# Rational Participation Revolutionizes Auction Theory*

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**Abstract**

Potential bidders respond to a seller's choice of auction mechanism for a common-value or affiliated-values asset by endogenous decisions whether to incur a participation cost (and observe a private signal), or forego competing. Privately informed participants decide whether to incur a bid-preparation cost and pay an entry fee, or cease competing. Auction rules and information flows are quite general; participation decisions may be simultaneous or sequential. The resulting revenue identity for any auction mechanism implies that optimal auctions are allocatively efficient; a nontrivial reserve price is revenue-inferior for any common-value auction. Optimal auctions are otherwise contentless: any auction that sells without reserve becomes optimal by adjusting any one of the continuous, spanning parameters, e.g., the entry fee. Seller's surplus-extracting tools are now substitutes, not complements. Many econometric studies of auction markets are seen to be flawed in their identification of the number of bidders.

D44; D82; C72; Keywords: optimal auctions, endegenous bidder participation, affiliated-values, common-value auctions, surplus-extracting devices

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# 1 Introduction

How should an owner or auctioneer select a selling procedure when bidders' value estimates for an asset are private information? That fundamental question has for centuries received a variety of answers from experienced auctioneers, who in different markets persist in conducting their business in quite different ways. In contrast, theoretical models of "optimal auctions" with rational risk-neutral bidders have tended to provide a unique answer.[1]

While the particular answer provided depends fragilely on the model assumed, optimal auctions in the literature share two common characteristics. First, the optimal auction is inefficient (unless surplus can be fully extracted), primarily due to a nontrivial *reserve price*.[2] Second, the optimal auction is a complicated mechanism. Depending on the particular assumptions, it has involved: distribution- and bidder-specific reserve prices, disjoint sets of prices at which seller refuses to sell, requiring payments from losing bidders that vary with their bids and rivals', requiring bidders to accept lotteries with unboundedly large losses, or to accept lotteries before their terms are specified.[3]

Expected-revenue comparisons across auction forms yield a similar picture: revenue rises as surplus-extracting tools are piled atop one another, since these tools are complements. Again theory suggests a complicated auction mechanism.[4]

Of course, an "optimal" auction may prove suboptimal outside a model's assumptions. Bulow and Klemperer [1996] surprisingly find an optimal auction selling to $n$ bidders reaches lower expected revenue than a basic English auction with $n+1$ bidders. In a way, this too is a specific prescription: a seller optimizes by a unique tactic, obtaining another bidder.

[1] In essence, the models cited in the following footnote each define a very narrow equivalence class of auctions, and show that optimal auctions all fall in a single equivalence class, which serves to characterize nearly all auction forms as necessarily suboptimal, even with adjustments in parameters of that auction form.

[2] Myerson [1981], Harris and Raviv [1981] and Riley and Samuelson [1981] derive optimal auctions when bidders' private information (their types) are independent. Of these models, Myerson's is most general. All revolve around a nontrivial reseve price (below which the seller prevents the asset from ever being sold); so do more recent papers (see surveys in Klemperer [2000] and Krishna [2002]). The only optimal auctions attaining efficiency are in models that use strong informational assumptions and correlated types to extract full surplus: Crémer and McLean [1985], [1988], McAfee, McMillan and Reny [1989] and McAfee and Reny [1992]. The criticism of these models in Robert [1991], that the weakest form of limited liability or infinitesimal risk aversion renders them discontinuously suboptimal, is similar in spirit to the present effort. Mares and Harstad [2005] provide an accessible treatment of necessary and sufficient conditions for full surplus extraction.

[3] Examples of these complications, in order: Harris and Raviv [1981], Myerson [1981], Crémer and McLean [1985], McAfee, McMillan and Reny [1989], McAfee and Reny [1992].

[4] Milgrom and Weber [1982] find higher expected revenue in a second-price than in a first-price sealed-bid auction, and higher still in an English (oral ascending-bid) auction. In any of these auctions, seller increases expected revenue by publicly announcing any information he possesses which is affiliated with asset value. (Even this prescription is further complicated if a seller has an option to privately provide an appraisal to a subset of bidders; cf. Mares and Harstad [2003].) In most circumstances, entry fees and reserve prices are also complications added to augment revenue. Further results in the same vein are surveyed in Klemperer [2000].

All these papers assume rational, risk-neutral bidders. Yet, critically, all analyze too narrow a scope for bidder rationality: none allow for a potential bidder's rational decision to participate.

