathbf{y})$ *and* $(\mathbf{m}', \mathbf{y}')$ *are equilibria in* $\mathbf{g}$ *and* $\mathbf{g}'$*, respectively.*

1. *If* $P(k|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ *first order stochastic dominates* $P(k|\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g}'))$ *then equilibrium* $(\mathbf{m}, \mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}', \mathbf{y}')$*.*
2. *If* $P(k|\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g}'))$ *is a mean preserving spread of* $P(k|\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ *then equilibrium* $(\mathbf{m}, \mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}', \mathbf{y}')$

To illustrate the first part of Corollary 1, consider an equilibrium in which i babbles with j and another equilibrium in which the only difference is that player i communicates truthfully with j. The presence of this additional truthful message only alters the equilibrium action of player j. In particular, player j's action becomes more precise and therefore the utility of each player increases. A direct consequence of this result is that if $(\mathbf{m}, \mathbf{y})$ and $(\mathbf{m}', \mathbf{y}')$ are two distinct equilibria in a communication network $\mathbf{g}$ and $\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g})$ is a subgraph of $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$, then equilibrium $(\mathbf{m}, \mathbf{y})$ Pareto dominates equilibrium $(\mathbf{m}', \mathbf{y}')$. In particular, whenever a fully revealing equilibrium in a communication network $\mathbf{g}$ exists, such equilibrium is the unique Pareto dominant equilibrium.

The second part of Corollary 1 allows to compare equilibria that have the same number of truthful communication links. It shows that equilibria in which truthful messages are distributed evenly across players Pareto dominate equilibria where few players receive many truthful messages, while others receive only a few. The reason is that the residual variance of θ associated to every player j is (decreasing) and *convex* in his in-degree. So, suppose we start from a situation where each player receives the same number of truthful messages, say k. Now, decrease by one the number of truthful messages received by i, and increase by one the number of truthful messages received by j. Since the expected residual variance associated to a player is convex in his in-degree, the increase in i's residual variance is larger in absolute value than the decrease in j's residual variance. So, the ex-ante expected utility of each player decreases.

Theorem 2 and Corollary 1 suggest the possibility that an equilibrium that sustains a low number of truthful messages may Pareto dominate an equilibrium with a high number of truthful messages, as long as its messages are distributed more evenly across players. We now develop an example in which this is the case.

Example 4: Even distribution of truthful messages vs. total number of truthful messages. Suppose $n = 5$ and that $b_{i+1} - b_i = \beta$, for $i = 1, 2, 3, 4$. Let $\mathbf{g}$ be a star network and player 3 be the center. When $\beta \leq 1/28$ the following two truthful networks are part of equilibrium. One equilibrium sustains four truthful links: each peripheral player sends a truthful message to the center, and there are no other truthful messages. The in-degree distribution of the equilibrium truthful network is then: $P(0) = 4/5$, $P(4) = 1/5$, and $P(k) = 0$, $k = 1, 2, 3$. The other equilibrium sustains three truthful links: the center sends a truthful message to players 1, 2 and 4, and there are no other truthful messages. The in-degree distribution associated to this equilibrium is: $\tilde{P}(0) = 2/5$, $\tilde{P}(1) = 3/5$ and $\tilde{P}(k) = 0$, $k = 2, 3, 4$.

Note that P and $\tilde{P}$ cannot be ranked in terms of first order or second order stochastic dominance relations. However, applying condition 3, it is easy to check that

$$\sum_{k=0}^{n-1} \tilde{P}(k) \frac{1}{k+3} = \frac{17}{60} < \frac{31}{105} = \sum_{k=0}^{n-1} P(k) \frac{1}{k+3}.$$

Hence, the second equilibrium Pareto dominates the former equilibrium, despite it sustains a lower number of truthful messages. ■

4 Illustrations

We consider three illustrations which may be analyzed within our framework. We first study strategic communication between communities. We then contribute to the study of optimal design of organizations. The last illustration analyzes the endogenous formation of truthful networks. In these illustrations we focus on top Pareto equilibria. A top Pareto equilibrium is an equilibrium which is not Pareto dominated by every other equilibrium.

4.1 Communication across groups

This section studies strategic communication between two communities. The set of players is partitioned in two groups, N_1 and N_2, with size n_1 and n_2, respectively. Without loss of generality, we assume that $n_1 > n_2 \geq 1$. Each member of group 1 has a bias which is normalized to 0; players in group 2 have a bias $b > 0$. Players can send a message to every other player.

We first note that since players within communities have the same preferences, top Pareto equilibrium networks have the property that players in the same group receives the same number of truthful messages. Let k_i be the in-degree of an arbitrary player in group i in a top Pareto equilibrium network. Then $k_i = k_{ii} + k_{ij}$, where k_{ii} is the number of truthful messages that a group i player receives from members of the same group, whereas k_{ij} is the number of truthful messages that a player in group i receives from members of the opposite community. So k_{ii} is a measure of the level of intra-group communication and k_{ij} is a natural measure of the level of cross-groups communication.

Our second observation is that, for every b, there always exists a top Pareto equilibrium in which intra-group communication is complete, i.e., $k_{ii} = n_i - 1$. To see this, consider a top Pareto equilibrium where $k_{11} < n_1 - 1$. This implies that there are two players i' and i'' who belong to group 1 and player i'' does not communicate truthfully with i'. If i' does not receive any truthful message from members of community 2, since i' and i'' have the same preferences, we can construct an equilibrium in which i'' communicates truthfully with i'. In light of Theorem 2 this new equilibrium Pareto dominates the original one, which contradicts our initial hypothesis. Suppose then that there is some player j' in community 2 who communicates truthfully with i'. In this case we can construct another equilibrium in which j' babbles with i', whereas i'' is truthful with i'. Note that the two equilibria generate the same in-degree distribution and therefore they induce the same *ex-ante* utility for all players. Our second observation then follows by iterating these two arguments for every player.

While in the appendix we provide a full characterization of top Pareto equilibrium networks, henceforth we will focus on the subclass where there is complete intra-group communication. We believe this is a natural class of equilibria in this environment. First, these equilibria are robust to the introduction of infinitesimal group-sensitive preferences.[7] Second, this class of top Pareto equilibria coincides with the set of top Pareto equilibria as long as the conflict of interest between the two groups is not too low.[8] Finally, within this class, the only parameters that must be determined are the levels of cross-groups communication. This feature allows us to parsimoniously describe cross-community communication as follows.

[7]For example, we can slightly modify the model so that the utility of every player l in group i is: $-(1+\epsilon)\sum_{l' \in N_i}(\hat{y}_{l'} - \theta - b_l)^2 - \sum_{l' \in N_j}(\hat{y}_{l'} - \theta - b_l)^2$, where ϵ is a small positive constant.

[8]A formal result is in Appendix, see Proposition A.

Consider a top Pareto equilibrium with complete intra-group communication. Since $k_{ii} = n_i - 1$ and since the in-degree within groups is the same across players, condition 2 in Theorem 1 implies that k_{ij} must satisfy

$$k_{ij} \leq \left\lfloor \frac{1}{2b} - n_i - 2 \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x. Furthermore, in view of Theorem 2, in a top Pareto equilibrium network k_{ij} will be the highest possible subject to the above equilibrium condition and that $k_{ij} \leq n_j$ and $k_{ij} \geq 0$. We can now state the following result.

Proposition 1 *In every top Pareto equilibrium network with complete intra-group communication, the levels of cross-communities communication are:*

$$k_{ij} = \max\left\{\min\left\{\left\lfloor \frac{1}{2b} - n_i - 2 \right\rfloor, n_j\right\}, 0\right\}, i, j = 1, 2, i \neq j.$$

If $b < \frac{1}{2(n+2)}$ there is complete cross-communities communication; If $b \in [\frac{1}{2(n+2)}, \frac{1}{2(n_2+3)}]$ the level of truthful communication from group j to group i, k_{ij}, declines with the size of group i and the level of communication from large group 1 to small group 2 is higher than the level of communication from group 2 to group 1, i.e., $k_{21} > k_{12}$; If $b > \frac{1}{2(n_2+3)}$, there is not communication across communities.

The proposition shows that as communities grow larger cross-groups communication declines and that, generally, cross-communities communication is more pervasive from large to small groups, than *vice-versa*. Both effects are a simple consequence of the congestion effect that is illustrated in Theorem 1.

4.2 Optimal Organization Network

This section explores the problem of optimal organization design in a context where decision rights are decentralized and members of the organization have different preferences. As a concrete example, consider a cooperative designing a new product and deciding its characteristics. The profitability of the new product depends on the unknown market demand for products with different characteristics. The different divisions of the cooperative may have counteracting incentives with respect to the characteristics of the product. For example, the engineering division may have a bias to launch a new model with the most advanced technological features, while the marketing division may prefer to design a product that appeals to a larger share of the market. Each division is privately informed of some features of the overall profitability of the new product, and undertakes tasks towards its design and completion. The collective problem of the cooperative is to design the optimal network of communication between the different divisions so as to ensure that as much information as possible is aggregated.

Motivated by the literature on organization design, we represent organizations as minimally connected networks. These are organizations in which there are $n-1$ undirected links and every pair of players is connected via a sequence of links. Our main insight is that the optimal organization network is the line where communication links are only built between agents with adjacent biases, i.e., *the ordered line communication network*. Hence, whenever the incentives of agents in organizations are misaligned, a "decentralized" communication architecture such as the line may outperform more "centralized" communication architectures.

Formally, let $\mathbf{G}$ be the set of undirected networks, i.e., every $\mathbf{g} \in \mathbf{G}$ is such that $g_{ij} = g_{ji}$. We say that there is a path in $\mathbf{g} \in \mathbf{G}$ between i and j if either $g_{ij} = 1$ or there exist players $j_1, ..., j_m$ distinct from each other and from i and j such that $\{g_{ij_1} = g_{j_1 j_2} = ... = g_{j_m j} = 1\}$. A network $\mathbf{g} \in \mathbf{G}$ is connected if there exists a path between every pair of players; $\mathbf{g}$ is minimally connected if it is connected and there exists only one path between every pair of players. Let $\tilde{\mathbf{G}} \subset \mathbf{G}$ be the set of minimally connected networks. The ordered line network is a minimally connected network $\mathbf{g}$ where $g_{ii+1} = 1$ for all $i = 1...n-1$.

Proposition 2 *For every equilibrium* $(\mathbf{m}, \mathbf{y})$ *in organization* $\mathbf{g} \in \tilde{\mathbf{G}}$*, there exists an equilibrium* $(\mathbf{m}^*, \mathbf{y}^*)$ *in the ordered line communication network such that all players' welfare in* $(\mathbf{m}^*, \mathbf{y}^*)$ *is weakly larger than in* $(\mathbf{m}, \mathbf{y})$*.*

The proof of Proposition 2 proceeds in two steps. In the first step we show that for every equilibrium $(\mathbf{m}, \mathbf{y})$ in an arbitrary organization $\mathbf{g} \in \tilde{\mathbf{G}}$, we can construct an equilibrium $(\mathbf{m}', \mathbf{y}')$, which can be sustained in the ordered line network and where the total number of truthful communication links in equilibrium $(\mathbf{m}', \mathbf{y}')$ is the same as in the original equilibrium $(\mathbf{m}, \mathbf{y})$. This step involves substituting truthful messages in equilibrium $(\mathbf{m}, \mathbf{y})$ between non-adjacent players, i.e., (i, j) with $|i-j| > 1$, with truthful communication links between adjacent agents. The basic intuition for which this is possible comes from Theorem 1: i 's ability to credibly communicate with j is higher when the in-degree of j is low and when the absolute bias difference between the two players is small.

As an illustration, suppose that in the original equilibrium i communicates with $i+2$. Since the starting network is minimally connected, it must be the case that either $g_{ii+1} = 0$ or $g_{i+1i+2} = 0$. Suppose $g_{ii+2} = 0$; then we can construct an equilibrium where we delete the link from i to $i+2$ and we let $i+1$ communicate truthfully with $i+2$. Note that since, by hypothesis, i had an incentive to communicate truthfully with $i+2$, also $i+1$ has an incentive to communicate truthfully with $i+2$, because in the two configurations the in-degree of $i+2$ is the same, and the difference in the bias between $i+1$ and $i+2$ is smaller.

