# Pricing Derivatives the Martingale Way

Pierre Collin Dufresne[1] William Keirstead[2] Michael P. Ross[3]

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[1]HEC School of Management, 78351 Jouy-en-Josas cedex, France. (d027@frhec1.hec.fr)

[2]Goldman, Sachs and Co., 85 Broad Street, New York, NY (bill.keirstead@gs.com)

[3]Haas School of Business, University of California, Berkeley, CA 94720 (mross@haas.berkeley.edu)

## Abstract

In recent years results from the theory of martingales has been successfully applied to problems in financial economics. In the present paper we show how efficient and elegant this "martingale technology" can be when solving for complex options. In particular we provide closed form solutions for several new classes of exotic options including the cliquet, the ladder, the discrete shout and the discrete lookback. We also provide a derivation of the price of an option on the maximum of $n$ assets to demonstrate the power of the multi-dimensional Girsanov theorem. Although some of the results presented are well known, the treatment of the material in this paper is new in that it focuses on the application of the martingale technology to concrete problems in option pricing, methods that until now have mostly been used for purely theoretical purposes.

# Pricing Derivatives the Martingale Way

## 1 Introduction

There are two main approaches to the pricing of derivative securities. The first, due to Black and Scholes (1973) and Merton (1973), is the "partial differential equation" (PDE) approach. This technique consists of constructing a PDE along with appropriate boundary conditions for the price of a derivative security. The PDE can then be solved using various analytical or numerical methods. The second approach, initiated by Cox and Ross (1976) and Harrison and Kreps (1979), is the "martingale method." This approach consists of writing the value of the security as the expected value of the discounted payoff under a risk neutral measure $\mathcal{Q}$ and calculating this expectation using probabilistic methods. In the present paper we show how to powerfully apply the second method when solving for complex options, and in particular we provide closed form solutions for several new classes of exotic options.

We begin the paper by deriving the Black and Scholes European call option formula using the martingale approach. The derivation here allows us to introduce the essential technology of the martingale method in an example well known to most readers. In the succeeding sections, we apply the martingale method to a series of more complicated exotic options, where we hope the elegance and computational simplicity of the approach will become readily apparent.

## 2 The Black and Scholes Case

From standard financial economics theory the value of the Black and Scholes European option can be written as

$$V_{\mathrm{BS}} = \mathrm{E}^{\mathcal{Q}}\left[e^{-rT}\max\left(\phi(S_T - K), 0\right)\right] \tag{2.1}$$

where $\phi = 1$ for a call option and $\phi = -1$ for a put option. Here $\mathrm{E}^{\mathcal{Q}}$ denotes the expectation, conditional on all information at time 0, with respect to the risk neutral probability measure $\mathcal{Q}$. The stock price $S_t$ has dynamics given by

$$d\log S_t = \alpha^{\mathcal{Q}}dt + \sigma dw_t^{\mathcal{Q}}, \tag{2.2}$$

where the risk-neutral drift is

$$\alpha^{\mathcal{Q}} = r - q - \frac{1}{2}\sigma^2. \tag{2.3}$$

In this equation, $r$ is the riskless instantaneous interest rate over the period considered, $q$ is the continuous dividend yield paid out by the stock, and $w_t^{\mathcal{Q}}$ is a Wiener process under the probability measure $\mathcal{Q}$.

Consider the valuation of the call option. The value of the call can be rewritten as:

$$\begin{aligned} C_{\mathrm{BS}} &= \mathrm{E}^{\mathcal{Q}}\left[e^{-rT}S_T\mathbf{1}_{\{S_T>K\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_T>K\}}\right] \\ &\equiv V_1 - V_2. \end{aligned} \tag{2.4}$$

From the definition of the indicator function, $\mathrm{E}^{\mathcal{Q}}[\mathbf{1}_{\{A\}}] = \mathcal{Q}[A]$. Hence, we have

$$V_2 = Ke^{-rT}\mathcal{Q}\left[S_T > K\right]. \tag{2.5}$$

Integrating the log-process for $S_t$ given above, we have

$$S_T = Se^{\alpha^{\mathcal{Q}}T + \sigma w_T^{\mathcal{Q}}} \tag{2.6}$$

where $S$ without subscript denotes the value of the stock price at time 0 (this notation will be used for the remainder of the paper).

Hence, we have

$$\begin{aligned} V_2 &= K e^{-rT} \mathcal{Q}\left[-\frac{w_T^{\mathcal{Q}}}{\sqrt{T}} < d^{\mathcal{Q}}(K,T)\right] \\ &= K e^{-rT} N\left(d^{\mathcal{Q}}(K,T)\right) \end{aligned} \tag{2.7}$$

where we have defined

$$d^{\mathcal{Q}}(K,T) = \frac{\log(S/K) + \alpha^{\mathcal{Q}} T}{\sigma\sqrt{T}} \tag{2.8}$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function. This result follows because $-w_T^{\mathcal{Q}}/\sqrt{T}$ is a standard normal random variable under $\mathcal{Q}$.

The first term in (2.4) can be calculated as follows. Define

$$\xi_T = \frac{S_T}{S} e^{-(r-q)T} = e^{\sigma w_T^{\mathcal{Q}} - \sigma^2 T/2}. \tag{2.9}$$

Note that $\xi_T$ is strictly positive and that $\mathrm{E}^{\mathcal{Q}}[\xi_T] = 1$. Hence, as shown in the Appendix, $\xi_T$ is a Radon-Nikodym derivative which can be used to define a probability measure $\mathcal{R}$ equivalent to $\mathcal{Q}$ such that

$$\mathrm{E}^{\mathcal{R}}[\mathbf{1}_{\{A\}}] = \mathrm{E}^{\mathcal{Q}}[\xi_T \mathbf{1}_{\{A\}}]. \tag{2.10}$$

Furthermore, if we define $w_t^{\mathcal{R}} = w_t^{\mathcal{Q}} - \sigma t$, then as also shown in the Appendix, $w_t^{\mathcal{R}}$ is a Wiener process under $\mathcal{R}$. Hence the dynamics for the stock process can be written as

$$d \log S_t = \alpha^{\mathcal{R}} dt + \sigma dw_t^{\mathcal{R}} \tag{2.11}$$

where the drift $\alpha^{\mathcal{R}}$ of the return under the $\mathcal{R}$ measure is

$$\alpha^{\mathcal{R}} = r - q + \frac{1}{2}\sigma^2 = \alpha^{\mathcal{Q}} + \sigma^2. \tag{2.12}$$

Hence the first term can be rewritten as:

$$\begin{aligned} V_1 &= Se^{-qT}\mathrm{E}^{\mathcal{Q}}\left[\xi_T \mathbf{1}_{\{S_T>K\}}\right] \\ &= Se^{-qT}\mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{S_T>K\}}\right] \\ &= Se^{-qT}\mathcal{R}\left[-\frac{w_T^{\mathcal{R}}}{\sqrt{T}} < d^{\mathcal{R}}(K,T)\right] \\ &= Se^{-qT}N\left(d^{\mathcal{R}}(K,T)\right) \end{aligned} \tag{2.13}$$

where we define

$$d^{\mathcal{R}}(K,T) = \frac{\log(S/K) + \alpha^{\mathcal{R}}T}{\sigma\sqrt{T}} = d^{\mathcal{Q}}(K,T) + \sigma\sqrt{T}. \tag{2.14}$$

Combining the expressions for $V_1$ and $V_2$ then leaves us with the usual Black-Scholes result. Similar calculations can be used to value the put option. Both formulas can be conveniently summarized in the following formula:

$$V_{\mathrm{BS}}(S,K,T,\sigma,r,q,\phi) = \phi Se^{-qT}N\left(\phi d^{\mathcal{R}}(K,T)\right) - \phi Ke^{-rT}N\left(\phi d^{\mathcal{Q}}(K,T)\right). \tag{2.15}$$

Note that the martingale methodology requires *no* integrals to be evaluated!

This example is meant to familiarize the reader with the probability measure $\mathcal{R}$ which will be used often in the following sections. Note that this change of measure is similar to the so-called "forward neutral" change of measure often used in fixed income securities pricing.[1] Note that the $\mathcal{R}$ measure is the equivalent martingale measure when the stock mutual fund $X_t = S_t e^{qt}$ is used as numeraire, as opposed to the $\mathcal{Q}$ measure which is the equivalent martingale measure when the money market fund $e^{rt}$ is used as numeraire. In particular, this means that for any derivative price $V_t$, we have

$$\frac{V_t}{X_t} = \mathrm{E}_t^{\mathcal{R}}\left[\frac{V_T}{X_T}\right], \tag{2.16}$$

or

$$V_t = S_t e^{-q(T-t)} \mathrm{E}_t^{\mathcal{R}} \left[ \frac{V_T}{S_T} \right]. \tag{2.17}$$

# 3 The Cliquet

Consider three dates, $0 < t_c < T$. At the terminal date $T$, the cliquet option pays

$$\max[\phi(S_T - K), \phi(S_{t_c} - K), 0] \tag{3.1}$$

where $\phi = 1$ for a call option and $\phi = -1$ for a put option. We refer to the intermediate date $t_c$ as the cliquet date.