Most auctions (and markets that can be stylized as auctions) share the characteristics that bidder participation is costly, and is motivated by the expected profitability of competing.[5] Auction theory employing an exogenously fixed number of bidders has been fruitful both in providing tools for analysis and in building our collective intuition about the forces that interact in equilibrium responses to auction rules (indeed, my analysis could not proceed without their building blocks). However, adjusting an auction model to incorporate rational decisions as to whether a potential bidder competes dramatically alters conclusions.

The model developed here analyzes the symmetric equilibrium that results when expected-profit maximizers rationally decide, first, whether to participate in an auction, and second, how to compete if participating.[6] A wide variety of auction forms and informational flows can be incorporated.

Optimal auctions then strike a sharp contrast with the prior literature. When bidder arrival is the result of sufficient expected profitability, drawing in an extra bidder is less attractive than when an additional competitor can exogenously be obtained (Bulow and Klemperer's model). In this model, discouraging potential bidder participation, by adopting an auction form where competition among relatively few bidders already extracts substantial surplus, is always part of optimizing by seller.

Allocative inefficiency no longer plays any role in optimal auctions: a seller's preferences between any two mechanisms now mirror those of an efficient social planner. Both prefer the same interior probabilities of selling the asset, and of selling to the highest-valuing bidder. In particular, "selling without reserve" characterizes optimal common-value auctions: a mechanism incorporating a nontrivial reserve price is strictly revenue-inferior.[7]

The starkest contrast arises in a previously unaddressed issue: the size and definitiveness of the set of optimal auctions. When bidder participation is rationally determined, optimal auctions are no longer a singleton, but now selling without reserve is the *entire content* of optimal auctions.

[5] Headline-grabbing auctions–airwaves licenses, privatization of governmental enterprises, offshore oil leases, museum-quality art, initial public offerings, acquisitions of new, established, and distressed corporations–all fit this mold. So do such mundane markets as used-car auctions, timber sales and routine art auctions. (Buying at auction and selling at retail is a sensible stylization of the art-gallery business.)

[6] The model treats situations where a seller offers an asset to potential bidders who decide whether to compete to buy. A corresponding model where a buyer details a contractual obligation, and potential bidders decide whether to compete to supply, has completely corresponding results.

[7] In a common-value model, the only efficiency issue is whether the asset is sold; any bidder is an equally efficient purchaser. An efficient social planner will never employ a nontrivial reserve price; below, unlike earlier models, an expected-revenue-maximizing seller shares this preference.

The principal result below is that any auction is within the setting of any one continuous, spanning variable of being optimal; a corollary: optimal auctions contain a subset of dimension one less than the dimension of the space of mechanisms with zero reserve prices.

This characterization accords better with the variety of auctions that have repeatedly been used in practice. If an unmodeled aspect favors one auction form over another (e.g., prior practice, or avoidable costs of congregating bidders), there is no reason within the model to overturn this preference.

The principal theorem is simply obtained, and its intuition straightforward. A seller is no longer interested in which surplus-extracting tools he employs, as these tools now acquire their natural role of substitutes, rather than complements. Instead, a seller focuses on the equilibrium participation probability that serves to make potential bidders indifferent over participating. That is, the new variable added to the traditional models, the probability that a given potential bidder participates, becomes the *only* variable of interest, the sole vehicle through which a seller's mechanism choices affect revenue. From any suboptimal auction (with a zero reserve price), any spanning continuous variable can be adjusted to move the equilibrium participation probability to its optimal level. Hence the starting point (e.g., first-price or English auction) only matters in how large (or small) an entry fee, or other continuously adjustable surplus-extracting device, attains optimality.

The presentation follows a natural order, beginning with an outline of a newly general model of common-value auctions (Appendix C extends nearly all results to general affiliated-values auctions, as modeled in Milgrom and Weber [1982]), and then assumptions. The analysis proceeds from specifying equilibrium mixed-strategy participation decisions by potential bidders to identifying the equilibrium expected revenue formula that characterizes any announcement of mechanism a seller might make. Allocative efficiency is a trivial corollary of the equilibrium revenue identity. Characterizing mechanism choices as capable of attaining any equilibrium participation probability without the use of a nontrivial reserve price implies reserve price inferiority. Comparative statics in section 6 imply that much of the literature empirically studying historical records of auction sales is fundamentally flawed, if bidders are believed to have arrived via equilibrium expected-profitability calculations. The main result, when a reduced-form concavity assumption is added, implies that revenue comparisons from the exogenous-bidders papers (e.g., a preference for English over second-price auctions) are extended with endogenous bidder participation to a half-space of underlying parameters, but reversed in another half-space. Concluding remarks assess the generality of these results.