The second step shows that from the new equilibrium $(\mathbf{m}', \mathbf{y}')$ it is possible to construct another equilibrium $(\mathbf{m}^*, \mathbf{y}^*)$, which is sustainable in the ordered line network and has the property that the in-degree distribution associated to the original equilibrium $(\mathbf{m}, \mathbf{y})$ is a mean preserving spread of the in-degree distribution induced by the new constructed equilibrium. The result then follows from Corollary 1.

4.3 Endogenous Communication Network Formation

We study the architectural properties of endogenous truthful networks in a model players where have equidistant bias: $b_{i+1} - b_i = \beta$, for all $i = 1, ..., n-1$. Suppose that every player can communicate to all other players. A link $g_{ij} = 1$ forms if and only if i truthfully communicates with j. This would be the case, for example, if a small cost is paid by the receiver of a link, or equivalently, ex-ante, by the sender, i.e. before knowing his signal realization.

In the previous section, we have shown that within the class of minimally connected networks the ordered line maximizes the welfare of all players. The ordered line network has two distinctive features. One feature is localization: communication links are built among players with adjacent bias. The other feature is decentralization: communication links are distributed evenly across players. We now show that there is a natural class of top Pareto equilibrium networks which display these two properties. Further, for a wide range of parameters, this class of equilibria coincides with top Pareto equilibria. We denote this class of equilibria $S^*(\beta)$. We first provide a formal definition of $S^*(\beta)$, we then report its welfare properties and finally describe localization and decentralization features.

Definition 1 *For every β, let $V(\beta) = \max\{V \in N : \beta \leq \frac{1}{2(2V-1+3)V}\}$. Let $S^*(\beta)$ be the set of strategies $(\mathbf{m}, \mathbf{y})$ where $\mathbf{m}$ is constructed as follows:*[9]

1. *Every player $j \in \{V(\beta)+1, ..., n-V(\beta)\}$ receives truthful information from i if $|i-j| < V(\beta)$ and from no players i such that $|i-j| > V(\beta)$; if $\beta > \frac{1}{2(2V(\beta)+3)V(\beta)}$, then j receives truthful information from one and only one player i such that $|i-j| = V(\beta)$; if $\beta \leq \frac{1}{2(2V(\beta)+3)V(\beta)}$, then j receives truthful information from both players i such that $|i-j| = V(\beta)$;*

2. *For all players $j \in \{1, ..., V(\beta)\} \cup \{n-V(\beta)+1, ..., n\}$, j receives truthful information from i if and only if $|i-j| \leq M(j, \beta)$, where $M(j,\beta) = \max\{M \in N : \beta \leq \frac{1}{2(\min[j-1,n-j]+M+3)M}\}$.*

We now show the welfare properties of the S^* equilibrium class. Let $B(\mathbf{m}, \mathbf{y}) = \{\beta : (\mathbf{m}, \mathbf{y}) \in S^*(\beta)\}$ and let $S^* = \cup_\beta S^*(\beta)$.

Proposition 3 *For every β, if $(\mathbf{m}, \mathbf{y}) \in S^*(\beta)$ then $(\mathbf{m}, \mathbf{y})$ is an equilibrium which maximizes the welfare of each player across all equilibria. Further, for every $(\mathbf{m}, \mathbf{y}) \in S^*$ there exists a $B' \subset B(\mathbf{m}, \mathbf{y})$ such that for every $\beta \in B'$ the set of top Pareto equilibria coincides with $S^*(\beta)$.*

[9] Recall that we are assuming that $\mathbf{y}$ satisfies sequential rationality, i.e., condition 1.

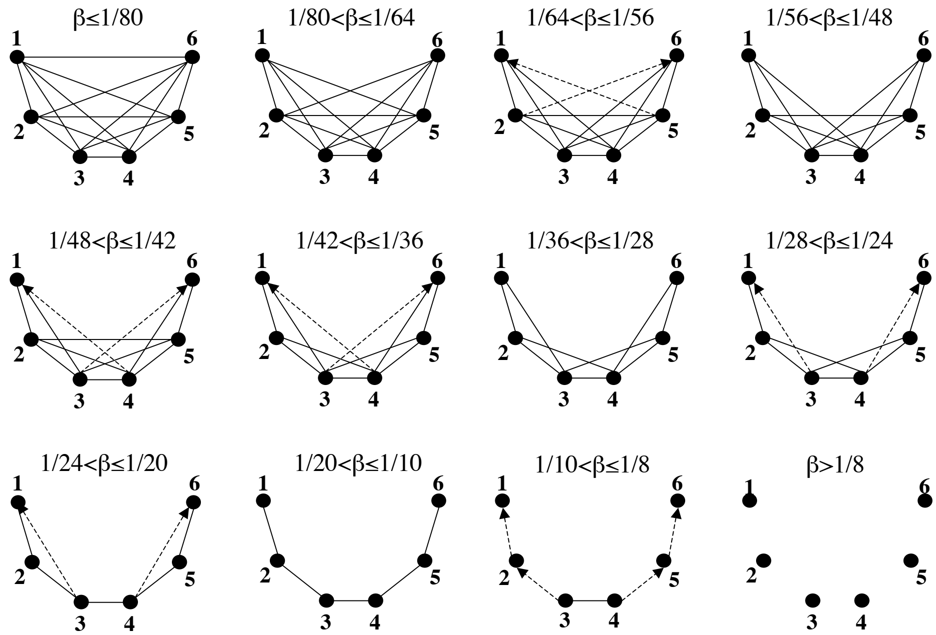

Figure 2: $S^*(\beta)$ Top Pareto Equilibrium Networks, $n = 6$.

Definition 1 determines for which threshold β there is truthful communication along any of the specific links in the class of equilibrium $S^*(\beta)$. For every receiver of a communication link j, it describes the set of players who truthfully communicate with j. The complete characterization of the network architecture requires considering all players j at the same time. As this is evidently cumbersome, we illustrate the $S^*(\beta)$ class of equilibria in figure 2, for different values of β when $n = 6$. In the figure a solid line linking i and j signifies that i and j communicate truthfully with each other; a dash line starting from i with an arrow pointing at j means that only player i truthfully communicates with j.

By inspection of figure 2, three features of S^*-equilibrium networks emerge. Information transmission is localized by bias differences: The individuals communicating to each one of the players have biases sufficiently close to the bias of the player. For example, when $\beta \in (1/36, 1/28]$ each player communicates with individuals of bias distance 2β. This localization property generates very decentralized network architectures. In particular, there is a set of moderate bias players who have the same in-degree, while the in-degree of the other players declines slowly as the bias becomes more extreme. For example, when $\beta \in (1/28, 1/24]$, players $2, 3, 4$ and 5 have in-degree of three, whereas players 1 and 6 have in-degree of two. Finally, information transmission may be *asymmetric*, from players with a moderate bias to players with an extreme bias players, but not *vice-versa*. For example, when $\beta \in (1/42, 1/36]$, player 3 truthfully communicates with 6 but player 6 does not truthfully communicate with 3. The basic intuition for this is that the property of localization implies that players with a moderate bias receive more truthful messages that players with an extreme bias. Because of the equilibrium congestion effect, this reduces the ability of extreme players to send truthful messages to moderate players. The following corollary formalizes these three equilibrium properties.

Corollary 2 *For every β, every $(\mathbf{m}, \mathbf{y}) \in S^*(\beta)$ has the following properties:*

1. *Localization. If a player i communicates with j, then so do all players l such that $|l - j| < |i - j|$;*

2. *Decentralization. Every player $j \in \{V(\beta)+1, ..., n-V(\beta)\}$ has the same in-degree, every player $j \in \{1, ..., V(\beta)\} \cup \{n - V(\beta) + 1, ..., n\}$ has in-degre* $\min[j-1, n-j] + M(j, \beta)$*;*

3. *Asymmetric Communication. For every $i < j \leq \lfloor \frac{n+1}{2} \rfloor$ or $i > j \geq \lfloor \frac{n+1}{2} \rfloor$, it cannot be the case that i truthfully communicates with j and yet j does not truthfully communicate with i. In constrast, it can be the case that j truthfully communicates with i and yet i does not truthfully communicates with j.*

5 General Modes of Communication

We now move beyond the case of private communication. For a given communication network $\mathbf{g}$ and for each agent i, let $\mathcal{N}_i(\mathbf{g})$ be a partition of the communication neighborhood of i, with the interpretation that player i must send the same message $m_{i,J}$ to all agents $j \in J$, for any group of agents $J \in \mathcal{N}_i(\mathbf{g})$. We refer to $\mathcal{N}_i(\mathbf{g})$, as the communication mode available to i. The strategy space in this model can be constructed from the strategy space of our basic model, imposing the following restriction: if $j \in J$ and $j' \in J$, then $m_{ij} = m_{ij'}$. The rest of the model is unchanged. Note that in a truthful network $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$, $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = c_{ij'}(\mathbf{y}, \mathbf{m}|\mathbf{g})$ for all j, j' that belong to the same element J of the partition $\mathcal{N}_i(\mathbf{g})$.

When for every player i the partition $\mathcal{N}_i(\mathbf{g})$ of $N_i(\mathbf{g})$ is composed of singleton sets, we obtain the model we have considered so far. In that case, each agent has the possibility to communicate privately with each of his neighbors– *private communication* –. The opposite polar case is when, for each player i, we have the trivial partition $\mathcal{N}_i(\mathbf{g}) = \{N_i(\mathbf{g})\}$. This is a model of *public communication*: each player i sends a public message to his communication neighbors.

The next result extends Theorem 1 to arbitrary communication modes.

Theorem 3 *Consider a communication network* $\mathbf{g}$ *and a collection of communication modes* $\{\mathcal{N}_i(\mathbf{g})\}_{i \in N}$. *The strategy profile $(\mathbf{m}, \mathbf{y})$ is an equilibrium if and only if for every truthful message from a player i to a group of players $J \in \mathcal{N}_i(\mathbf{g})$, i.e. for all i, J such that $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1$ for all $j \in J$, the following condition holds:*

$$\left| \sum_{j \in J} \frac{b_j - b_i}{k_j(\mathbf{m}, \mathbf{y}|\mathbf{g}) + 3} \right| \leq \sum_{j \in J} \frac{1}{2\left(k_j(\mathbf{m}, \mathbf{y}|\mathbf{g}) + 3\right)^2}. \tag{4}$$

It is easy to verify that in the case of private communication the equilibrium conditions 4 in Theorem 3 degenerate to the conditions 2 in Theorem 1. We also note that the welfare results we derived for private communication extends to any general mode of communication.

Recall that under private communication the ability of a player to credibility communicate with a neighbor decreases with the neighbor's in-degree. The following example shows that for arbitrary modes of communication this may not be the case.

Example 5. Let $N = \{1, 2, 3, 4\}$ and $\mathbf{b} = \{-1, 0, \beta, \beta + c\}$, where $\beta > 1$ and c is a small positive constant. Consider the following communication network $\mathbf{g}$: $g_{21} = g_{23} = g_{43} = 1$ and no other communication links. Suppose also that player 2 must send the same message to his neighbors $\{1, 3\}$. Let us suppose that player 4 babbles to player 3. Then, the communication strategy in which player 2 sends a truthful public message to $\{1, 3\}$ is equilibrium whenever $\beta \leq 5/4$. Next, consider that player 4 communicates truthfully with 3 (which is always possible in equilibrium for sufficiently small c). In this case the condition for player 2 to be able to send a truthful public message to $\{1, 3\}$ is

$$\left|\frac{4\beta - 5}{20}\right| \leq \frac{1}{32} + \frac{1}{50},$$

which holds whenever $\beta \leq 241/160$.

Hence, whenever $\beta \in (5/4, 241/160]$ player 2 is able to report a truthful public message to his neighbors $\{1, 3\}$ only if player 4 also communicates truthfully with 3.

An intuitive explanation of this result is as follows. Suppose that the information held by both player 1 and 3 consists only in their signal. Then, for $\beta > 5/4$, player 2 cannot truthfully report a high signal because he would like to deviate by biasing downwards the action of players 1 and 3. Indeed, since the bias difference between 3 and 2 is sufficiently larger than the one between 1 and 2, player 2 is willing to suffer the loss incurred by lowering the action of player 1 in order to achieve the gain obtained by lowering player 3's action. Instead, suppose that player 3 is more informed than player 1, because he receives a truthful message from player 4. Now, if player 2 misreports a high signal, he lowers player 3's action less than he lowers player 1's action. This reduces the gain for misreporting, without changing the loss. When $\beta \leq 241/160$, the loss reduction is sufficiently large to deter player 2 from misreporting a high signal. ■

In what follows we investigate the efficiency properties of private communication relative to public communication. We first focus on the case of a single sender and multiple receivers. We then allow all players to simultaneously act as senders and receivers.