Consider then the evaluation at date 0 of the cliquet call option. By the usual risk-neutral argument we have

$$\begin{aligned} C_{\mathrm{CL}} &= e^{-rT} \mathrm{E}^{\mathcal{Q}} \left[ \mathbf{1}_{\{S_{t_c} > S_T, S_{t_c} > K\}} (S_{t_c} - K) \right] + e^{-rT} \mathrm{E}^{\mathcal{Q}} \left[ \mathbf{1}_{\{S_T > S_{t_c}, S_T > K\}} (S_T - K) \right] \\ &\equiv V_1 + V_2. \end{aligned} \tag{3.2}$$

Using the law of iterated expectations the first term can be written as

$$V_1 = e^{-rT} \mathrm{E}^{\mathcal{Q}} \left[ \mathbf{1}_{\{S_{t_c} > K\}} (S_{t_c} - K) \mathrm{E}_{t_c}^{\mathcal{Q}} \left[ \mathbf{1}_{\{S_{t_c} > S_T\}} \right] \right]. \tag{3.3}$$

But

$$S_T = S_{t_c} e^{\alpha^{\mathcal{Q}} (T - t_c) + (w_T^{\mathcal{Q}} - w_{t_c}^{\mathcal{Q}})} \tag{3.4}$$

and hence we have

$$\mathrm{E}_{t_c}^{\mathcal{Q}} \left[ \mathbf{1}_{\{S_{t_c} > S_T\}} \right] = \mathcal{Q} \left( \frac{w_T^{\mathcal{Q}} - w_{t_c}^{\mathcal{Q}}}{\sqrt{T - t_c}} < -\frac{\alpha^{\mathcal{Q}}}{\sigma} \sqrt{T - t_c} \right) = N \left( -\frac{\alpha^{\mathcal{Q}}}{\sigma} \sqrt{T - t_c} \right) \tag{3.5}$$

Since this expression is a constant it can be taken out of the expectation. We are thus left with:

$$\begin{aligned} V_1 &= e^{-r(T-t_c)} N\left(-\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}\right) \mathrm{E}^{\mathcal{Q}}\left[e^{-rt_c}\mathbf{1}_{\{S_{t_c}>K\}}(S_{t_c}-K)\right] \\ &= e^{-r(T-t_c)} N\left(-\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}\right) C_{\mathrm{BS}}(S,K,t_c,\sigma,r,q) \end{aligned} \tag{3.6}$$

where $C_{\mathrm{BS}}$ is the familiar Black-Scholes European call option formula as given in (2.15).

For the second term we split the expectation into two parts and use the change of measure introduced in the previous section to find

$$V_2 = Se^{-qT}\mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{S_{t_c}<S_T, S_T>K\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_{t_c}<S_T, S_T>K\}}\right]. \tag{3.7}$$

As above, we find

$$\begin{aligned} \mathrm{E}^{\mathcal{Q}}[\mathbf{1}_{\{S_{t_c}<S_T, S_T>K\}}] &= \mathcal{Q}\left(-\frac{w_T^{\mathcal{Q}}-w_{t_c}^{\mathcal{Q}}}{\sqrt{T-t_c}} < \frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}, -\frac{w_T^{\mathcal{Q}}}{\sqrt{T}} < d^{\mathcal{Q}}(K,T)\right) \\ &= N_2\left(\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}, d^{\mathcal{Q}}(K,T); \sqrt{\frac{T-t_c}{T}}\right) \end{aligned} \tag{3.8}$$

where $N_2(a,b;\rho)$ is the bivariate cumulative normal distribution function with correlation $\rho$. Note that to get the second equality we have made use of the fact that at date 0, $-\frac{w_T^{\mathcal{Q}}-w_{t_c}^{\mathcal{Q}}}{\sqrt{T-t_c}}$ and $-\frac{w_T^{\mathcal{Q}}}{\sqrt{T}}$ are standard normal random variables. The correlation between them is found by noting that $\mathrm{Cov}(-\frac{w_T^{\mathcal{Q}}-w_{t_c}^{\mathcal{Q}}}{\sqrt{T-t_c}}, -\frac{w_T^{\mathcal{Q}}}{\sqrt{T}}) = \sqrt{\frac{T-t_c}{T}}$.

Similarly, the first term of $V_2$ is

$$\begin{aligned} \mathrm{E}^{\mathcal{R}}[\mathbf{1}_{\{S_{t_c}<S_T, S_T>K\}}] &= R\left(-\frac{w_T^{\mathcal{R}}-w_{t_c}^{\mathcal{R}}}{\sqrt{T-t_c}} < \frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_c}, -\frac{w_T^{\mathcal{R}}}{\sqrt{T}} < d^{\mathcal{R}}(K,T)\right) \\ &= N_2\left(\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_c}, d^{\mathcal{R}}(K,T); \sqrt{\frac{T-t_c}{T}}\right) \end{aligned} \tag{3.9}$$

where $d^{\mathcal{Q}}$ and $d^{\mathcal{R}}$ are as defined in (2.8) and (2.14). Hence we have:

$$V_2 = Se^{-qT}N_2\left(\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_c}, d^{\mathcal{R}}(K,T); \sqrt{\frac{T-t_c}{T}}\right) - Ke^{-rT}N_2\left(\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}, d^{\mathcal{Q}}(K,T); \sqrt{\frac{T-t_c}{T}}\right). \tag{3.10}$$

Similar calculations can be used to value the cliquet put option. Both the put and the call can be conveniently summarized by the following expression:

$$\begin{aligned} V_{\mathrm{CL}}(S,K,T,\sigma,r,q,t_c,\phi) &= e^{-r(T-t_c)}N\left(-\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c}\right)V_{\mathrm{BS}}(S,K,t_c,\sigma,r,q,\phi) \\ &\quad +\phi S e^{-qT}N_2\left(\phi\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_c},\phi d^{\mathcal{R}}(K,T);\sqrt{\frac{T-t_c}{T}}\right) \\ &\quad -\phi K e^{-rT}N_2\left(\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_c},\phi d^{\mathcal{Q}}(K,T);\sqrt{\frac{T-t_c}{T}}\right). \end{aligned} \tag{3.11}$$

# 4 The Ladder

Defining $L > K$ to be the ladder price the payoff to a Ladder option is:

$$\begin{cases} \max[S_T-K,L-K], & \text{if S has reached } L \text{ before maturity, or} \\ \max[S_T-K,0], & \text{otherwise.} \end{cases}$$

If $L \leq K$ the problem is trivial and the value of the ladder call option reduces to the value of a simple Black Scholes call.

Since the payoff of the ladder option will depend on the value of the maximum price attained by the stock price during the life of the option (which we define by $\overline{S}(T)$), its price will depend on the joint distribution of the stock price at maturity $S_T$ and $\overline{S}(T)$. More formally let us introduce the general notation for the running maximum of a process $X_t$:

$$\overline{X}(t) = \max_{0\leq s\leq t} X_s \tag{4.1}$$

Using the martingale approach developed in the previous section the value of the ladder call option can be expressed as follows:

$$C_L(S,K,T,\sigma,r,q,L) \quad = \quad e^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{\overline{S}(T)<L,S_T>K\}}(S_T-K)\right]$$

$$
\begin{aligned}
&+e^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\left\{\overline{S}(T)>L,S_T>L\right\}}(S_T-K)\right] \\
&+e^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\left\{\overline{S}(T)>L,S_T<L\right\}}(L-K)\right]
\end{aligned} \tag{4.2}
$$

Using the same trick, namely splitting the expectations and applying Girsanov's theorem when necessary we can rewrite the value of the option as follows:

$$
\begin{aligned}
C_L(S,K,T,\sigma,r,q,L) \;=\;& Se^{-qT}\mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\left\{\overline{S}(T)<L,S_T>K\right\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\left\{\overline{S}(T)<L,S_T>K\right\}}\right] \\
&+Se^{-qT}\mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\left\{\overline{S}(T)>L,S_T>L\right\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\left\{\overline{S}(T)>L,S_T>L\right\}}\right] \\
&+e^{-rT}(L-K)\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\left\{\overline{S}(T)>L,S_T<L\right\}}\right]
\end{aligned} \tag{4.3}
$$

But this can be rewritten with our previously defined notation as:

$$
\begin{aligned}
&C_L(S,K,T,\sigma,r,q,L) = \\
&\quad Se^{-qT}\mathcal{R}\left(\frac{\overline{X}_T}{\sqrt{T}}<l,\frac{X_T}{\sqrt{T}}>k)\right) - Ke^{-rT}\mathcal{Q}\left(\frac{\overline{X}_T}{\sqrt{T}}<l,\frac{X_T}{\sqrt{T}}>k\right) \\
&\quad +Se^{-qT}\mathcal{R}\left(\frac{\overline{X}_T}{\sqrt{T}}>l,\frac{X_T}{\sqrt{T}}>l\right) - Ke^{-rT}\mathcal{Q}\left(\frac{\overline{X}_T}{\sqrt{T}}>l,\frac{X_T}{\sqrt{T}}>l\right) \\
&\quad +(L-K)\mathcal{Q}\left(\frac{\overline{X}_T}{\sqrt{T}}>l,\frac{X_T}{\sqrt{T}}<l\right)
\end{aligned} \tag{4.4}
$$