The model presented treats potential bidders' decisions to become privately informed as simultaneous. Appendix A develops an alternative model allowing these decisions to be sequential. The principal difference is that symmetry is no longer a sufficient equilibrium selection criterion.

I emphasize at the outset that the model avoids any assumption of monotone equilibrium in common-value auctions, in particular avoiding the assumption of a screening level (a threshold level of private information above which a participant chooses to pay the entry fee). A recent impossibility theorem by Landsberger and Tsirelson [2002] makes this complication critical. This aspect, together with allowing for general affiliated valuations, for resource costs facing potential bidders both *before* (a participation cost) and *after* becoming privately informed (a bid-preparation cost), allowing seller the widest variety of surplus-extracting tools, as well as allowing for players either to observe or not observe the number of players still competing at each stage, greatly distinguish the generality of this model from prior auction models which endogenize the number of bidders.[8]

## 2 A General Model with Endogenous Participation

Begin with the notion that potential bidders choose among a variety of auctions, and other uncertain economic opportunities, in which to invest their attention, time and money. Only a segment of the extensive form of such a game, that relating to a particular auction, appears explicitly here. One indivisible asset is sold in the explicit model. A subset of the (exogenously determined) $N$ potential bidders will participate, and a subset of the $n$ participants will become the $a$ actual bidders.

The game segment unfolds as follows, cf. Figure 1. First, seller announces an *auction mechanism* $M := (m, \varphi, r) \in \mathbb{M} \subset \mathcal{M} \times \Re \times \Re_+$, where $m$ is an *auction form*, $\mathcal{M}$ the set of auction forms, $\varphi$ an *entry fee*, and $r$ a *reserve price*. An auction form $m$ specifies not just pricing rules, but the entire flow of information and hence the nature of the extensive form continuation. For example, $m = m_0$ might specify a second-price auction with seller releasing an uncensored independent appraisal to all participants, as well as specifying that neither participants nor active bidders learn their number before bidding. Or $m = m_1$ might specify an English (oral ascending) auction without any seller-released public information, but with an appraisal privately revealed to one actual bidder chosen

[8] Harstad [1990] introduces the notion that the number of bidders ought to be considered an endogenous variable, in a simpler model employing monotone equilibrium and with a smaller set of surplus-extracting tools. Levin and Smith [1994] also depend on monotone equilibrium, and critically on assumptions that the seller [i] cannot disclose an appraisal, and [ii] cannot impose an entry fee after bidders have private information, for their result supporting a positive reserve price. Chakraborty and Kosmopoulou [2001] partially specify a simpler model employing monotone equilibrium and with a smaller set of surplus-extracting tools, and argue for a zero entry fee when negative entry fees are impossible.

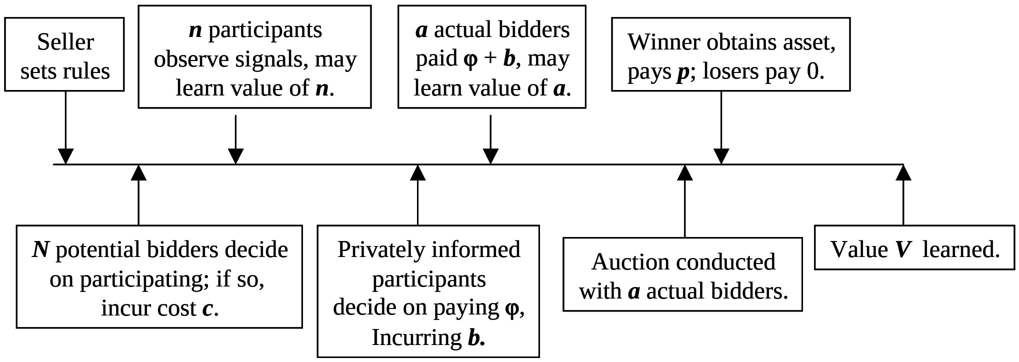


Figure 1: Time Line

at random, in which the number of participants is not learned but the number of actual bidders is, and alternating recognition rules determine the probabilistic revelation of bidders' exit prices to remaining bidders.[9] A particular seller in a particular situation may face additional constraints: he may, for example, find credibly imposing a nontrivial reserve price impossible, or may be unable to inform participants of the number of competitors who acquired private information, or may not have a reputation that would allow using a second-price auction without bidders assuming he could well insert a fake bid just below the highest bid;[10] all such constraints are treated via making $\mathbb{M}$ the feasible set of auction mechanisms for a particular auction. Note that when a seller has the option of credibly announcing how many participants are still competing (or how many actual bidders) before continuing, or of preventing the participants (or actual bidders) from knowing $n$ (or $a$), seller's choice is simply modeled as a choice between two (otherwise identical) auction forms, just as if it were a choice between first- and second-price auction rules.