5.1 One Sender and Multiple Receivers.

Farrell and Gibbons [1989] compare the implication for welfare of private and public communication in a cheap talk game where one sender can communicate with two receivers, and each receiver chooses between two actions. Their study shows the existence of two conflicting effects. The first effect favors private communication. In this case, a sender can discriminate among her receivers and send a truthful message only to those who have incentives more closely aligned. The other effect favors public communication. In that case, the sender's incentive to misreport a low signal in order to raise the action of lower bias players is tempered by the loss incurred from the increase in actions of all higher bias players. We now investigate under which conditions private communication fares better than public communication in a well structured environment with a single sender and multiple receivers. We will show that a key element to pin down the implication for welfare of the two modes of communication is the sender's bias relative to the average bias of the receivers.

We begin by formulating a general result on public communication by a single sender to many receivers. Consider a communication network $\mathbf{g}$ where $g_{sj} = 1$ for all $j \in N \setminus \{s\}$, and there are no other links, and suppose that player s can only send a common message to all other players: player s broadcasts a message to his audience, composed of all other players. Without loss of generality, let $b_1 \geq 0$ and let $\hat{b}$ be the average bias in the population, i.e., $\hat{b} = \sum_{i \in N} b_i / n$.

Corollary 3 *Suppose that sender s can only send a common message to all other players. An equilibrium where the public message of s is truthful exists if and only if*

$$|\hat{b} - b_s| \leq \frac{n-1}{8n}$$

Corollary 3 illustrates how the credibility of public messages depends on the size of the audience and its bias composition. First, a sender with a bias similar to the average bias of the receivers has higher incentives to communicate truthfully. Second, an increase in the number of receivers (holding constant the average bias) increases the incentives of the sender to communicate truthfully.

For the sake of expositional clarity, we now specialize the model. Suppose that $n + 1$ players have equally distant biases in the interval $[0, 1/2]$. Hence for all players $l = 0, 1, ..., n$, the associated bias is $b_l = \frac{l}{n}$. Player s is selected to be the sender, the other players are receivers. Under private communication, in equilibrium s communicates with every player l such that $|b_s - b_l| \leq 1/8$. Hence, there is some information privately exchanged in the game for all $n \geq 4$, but the largest number of truthful messages that is possible to send is $2 \lfloor n/4 \rfloor \leq n/2 < n$. The equilibrium with public communication is such that player s truthfully communicates with all n players if and only if $\left| b_s - \hat{b} \right| \leq n/(8(n+1))$, where $\hat{b} = 1/4$. Solving out, s communicates if and only if $1/4 + n/(8(n+1)) \leq b_s \leq 1/4 - n/(8(n+1))$.

Note that the function $n/(8(n+1)) = \frac{1}{12}$ for the minimal value $n = 2$ and $n/(8(n+1)) \to 1/8$ for $n \to \infty$.

These observations are summarized in the following proposition.

Proposition 4 *Public communication dominates private communication if the sender is* moderate, *i.e.* $1/4 + n/(8(n+1)) \leq b_s \leq 1/4 - n/(8(n+1))$. *On the other hand, for* $n \geq 4$, *private communication dominates public communication if the sender is extremist i.e.* $|b_s - 1/4| > n/(8(n+1))$.

The result that public communication dominates private communication whenever the sender has a moderate bias can be generalized beyond the case of equidistant biases. Indeed, suppose that the number of players is sufficiently large and that the distribution of biases is sufficiently diverse. Then there will be players i for whose bias b_i the condition in Corollary 3 fails. Yet, there are some other players j such that $|b_i - b_j|$ is sufficiently small that some truthful private communication takes place. When the sender is one of such extreme bias players, then private communication dominates public communication. Conversely, there will also exist moderate bias players, whose bias b_i is sufficiently close to the average bias $\hat{b}$ so that the condition in corollary 3 holds, and yet, for some extreme bias player j the bias difference $|b_i - b_j|$ is too large for i to privately communicate to j. When the sender is one of such moderate bias players, then public communication dominates private communication.

5.2 Endogenous communication network formation under public communication

We now compare the welfare of private and public communication in a model analogous to the one studied in section 4.1: each player can communicate to all other individuals and players have equidistant bias, i.e., $b_{i+1} - b_i = \beta$, for all $i = 1, ..., n-1$. Specifically, we compare the welfare properties of top Pareto equilibria under private communication and under public communication for different values of β. Because all top Pareto equilibria yield the same ex-ante utility to all players, it is sufficient to calculate the welfare of well-defined subclasses of top Pareto equilibria. So, for any β, the welfare induced by private communication is calculated as the sum of expected residual variances $\sum_{j \in N} \sigma_j^2(\mathbf{m}, \mathbf{y})$ associated to an equilibrium $(\mathbf{m}, \mathbf{y})$ that belongs to the equilibrium class $S^*(\beta)$ calculated in section 4.1. The welfare induced by public communication is calculated as the sum of expected residual variances associated to equilibria that belong to a subclass of top Pareto equilibrium that we describe in proposition 5 below, where we use the functions f and g defined as follows.

For any index $l = 1, ..., \lfloor n/2 \rfloor$,

$$f(l,n) = \frac{\frac{n-2l+1}{2(n-2l+4)^2} + \frac{l-1}{(n-2l+5)^2}}{(n-2l+1)\left[\frac{n+2-2l}{2(n-2l+4)} + \frac{l-1}{n-2l+5}\right]}$$

and

$$g(l,n) = \frac{\frac{n-2l}{2(n-2l+3)^2} + \frac{2l-1}{2(n-2l+4)^2}}{(n-2l+1)\left[\frac{n-2l}{2(n-2l+3)} + \frac{l}{n-2l+4}\right]}.$$

For $l = 0$ the function g is defined by $g(0,n) = 0$ for all n.

Proposition 5 *Suppose that $b_{i+1} - b_i \equiv \beta$ for all players $i = 1, ..., n-1$. For any $l = 1, ..., \lfloor n/2 \rfloor$, if $g(l-1,n) < \beta \leq f(l,n)$ there exists a top Pareto equilibrium where the players who communicate truthfully are $\{l, ..., n-l+1\}$; if $f(l,n) < \beta \leq g(l,n)$ there exists a top Pareto equilibrium where the players who communicate truthfully are $\{l, ..., n-l\}$. If $\beta > g(\lfloor n/2 \rfloor, n)$, and n is even, no player truthfully communicates in equilibrium, if n is odd, player $(n+1)/2$ truthfully communicates in a top Pareto equilibrium.*

We now use Proposition 5 and Proposition 3 to compare numerically the welfare of top Pareto equilibria under private and public communication, for different values of β and different number of players.

Figure 3 summarizes some of these numerical simulations, which we now comment. The first two figures consider the case of an even number of players, $n = 6$ and $n = 16$. In this case public communication always dominates private communication in terms of welfare. The last two figures take the case of an odd number of players, $n = 5$ and $n = 15$. Here, for low and high values of β public communication is still always dominant. For intermediate values of β private communication only occasionally outperforms public communication. Overall, we conclude that public communication fares better than private communication in the model with equidistant bias.

6 Conclusion

This paper provides a tractable model to study multi-person environments where players can strategically transmit their private information to individuals who are connected to them in a communication network. An important insight that emerges from the analysis is that whether truthful communication can be sustained in equilibrium or not does not only depend on the conflict of interest between players, but also on the architecture of the communication network and the allocation of players within the network. In particular, the willingness of a player to communicate with another individual decreases with the number of players communicating with the individual. Another general insight is that, in a multi-person environment, equilibrium welfare does not only depend on the amount of information aggregated in the network, but also on how evenly truthful information transmission is distributed across players. These congestion effects are driven by a general feature of statistical Bayesian models: the effect of a signal on the posterior update decreases with the precision of prior.

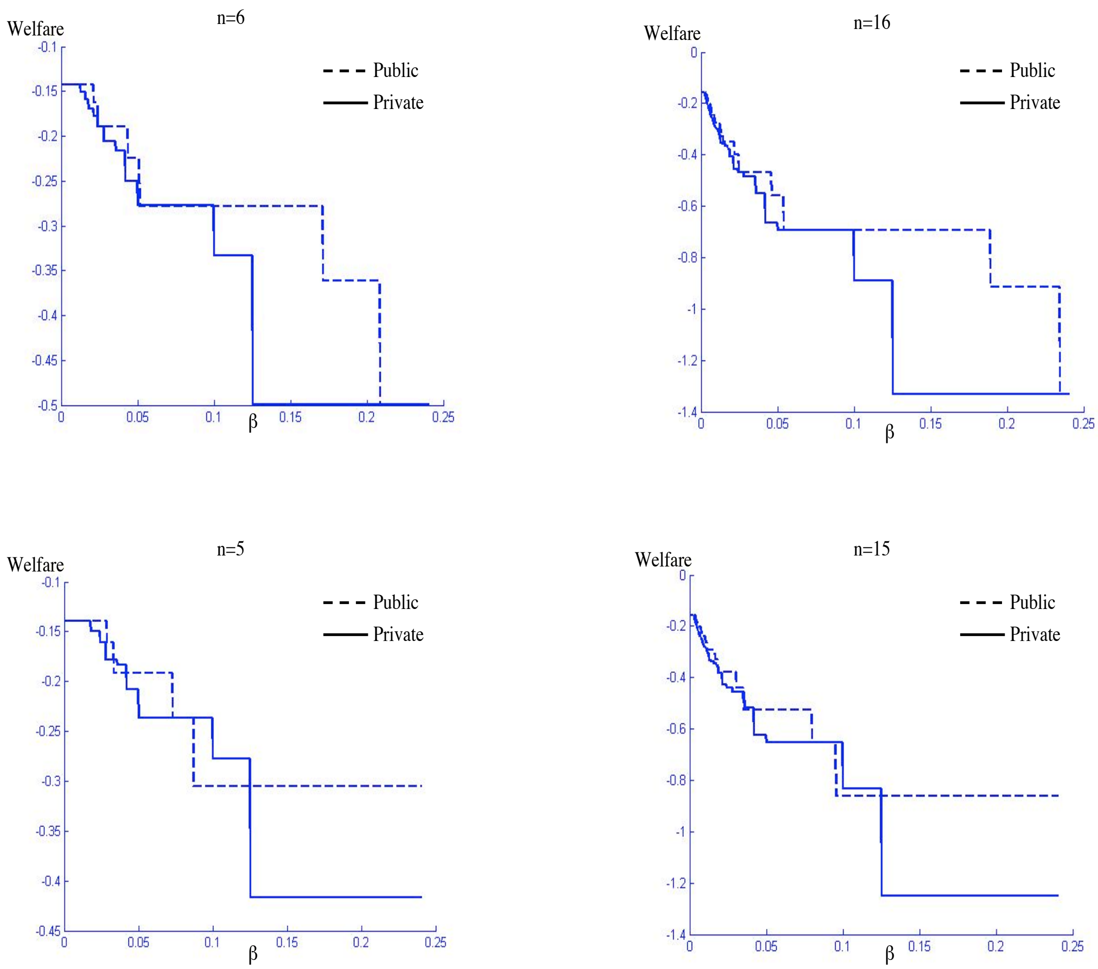

Figure 3: Welfare under Private Communication *vs.* Public Communication

In a multi-player communication model, this implies that the marginal equilibrium effect and welfare value of a players truthfully reported signal decreases with the number of truthful reports received from other players.

We demonstrate the wide applicability of our general results in a number of illustrations. We provide a new perspective to the study of homophily and segregation in communities, by studying equilibrium information transmission within and across groups with different preferences. We show that communication across communities decreases as communities become larger, and that communication may be asymmetric: From large communities to small ones. We investigate how to organize cheap talk in a minimally connected network, so as to explore the implications of our findings for the study of organization design. In our set up, fully decentralized organizations maximize all players' welfare. We examine equilibrium communication in a model where each player can communicate with any other player at a small cost paid ex-ante, thereby providing a natural counterpart to the existing literature, which studies endogenous network formation where communication is assumed to be truthful. In our model, decentralized networks, where information may flow asymmetrically, endogenously form in equilibrium.