Where we use the following notations:

$$X_t = \log(S_t/S) \tag{4.5}$$

$$l = \log(L/S) \tag{4.6}$$

$$k = \log(K/S) \tag{4.7}$$

Note from our previous results that $X_t$ is a $(\alpha^{\mathcal{Q}},\sigma)$ standard (i.e starting at 0) brownian motion under $\mathcal{Q}$ and $(\alpha^{\mathcal{R}},\sigma)$ standard brownian motion under $\mathcal{R}$. Whence, using the formulas for the joint distribution of the running maximum of a brownian motion and the value itself given in the appendix B, we get the following result:

$$
\begin{aligned}
&V_L(S,K,T,\sigma,r,q,L) = \\
&\quad V_{\mathrm{BS}}(S,K,T,\sigma,r,q) + \phi(L-K)e^{-rT}\left(\frac{L}{S}\right)^{\frac{2\alpha^{\mathcal{Q}}}{\sigma^2}} N\left(-\phi\frac{l+\alpha^{\mathcal{Q}}T}{\sigma\sqrt{T}}\right) \\
&\quad - Se^{-qT}\left(\frac{L}{S}\right)^{\frac{2\alpha^{\mathcal{R}}}{\sigma^2}}\left[N\left(-\phi\frac{l+\alpha^{\mathcal{R}}T}{\sigma\sqrt{T}}\right) - N\left(\phi\frac{k-2l-\alpha^{\mathcal{R}}T}{\sigma\sqrt{T}}\right)\right] \\
&\quad + Ke^{-rT}\left(\frac{L}{S}\right)^{\frac{2\alpha^{\mathcal{Q}}}{\sigma^2}}\left[N\left(-\phi\frac{l+\alpha^{\mathcal{Q}}T}{\sigma\sqrt{T}}\right) - N\left(\phi\frac{k-2l-\alpha^{\mathcal{Q}}T}{\sigma\sqrt{T}}\right)\right]
\end{aligned}
\tag{4.8}
$$

# 5 The Discrete Shout

The continuous shout is an option that allows its holder to "shout" at any one date before expiration and obtain the following payoff at expiration:

$$\max\left[\phi(S_s-K), \phi(S_T-K), 0\right] \tag{5.1}$$

where $S_s$ is the value of the stock on the shout date $t_s$, and $\phi = 1$ $(-1)$ for a call (put). A discrete shout is like the continuous version, except that shouting is allowed on only a discrete set of dates. Note that the option holder can shout at most once during the life of the option. If only one intermediate shout date is allowed, then the discrete shout reduces to the cliquet option studied in the previous section.

In this section we apply the martingale methodology to evaluate the discrete shout option. The continuous shout can be obtained as the limit of the discrete shout as the maximum time interval between shout dates tends to zero, and hence can be arbitrarily closely approximated by the discrete shout.[2]

Consider now the evaluation of the $n$-date discrete shout call option. Assume that the allowed shout dates are $0 < t_1 < t_2 < \cdots < t_n < t_{n+1} \equiv T$. Let $C^n_{\mathrm{SH}}(S,K,T,\sigma,r,q;t_1,\ldots,t_n)$ denote the value of the shout with maturity $T$ and $n$ remaining shout dates. Suppose that at date $t_j$ the option holder has not yet shouted. If he shouts at $t_j$, the value of the option at this date is given by

$$\begin{aligned} C_{\mathrm{SH}}(t_j) &= e^{-r(T-t_j)}\mathrm{E}_j^{\mathcal{Q}}\left[\max(S_j-K, S_T-K, 0)\right] \\ &= e^{-r(T-t_j)}\mathrm{E}^{\mathcal{Q}}\left[(S_j-K)\right] + e^{-r(T-t_j)}\mathrm{E}^{\mathcal{Q}}\left[(S_T-S_j)\mathbf{1}_{\{S_T>S_j\}}\right] \\ &= (S_j-K)e^{-r(T-t_j)} + C_{\mathrm{BS}}(S_j, S_j, T-t_j, \sigma, r, q). \end{aligned} \tag{5.2}$$

If he does not shout at $t_j$, he is left with $C_{\mathrm{SH}}^{n-j}(S, K, T-t_j, \sigma, r, q; t_{j+1}, \cdots, t_n)$. Thus the option holder will shout at date $t_j$, $j = 1, \ldots, n$, if and only if the stock price is greater than a critical value $S_j^*$ determined recursively by

$$(S_j^*-K)e^{-r(T-t_j)} + C_{\mathrm{BS}}(S_j^*, S_j^*, T-t_j, \sigma, r, q) = C_{\mathrm{SH}}^{n-j}(S_j^*, K, T-t_j, \sigma, r, q; t_{j+1}-t_j, \ldots, t_n-t_j) \tag{5.3}$$

where we use the convention that $C_{\mathrm{SH}}^0 = C_{\mathrm{BS}}$.

Using the shouting criterion derived above, the value of the discrete shout is given by

$$\begin{aligned} &C_{\mathrm{SH}}^n(S, K, T, \sigma, r, q; t_1, \cdots, t_n) = \\ &\quad \sum_{j=1}^{n+1} e^{-rt_j}\mathrm{E}^{\mathcal{Q}}\left[\left\{(S_j-K)e^{-r(T-t_j)} + C_{\mathrm{BS}}(S_j, S_j, T-t_j, \sigma, r, q)\right\}\mathbf{1}_{\{S_1<S_1^*, S_2<S_2^*, \cdots, S_{j-1}<S_{j-1}^*, S_j\geq S_j^*\}}\right] \end{aligned} \tag{5.4}$$

with

$$S_{n+1}^* \equiv K \tag{5.5}$$

and $S_1^*, \ldots, S_n^*$ are defined by (5.3).

We proceed to solve this as in the previous sections. Define functions $\mathcal{I}^{\mathcal{Q}}{}_j$ and $\mathcal{I}^{\mathcal{R}}{}_j$ as

$$\mathcal{I}^{\mathcal{Q}}{}_j = \mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_1<S_1^*, \cdots, S_j<S_j^*\}}\right] \tag{5.6}$$

$$\mathcal{I}^{\mathcal{R}}{}_j = \mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{S_1<S_1^*, \cdots, S_j<S_j^*\}}\right]. \tag{5.7}$$

Using the Black-Scholes formula, simple algebra leads directly to

$$C^n_{\mathrm{SH}}(S,K,T,\sigma,r,q;t_1,\cdots,t_n)=\sum_{j=1}^{n+1}\left\{Se^{-qt_j}({\mathcal{I}^{\mathcal{R}}}_{j-1}-{\mathcal{I}^{\mathcal{R}}}_{j})A_j-Ke^{-rT}({\mathcal{I}^{\mathcal{Q}}}_{j-1}-{\mathcal{I}^{\mathcal{Q}}}_{j})\right\} \tag{5.8}$$

where $A_j$ is defined by:

$$A_j=e^{-q(T-t_j)}N\left(\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_j}\right)+e^{-r(T-t_j)}N\left(-\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_j}\right). \tag{5.9}$$

Note that ${\mathcal{I}^{\mathcal{Q}}}_j$ and ${\mathcal{I}^{\mathcal{R}}}_j$ can be rewritten more explicitly as follows

$$\begin{aligned}{\mathcal{I}^{\mathcal{Q}}}_j &= \mathcal{Q}\left(\frac{w_1^{\mathcal{Q}}}{\sqrt{t_1}}<-d^{\mathcal{Q}}(S_1^*,t_1),\ldots,\frac{w_j^{\mathcal{Q}}}{\sqrt{t_j}}<-d^{\mathcal{Q}}(S_j^*,t_j)\right)\\ &= N_j\left(-d^{\mathcal{Q}}(S_1^*,t_1),\cdots,-d^{\mathcal{Q}}(S_j^*,t_j);\{C_{ik}\}\right)\end{aligned} \tag{5.10}$$

where

$$C_{ik}=\sqrt{\frac{t_{i\wedge k}}{t_{i\vee k}}} \tag{5.11}$$

Here, $N_j(a_1,\cdots,a_j;\{\rho_{i,k}\})$ denotes the joint $j$-nomial standard normal cumulative distribution function with correlation matrix $\{\rho_{ik}\}$. Similarly we find

$${\mathcal{I}^{\mathcal{R}}}_j=N_j\left(-d^{\mathcal{R}}(S_1^*,t_1),\ldots,-d^{\mathcal{R}}(S_j^*,t_j);\{C_{ik}\}\right). \tag{5.12}$$