Second, a pool of potential bidders $\mathbb{N} := \{1, \ldots, N\}$ simultaneously select probabilities $\pi_i$ of becoming a *participant* in this auction, basing those decisions on $M$.[11] Participation has two consequences: each participant $j$ obtains some private information $X_j \in \mathcal{X} \subset \Re$ about the asset's value to him (call this $j$'s *signal*), and each incurs a *participation cost*, $c > 0$. The participation cost is exogenously specified, and does not generate revenue for seller; it represents foregone profitable opportunities (e.g., inability to participate in another auction occurring elsewhere).[12] This cost is

[9] The impact of privately revealed information is considered in Mares and Harstad [2002]; alternating recognition rules for English auctions are analyzed in Harstad and Rothkopf [2000].

[10] Impacts of such bidtaker cheating are considered in Rothkopf and Harstad [1995].

[11] An alternative model, in which bidders sequentially decide whether to participate, is outlined in Appendix A. Similar results to those in the main text depend on an equilibrium selection favorable to seller.

[12] This consideration is missed if a view of substitute auctions is not at least implicitly present. Unwillingness of

likely to vary across auctions, but $c$ is the same for all potential bidders in a given auction, and invariant to the mechanism by which the auction is run. The payoff of a potential bidder who does not participate is normalized to 0.

Third, each participant $j = 1, \ldots, n$ decides whether or not to incur a *bid-preparation cost* $b \geq 0$ (which does not accrue to seller),[13] plus pay the entry fee $\varphi$ to seller and thereby become an *actual bidder*, based on information available at the time. This information includes the auction mechanism $M$, and participant $j$'s private signal $X_j$. If the component $m$ of $M$ characterizing the auction form specifies that participants are informed of the number $n$ of participants, then $n$ is taken into account. If $n$ is not known, then the vector $(\pi_1, \ldots, \pi_N)$ of rational participation probabilities of potential bidders is taken into account. A participant who chooses not to continue attains a payoff of $-c$.

Fourth, each actual bidder $k = 1, \ldots, a$ selects a bidding strategy for the auction form $m$ with reserve price $r$. In addition to $M$ and $X_k$, $n$ if known, and $(\pi_1, \ldots, \pi_N)$ if $n$ is not known, this decision takes into account the number $a$ of actual bidders if the auction form $m$ releases this information. If not, the bidding decision takes into account the functional structure of participants' decisions on whether to pay the entry fee, and includes strategizing to learn about $a$ (and perhaps useful inferences about rivals' private information) as soon as information flows permit.

The winning bidder pays a price $p$ for the asset, if this price is no less than the reserve price $r$; otherwise the asset goes unsold, which implies that all actual bidders would then be losers.[14] Losing actual bidders attain a payoff of $-\varphi - b - c$.

I present and analyze in the main text the special case in which the ultimate value of the asset is common across potential bidders, represented by a random variable $V$. Appendix C indicates how nearly all results can be extended to an affiliated-values auction model (the "General Symmetric Model" of Milgrom and Weber [1982]), in which $V$ is the "underlying asset value" and the value to any particular participant is a function $t(V, X_i)$ of underlying asset value and his own signal. That extension is interesting in that both failing to sell the asset and selling to an actual bidder

an additional potential bidder to participate need not imply zero (gross) expected profit.

[13] The bid-preparation cost is treated as the same no matter what auction mechanism is employed. This assumption is not innocuous; I return to it in Concluding Remarks.

[14] Notation that is already more cumbersome than might be hoped for would find significant additional complication if the seller were allowed to announce a vector of reserve prices $r_a$, with the $r_a$ that corresponded to the number $a$ of bids actually submitted enforced after bids were submitted. That complication would not affect any of the results below. In particular, the result in Levin and Smith [1994] that a nontrivial reserve price would be employed at least in the case where only 1 actual bidder showed up is still seen to depend on their assumptions that the seller cannot utilize an entry fee, and cannot publicly reveal such information as an appraisal.

who values the asset less highly than some other participant are possible inefficiencies.