Finally, we introduce the possibility of public communication in networks and determine under which conditions it Pareto dominates private communication. This is particularly important because of the emergence of many web-applications which aim at aggregating information among users. This advance in new technologies requires an understanding of how different designs of such institutions shape the incentive of players to transmit their private information, and, in turn, to determine the efficiency of the system.

Our model can be extended in several directions. A particularly promising extension consists in letting the set of decision makers be a possibly proper subset of the set of players. In fact, our equilibrium and welfare results (Theorems 1 and 2) still hold in this extended environment, with minimal modifications. This extension would allow to study how the equilibrium communication strategies and welfare change when varying the allocation of decision making within a network of players. As this question is a main concern of the organization design literature, such an extension may uncover further exciting insights on the optimal organization of the firms.

Appendix A.

Proof of Theorem 1. Theorem 1 is a special case of theorem 3, in which for every $i \in N$ the partition $\mathcal{N}_i(\mathbf{g})$ of i's communication neighborhood, $N_i(\mathbf{g})$, is composed of singleton sets. ■

Proof of Theorem 2. Assume $(\mathbf{m}, \mathbf{y})$ is equilibrium in communication network $\mathbf{g}$. Select an arbitrary player i. The ex-ante expected utility of i is:

$$\begin{aligned} EU_i(\mathbf{m},\mathbf{y}) &= -E\left[\sum_{j=1}^{n}(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c}))+1}\right] && (5)\\ &= -\sum_{j=1}^{n}E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right], && (6)\end{aligned}$$

where, with some abuse of notation, $k_j(\mathbf{c})$ indicates j's in-degree in truthful network $\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})$.

Consider an arbitrary j with in-degree $k_j(\mathbf{c})$ and let l be the number of digits equal to one in a realized information vector $\{0,1\}^{k_j(\mathbf{c})+1}$. Then, we obtain:

$$\begin{aligned} E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right] &= \int_0^1\sum_{l=0}^{k_j(\mathbf{c})+1}\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]+b_j-\theta-b_i\right)^2 f(l|k_j(\mathbf{c})+1,\theta)d\theta\\ &= \int_0^1\sum_{l=0}^{k_j(\mathbf{c})+1}\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]+b_j-\theta-b_i\right)^2 \frac{f\left(\theta|l,k_j(\mathbf{c})+1\right)}{k_j(\mathbf{c})+1+1}d\theta,\end{aligned}$$

where the second equality follows from $f(l|k_j(\mathbf{c})+1,\theta)=f(\theta|l,k_j(\mathbf{c})+1)/(k_j(\mathbf{c})+2)$. Let $\Pi=\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)^2$. Then we have:

$$\begin{aligned} & E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right]\\ =\; & \frac{1}{k_j(\mathbf{c})+2}\int_0^1\sum_{l=0}^{k_j(\mathbf{c})+1}\Big(\Pi+(b_j-b_i)^2+2(b_j-b_i)\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)\Big)f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\\ =\; & (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\int_0^1\sum_{l=0}^{k_j(\mathbf{c})+1}\left(\Pi+2(b_j-b_i)\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)\right)f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\right]\\ =\; & (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\sum_{l=0}^{k_j(\mathbf{c})+1}\left(\int_0^1\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)^2 f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\right)\right].\end{aligned}$$

Next, let $V(\theta|l,k)$ be the variance of a beta distribution of parameters l and k, i.e.,

$$V(\theta|l,k)=\int_0^1(E[\theta|l,k]-\theta)^2 f(\theta|l,k)d\theta.$$

It is well known that:

$$V(\theta|l,k)=\frac{(l+1)(k-l+1)}{(k+2)^2(k+3)}.$$

Hence,

$$\begin{aligned}
E\left[(y_j-\theta-b_i)^2\,|\{0,1\}^{k_j(\mathbf{c})+1}\right] &= (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\sum_{l=0}^{k_j(\mathbf{c})+1}V\left(\theta|l,k_j(\mathbf{c})+1\right)\right]\\
&= (b_j-b_i)^2+\sum_{l=0}^{k_j(\mathbf{c})+1}\frac{(l+1)\left(k_j(\mathbf{c})-l+2\right)}{\left(k_j(\mathbf{c})+2\right)\left(k_j(\mathbf{c})+3\right)^2\left(k_j(\mathbf{c})+4\right)}\\
&= (b_j-b_i)^2+\frac{1}{6(k_j(\mathbf{c})+3)}.
\end{aligned}$$

We can then write the ex-ante expected utility of player i in equilibrium $(\mathbf{m},\mathbf{y})$ as follows:

$$\begin{aligned}
EU_i(\mathbf{m},\mathbf{y}) &= -\sum_{j=1}^{n}\left[(b_j-b_i)^2+\frac{1}{6(k_j(\mathbf{c})+3)}\right]\\
&= -\sum_{j=1}^{n}(b_j-b_i)^2-\frac{1}{6}\sum_{j=1}^{n}\frac{1}{k_j(\mathbf{c})+3}\\
&= -\sum_{j=1}^{n}(b_j-b_i)^2-\frac{1}{6}\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|}{k+3},
\end{aligned}$$

where $|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|$ is the set of players with in-degree k, i.e., $I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})=\{i\in N:k_i(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))=k\}$. Therefore,

$$EU_i(\mathbf{m},\mathbf{y})\geq EU_i(\mathbf{m}',\mathbf{y}')$$

if and only if:

$$\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|}{k+3}\leq\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}'))|}{k+3},$$

which is equivalent to

$$\sum_{k=0}^{n-1}P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))\frac{1}{k+3}\leq\sum_{k=0}^{n-1}P(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}'))\frac{1}{k+3}.$$

This concludes the proof of Theorem 2. ■.

Proof of Corollary 1. The proof of Corollary 1 follows from standard arguments of stochastic dominance and therefore the details are omitted. ■.

Appendix B. Proof of Proposition 3. The proof proceeds in three steps. The first step shows that a strategy profile $(\mathbf{m},\mathbf{y})\in S^*(\beta)$ is equilibrium. The second step shows that $S^*(\beta)$ is a subset of the set of top Pareto equilibria. The last step shows the second part of the proposition.

Step I. Let $(\mathbf{m},\mathbf{y}) \in S^{*}(\beta)$. We show that $(\mathbf{m},\mathbf{y})$ is an equilibrium. Select $l = 1,...,2j-n-1$. When $\beta \leq [2(j-l)(n-l+3)]^{-1}$, the strategy profile such that the $n-l$ players $\{l,...,j-1,j+1,...,n\}$ truthfully communicate with j is part of an equilibrium. Indeed, as $j-l > n-j$, i.e., $l < 2j-n$, it follows that

$$\{l\} = \arg \max_{i\in\{l,...,j-1,j+1,...,n\}} |b_i - b_j| \text{ and that } j-l = \max_{i\in\{l,...,j-1,j+1,...,n\}} |b_i - b_j|,$$

and Theorem 1 implies that the requirement for the strategy profile to be an equilibrium is exactly $\beta \leq [2(j-l)(n-l+3)]^{-1}$.

Next, select $l = 2j-n,...j-1$. For $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$ the profile such that the $2(j-l)$ players who truthfully communicate with j are $\{l,...,j-1,j+1,2j-l\}$ or $\{l+1,...,j-1,j+1,2j-l+1\}$ is part of an equilibrium. Indeed, suppose the players who truthfully communicate with j are $\{l,...,j-1,j+1,2j-l\}$ (the other case being symmetric). As $j-l = (2j-l)-j$ it follows that:

$$\{l,2j-l\} = \arg \max_{i\in\{l,...,j-1,j+1,...,2j-l\}} |b_i - b_j| \text{ and that } j-l = \max_{i\in\{l,...,j-1,j+1,...,2j-l\}} |b_i - b_j|,$$

and Theorem 1 implies that the requirement for the profile to be an equilibrium is exactly $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$.

To conclude the first step, note that for $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$, the profile such that the $2(j-l)-1$ players who truthfully communicate with player j are $\{l+1,...,j-1,j+1,2j-l\}$ is part of an equilibrium. Indeed, as $j-l = (2j-l)-j$ it follows that:

$$\{2j-l\} = \arg \max_{i\in\{l+1,...,j-1,j+1,...,2j-l\}} |b_i - b_j| \text{ and that } j-l = \max_{i\in\{l+1,...,j-1,j+1,...,2j-l\}} |b_i - b_j|,$$

and theorem 1implies that the requirement for the strategy profile to be an equilibrium is exactly $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$.

Step II. We now show that $(\mathbf{m},\mathbf{y}) \in S^{*}(\beta)$ is a top Pareto equilibrium. We start by noting that for every $l = 1,...,2j-n-1$ if a set of players C_j communicates with j and $|C_j| = n-l$, then $\beta \leq [2(j-l)(n-l+3)]^{-1}$. Indeed, since $n-l$ players communicate with j, there must be a player $i \in C_j$ such that $i \leq l$, and the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(n-l+3)]^{-1} \leq [2(j-l)(n-l+3)]^{-1},$$

where the inequality follows because $i \leq l$. Because $[2(j-l)(n-l+3)]^{-1}$ increases in l, it follows that if a set of players C_j truthfully communicates with j and $|C_j| = n-v \geq n-l$, then $\beta \leq [2(j-v)(n-v+3)]^{-1} \leq [2(j-l)(n-l+3)]^{-1}$. Hence, for every $l = 1,...,2j-n-1$, if $\beta > [2(j-l)(n-l+3)]^{-1}$, then there is no equilibrium where $n-l$ players truthfully communicate to

j. So, the proposed profile where $n-l-1$ players truthfully communicate to j, achieves the maximal number of communication links to player j and it is part of a top Pareto equilibrium.

We now turn to the case of $l = 2j-n, ..., j-1$, and show equilibrium communication by $2(j-l)$ players to j requires that $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$. To see this, suppose that a set C_j of $2(j-l)$ players communicate with j. Then, there must be a player $i \in C_j$ such that $|j-i| \geq j-l$. Consequently, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)+3)]^{-1} \leq [2(j-l)(2(j-l)+3)]^{-1}.$$

Because $[2(j-l)(2(j-l)+3)]^{-1} < [2(j-l)(2(j-l)-1+3)]^{-1}$ holds for all l and the fact that $[2(j-l)(2(j-l)+3)]^{-1}$ increases in l, we can conclude that communication by at least $2(j-l)$ players to j requires $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$. Hence, for $\beta > [2(j-l)(2(j-l)+3)]^{-1}$, the specified strategy profile where $2(j-l)-1$ players communicate with j, is part of a top Pareto equilibrium.

To conclude this second step we need to show that equilibrium communication by $2(j-l)-1$ players to j requires that $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$. Indeed, if a set C_j of $2(j-l)-1$ players communicates with j, then, there must be a player $i \in C_j$ such that $|j-i| \geq j-l$. Then, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)-1+3)]^{-1} \leq [2(j-l)(2(j-l)-1+3)]^{-1}.$$

Because $[2(j-l)(2(j-l)-1+3)]^{-1} < [2(j-(l+1))(2(j-(l+1))+3)]^{-1}$ holds and the fact that $[2(j-l)(2(j-l)-1+3)]^{-1}$ increases in l, we can conclude that communication by $2(j-l)-1$ players with j requires $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$. Hence, for $\beta > [2(j-l)(2(j-l)-1+3)]^{-1}$, the specified strategy profile where $2(j-(l-1))$ players communicate with j, is part of a top Pareto equilibrium.

Step III. We now show the last part of the proposition. We have already shown that for any $l = 1, ..., 2j-n-1$ if a set of players C_j communicates to j and $|C_j| = n-l$, then $\beta \leq [2(j-l)(n-l+3)]^{-1}$. If the $n-l$ players who communicate are not $\{l, ..., j-1, j+1, ..., n\}$, then there must be a player $i < l$, and the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(n-l+3)]^{-1} < [2(j-l)(n-l+3)]^{-1},$$

where the inequality follows because $i < l$. Therefore for every configuration where the $n-l$ players who communicate are not $\{l, ..., j-1, j+1, ..., n\}$ there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ (recall $\mathbf{m}, \mathbf{y} \in S^*(\beta)$) such that such configuration is not an equilibrium.