The same method can be used to derive the price of the discrete shout put option. Setting $\phi=1$ $(-1)$ if the discrete shout is a call (put), the general formula can then be written as

$$V^n_{\mathrm{SH}}(S,K,T,\sigma,r,q,\phi;t_1,\cdots,t_n)=\sum_{j=1}^{n+1}\left\{\phi Se^{-qt_j}({\mathcal{I}^{\mathcal{R}}}_{j-1}-{\mathcal{I}^{\mathcal{R}}}_{j})A_j\right\}-\phi Ke^{-rT}(1-{\mathcal{I}^{\mathcal{Q}}}_{n+1}) \tag{5.13}$$

where $I_0^{\mathcal{R}}=1$ and for $j=1,\ldots,n+1$

$${\mathcal{I}^{\mathcal{Q}}}_j = N_j\left(-\phi d^{\mathcal{Q}}(S_1^*,t_1),\ldots,-\phi d^{\mathcal{Q}}(S_j^*,t_j);\{C_{ik}\}\right) \tag{5.14}$$

$${\mathcal{I}^{\mathcal{R}}}_j = N_j\left(-\phi d^{\mathcal{R}}(S_1^*,t_1),\ldots,-\phi d^{\mathcal{R}}(S_j^*,t_j);\{C_{ik}\}\right) \tag{5.15}$$

$$A_j = e^{-q(T-t_j)}N\left(\phi\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{T-t_j}\right)+e^{-r(T-t_j)}N\left(-\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{T-t_j}\right). \tag{5.16}$$

The critical prices $S_1^*, S_2^*, \ldots, S_{n+1}^*$ are defined recursively by

$$S_{n+1}^* = K \tag{5.17}$$

$$\phi(S_j^* - K)e^{-r(T-t_j)} + V_{\mathrm{BS}}(S_j^*, S_j^*, T-t_j, \sigma, r, q, \phi) = V_{\mathrm{SH}}^{n-j}(S_j^*, K, T-t_j, \sigma, r, q, \phi; t_{j+1}-t_j, \ldots, t_n-t_j). \tag{5.18}$$

## 6 Option on a Discrete Maximum

In this section we apply the same methodology to a new option—an option on a discrete maximum. Consider a set of dates, $0 \le t_1 < t_2 < \cdots < t_n \le T$. The payoff at maturity of this option defined to be

$$V_{\mathrm{DM}}^n(T) = \max\left[\phi\left(\max[S_1, S_2, \ldots, S_n] - K\right), 0\right] \tag{6.1}$$

where $S_i$ is the value of the stock at date $t_i$, and $\phi = 1$ for a call and $\phi = -1$ for a put.

The value of a call option on the discrete maximum at time 0 is given by

$$\begin{aligned} C_{\mathrm{DM}}^n &= e^{-rT} \mathrm{E}^{\mathcal{Q}}\left[\max(S_1, \ldots, S_n) \mathbf{1}_{\{\max(S_1,\ldots,S_n)>K\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{\max(S_1,\ldots,S_n)>K\}}\right] \\ &\equiv V_1 + V_2. \end{aligned} \tag{6.2}$$

The second term can be simplified by noting that

$$\begin{aligned} V_2 &= Ke^{-rT}\mathcal{Q}\left(\max(S_1, \ldots, S_n) > K\right) \\ &= Ke^{-rT}\left[1 - \mathcal{Q}\left(\max(S_1, \ldots, S_n) < K\right)\right] \\ &= Ke^{-rT}\left[1 - \mathcal{Q}\left(S_1 < K, S_2 < K, \ldots, S_n < K\right)\right] \\ &= Ke^{-rT}\left[1 - N_n\left(-d^{\mathcal{Q}}(K, t_1), \ldots, -d^{\mathcal{Q}}(K, T_n); \{C_{ik}^{(1)}\}\right)\right], \end{aligned} \tag{6.3}$$

where the correlation matrix is

$$C_{ik}^{(1)} = \sqrt{\frac{t_{i\wedge k}}{t_{i\vee k}}}. \tag{6.4}$$

The first term can be reexpressed in the following form:

$$\begin{aligned} V_1 &= e^{-rT}\sum_{j=1}^{n} \mathrm{E}^{\mathcal{Q}}\left[S_j \mathbf{1}_{\{S_j>S_1,\ldots,S_j>S_n,S_j>K\}}\right] \\ &\equiv \sum_{j=1}^{n} V_{1j}. \end{aligned} \tag{6.5}$$

Applying the law of iterated expectations, the $V_{1j}$ term simplifies to

$$V_{1j} = e^{-rT}\mathrm{E}^{\mathcal{Q}}\left[S_j \mathbf{1}_{\{S_j>S_1,\ldots,S_j>S_{j-1},S_j>K\}}\mathrm{E}_{t_j}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_j>S_{j+1},\ldots,S_j>S_n\}}\right]\right]. \tag{6.6}$$

We now define

$$\begin{aligned} I_{n-j} &= \mathrm{E}_{t_j}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_j>S_{j+1},\ldots,S_j>S_n\}}\right] \\ &= N_{n-j}\left(-\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_{j+1}-t_j}, -\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_{j+2}-t_j}, \ldots, -\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_n-t_j}; \{C_{ik}^{(2)}\}\right) \end{aligned} \tag{6.7}$$

where

$$C_{ik}^{(2)} = \sqrt{\frac{t_{j+i\wedge k}-t_j}{t_{j+i\vee k}-t_j}}, \quad 1 \le i,j \le n-j. \tag{6.8}$$

Notice that $I_{n-j}$ is nonstochastic and thus can be taken out of the expectation. We are then left with

$$V_{1j} = I_{n-j}\mathrm{E}^{\mathcal{Q}}\left[e^{-rT}S_j \mathbf{1}_{\{S_j>S_1,\ldots,S_j>S_{j-1},S_j>K\}}\right]. \tag{6.9}$$

Applying the same change of measure from $\mathcal{Q}$ to $\mathcal{R}$ as in the previous sections, we obtain

$$V_{1j} = I_{n-j}e^{-rT}Se^{(r-q)t_j}\mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{S_j>S_1,\ldots,S_j>S_{j-1},S_j>K\}}\right]. \tag{6.10}$$

Defining the expectation to be $H_j$ and evaluating, we find

$$\begin{aligned} H_j &= \mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{S_j>S_1,\cdots,S_j>S_{j-1},S_j>K\}}\right] \\ &= N_j\left(\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j-t_1}, \frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j-t_2}, \ldots, \frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j-t_{j-1}}, d^{\mathcal{R}}(K,t_j); \{C_{ik}^{(3)}\}\right) \end{aligned} \tag{6.11}$$

where the correlation matrix is given by

$$\begin{aligned} C_{ik}^{(3)} &= \sqrt{\frac{t_j - t_{i\vee k}}{t_j - t_{i\wedge k}}}, \quad i,k \neq j \\ C_{ij}^{(3)} &= C_{ji}^{(3)} = \sqrt{\frac{t_j - t_i}{t_j}}, \quad i \neq j. \end{aligned} \tag{6.12}$$

Combining the above results yields the final answer for the call. The put option on the discrete maximum can be calculated in a similar manner. We can summarize both results in the following formula:

$$V_{\mathrm{DM}}^n(S,K,T,\sigma,r,q,\phi;t_1,\ldots,t_n) = \tag{6.13}$$

$$\sum_{j=1}^{n} \phi H_j I_{n-j} S e^{(r-q)t_j - rT} - \phi K e^{-rT}\left[1 - N_n\left(-\phi d^{\mathcal{Q}}(K,t_1),\ldots,-\phi d^{\mathcal{Q}}(K,T_n);\{C_{ik}^{(1)}\}\right)\right] \tag{6.14}$$

with

$$H_j = N_j\left(\phi\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j - t_1}, \phi\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j - t_2},\ldots,\phi\frac{\alpha^{\mathcal{R}}}{\sigma}\sqrt{t_j - t_{j-1}}, \phi d^{\mathcal{R}}(K,t_j);\{C_{ik}^{(3)}\}\right) \tag{6.15}$$

$$I_{n-j} = N_{n-j}\left(-\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_{j+1} - t_j}, -\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_{j+2} - t_j},\ldots,-\phi\frac{\alpha^{\mathcal{Q}}}{\sigma}\sqrt{t_n - t_j};\{C_{ik}^{(2)}\}\right). \tag{6.16}$$

The above results can also be used to price a *discrete lookback put* option. The continuous lookback put option (Goldman, Sosin, and Gatto (1979)) has terminal payout

$$P_{\mathrm{CLB}}(T) = \max_{0 \leq t \leq T} S(t) - S(T). \tag{6.17}$$

The discrete version we consider here has a similar terminal payout, namely

$$P_{\mathrm{DLB}}(T) = \max(S_1, S_2, \ldots, S_n) - S(T). \tag{6.18}$$

The value of the discrete lookback put can be obtained using our above results, setting $K = 0$ and subtracting the present value of the terminal stock price. The procedure is straightforward and yields the final answer

$$P_{\mathrm{DLB}}(0) = V_{\mathrm{DM}}^n(S, K=0, T, \sigma, r, q, \phi = 1; t_1, \ldots, t_n) - Se^{-qT}. \tag{6.19}$$

# 7 Options on the Maximum of $n$ Assets

In Stulz (1982) and Johnson (1987), closed form solutions are derived for the option on the maximum of, respectively, two and $n$ correlated assets. We show here that these solutions can be found easily using the martingale approach. Besides showing how computationally efficient this method can be, our example also shows how the Girsanov theorem can be applied in a multidimensional framework (that is with $n$ correlated sources of risk).

Consider a generalized Black-Scholes economy with $n$ correlated stocks and a money market fund. The interest rate $r$, the dividend yields $q_i$, and the correlations $\rho_{ij}$, are all assumed constant. Such an economy is known to be viable and dynamically complete, hence any derivative security can be priced by arbitrage in a risk-neutral setting. In particular, under the risk-neutral measure $\mathcal{Q}$, the stock prices are given by

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dw_i^{\mathcal{Q}}(t), \quad i = 1, \ldots, n. \tag{7.1}$$

By definition, the correlation between $w_i^{\mathcal{Q}}$ and $w_j^{\mathcal{Q}}$ is $\rho_{ij}$.

Consider now an option on the maximum terminal value of these $n$ stocks. The payoff of this option is

$$V_{\mathrm{MX}}^n(T) = \max\left[\phi\left(\max[S_1(T), \cdots, S_n(T)] - K\right), \, 0\right], \tag{7.2}$$

where as usual $\phi = 1$ for a call and $\phi = -1$ for a put. Note that the payout of this option is superficially similar to that of the option on the discrete maximum studied in the previous section. But here, $S_i$ refers to the terminal value of stock $i$, while previously it referred to the value of a single stock at a previous date $t_i$.

Consider the value at date 0 of a call option:

$$C_{\mathrm{MX}}^n \;=\; \mathrm{E}^{\mathcal{Q}}\left[e^{-rT}\max[S_1(T), \cdots, S_n(T)]\mathbf{1}_{\{\max[S_1(T),\cdots,S_n(T)]>K\}}\right] - Ke^{-rT}\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{\max[S_1(T),\cdots,S_n(T)]>K\}}\right]$$

$$\equiv \quad V_1 + V_2. \tag{7.3}$$

The second term can easily be computed:

$$\begin{aligned} V_2 &= K e^{-rT} \mathcal{Q}\left(\max[S_1, S_2, \ldots, S_n] > K\right) \\ &= K e^{-rT} \left[1 - \mathcal{Q}\left(S_1(T) < K, \ldots, S_n(T) < K\right)\right] \\ &= K e^{-rT} \left[1 - N_n\left(-d_1^{\mathcal{Q}}(K,T), \cdots, -d_n^{\mathcal{Q}}(K,T); \{\rho_{ij}\}\right)\right] \end{aligned} \tag{7.4}$$

with $d_i^{\mathcal{Q}}$ defined for the stock $S_i$ as $d^{\mathcal{Q}}$ is for stock $S$ in the previous sections.

The first term can be rewritten as the following sum:

$$\begin{aligned} V_1 &= \sum_{j=1}^{n} \mathrm{E}^{\mathcal{Q}}\left[e^{-rT} S_j(T) \mathbf{1}_{\{S_j(T) > S_1(T), \ldots, S_j(T) > S_n(T), S_j(T) > K\}}\right] \\ &\equiv \sum_{j=1}^{n} V_{1j} \end{aligned} \tag{7.5}$$

To determine $V_{1j}$, begin by defining

$$\xi_T^{(j)} = e^{-(r-q_j)T} S_j(T) / S_j(0). \tag{7.6}$$

Note that $\xi_T^{(j)}$ is strictly positive and has $\mathcal{Q}$-expectation one. Hence we can define a new probability measure $\mathcal{R}^{(j)}$, equivalent to $\mathcal{Q}$, by

$$\mathcal{R}^{(j)}[A] \equiv \mathrm{E}^{\mathcal{Q}}[\xi_T^{(j)} \mathbf{1}_{\{A\}}]. \tag{7.7}$$

The likelihood ratio $\xi_t^{(j)} = \mathrm{E}_t^{\mathcal{Q}}[\xi_T^{(j)}]$ is an exponential $\mathcal{Q}$-martingale with the following dynamics

$$\frac{d\xi_t^{(j)}}{\xi_t^{(j)}} = \sigma_j dw_j^{\mathcal{Q}}(t) \equiv \beta^{(j)\top} dw^{\mathcal{Q}}(t) \tag{7.8}$$

where $\beta^{(j)} = (0, \ldots, 0, \sigma_j, 0, \ldots, 0)^\top$ and $w(t)$ is the $n$-dimensional vector of correlated Wiener processes.

As shown in the Appendix, the vector process

$$dw^{(j)} = dw^{\mathcal{Q}} - \rho\beta^{(j)}dt \tag{7.9}$$

is a vector of Wiener processes under the equivalent measure $\mathcal{R}^{(j)}$ with the same correlation structure as the $w^{\mathcal{Q}}$'s have under $Q$. That is,

$$\mathrm{Cov}(dw_i^{(j)}, dw_k^{(j)}) = \rho_{ik}dt. \tag{7.10}$$

For our case here, this simplifies to

$$dw_i^{(j)} = dw_i^{\mathcal{Q}} - \rho_{ij}\sigma_j dt. \tag{7.11}$$

Hence we may rewrite the stock processes relative to the $\{w^{(j)}\}$ as follows:

$$d\log S_i = \alpha_i^{(j)}dt + \sigma_i dw_i^{(j)}, \quad i = 1, \ldots, N, \tag{7.12}$$

where

$$\alpha_i^{(j)} = r - q_i + \rho_{ij}\sigma_i\sigma_j - \frac{1}{2}\sigma_i^2, \quad i = 1, \ldots, N. \tag{7.13}$$

Hence note that under the new measure the terminal stock prices are still lognormally distributed with only a constant shift in the drift term.

With these results in mind, then, we can write $V_{1j}$ as

$$\begin{aligned} V_{1j} &= S_j e^{-q_j T}\mathrm{E}^{\mathcal{R}_j}\left[\mathbf{1}_{\{S_j(T)>S_1(T),\ldots,S_j(T)>S_n(T),S_j(T)>K\}}\right] \\ &= S_j e^{-q_j T}\mathcal{R}_j\left(-\frac{\sigma_j w_j^{(j)}(T) - \sigma_k w_k^{(j)}(T)}{\Omega_{jk}\sqrt{T}} < e_{jk}, \forall k \neq j, -\frac{w_j^{(j)}(T)}{\sqrt{T}} < d_j^{(j)}(K,T)\right) \\ &= N_n\left(e_{j1}, \ldots, e_{jj-1}, d_j^{(j)}(K,T), e_{jj+1}, \ldots, e_{jn}; \{C_{ik}\}\right) \end{aligned} \tag{7.14}$$

where

$$\Omega_{jk} = \sqrt{\sigma_j^2 - 2\rho_{jk}\sigma_j\sigma_k + \sigma_k^2} \tag{7.15}$$

$$
\begin{aligned}
e_{jk} &= \frac{\log(S_j/S_k) + (q_k - q_j + \Omega_{jk}^2/2)T}{\Omega_{jk}\sqrt{T}} && (7.16)\\
d_j^{(j)}(K,T) &= \frac{\log(S_j/K) + (r - q_j + \sigma_j^2)T}{\sigma_j\sqrt{T}} && (7.17)
\end{aligned}
$$

and the correlation matrix $C$ is

$$
\begin{aligned}
C_{ik} &= \frac{\sigma_j^2 + \rho_{ik}\sigma_i\sigma_k - \rho_{ij}\sigma_i\sigma_j - \rho_{jk}\sigma_j\sigma_k}{\Omega_{ij}\Omega_{jk}}, \quad i,k \neq j && (7.18)\\
C_{jk} &= \frac{\sigma_j - \rho_{jk}\sigma_k}{\Omega_{jk}}, \quad j \neq k. && (7.19)
\end{aligned}
$$

The final answer can be expressed for both the put and call using the above notation. We find

$$
\begin{aligned}
V_{\mathrm{MX}}^n(\{S_i\},\{\sigma_i\},\{\rho_{ij}\},r,q,\phi) &= \sum_{j=1}^{n} \phi S_j e^{-q_j T} N_n\left(\phi e_{j1},\ldots,\phi e_{jj-1},\phi d_j^{(j)}(K,T),\phi e_{jj+1},\ldots,\phi e_{jn};\{C_{ik}\}\right)\\
&\quad - \phi K e^{-rT}\left[1 - N_n\left(-\phi d_1^{\mathcal{Q}}(K,T),\ldots,-\phi d_n^{\mathcal{Q}}(K,T);\{\rho_{ik}\}\right)\right]. \qquad (7.20)
\end{aligned}
$$

## 8 Conclusion

Our goal in this paper has been to show the power and simplicity of the martingale method in pricing several classes of complex exotic options. We have also introduced closed-form solutions for the cliquet, the ladder, the shout and the discrete lookback. We hope that our results help convince wary readers that the martingale technology is an important practical tool in derivative pricing, and that readers previously unfamiliar with these tools can profit from them in the future.

# A Girsanov's Theorem

In this Appendix we present an intuitive overview of the body of results generally referred to as Girsanov's theorem. There are many rigorous treatments of this material, and the mathematically inclined reader is encouraged to refer to these works.[3] Here, we purposely keep the presentation simple in order that the martingale methodology may reach a wider audience.