Consider now the case of $l = 2j-n, ...j-1$. Suppose that a set C_j of $2(j-l)$ players communicates with j, other than the specified configurations. Then, there must be a player $i \in C_j$ such that $|j-i| > j-l$, so that the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)+3)]^{-1} < [2(j-l)(2(j-l)+3)]^{-1}.$$

Consequently, there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ such that such configuration is not an equilibrium.

Finally, if a set C_j of $2(j-l)-1$ players communicate with j, other than the specified configurations, then, there must be a player $i \in C_j$ such that $|j-i| > j-l$. Hence, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)-1+3)]^{-1} < [2(j-l)(2(j-l)-1+3)]^{-1}.$$

Again there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ such that such configuration is not an equilibrium. This concludes the proof of Proposition 3. ■

Proof of Corollary 2. The first part of the corollary is obvious. We prove the second part. For all β, we show that $M(j,\beta)$ is weakly decreasing in j for $j \in \{1, ..., V(\beta)\}$. In fact, solving $2\beta(j-1+M+3)M-1=0$, we obtain that $M(j,\beta) = \left\lfloor \frac{1}{2}\left(-(j+2)+\sqrt{2/\beta+(j+2)^2}\right)\right\rfloor$ and that

$$\frac{dM(j,\beta)}{dj} = \frac{1}{2}\left(\frac{j+2}{\sqrt{2/\beta+(j+2)^2}}-1\right) < 0$$

Consequently, $d(j)$ decreases in j. Furthermore, j's in-degree is equal to $j-1+M(j,\beta)$, and it is easy to check that it increases in j. Also, we note that, by construction, $M(V(\beta),\beta) = V(\beta)$. Hence, when $\beta > \frac{1}{2(2V(\beta)+3)V(\beta)}$, j's in-degree increases in j until reaching $2V(\beta)-1$ for $j = V(\beta)$. On the other hand, when $\beta \leq \frac{1}{2(2V(\beta)+3)V(\beta)}$, j's in-degree increases in j until reaching $2V(\beta)$ for $j = V(\beta)+1$ and then stays constant. ■

Proof of Proposition 2. Let $\mathbf{g} \in \tilde{\mathbf{G}}$ be a minimally connected network and let $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ be part of an equilibrium. Hereafter, when there is no confusion we write $\mathbf{c}$ to indicate $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$. We say that the link $c_{ij} = 1$ is a jump link if $|i-j| > 1$. The set of jump links in network $\mathbf{c}$ is $P(\mathbf{c}) = \{(i,j) : c_{ij} = 1 \text{ and } |i-j| > 1\}$ and we partition it in two sets: $P_1(\mathbf{c}) = \{(i,j) \in P(\mathbf{c}) : k_j(\mathbf{c}) = 1\}$ and $\tilde{P}_2(\mathbf{c}) = \{(i,j) \in P(\mathbf{c}) : k_j(\mathbf{c}) > 1\}$. We also single out two subsets of $\tilde{P}_2(\mathbf{c})$: $P_{2A}(\mathbf{c}) = \{(i,j) \in \tilde{P}_2(\mathbf{c}) : c_{ji} = 0\}$ and $P_{2B}(\mathbf{c}) = \{(i,j) \in \tilde{P}_2(\mathbf{c}) : c_{ji} = 1 \text{ and } i < j\}$. Define $P_2 = P_{2A} \cup P_{2B}$. Let $A(\mathbf{c}) = \{l : c_{ll-1} = c_{l-1l} = 0\}$ and, with some abuse of terminology, we term this as the set of unused adjacent links in network $\mathbf{c}$.

We now provide a procedure which substitutes jump links in $\mathbf{c}$ with unused adjacent links. This procedure leads to a network $\mathbf{c}'$ such that there exists a strategy profile $(\mathbf{m}', \mathbf{y}')$ which is equilibrium in the ordered line communication network $\mathbf{g}'$ and $\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g}') = \mathbf{c}'$.

We start with two claims, which are key for the proof.

Claim 1. For every jump link $(i, j) \in P_2$ there exists a $l \in A(\mathbf{c})$ where $\min\{i, j\} < l \leq \max\{i, j\}$. This defines a non-empty correspondence $\Sigma : P_2 \to A$.

Proof of Claim 1: Suppose, by contradiction, that such l does not exist. Then the closure of $\mathbf{g}$, $\bar{\mathbf{g}}$, cannot be minimal, because there would be a cycle $\{(\min\{i, j\}, \min\{i, j\} + 1), (\min\{i, j\} + 1, \min\{i, j\} + 2, ..., (\max\{i, j\} - 1, \max\{i, j\}), (i, j)\}$. ■

Claim 2. There exists a selection σ of Σ with a well defined inverse σ^{-1}.

Proof of Claim 2: We proceed by contradiction. Suppose that there are two pairs $(i, j) \in P_2$ and $(i', j') \in P_2$ such that $\sigma(i, j) = \sigma(i', j')$ for all selections σ of Σ where $\sigma(i, j)$ is a singleton $l \in A(\mathbf{c})$. Suppose without loss of generality that $\min\{i, j\} \leq \min\{i', j'\}$. Further, because $\min\{i, j\} < l \leq \max\{i, j\}$ and $\min\{i', j'\} < l \leq \max\{i', j'\}$, it must be that $\min\{i', j'\} < \max\{i, j\}$. We distinguish two cases.

First, suppose $\max\{i, j\} \leq \max\{i', j'\}$. Then, because $\min\{i, j\} < l \leq \max\{i, j\}$ and $\min\{i', j'\} < l \leq \max\{i', j'\}$, it must be that $\min\{i', j'\} < l \leq \max\{i, j\}$. But this means that $\bar{\mathbf{g}}$ has the cycle $\bar{g}_{i,j} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\min\{i, j\} < l' \leq \min\{i', j'\}$, $\bar{g}_{i',j'} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\max\{i, j\} < l' \leq \max\{i', j'\}$.

In the second case, $\max\{i', j'\} < \max\{i, j\}$. Then, because $\min\{i, j\} < l \leq \max\{i, j\}$ and $\min\{i', j'\} < l \leq \max\{i', j'\}$, it must be that $\min\{i', j'\} < l \leq \max\{i', j'\}$. Because $\sigma(i, j) = \sigma(i', j')$ is a singleton, it must be that $\bar{g}_{l',l'-1} = 1$ for all $\min\{i, j\} < l' \leq \min\{i'.j'\}$ and $\bar{g}_{l',l'-1} = 1$ for all $\max\{i', j'\} < l' \leq \max\{i, j\}$. But this means that $\bar{\mathbf{g}}$ has the cycle $\bar{g}_{i,j} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\min\{i, j\} < l' \leq \min\{i', j'\}$, $\bar{g}_{i',j'} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\max\{i, j\} < l' \leq \max\{i', j'\}$. ■

We are now ready to prove Proposition 2.

Part A. Jump links in $P_1(\mathbf{c})$. Substitute any jump link $(i, j) \in P_1(\mathbf{c})$ such that $i < j$, with the unused adjacent link $(j - 1, j)$. This is possible because, since $(i, j) \in P_1(\mathbf{c})$, then $k_j(\mathbf{c}) = 1$ and, since $c_{ij} = 1$, it follows that $c_{j-1j} = 0$. Analogously, substitute any jump link $(i, j) \in P_1(\mathbf{c})$ such that $i > j$, with the unused adjacent link $(j + 1, j)$.

Part B: jump links in $\mathbf{P}_2(\mathbf{c})$. We now take up jump links in the set $\mathbf{P}_2(\mathbf{c})$. Here, we use extensively claim 1 and claim 2. Note that the two claims imply that there is an invertible function, σ, which maps for every jump link in P_2, say jump link (i, j), to an unused adjacent link $l \in A(\mathbf{c})$ with $c_{ll-1} = c_{l-1l} = 0$ and $\min\{i, j\} < l \leq \max\{i, j\}$. We first consider jump links in P_{2A} and then jump links in P_{2B}.

Jump links in P_{2A}. *First,* substitute any $(i,j) \in P_{2A}$ such that there is no jump link $(i',j') \in P_1$ where $j' = \sigma(i,j) - 1$ and $i' > j'$, with the unused adjacent link $(\sigma(i,j), \sigma(i,j) - 1)$ if $i > j$, while, if $i < j$, with the unused adjacent link $(\sigma(i,j) - 1, \sigma(i,j))$. Make the same substitution for $(i,j) \in P_{2A}$ such that there is no jump link $(i',j') \in P_1$, with $j' = \sigma(i,j)$ and $i' < j'$. *Second,* Substitute any $(i,j) \in P_{2A}$ such that there is a jump link $(i',j') \in P_1$ where $j' = \sigma(i,j) - 1$ and $i' > j'$, with the unused adjacent link $(\sigma(i,j) - 1, \sigma(i,j))$. *Third,* substitute any $(i,j) \in P_{2A}$ such that there is a $(i',j') \in P_1$ with $j' = \sigma(i,j)$ and $i' < j'$, with the unused adjacent link $(\sigma(i,j) + 1, \sigma(i,j))$. *Fourth,* substitute any $(i,j) \in P_{2A}$ such that there is a $(i',j') \in P_1$ with $j' = \sigma(i,j) - 1$ and $i' > j'$ as well as a jump link $(i',j') \in P_1$ with $j' = \sigma(i,j)$ and $i' < j'$, with unused adjacent link $(\sigma(i,j) + 1, \sigma(i,j))$.

Jump links in P_{2B}. *First,* for any $(i,j) \in P_{2B}$ such that there is no jump link $(i',j') \in P_1$ where $j' = \sigma(i,j) - 1$ and $i' > j'$, substitute the link $c_{ij} = 1$ with the unused adjacent link $(\sigma(i,j) - 1, \sigma(i,j))$ and the link $c_{ji} = 1$ with the unused adjacent link $(\sigma(i,j), \sigma(i,j) - 1)$. Make the same substitutions for $(i,j) \in P_{2B}$ such that there is no jump link $(i',j') \in P_1$, with $j' = \sigma(i,j)$ and $i' < j'$. *Second,* for any $(i,j) \in P_{2B}$ such that there is a jump link $(i',j') \in P_1$ where $j' = \sigma(i,j) - 1$ and $i' > j'$, substitute the jump link $c_{ij} = 1$ with $(\sigma(i,j) - 1, \sigma(i,j))$ and the jump link $c_{ji} = 1$ with $(\sigma(i,j) - 2, \sigma(i,j) - 1)$. Here note that $c_{\sigma(i,j)-2, \sigma(i,j)-1} = 0$ because, since $(i',j') \in P_1$, $k_{\sigma(i,j)-1} = 1$. *Third,* for any $(i,j) \in P_{2B}$ such that there is a jump link $(i',j') \in P_1$ where $j' = \sigma(i,j)$ and $i' < j'$, substitute the jump link $c_{ij} = 1$ with $(\sigma(i,j), \sigma(i,j) - 1)$ and the jump link $c_{ji} = 1$ with $(\sigma(i,j) + 1, \sigma(i,j))$. *Fourth,* for any $(i,j) \in P_{2B}$ such that there is a jump link $(i',j') \in P_1$ with $j' = \sigma(i,j) - 1$ and $i' > j'$ as well as a jump link $(i',j') \in P_1$ with $j' = \sigma(i,j)$ and $i' < j'$, substitute the jump link $c_{ij} = 1$ with $(\sigma(i,j) - 2, \sigma(i,j) - 1)$ and the jump link $c_{ji} = 1$ with $(\sigma(i,j) + 1, \sigma(i,j))$.

By construction, when applying simultaneously to $\mathbf{c}$ all these substitutions we obtain a new network $\mathbf{c}'$, which can be supported in equilibrium in a ordered line communication network. Note also that, by construction, the total number of (directed) links in $\mathbf{c}$ is the same as the total number of (directed) links in $\mathbf{c}'$.

We now show that from $\mathbf{c}'$ we can construct a new equilibrium $\mathbf{c}''$, which can be supported in the ordered line communication network and in which the expected utility of each player is higher than in the original equilibrium $\mathbf{c}$.