## A.1 One-Dimensional Case

We begin by defining a Wiener process and the probability space it generates in a discrete approximation to the usual continuous-time economy. Consider a time interval $[0,T]$ partitioned into $N$ equal subintervals, $0 \equiv t_0 < t_1 < \cdots < t_N \equiv T$, where $t_i = i\Delta t$ and $\Delta t = T/N$. For notational convenience, for an arbitrary process $x(t)$ define $x_i = x(t_i)$ and $\Delta x_i = x_i - x_{i-1}$. By definition, a process $w(t)$ is a Wiener process if and only if its increments $(\Delta w_1, \Delta w_2, \ldots, \Delta w_N)$ are independent and identically distributed (IID) normal random variables with mean 0 and variance $\Delta t$. Define a probability space $\Omega$ to consist of all possible $N$-tuples of increments $\{(\Delta w_1, \Delta w_2, \ldots, \Delta w_N)\}$, and define a corresponding probability measure $\mathcal{P}$ by

$$\mathcal{P}\left(\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N\right) = \prod_{i=1}^{N} \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \cdot dx_i. \tag{A.1}$$

(The notation $\Delta w_i \in dx_i$ means that $x_i < \Delta w_i < x_i + dx_i$.) Note that this is just the distribution function for $N$ IID normal variables with mean 0 and variance $\Delta t$.

A random variable $Z$ defined on $(\Omega, \mathcal{P})$ is a function from $\Omega \to$ R. Hence, for a point $\omega = (\Delta w_1, \ldots, \Delta w_N) \in \Omega$, we may write

$$Z(\omega) = z(\Delta w_1, \ldots, \Delta w_N) \tag{A.2}$$

for some function $z : \mathrm{R}^N \to \mathrm{R}$. The expectation of $Z$ under $\mathcal{P}$ can thus be expressed as

$$\mathrm{E}^{\mathcal{P}}[Z] = \int \cdots \int \left( \prod_{i=1}^{N} dx_i \, \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \right) z(x_1, \ldots, x_N). \tag{A.3}$$

Similarly, the conditional expectation of $Z$ at time $t = t_n$ can be written as

$$\mathrm{E}_t^{\mathcal{P}}[Z] = \int \cdots \int \left( \prod_{i=n+1}^{N} dx_i \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \right) z(x_1, \ldots, x_N). \tag{A.4}$$

Stochastic integrals of the Ito form can be approximated by sums in the following way:

$$\int_0^T f(t) dw(t) \sim \sum_{i=1}^{N} f_{i-1} \, \Delta w_i. \tag{A.5}$$

Note here that the function $f$ is evaluated at the left endpoint of the time interval, as required in the definition of the Ito integral.

Let us now turn to the main assertion of this section. Consider an adapted process $\beta(t)$, where here adapted means that $\beta(t)$ depends at most on the path of $w$ up to date $t$, and perhaps $t$ explicitly. That is, for $t = t_n$, we may write

$$\beta(t) = \beta_n(\Delta w_1, \Delta w_2, \ldots, \Delta w_n). \tag{A.6}$$

Suppose we define a new stochastic process $w'(t)$ by

$$w'(t) = w(t) - \int_0^t \beta(s) \, ds. \tag{A.7}$$

*We claim that there exists a probability measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, under which $w'(t)$ is itself a standard Wiener process.*

To prove this assertion, and to find an explicit form for $\mathcal{Q}$, begin by writing the differential of (A.7) in discrete form as

$$\Delta w_i' = \Delta w_i - \beta_{i-1} \Delta t. \tag{A.8}$$

Note that because $\beta$ is an adapted process, there is an invertible relation between the paths $(\Delta w_1, \Delta w_2, \ldots, \Delta w_N)$ and $(\Delta w'_1, \Delta w'_2, \ldots, \Delta w'_N)$. Furthermore, the Jacobian of the transformation between the primed and unprimed spaces is

$$J = \left| \frac{\partial \Delta w'_i}{\partial \Delta w_j} \right| = 1. \tag{A.9}$$

Now, let us define a candidate probability measure $\mathcal{Q}$ by

$$\mathcal{Q}(\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N) = \mathcal{P}\left(\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N\right) \cdot \xi(x_1, \ldots, x_N) \tag{A.10}$$

where $\xi$ is given by

$$\xi(\Delta w_1, \ldots, \Delta w_N) = e^{\sum_{i=1}^N \beta_{i-1} \Delta w_i - \frac{1}{2} \sum_{i=1}^N \beta_{i-1}^2 \Delta t}. \tag{A.11}$$

Note that $\xi$ is the discrete version of the continuous variable $\exp\left( \int_0^T \beta(t) dw(t) - \frac{1}{2} \int_0^T \beta(t)^2(t) dt \right)$. We claim that $\mathcal{Q}$ is in fact a probability measure on $\Omega$, that $\mathcal{Q}$ is equivalent to $\mathcal{P}$, and that $w'(t)$ is a Wiener process under $\mathcal{Q}$.

To show that $\mathcal{Q}$ is a probability measure, note only that

$$\begin{aligned}
\mathcal{Q}(\Omega) &= \int \cdots \int \mathcal{P}(\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N) \xi(x_1, \ldots, x_N) \\
&= \int \cdots \int \left( \prod_{i=1}^N \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi \Delta t}}\, dx_i \right) e^{\sum_{i=1}^N \beta_{i-1} x_i - \frac{1}{2} \beta_{i-1}^2 \Delta t} \\
&= \int \cdots \int \left( \prod_{i=1}^N \frac{e^{-x_i'^2/2\Delta t}}{\sqrt{2\pi \Delta t}}\, dx'_i \right) \\
&= 1,
\end{aligned} \tag{A.12}$$

where $x'_i = x_i - \beta_{i-1} \Delta t$. Note that on the third line we have made use of the fact that the Jacobian of the transformation between the primed and and unprimed variables is one. Equivalence of $\mathcal{P}$ and $\mathcal{Q}$ follows because $\xi$ is strictly positive, hence $\mathcal{Q}(A) = 0$ if and only if $\mathcal{P}(A) = 0$. To show that

$w'(t)$ is a Wiener process under $\mathcal{Q}$, note that

$$\begin{aligned}
\mathcal{Q}(\Delta w_1' \in dx_1', \ldots, \Delta w_N' \in dx_N') &= \mathcal{Q}(\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N) \\
&= \left( \prod_{i=1}^{N} \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \, dx_i \right) \cdot \xi(x_1, \ldots, x_N) \\
&= \prod_{i=1}^{N} \frac{e^{-x_i'^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \, dx_i' \qquad \text{(A.13)}
\end{aligned}$$

where $x_i' = x_i - \beta_{i-1}\Delta t$. We recognize the final line of this equation as the formula for the joint density of $N$ IID normal variables with mean 0 and variance $\Delta t$, hence $w'(t)$ is a Wiener process under $\mathcal{Q}$ as claimed.

Note that the above results imply that expectations under $\mathcal{Q}$ are related to expectations under $\mathcal{P}$ by

$$\mathrm{E}^{\mathcal{Q}}[Z] = \mathrm{E}^{\mathcal{P}}[\xi Z]. \qquad \text{(A.14)}$$

We may derive how conditional expectations are related as follows. Define the *likelihood ratio*

$$\xi(t) = \mathrm{E}_t^{\mathcal{P}}[\xi]. \qquad \text{(A.15)}$$

Note that $\xi(0) = 1$ and $\xi(T) = \xi$. We may derive an explicit expression for $\xi(t)$ as follows. Note that for $t = t_n$, we have

$$\begin{aligned}
\xi(t) &= \int \cdots \int \mathcal{P}(\Delta w_{n+1} \in dx_{n+1}, \ldots, \Delta w_N \in dx_N) \xi(\Delta w_1, \ldots, \Delta w_n, \Delta x_{n+1}, \ldots, \Delta x_N) \\
&= e^{\sum_{i=1}^{n} \beta_{i-1}\Delta w_i - \frac{1}{2}\sum_{i=1}^{n} \beta_{i-1}^2 \Delta t} \int \cdots \int \left( \prod_{i=n+1}^{N} \frac{e^{-x_i'^2/2\Delta t_i}}{\sqrt{2\pi\Delta t}} \, dx_i' \right) \\
&= e^{\sum_{i=1}^{n} \beta_{i-1}\Delta w_i - \frac{1}{2}\sum_{i=1}^{n} \beta_{i-1}^2 \Delta t}, \qquad \text{(A.16)}
\end{aligned}$$

and hence in continuous form

$$\xi(t) = e^{\int_0^t \beta(t) dw(t) - \frac{1}{2}\int_0^t \beta(t)^2 dt}. \qquad \text{(A.17)}$$

The expectation under $\mathcal{Q}$ of some random variable $Z$, conditional on information at time $t = t_n$, can then be written as

$$\begin{aligned}
\mathrm{E}_t^{\mathcal{Q}}[Z] &= \int \cdots \int \left( \prod_{i=n+1}^{N} \frac{e^{-x_i'^2/2\Delta t}}{\sqrt{2\pi\Delta t}}\, dx_i' \right) \\
&\quad z(\Delta w_1' + \beta_0 \Delta t, \ldots, \Delta w_n' + \beta_{n-1}\Delta t, x_{n+1}' + \beta_n \Delta t, \ldots, x_N' + \beta_{N-1}\Delta t) \\
&= \int \cdots \int \left( \prod_{i=n+1}^{N} \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} e^{\beta_{i-1}x_i - \frac{1}{2}\beta_{i-1}^2 \Delta t}\, dx_i \right) z(\Delta w_1, \ldots, \Delta w_n, x_{n+1}, \ldots, x_N) \\
&= \left( \prod_{i=1}^{n} e^{\beta_{i-1}\Delta w_i - \frac{1}{2}\beta_{i-1}^2 \Delta t} \right)^{-1} \int \cdots \int \left( \prod_{i=n+1}^{N} \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}}\, dx_i \right) \xi(\Delta w_1, \ldots, \Delta w_n, x_{n+1}, \ldots, x_N) \\
&\quad z(\Delta w_1, \ldots, \Delta w_n, x_{n+1}, \ldots, x_N), \qquad \text{(A.18)}
\end{aligned}$$

and hence

$$\mathrm{E}_t^{\mathcal{Q}}[Z] = \frac{1}{\xi(t)} \mathrm{E}_t^{\mathcal{P}}[\xi Z]. \tag{A.19}$$

Consider now a slight twist on the above problem. Suppose we have a probability space $\mathcal{P}$ generated by a Wiener process $w(t)$, $0 \le t \le T$. Suppose that $\xi$ is a strictly positive random variable defined on this probability space such that $\mathrm{E}^{\mathcal{P}}[\xi] = 1$. Then we can create a new probability measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, by defining for any event $A$

$$\mathcal{Q}(A) = \mathrm{E}^{\mathcal{P}}\left[\xi \mathbf{1}_{\{A\}}\right]. \tag{A.20}$$

The fact that $\mathrm{E}^{\mathcal{P}}[\xi] = 1$ insures that $\mathcal{Q}$ is a probability measure, and the positivity of $\xi$ assures us that $\mathcal{P}$ and $\mathcal{Q}$ are equivalent. We want to find a process $w'(t)$ which is a Wiener process under $\mathcal{Q}$. Note that this problem is in effect the inverse of the problem studied above. There we consider a process $w'(t)$ and find an equivalent measure under which it is a Wiener process. Here we are given an equivalent measure $\mathcal{Q}$ and want to find a process $w'(t)$ which is a Wiener process under $\mathcal{Q}$.

To find $w'(t)$, proceed as follows. The likelihood ratio $\xi(t) = \mathrm{E}_t^{\mathcal{P}}[\xi]$ is a positive $\mathcal{P}$-martingale, and hence its dynamics can be written as

$$d\xi(t) = \beta(t)\xi(t)dw(t) \tag{A.21}$$

for some adapted process $\beta(t)$. Define $w'(t) = w(t) - \int_0^t \beta(s)\,ds$. We claim that $w'(t)$ is a Wiener process under $\mathcal{Q}$. This can be easily proven by noting that

$$\begin{aligned}
\mathcal{Q}(\Delta w'_1 \in dx'_1, \ldots, \Delta w'_N \in dx'_N) &= \mathrm{E}^{\mathcal{P}}\left[\xi \mathbf{1}_{\{\Delta w'_1 \in dx'_1, \ldots, \Delta w'_N \in dx'_N\}}\right] \\
&= \mathrm{E}^{\mathcal{P}}\left[\xi \mathbf{1}_{\{\Delta w_1 \in dx_1, \ldots, \Delta w_N \in dx_N\}}\right] \\
&= \prod_{i=1}^{N} \frac{e^{-x_i^2/2\Delta t}}{\sqrt{2\pi\Delta t}} e^{\beta_{i-1}x_i - \frac{1}{2}\beta_{i-1}^2\Delta t} \\
&= \prod_{i=1}^{N} \frac{e^{-x_i'^2/2\Delta t}}{\sqrt{2\pi\Delta t}}\, dx'_i
\end{aligned} \tag{A.22}$$

Hence, the increments of $w'(t)$ are IID normal variables with mean 0 and variance $\Delta t_i$, and therefore $w'(t)$ is a Wiener process under $\mathcal{Q}$.

## A.2 Multidimensional Case

The results of the previous section can be generalized to multiple dimensions in a straightforward manner. Consider a $K$-dimensional vector of Wiener processes $\mathbf{w}(t)$, with correlation matrix $\rho(t)$. The probability space $\Omega$ in the discrete-time approximation will again consist of all paths $\{(\Delta\mathbf{w}_1, \Delta\mathbf{w}_2, \ldots, \Delta\mathbf{w}_N)\}$, and the probability measure $\mathcal{P}$ is defined by

$$\mathcal{P}(\Delta\mathbf{w}_1 \in d\mathbf{x}_1, \ldots, \Delta\mathbf{w}_N \in d\mathbf{x}_N) = \prod_{i=1}^{N} \frac{e^{-\mathbf{x}_i^\top \rho^{-1}\mathbf{x}_i/2\Delta t}}{\sqrt{(2\pi\Delta t)^K \det\rho}} \cdot d\mathbf{x}_i. \tag{A.23}$$

Now, suppose we have an adapted $K$-vector process $\boldsymbol{\beta}(t)$, and we define a new stochastic process $\mathbf{w}'(t)$ by

$$\mathbf{w}'(t) = \mathbf{w}(t) - \int_0^t \boldsymbol{\beta}(s)\,ds. \tag{A.24}$$

We claim that there exists a probability measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, under which $\mathbf{w}'(t)$ is a vector of Wiener processes with correlation $\rho$.

We prove this assertion along the lines established in the previous section. In discrete differential form, we have

$$\Delta \mathbf{w}'_i = \Delta \mathbf{w}_i - \boldsymbol{\beta}_{i-1} \Delta t. \tag{A.25}$$

Again, because $\boldsymbol{\beta}$ is adapted, there is an one-to-one relationship between the paths $(\Delta \mathbf{w}_1, \Delta \mathbf{w}_2, \ldots, \Delta \mathbf{w}_N)$ and $(\Delta \mathbf{w}'_1, \Delta \mathbf{w}'_2, \ldots, \Delta \mathbf{w}'_N)$ and the Jacobian of the transformation is

$$J = \left| \frac{\partial \Delta \mathbf{w}'_i}{\partial \Delta \mathbf{w}'_j} \right| = 1. \tag{A.26}$$

Now, define the candidate probability measure $\mathcal{Q}$ by

$$\mathcal{Q}(\Delta \mathbf{w}_1 \in d\mathbf{x}_1, \ldots, \Delta \mathbf{w}_N \in d\mathbf{x}_N) = \mathcal{P}(\Delta \mathbf{w}_1 \in d\mathbf{x}_1, \ldots, \Delta \mathbf{w}_N \in d\mathbf{x}_N) \cdot \xi(\mathbf{x}_1, \ldots, \mathbf{x}_N) \tag{A.27}$$

where $\xi$ is given by

$$\xi(\Delta \mathbf{w}_1, \ldots, \Delta \mathbf{w}_N) = e^{\sum_{i=1}^N \boldsymbol{\beta}_{i-1}^\top \rho_i^{-1} \Delta \mathbf{w}_i - \frac{1}{2} \sum_{i=1}^N \boldsymbol{\beta}_i^\top \rho_i^{-1} \boldsymbol{\beta}_i \Delta t}. \tag{A.28}$$

Note that in the continuous limit, we have

$$\xi = e^{\int_0^T \boldsymbol{\beta}(t)^\top \rho^{-1} d\mathbf{w}(t) - \frac{1}{2} \int_0^T \boldsymbol{\beta}(t)^\top \rho(t)^{-1} \boldsymbol{\beta}(t)\, dt}. \tag{A.29}$$

We claim that $\mathcal{Q}$ is a probability measure on $\Omega$, that $\mathcal{Q}$ is equivalent to $\mathcal{P}$, and that $\mathbf{w}'(t)$ is a $K$-vector of Wiener processes under $\mathcal{Q}$ with correlation matrix $\rho$.

The proof of this assertion consists of completing the square of the multivariate normal distributions, just as in the one-dimensional case above, and is omitted. Also, as in the one-dimensional case, we have the following relationships between expectations under $\mathcal{P}$ and $\mathcal{Q}$:

$$\begin{aligned} \mathrm{E}^{\mathcal{Q}}[Z] &= \mathrm{E}^{\mathcal{P}}[\xi Z] \\ \mathrm{E}_t^{\mathcal{Q}}[Z] &= \frac{1}{\xi(t)} \mathrm{E}_t^{\mathcal{P}}[\xi Z], \end{aligned} \tag{A.30}$$

where the likelihood ratio $\xi(t) = \mathrm{E}_t^{\mathcal{P}}[\xi]$ is

$$\xi(t) = e^{\int_0^t \boldsymbol{\beta}(s)^\top \rho(s)^{-1} d\mathbf{w}(s) - \frac{1}{2}\int_0^t \boldsymbol{\beta}(s)^\top \rho(s)^{-1} \boldsymbol{\beta}(s) ds}. \tag{A.31}$$

We may also consider the inverse problem in multiple dimensions. Suppose we have a probability space $\mathcal{P}$ generated by a $K$-vector of correlated Wiener processes $\mathbf{w}(t)$, $0 \le t \le T$. Suppose $\xi$ is a strictly positive random variable on this probability space such that $\mathrm{E}^{\mathcal{P}}[\xi] = 1$. Then we can define a new measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, by defining for any event $A$

$$\mathcal{Q}(A) = \mathrm{E}^{\mathcal{P}}[\xi \mathbf{1}_{\{A\}}]. \tag{A.32}$$

We want to find a vector process $\mathbf{w}'(t)$ which is a Wiener process under $\mathcal{Q}$ with correlation matrix $\rho$.