Let $N^+(\mathbf{c}') = \{i \in N : k_i(\mathbf{c}') > k_i(\mathbf{c})\}$ and $N^-(\mathbf{c}') = \{j \in N : k_j(\mathbf{c}') < k_j(\mathbf{c})\}$, and we recall that $k_j(\mathbf{c})$ denotes the in-degree of j in network $\mathbf{c}$. Define $S(\mathbf{c}') = N \setminus \{N^+(\mathbf{c}) \cup N^-(\mathbf{c})\}$. Clearly, if $S(\mathbf{c}') = N$, then $EU_i(\mathbf{c}) = EU_i(\mathbf{c}')$, for all $i \in N$, and the claim follows.

Suppose instead that $S(\mathbf{c}') \subset N$. By construction of $\mathbf{c}'$, it follows that $\sum_{i \in N^+(\mathbf{c}')}[k_i(\mathbf{c}') - k_i(\mathbf{c})] = \sum_{j \in N^i(\mathbf{c}')}[k_j(\mathbf{c}) - k_j(\mathbf{c})]$, and therefore $S(\mathbf{c}') \subset N$ if and only if $N^+(\mathbf{c}')$ and $N^-(\mathbf{c}')$ are both non-empty sets. Furthermore, each player in $i \in N^+(\mathbf{c}')$ is such that: (a) $k_i(\mathbf{c}') = 1$ and $k_i(\mathbf{c}) = 0$, or, (b) $k_i(\mathbf{c}') = 2$ and $k_i(\mathbf{c}) = 1$, or, (c) $k_i(\mathbf{c}') = 2$ and $k_i(\mathbf{c}') = 0$.

Take a player $i \in N^+(\mathbf{c}')$ with $k_i(\mathbf{c}') = 1$ and $k_i(\mathbf{c}) = 0$. Select, if there exists, a $j \in N^-(\mathbf{c}')$ with $k_j(\mathbf{c}') < 2$. Delete the link that i receives, and add an adjacent link to j which can be sustained in equilibrium. Clearly, such link exists because $k_j(\mathbf{c}) > k_j(\mathbf{c}') \in \{0, 1\}$. Call this new profile $\mathbf{c}''$. This is equilibrium and note that $S(\mathbf{c}') \subset S(\mathbf{c}'')$. By repeating this procedure, we end up with an equilibrium, say $\hat{\mathbf{c}}$ such that if there exists $i \in N^+(\hat{\mathbf{c}})$ with $k_i(\hat{\mathbf{c}}) = 1$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\hat{\mathbf{c}})$ has $k_j(\hat{\mathbf{c}}) = 2$.

Take a player $i \in N^+(\hat{\mathbf{c}})$ with $k_i(\hat{\mathbf{c}}) = 2$ and $k_i(\mathbf{c}) = 1$. Select, if there exists, a $j \in N^-(\hat{\mathbf{c}})$ with $k_j(\hat{\mathbf{c}}) < 2$. Delete a link that i receives, and add an adjacent link to j which can be sustained in equilibrium. Clearly, such link exists because $k_j(\mathbf{c}) > k_j(\hat{\mathbf{c}}) \in \{0, 1\}$. Call this new profile $\hat{\mathbf{c}}'$. This is equilibrium and note that $S(\hat{\mathbf{c}}) \subset S(\hat{\mathbf{c}}')$. By repeating this procedure, we end up with an equilibrium, say $\tilde{\mathbf{c}}$ such that if there exists $i \in N^+(\tilde{\mathbf{c}})$ with $k_i(\tilde{\mathbf{c}}) = 1$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\tilde{\mathbf{c}})$ has $k_j(\tilde{\mathbf{c}}) = 2$.

Take a player $i \in N^+(\tilde{\mathbf{c}})$ with $k_i(\tilde{\mathbf{c}}) = 2$ and $k_i(\mathbf{c}) = 0$. Select, if there exists, a $j \in N^-(\tilde{\mathbf{c}})$ with one of the following property: (1) $k_j(\tilde{\mathbf{c}}) = 0$ and $k_j(\mathbf{c}) = 1$, (2) $k_j(\tilde{\mathbf{c}}) = 0$ and $k_j(\mathbf{c}) \geq 2$, (3) $k_j(\tilde{\mathbf{c}}) = 1$ and $k_j(\mathbf{c}) = 2$. In case (1) delete a link that i receives and add an adjacent link to j; in case (2), delete the two links that i receives and add two adjacent links to j; in case (3) delete a link that i receives and add an adjacent link to j. Call this new profile $\tilde{\mathbf{c}}'$. This is equilibrium and note that $S(\tilde{\mathbf{c}}) \subset S(\tilde{\mathbf{c}}')$. By repeating this procedure, we end up with an equilibrium, say $\tilde{\tilde{\mathbf{c}}}$, such that if there exists $i \in N^+(\tilde{\tilde{\mathbf{c}}})$ with $k_i(\tilde{\tilde{\mathbf{c}}}) = 2$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\tilde{\tilde{\mathbf{c}}})$ has $k_j(\tilde{\tilde{\mathbf{c}}}) \in \{1, 2\}$ and $k_j(\mathbf{c}) \geq 3$.

For a given $\mathbf{c}$, the three procedures given above transform a profile on the line with the same total number of (directed) links as in $\mathbf{c}$ into another profile on the line with the same number of (directed) links as in $\mathbf{c}$ and $\mathbf{c}'$. We note this transformation Φ, so that $\tilde{\tilde{\mathbf{c}}} = \mathbf{\Phi}(\mathbf{c}^{'})$.The procedure can be iterated until a profile $\mathbf{c}^*$ such that $S(\mathbf{c}^*) = S(\mathbf{\Phi}(\mathbf{c}^*))$ is reached. If $S(\mathbf{c}^*) = N$, then clearly the in-degree distribution in $\mathbf{c}^*$ equals the in-degree distribution in $\mathbf{c}$. Hence, the expected utility of each player is the same in the two equilibria and the proof follows. Suppose that $S(\mathbf{c}^*) \subset N$. Let $N_a^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 2 \text{ and } k_i(\mathbf{c}) = 1\}$, $N_b^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 2 \text{ and } k_i(\mathbf{c}) = 0\}$ and $N_b^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 1 \text{ and } k_i(\mathbf{c}) = 0\}$. Let also $n_x = |N_x^+(\mathbf{c}^*)|$, $x = a, b, c$. By construction of $\mathbf{c}^*$ we have that $N^+(\mathbf{c}^*) = \cup_{x \in \{a,b,c\}} N_x^+(\mathbf{c}^*)$. Furthermore, note that

$$\sum_{i \in N^+(\mathbf{c}^*)} [k_i(\mathbf{c}^*) - k_i(\mathbf{c})] = \sum_{j \in N^-(\mathbf{c}^*)} [k_j(\mathbf{c}) - k_j(\mathbf{c}')]$$

and since

$$\sum_{i \in N^+(\mathbf{c}^*)} [k_i(\mathbf{c}^*) - k_i(\mathbf{c})] = n_a + 2n_b + n_c$$

it follows that

$$n_a + 2n_b + n_c = \sum_{j \in N^-(\mathbf{c}^*)} [k_j(\mathbf{c}) - k_j(\mathbf{c}^*)]. \tag{7}$$

Using the expression 3, we can see that the expected utility of an arbitrary i in $\mathbf{c}^*$ is at least as high as her expected utility in equilibrium $\mathbf{c}$ if and only if

$$\frac{n_a + n_b}{5} + \frac{n_c}{4} + \sum_{j \in N^-(\mathbf{c}^*)} \frac{1}{k_j(\mathbf{c}^*) + 3} \leq \frac{n_a}{4} + \frac{n_b + n_c}{3} + \sum_{j \in N^-(\mathbf{c}^*)} \frac{1}{k_j(\mathbf{c}) + 3}.$$

This is satisfied if only if

$$\sum_{j \in N^-(\mathbf{c}^*)} \frac{k_j(\mathbf{c}) - k_j(\mathbf{c}^*)}{(k_j(\mathbf{c}) + 3)(k_j(\mathbf{c}^*) + 3)} \leq \frac{3n_a + 8n_b + 5n_c}{60}.$$

Note that by construction $j \in N^-(\mathbf{c}^*)$ if and only if $k_j(\mathbf{c}^*) \in \{1, 2\}$ and $k_j(\mathbf{c}) \geq 3$. Hence,

$$\begin{aligned}
\sum_{j \in N^-(\mathbf{c}^*)} \frac{k_j(\mathbf{c}) - k_j(\mathbf{c}^*)}{(k_j(\mathbf{c}) + 3)(k_j(\mathbf{c}^*) + 3)} &\leq \sum_{j \in N^-(\mathbf{c}^*)} \frac{k_j(\mathbf{c}) - k_j(\mathbf{c}^*)}{(3 + 3)(1 + 3)} \\
&= \sum_{j \in N^-(\mathbf{c}^*)} \frac{k_j(\mathbf{c}) - k_j(\mathbf{c}^*)}{24} \\
&= \frac{n_a + 2n_b + n_c}{24} \\
&< \frac{3n_a + 8n_b + 5n_c}{60},
\end{aligned}$$

where the second equality follows by using (7), while the last inequality is easily verified. This completes the proof of the proposition. ■.

We now provide Proposition A which characterizes all top Pareto equilibria in the two communities case developed in Section 4.2. The proof of Proposition 2 follows simply from Proposition A and the details are omitted.

Before stating the result we need to introduce some definitions. A $k^1 \times k^2$-network is a network where k^x is the in-degree of players in group x, $x = 1, 2$. A segregated network is a $(n_1 - 1) \times (n_2 - 1)$-network with no links across communities. A partially segregated network is a $(n_1 - 1) \times k^2$-network where there are no links going from players in community 2 to players in community 1 and there are some links going from community 1 to community 2, i.e., $k^2 \in \{n_2, ..., n_1 - 1\}$. A complete network is a $(n-1)\times(n-1)$-network.

Proposition A. Consider the two-communities model.

I. The complete network is a top Pareto equilibrium network if and only if $b \leq \frac{1}{2(n+2)}$;

II. A $k\times k$–network with $k \in \{n_1, ..., n-2\}$ is is a top Pareto equilibrium network if and only if $b \in \left(\frac{1}{2(k+4)}, \frac{1}{2(k+3)}\right]$;

III. A partially segregated network with $k^2 \in \{n_2, ..., n_1-1\}$ is a is a top Pareto equilibrium network if and only if $b \in \left(\frac{1}{2(k+4)}, \frac{1}{2(k+3)}\right]$;

IV. A segregated network is a is a top Pareto equilibrium network if and only if $b > \frac{1}{2(n_2+3)}$.

Proof of Proposition A. We first need to show the following Lemma.

Lemma 1 *Suppose* $\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})$ *is a top Pareto equilibrium network (TPEN). Then,* $\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})$ *is a* $k^1\times k^2$*-communication network, i.e. all players in a group have same degree and this is larger or equal than the size of the group minus one.*

Proof of Lemma 1. With some abuse of notation we indicate a TPEN $\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})$ as to $\mathbf{c}$, and the in-degree of a player i in $\mathbf{c}$ as to k_i. Let $M(\mathbf{c})$ be the number of (directed) links in $\mathbf{c}$, i.e., the total number of truthful communications. First, note that the segregate communication network, $\mathbf{c}^s$, is always equilibrium and that $M(\mathbf{c}^s) = n_1(n_1-1)+n_2(n_2-1)$. Since $\mathbf{c}$ is a TPEN equilibrium, and each player in $\mathbf{c}^\mathbf{s}$ has the same in-degree, Theorem 2 implies that $M(\mathbf{c}) \geq M(\mathbf{c}^s)$. We now divide the analysis in two parts.

Part A. If $b > \frac{1}{2(n_2+3)}$ then $\mathbf{c} = \mathbf{c}^s$. To see this, suppose, for a contradiction, that $\mathbf{c} \neq \mathbf{c}^s$. Let $I_{12} = \{i \in N_1 : c_{ji} = 1 \text{ for some } j \in N_2\}$ and $I_{21} = \{j \in N_2 : c_{ij} = 1 \text{ for some } i \in N_1\}$. If $|I_{12}| = |I_{21}| = 0$, then, since $\mathbf{c} \neq \mathbf{c}^s$, $M(\mathbf{c}) < M(\mathbf{c}^s)$, a contradiction.

Next, assume that $|I_{12}| \neq 0$ and that $|I_{21}| \neq 0$. Since $\mathbf{c}$ is a TPEN equilibrium, it cannot be the case that $k_i < n_1-1$, for all $i \in I_{12}$ and $k_j < n_2-1$ for all $j \in I_{21}$; for otherwise $M(\mathbf{c}) < M(\mathbf{c}^s)$. Note also that for all $i \in I_{12}$ it must be the case that $k_i < n_1-1$. Indeed, if it exists some $i \in I_{12}$ with $k_i \geq n_1-1$, then, since $\mathbf{c}$ is equilibrium, it must hold that $b(k_i+3) \leq 1/2$, which contradicts our initial hypothesis that $b(n_2+3) > 1/2$, because $k_i+3 \geq n_1-1+3 \geq n_2+3$. These two observations imply that there must exist $j \in I_{21}$ such that $k_j \geq n_2-1$. Furthermore, if all these players like j have $k_j = n_2-1$, then $M(\mathbf{c}) < M(\mathbf{c}^s)$. So, there exists $j \in I_{21}$ such that $k_j > n_2-1$. In such a case, equilibrium implies that $b[k_j+3] \leq 1/2$. But, since $k_j+3 \geq n_2+3$, this contradicts our initial hypothesis that $b[n_2+3] > 1/2$.

Hence, it must be the case that either $|I_{12}| \neq 0$ and $|I_{21}| = 0$ or $|I_{12}| = 0$ and $|I_{21}| \neq 0$. Each of these two cases can be ruled out using the same arguments adopted for the case in which $|I_{12}| \neq 0$ and that $|I_{21}| \neq 0$; details are omitted. This completes the proof of part A.

Part B. Suppose that $b(n_2 + 3) < 1/2$. We first prove that each player in group 2 must have the same in-degree, i.e., $k_i = k^2$ for all $i \in N_2$. Given $\mathbf{c}$, without loss of generality, all players in group 2 are ordered according to their in-degrees, i.e., $k_1 \leq k_2 \leq .... \leq k_{n_2}$. Assume, for a contradiction, that $k_1 < k_{n_2}$. We consider three sub-cases.

Part B, Case 1. Suppose $k_{n_2} > n_2 - 1$. This implies that $c_{jn_2} = 1$ for some $j \in N_1$, and since $\mathbf{c}$ is equilibrium, it must hold that $b[k_{n_2} + 3] \leq 1/2$. Next, since $k_1 < k_{n_2}$, it must exist a $j \in N$ such that $c_{j1} = 0$. But then the network $\mathbf{c}' = \mathbf{c} + c_{j1}$ is also equilibrium. In fact, every agent communicating in $\mathbf{c}$ with a player different from player 1 can still communicate in $\mathbf{c}'$, because the in-degrees of these players have not changed, and every agent l that was communicating with 1 in $\mathbf{c}$ still communicates in $\mathbf{c}'$ because $k_1(\mathbf{c}') = k_1(\mathbf{c} + 1) \leq k_{n_2}$ and $b[k_{n_2} + 3] \leq 1/2$. But then $\mathbf{c}$ is a subgraph of $\mathbf{c}'$, which, in view of Theorem 2, contradicts our initial hypothesis that $\mathbf{c}$ is a TPEN.

Part B, Case 2. Suppose $k_{n_2} = n_2 - 1$. We first note that $c_{jn_2} = 0$ for all $j \in N_1$; otherwise, we can replicate the argument developed in Part A, Case 1 to show a contradiction. Next, let player $l \in N_2$ such that $k_l < k_{n_2}$ and $k_{l+1} = k_{n_2}$. Note that for all $l' \in N_2$ with $l' \leq l$, there must exist some $j \in N_1$ such that $c_{jl'} = 1$. Indeed, if there exists a $l' \in N_2$ with $l' \leq l$, such that $c_{jl'} = 0$ for all $j \in N_1$, then, since $k_{l'} < n_2 - 1$, there exists a $i \in N_2$ such that $c_{il'} = 0$. But then, the network $\mathbf{c}' = \mathbf{c} + c_{il'}$ is also equilibrium, and in view of Theorem 2, this contradicts that $\mathbf{c}$ is a TPEN.

Now, for an arbitrary $l' \in N_2$ with $l' \leq l$, define $A(l')$ as the number of links that l' receives from players in group N_1. Define also $W(l')$ as the number of links that l' receives from players in group N_2. Then, the number of players in group N_2 who do not communicate with l' is

$$\bar{W}(l') = n_2 - 1 - k_{l'} + A(l') > A(l').$$

From network $\mathbf{c}$, construct $\mathbf{c}'$ in the following way: one, delete all links from group 1 to players $l' \in N_2$, with $l' \leq l$, and, two, for each $j \in N_2$ such that $c_{jl'} = 0$, $l' \in N_2$, $l' \leq l$, set $c'_{jl'} = 1$. Note that since $\mathbf{c}$ is equilibrium, then $\mathbf{c}'$ is also equilibrium, because each of the new links in $\mathbf{c}'$ are between members of the same community. Note also that

$$M(\mathbf{c}') - M(\mathbf{c}) = \sum_{l' < l, l' \in N_2} \left(\bar{W}(l') - A(l') \right) > 0,$$

and, by construction, the in-degree distribution in $\mathbf{c}'$ first order stochastic dominates the in-degree distribution of $\mathbf{c}$. Corollary 2 then implies that $\mathbf{c}'$ Pareto dominates $\mathbf{c}$, which contradicts that $\mathbf{c}$ is a TPEN equilibrium.

The final case in which $k_{n_2} < n_2 - 1$ is easy to rule out and details are omitted. We have shown that players in group 2 must have the same in-degree. The arguments developed here, can then be used to show that all players in group 1 must have the same in-degree. This concludes the proof of Lemma 1. ■.

Finally, note that part IV of Proposition A follows from the proof of Part A of Lemma 1. Parts I-III of Proposition A simply follows by comparing the total number of links that can be sustained in $k^1 \times k^2$-communication network equilibrium. This concludes the proof of Proposition A. ■.

Appendix C.

Proof of Theorem 3 Suppose that all agents in J believe that agent i reports his signal s truthfully. Let s_R be a vector containing the (truthful) signals that each j has received from his communication neighbors, i.e, from every $j' \in C_j(\mathbf{c}) \setminus \{i\}$, and his own signal. With some abuse of notation, we denote the in-degree of j in truthful network $\mathbf{c}$ by $k_j = |C_j(\mathbf{c})|$. Let also $y_{s_R,s}$ be the action that j would take if he has information s_R and player i has sent signal s; analogously, $y_{s_R,1-s}$ is the action that j would take if he has information s_R and player i has sent signal $1-s$. Agent i reports truthfully signal s to a collection of agents J if and only if

$$-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[(y_{s_R,s} - \theta - b_i)^2 - (y_{s_R,1-s} - \theta - b_i)^2\right] f(\theta, s_R|s) \geq 0,$$

and using the identity $a^2 - b^2 = (a-b)(a+b)$ we get:

$$-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[(y_{s_R,s} - y_{s_R,1-s})\left(\frac{y_{s_R,s} + y_{s_R,1-s}}{2} - (\theta + b_i)\right)\right] f(\theta, s_R|s) \geq 0.$$

Next, observing that

$$y_{s_R,s} = E\left[\theta + b_j | s_R, s\right],$$

we obtain

$$\begin{aligned}-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} &[(E[\theta + b_j|s_R, s] - E[\theta + b_j|s_R, 1-s]) \\ &\cdot \left(\frac{E[\theta + b_j|s_R, s] + E[\theta + b_j|s_R, 1-s]}{2} - (\theta + b_i)\right)\Bigg] f(\theta, s_R|s) \geq 0.\end{aligned}$$

Denote

$$\Omega = \left(E\left[\theta|s_R, s\right] - E\left[\theta|s_R, 1-s\right]\right).$$

Observing that:

$$f(\theta, s_R|s) = f(\theta|s_R, s)P(s_R|s),$$

and simplifying, we get:

$$-\sum_{j\in J}\sum_{s_R\in\{0,1\}^{k_j}}\int_0^1\left[\Omega\left(\frac{E\left[\theta|s_R,s\right]+E\left[\theta|s_R,1-s\right]}{2}+b_j-b_i-\theta\right)\right]f(\theta|s_R,s)P(s_R|s)\geq 0.$$

Furthermore,

$$\int_0^1 \theta f(\theta|s_R, s)d\theta = E[\theta|s_R, s],$$

and

$$\int_0^1 P(\theta|s_R, 1)E[\theta|s_R, s]d\theta = E[\theta|s_R, s],$$

because $E[\theta|s_R, s]$ does not depend on θ. Therefore, we obtain:

$$\begin{aligned}
& -\sum_{j\in J}\sum_{s_R\in\{0,1\}^{k_j}}\left[\Omega\left(\frac{E\left[\theta|s_R,s\right]+E\left[\theta|s_R,1-s\right]}{2}+b_j-b_i-E\left[\theta|s_R,s\right]\right)\right]P(s_R|s) \\
= & -\sum_{j\in J}\sum_{s_R\in\{0,1\}^{k_j}}\left[\Omega\left(-\frac{E\left[\theta|s_R,s\right]-E\left[\theta|s_R,1-s\right]}{2}+b_j-b_i\right)\right]P(s_R|s)\geq 0.
\end{aligned}$$

Now, note that:

$$\begin{aligned}
\Omega &= E[\theta|s_R, s] - E[\theta|s_R, 1-s] \\
&= E\left[\theta|l+s, k_j+1\right] - E\left[\theta|l+1-s, k_j+1\right] \\
&= (l+1+s)/(k_j+3) - (l+2-s)/(k_j+3) \\
&= \begin{cases} -1/(k_j+3) & \text{if } s=0 \\ 1/(k_j+3) & \text{if } s=1. \end{cases}
\end{aligned}$$

where l is the number of digits equal to one in s_R. Hence, we obtain that agent i is willing to communicate to agent j the signal $s = 0$ if and only if:

$$-\sum_{j\in J}\left(\frac{-1}{k_j+3}\right)\left(-\frac{-1}{2(k_j+3)}+b_j-b_i\right)\geq 0,$$

or

$$\sum_{j\in J}\frac{b_j-b_i}{k_j+3}\geq-\sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}$$

Note that this condition is redundant if $\sum_{j\in J}b_j-b_i>0$. On the other hand, she is willing to communicate to agent j the signal $s=1$ if and only if:

$$-\sum_{j\in J}\left(\frac{1}{k_j+3}\right)\left(-\frac{1}{2(k_j+3)}+b_j-b_i\right)\geq 0,$$

or

$$\sum_{j\in J}\frac{b_j-b_i}{k_j+3}\leq\sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}.$$

Note that this condition is redundant if $\sum_{j\in J}b_j-b_i<0$. Collecting the two conditions:

$$\left|\sum_{j\in J}\frac{b_j-b_i}{k_j+3}\right|\leq\sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}.$$

This completes the proof of Theorem 3. ■

Proof of Proposition 5 The proof proceeds in two steps. In the first step we show that the described profile of strategies is equilibrium. The second step shows that the constructed equilibria are top Pareto equilibria. In what follows $(\mathbf{m},\mathbf{y})$ denotes the equilibrium, $\mathbf{c}(\mathbf{m},\mathbf{y})$ the truthful communication network and k_j is the in-degree of j in truthful communication network $\mathbf{c}(\mathbf{m},\mathbf{y})$. Note that, with some abuse of notation, we have suppressed the qualification that the communication network $\mathbf{g}$ is complete.

First Step. We show that the described strategy profiles are equilibria. First, note that Theorem 3 implies that, when $\beta\leq f(k,n)$, the profile $(\mathbf{m},\mathbf{y})$ such that $c_{i,j}(\mathbf{m},\mathbf{y})=1$ if and only if $i\in\{l,...,n-l+1\}$ is equilibrium if and only if

$$\left|\sum_{j\in N\backslash\{i\}}\frac{b_j-b_i}{k_j+3}\right|\leq\sum_{j\in N\backslash\{i\}}\frac{1}{2\left(k_j+3\right)^2}$$

for all $i\in\{l,...,n-l+1\}$. To see this note that in $\mathbf{c}(\mathbf{m},\mathbf{y})$ there are $n-2l+2$ players communicating truthfully, $k_j=n-2l+1$ for all $j\in\{l,...,n-l+1\}$, whereas $k_j=n-2l+2$ for all $j\notin\{l,...,n-l+1\}$.