We solve this problem in a manner similar to the one-dimensional problem. Since the likelihood ratio $\xi(t)$ is a positive $\mathcal{P}$-martingale, there exists an adapted process $\boldsymbol{\beta}(t)$ such that

$$d\xi(t) = \xi(t)\boldsymbol{\beta}(t)^\top d\mathbf{w}(t). \tag{A.33}$$

Let us define

$$\mathbf{w}'(t) = \mathbf{w}(t) - \int_0^t \rho(s)\boldsymbol{\beta}(s)\, ds. \tag{A.34}$$

It is straightforward to show that $\mathbf{w}'$ has the desired properties.

# B Appendix B

Throughout the appendix we consider a $(\alpha, \sigma)$ Brownian motion starting at zero at time $t = 0$ which we call $X(t)$. $X(t)$ is defined by the following equation:

$$X(t) = \alpha t + \sigma w(t) \tag{B.35}$$

where $w(t)$ is a wiener process. We also define $\mathcal{P}(X(t) \in dx)$ to be the density function for the random variable $X$. Similarly $\mathcal{P}(X(t) \in dx, X_0(t) < l)$is the density of a brownian motion with absorbing barrier at $l$. Capital letters are used in general when we denote the random variable (as in $X$) and small letters when we denote values taken by this random variable. Finally $\mathbf{1}_{\{\}}A$ denotes the indicator function of set $A$,which is equal to one if event $A$ is realized and zero else.

## B.1 Maximum of a Brownian Motion

We define the running maximum of a process to be:

$$X_0(t) = \max_{0 \leq s \leq t} X(s)$$

Moreover, we consider in this section a constant $l$ such that:

$$l > 0$$

However, all the results can be straightforwardly extended to the case where $l < 0$, using the fact that for a Brownian motion starting at zero there is a zero probability that the maximum $X_0(t)$ attained by $x$ during the time $t$ is negative.

By defintion of the wiener process, $X(t+s) - X(t)$ is normally distributed with mean $\mu s$ and variance $\sigma^2 s$. Thus the transition density $p(t, t+s, x, y)dy$ (the probability density of $X(t+s) = y$

when $X(t) = x$) is given by:

$$p(t, t+s, x, y) = \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{\frac{(y-x-\mu s)^2}{2\sigma^2 s}}$$

and verifies the backward equation:[4]

$$\frac{\partial}{\partial s} p(t, t+s, x, y) = \left( \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) p(t, t+s, x, y)$$

with initial conditions

$$p(t, t, x, y) = \delta(x - y)$$

where $\delta(x - y)$ is the dirac delta function.

The transition densities of regulated brownian motion verify the same backward equation, but with modified boundary conditions. Thus for example the transition density for a brownian motion with absorbing boundary at $l > 0$ is the solution of the following partial differential equation:

$$\begin{aligned}
\frac{\partial}{\partial s} p(t, t+s, x, y) &= \left( \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) p(t, t+s, x, y) \\
p(t, t, x, y) &= \delta(x - y) \\
p(t, t+s, l, y) &= 0
\end{aligned}$$

We can then calculate the following joint densities:

$$\mathcal{P}(X(t) \in dx) = \frac{1}{\sqrt{2\pi\sigma^2 t}} [e^{-\frac{(x-\alpha t)^2}{2\sigma^2 t}}] \tag{B.36}$$

$$\mathcal{P}(X(t) \in dx, X_0 < l) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2 t}} [e^{-\frac{(x-\alpha t)^2}{2\sigma^2 t}} - e^{\frac{2\alpha l}{\sigma^2}} e^{-\frac{(x-\alpha t-2l)^2}{2\sigma^2 t}}] & \text{if } x < l \\ 0 & \text{if } x > l \end{cases} \tag{B.37}$$

$$\begin{aligned}
\mathcal{P}(x, X_0 > l) &= \mathcal{P}(x) - \mathcal{P}(x, X_0 < l) \qquad\qquad (B.38) \\
&= \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{\frac{2\alpha l}{\sigma^2}} e^{-\frac{(x-\alpha t-2l)^2}{2\sigma^2 t}} & \text{if } x < l \\ \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\alpha t)^2}{2\sigma^2 t}} = \mathcal{P}(x) & \text{if } x > l \end{cases}
\end{aligned}$$

By simple integration we get:

$$\mathcal{P}(X(t) < k) = N(\frac{k - \alpha t}{\sigma\sqrt{t}}) \tag{B.39}$$

$$\mathcal{P}(X(t) < k, X(t+s) < l) = N_2(k, l, s) \tag{B.40}$$

$$\begin{aligned}\mathcal{P}(X_0 < l) &= \int_{-\infty}^{l} \mathcal{P}(x, X_0 < l)dx \\ &= N(\frac{l - \alpha t}{\sigma\sqrt{t}}) - e^{\frac{2\alpha l}{\sigma^2}} N(\frac{-l - \alpha t}{\sigma\sqrt{t}})\end{aligned} \tag{B.41}$$

$$\begin{aligned}\mathcal{P}(X_0 > l) &= 1 - \mathcal{P}(X_0 < l) \\ &= N(\frac{-l + \alpha t}{\sigma\sqrt{t}}) + e^{\frac{2\alpha l}{\sigma^2}} N(\frac{-l - \alpha t}{\sigma\sqrt{t}})\end{aligned} \tag{B.42}$$

$$\begin{aligned}\mathcal{P}(x < k, X_0 < l) &= \int_{-\infty}^{k} \mathcal{P}(x, X_0 < l)dx \\ &= \begin{cases} N(\frac{k-\alpha t}{\sigma\sqrt{t}}) - e^{\frac{2\alpha l}{\sigma^2}} N(\frac{k-2l-\alpha t}{\sigma\sqrt{t}}) & \text{if } k < l \\ N(\frac{-l+\alpha t}{\sigma\sqrt{t}}) + e^{\frac{2\alpha l}{\sigma^2}} N(\frac{-l-\alpha t}{\sigma\sqrt{t}}) = \mathcal{P}(X_0 < l) & \text{if } k > l \end{cases}\end{aligned} \tag{B.43}$$

$$\begin{aligned}\mathcal{P}(x < k, X_0 > l) &= Pr(x < k) - \mathcal{P}(x < k, X_0 < l) \\ &= \begin{cases} e^{\frac{2\alpha l}{\sigma^2}} N(\frac{k-2l-\alpha t}{\sigma\sqrt{t}}) & \text{if } k < l \\ N(\frac{k-\alpha t}{\sigma\sqrt{t}}) - N(\frac{l-\alpha t}{\sigma\sqrt{t}}) + e^{\frac{2\alpha l}{\sigma^2}} N(\frac{-l-\alpha t}{\sigma\sqrt{t}}) & \text{if } k > l \end{cases}\end{aligned} \tag{B.44}$$

$$\begin{aligned}\mathcal{P}(x > k, X_0 < l) &= Pr(X_0 < l) - \mathcal{P}(x < k, X_0 < l) \\ &= \begin{cases} N(\frac{l-\alpha t}{\sigma\sqrt{t}}) - N(\frac{k-\alpha t}{\sigma\sqrt{t}}) - e^{\frac{2\alpha l}{\sigma^2}} \left[N(\frac{-l-\alpha t}{\sigma\sqrt{t}}) - N(\frac{k-2l-\alpha t}{\sigma\sqrt{t}})\right] & \text{if } k < l \\ 0 & \text{if } k > l \end{cases}\end{aligned} \tag{B.45}$$

$$\begin{aligned}\mathcal{P}(x > k, X_0 > l) &= Pr(X_0 > l) - \mathcal{P}(x < k, X_0 > l) \\ &= \begin{cases} N(\frac{-l+\alpha t}{\sigma\sqrt{t}}) + e^{\frac{2\alpha l}{\sigma^2}} \left[N(\frac{-l-\alpha t}{\sigma\sqrt{t}}) - N(\frac{k-2l-\alpha t}{\sigma\sqrt{t}})\right] & \text{if } k < l \\ N(\frac{-k+\alpha t}{\sigma\sqrt{t}}) = \mathcal{P}(x > k) & \text{if } k > l \end{cases}\end{aligned} \tag{B.46}$$

## Notes

[1]See, for example, El Karoui and Rochet (1989) and Jamshidian (1991).

[2]As in Geske and Johnson (1984), an efficient approximation algorithm for the continuous shout can be obtained using a Richardson extrapolation procedure on the discrete shout. Thomas (1993) describes a binomial tree approximation for the continuous shout.

[3] See, for example, Karatzas and Shreve (1991), Section 3.3.5.

[4]as can be verified by direct calculation.

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