Because $b_j - b_i = \beta(j-i)$, the above equilibrium condition simplifies to:

$$\left| \sum_{j \in \{l,...,n-l+1\} \setminus \{i\}} \frac{\beta(j-i)}{n-2l+1+3} + \sum_{j=1}^{l-1} \frac{\beta(j-i)}{n-2l+2+3} + \sum_{j=n-l+2}^{n} \frac{\beta(j-i)}{n-2l+2+3} \right|$$
$$\leq \sum_{j \in \{l,...,n-l+1\} \setminus \{i\}} \frac{1}{2(n-2l+1+3)^2} + \sum_{j=1}^{l-1} \frac{1}{2(n-2l+2+3)^2} + \sum_{j=n-l+2}^{n} \frac{1}{2(n-2l+2+3)^2},$$

for all $i \in \{l, ..., n-l+1\}$. This condition can be further simplified as follows: $\beta \leq \min_{i \in \{l,...,n-l+1\}} \phi(i,l,n)$, where

$$\begin{aligned} \phi(i,l,n) &= \frac{\frac{n-2l+1}{2(n-2l+4)^2} + \frac{2(l-1)}{2(n-2l+5)^2}}{\left| \frac{\sum_{j \in \{l,...,n-l+1\} \setminus \{i\}}(j-i) + (n-2l+4)\left[\sum_{j \neq i}(j-i)\right]}{(n-2l+4)(n-2l+5)} \right|} \\ &= \frac{\frac{n-2l+1}{2(n-2l+4)^2} + \frac{2(l-1)}{2(n-2l+5)^2}}{\frac{1}{2}|n+1-2i| \frac{(n+(n-2l+4)(n-2l+2))}{(n-2l+4)(n-2l+5)}}. \end{aligned}$$

The numerator of this expression does not depend on i, whereas the denominator is decreasing for $i < (n+1)/2$, it is increasing for $i > (n+1)/2$ and symmetric around $(n+1)/2$. Thus, the denominator is maximized for $i = l$ and $i = n-l+1$. This implies that $\min_{i \in \{l,...,n-l+1\}} \phi(i,l,n) = \phi(l,l,n)$ and, by definition, $f(l,n) = \phi(l,l,n)$. Hence we have recovered the condition that $\beta \leq f(l,n)$. For future reference, we stress that $\min_{i \in \{l,...,n-l+1\}} \phi(i,l,n) = \phi(n-l+1,l,n) = \phi(l,l,n)$.

Next, using a similar approach, we note that when $\beta \leq g(l,n)$, the strategy profile $(\mathbf{m}, \mathbf{y})$ such that $c_{i,j}(\mathbf{m}, \mathbf{y}) = 1$ if and only if $i \in \{l, ..., n-l\}$ is equilibrium if and only if

$$\left| \sum_{j \in \{l,...,n-l\} \setminus \{i\}} \frac{\beta(j-i)}{n-2l+3} + \sum_{j=1}^{l-1} \frac{\beta(j-i)}{n-2l+1+3} + \sum_{j=n-l+1}^{n} \frac{\beta(j-i)}{n-2l+1+3} \right|$$
$$\leq \sum_{j \in \{l,...,n-l\} \setminus \{i\}} \frac{1}{2(n-2l+3)^2} + \sum_{j=1}^{l-1} \frac{1}{2(n-2l+1+3)^2} + \sum_{j=n-l+1}^{n} \frac{1}{2(n-2l+1+3)^2},$$

for all $i \in \{l, ..., n-l\}$. The above condition simplifies as: $\beta \leq \min_{i \in \{l,...,n-l\}} \gamma(i,l,n)$, where

$$\begin{aligned} \gamma(i,l,n) &= \frac{\frac{n-2l}{2(n-2l+3)^2} + \frac{2l-1}{2(n-2l+4)^2}}{\left| \frac{\sum_{j \in \{l,...,n-l\} \setminus \{i\}}(j-i) + (n-2l+3)\sum_{j \neq i}(j-i)}{(n-2l+3)(n-2l+4)} \right|} \\ &= \frac{\frac{n-2l}{2(n-2l+3)^2} + \frac{2l-1}{2(n-2l+4)^2}}{\frac{1}{2} \frac{|(n-2i)(n-2l+1)+(n-2l+3)n(n+1-2i)|}{(n-2l+3)(n-2l+4)}}. \end{aligned}$$

In $\gamma(i,l,n)$, the numerator does not depend on i; the denominator is maximal for $i = l$, because $\{l, n-l\} = \arg\max_{i\in\{l,...,n-l\}} |n-2i|$ and $\{l\} = \arg\max_{i\in\{l,...,n-l\}} |n+1-2i|$. Hence, $\min_{i\in\{l,...,n-l\}} \gamma(i,l,n) = \gamma(l,l,n)$ and, by definition, $g(l,n) = \gamma(l,l,n)$. Hence we have recovered the condition that $\beta \le g(l,n)$.

Second Step. We now show that the equilibria described are top Pareto equilibria. This amounts to show that: 1) when $g(l,n) < \beta$, there is no equilibrium where strictly more than $n-2l$ players truthfully communicates, and 2) when $f(l,n) < \beta$, there is no equilibrium where strictly more than $n-2l+1$ players truthfully communicate. To see that this is sufficient, note that the welfare of each player i when L players communicate truthfully is: $W_i(L) = -\sum_{j\in N}(b_i-b_j)^2 - (n-L)\frac{1}{6(L+2)} - L\frac{1}{6(L-1+2)}$. Indeed, each of the L players who communicate truthfully receives $L-1$ truthful messages, whereas each of the remaining players who do not communicate truthfully receives L messages. It is easy to see that $W_i(L)$ is increasing in L, i.e., $W_i'(L) = \frac{1}{6}\frac{n(1+L)^2+L^2-2}{(L+1)^2(L+2)^2} > 0$ for $n > 2$.

We start by noting that because $g(v-1,n) < f(v,n) < g(v,n)$ for all $v = 1, ..., \lfloor \frac{n}{2} \rfloor$, it follows that for $\beta > f(l,n)$ there are no equilibria where strictly more than $n-2l+2$ players communicate, and that for $\beta > g(l,n)$ there are no equilibria where strictly more than $n-2l$ players communicate.

Next, suppose that $n-2l+2$ players communicate in an equilibrium $(\mathbf{m}', \mathbf{y}')$. Let the set of players who truthfully communicate in $(\mathbf{m}', \mathbf{y}')$ be C', so that $|C'| = n-2l+2$. Then, since $(\mathbf{m}', \mathbf{y}')$ is equilibrium, Theorem 3 implies that for all $i \in C'$ it must be that

$$\beta \le \min_{i\in C'} \phi\left(i, |C'|, n\right) \text{ where } \phi\left(i, |C'|, n\right) = \frac{\frac{n-2l+1}{2(n-2l+4)^2} + \frac{2l-2}{2(n-2l+5)^2}}{\left|\frac{\left[\sum_{j\in N\setminus\{i\}}(j-i)\right][n-2l+4]+\left[\sum_{j\in C'\setminus\{i\}}(j-i)\right]}{[n-2l+4][n-2l+5]}\right|}.$$

We now claim that the set $C^* = \{l, ..., n-l+1\}$ has the property that

$$\{C^*\} = \arg\max_{C:|C|=n-2l+2} \min_{i\in C} \phi(i, C, n).$$

Note that this claim would imply that an equilibrium where $n-2l+2$ players communicate truthfully exists if and only if $\beta \le \min_{i\in C^*} \phi(i, C^*, n)$. But, since we have earlier proved that $l \in \arg\min_{i\in C^*} \phi(i, |C^*|, n)$ and that $f(l,n) = \min_{i\in C^*} \phi(i, |C^*|, n)$, this implies that if $\beta > f(l,n)$ then there are no equilibria where $n-2l+2$ players communicate.

To prove the claim, first note that the numerator of $\phi(i, |C|, n)$ depend neither on i nor on $|C|$. Consider the denominator of $\phi(i, |C|, n)$, and suppose that $C \ne \{l, ..., n-l+1\}$. Let v be one of the most extreme players in C, i.e., $v \in \arg\max_{i\in C} |i-(n+1)/2|$. We must consider two sub-cases.

The first sub-case is when $v < (n+1)/2$. Here, note that

$$\sum_{j\in N\backslash\{v\}}(j-v) > \sum_{j\in N\backslash\{l\}}(j-l) > 0 \textit{ and } \sum_{j\in C\backslash\{v\}}(j-v) \geq \sum_{j\in C^*\backslash\{l\}}(j-l) > 0.$$

These inequalities follow from noticing that: 1) since $C \neq \{l,...,n-l+1\}$ and $v \in \arg\max_{i\in C}|i-(n+1)/2|$, it must be the case that $v < l$, and, 2) since because $l = \min\{i : i \in C^*\}$ and $v = \min\{i : i \in C\}$, we have then $j - v > 0$ for all $j \in C\backslash\{v\}$ and $j - l > 0$ for all $j \in C^*\backslash\{l\}$. Hence, we can now conclude that:

$$f(l,n) = \phi(l,|C^*|,n) = \min_{i\in C^*}\phi(i,|C^*|,n) > \phi(v,|C|,n) \geq \min_{i\in C}(i,|C|,n).$$

The sub-case when $v > (n+1)/2$, can be ruled out using similar arguments, and therefore details are omitted. Hence, we can conclude that an equilibrium where $n-2l+2$ players communicate truthfully exists if and only if $\beta \leq f(l,n)$.

Suppose now that $n-2l+1$ players communicate in equilibrium $(\mathbf{m}',\mathbf{y}')$; again, C' is the set of players communicating truthfully and $|C'| = n-2l+1$. Since $(\mathbf{m}',\mathbf{y}')$ is equilibrium, Theorem 3 implies that for all $i \in C\,(m,y)$ it must be that:

$$\beta \leq \min_{i\in C'}\gamma\left(i,|C'|,n\right) \text{ where } \gamma\left(i,|C'|,n\right) = \frac{\frac{n-2l}{2(n-2l+3)^2}+\frac{2l-1}{2(n-2l+4)^2}}{\left|\frac{\left[\sum_{j\in N\backslash\{i\}}(j-i)\right][n-2l+3]+\left[\sum_{j\in C'\backslash\{i\}}(j-i)\right]}{[n-2l+3][n-2l+4]}\right|}.$$

Consider the sets $C^* = \{l,...,n-l\}$ and its symmetric counterpart around $(n+1)/2$, denoted $C^{**} = \{l+1,...,n-l+1\}$. Let $h = n-l+1$. By symmetry, it is easy to see that

$$\min_{i\in C^*}\phi(i,|C^*|,n) = \phi(l,|C^*|,n) = \phi(h,|C^{**}|,n) = \min_{i\in C^{**}}\phi(i,|C^{**}|,n).$$

We now claim that

$$\{C^*,C^{**}\} = \arg\max_{C:|C|=n-2l+1}\min_{i\in C}\phi(i,|C|,n).$$

As in the case covered above for $f(l,n)$, this result concludes that if $\beta > g(l,n)$ then there are no equilibria where $n-2l+1$ players communicate.

To prove the claim, note that the numerator of $\gamma(i,|C|,n)$ does not depend on i nor on $|C|$. Consider the denominator of $\gamma(i,|C|,n)$. Suppose that $C \notin \{\{l,...,n-l\},\{l+1,...,n-l+1\}\}$. Let v be one of the most extreme players in C, i.e., $v \in \arg\max_{i\in C}|i-(n+1)/2|$. Proceeding in exactly the same way as for the case of $f(k,n)$, we show that for $v < (n+1)/2$, $g(l,n) = \gamma(l,|C^*|,n) = \min_{i\in C^*}\gamma(i,|C^*|,n) > \gamma(v,|C|,n) \geq \min_{i\in C}\gamma(i,|C|,n)$; and that for $v > (n+1)/2$, $g(l,n) = \gamma(h,|C^*|,n) = \min_{i\in C^*}\gamma(i,|C^*|,n) > \gamma(v,|C|,n) \geq \min_{i\in C}\gamma(i,|C|,n)$. Because $g(l,n) \geq \min_{i\in C}(i,|C|,n)$ for all C such that $|C| = n-2l+1$, we conclude that an equilibrium where $n-2l+1$

players communicate truthfully exists if and only if $\beta \leq g(l, n)$. ■

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