A sale yields the winning bidder a payoff of $V - p - \varphi - b - c$. Seller's payoff is

$$u^S = \begin{cases} 0, & \text{if } a = 0 \text{ or } p < r, \\ p + a\varphi, & \text{if } a \geq 1 \text{ and } p \geq r. \end{cases} \quad {}^{15}$$

## 2.1 Assumptions: Auction Environment

$\mathcal{A}.1$. The infinite sequence $\{X_1, X_2, \ldots\}$ from which participants will observe signals is a sequence of exchangeable, positively affiliated, real-valued random variables with nonatomic measure $\mathfrak{B}$, and marginal $\mathfrak{B}_1$ onto support $\mathcal{X}$.

Affiliation is defined and characterized in Milgrom and Weber [1982], pp. 1098-1100 and 1118-1121; it is referred to as the MLRP (monotone likelihood ratio property) in several auction models. Roughly, affiliation means that higher realizations for any subset of the variables $\{X_1, X_2, \ldots\}$ make higher realizations for any disjoint subset more likely. Exchangeability means that the joint distribution is unaffected by any finite permutation of the indices. Let $\mathbf{X}_n = \{X_1, \ldots, X_n\}$, and $V_z = \left(\frac{1}{z}\right) \sum_{i=1}^{z} X_i$.

$\mathcal{A}.2$. Asset value $V = \lim_{z\to\infty} V_z$; $c + b < E[V] < \infty$.

A variant of DiFinetti's Theorem justifies the use of a limit in $\mathcal{A}.2$:

**Theorem 1** *(Kingman [1980]) Given $\mathcal{A}.1$, the sequence $\{V_1, V_2, \ldots\}$ almost surely converges pointwise. Moreover, conditional on $V$, the $\{X_i\}$ are mutually independent.*

Letting the common value equal the asymptotic mean is without loss of generality (Milgrom and Weber [1986]).

## 2.2 Assumptions: Auction Rules

$\mathcal{A}.3$. The price paid is an anonymous, nondecreasing, continuous function of the profile of actual bids submitted.[16]

[15] Considering the value to seller of an unsold asset to be 0 is, as usual, a harmless normalization. It bears emphasis, however, that failing to meet the reserve price implies that the seller is irrevocably constrained from ever offering this asset to this set of potential bidders in the future (this assumption is nearly ubiquitous in auction theory, though seldom mentioned). I return to this consideration in Concluding Remarks.

[16] The sort of revenue-maximizing, non-capricious discriminaton across bidders in Myerson [1981] has already been ruled out by exchangeability (in $\mathcal{A}.1$). The sort of capricious discriminaton contemplated in McAfee, McMillan and Reny [1989] is ruled out here, solely for notational ease. Footnote 27 below explains how their mechanism, which extracts full surplus whenever the exogenous number of bidders is at least two, becomes revenue-inferior with endogenous bidder participation.

$\mathcal{A}$.4. Each auction form $m$ determines a winning bidder,[17] and attains a unique symmetric equilibrium continuation for any exogenously specified binomial distribution of the number of actual bidders (including degenerate).

Allowing for payments to or from losing bidders greatly complicates the notation, but would not change any results below. Uniqueness of the symmetric equilibrium continuation is critical to being able to predict the profitability of participating and actually bidding; it is satisfied for a wide variety of auction forms.[18]

### 2.3 Assumptions: Behavior

$\mathcal{A}$.5. All $N \geq 2$ potential bidders are risk-neutral.

$\mathcal{A}$.6. Symmetric behavior: each potential bidder selects the same probability $\pi$ of participating, each participant selects the same function of known and inferred information to determine whether to actually bid, and each actual bidder selects the same bid function. These selections constitute a Bayesian equilibrium continuation.

If, in addition, seller selects the mechanism $M$ to maximize expected revenue given the assumed behavior of bidders, a full Bayesian equilibrium is attained. As our focus is on the behavior that various announcements of $M$ will induce, and thus upon the expected revenue attained, equilibrium continuation is the key assumption.[19]

## 3 The Participation Decision

In this model, the equilibrium expected number of bidders is not invariant to seller's choice of auction mechanism. Rather, it adjusts to the auction mechanism $M$ so that expected profit equals participation cost. The straightforward logic is, ultimately, independent of many details of the mechanism.

A fair bit of notation is needed, and some equations in this section may appear untidy. However, Conclusion 2 ending this section is conceptually easy, and the general revenue formulation in the next section will be strikingly simple. Recall that $N$ is the exogenous number of potential bidders,

[17] Though it may be natural, nothing depends on this being the highest bidder.

[18] Cf. Levin and Harstad [1986], Bikhchandani and Riley [1991], Pesendorfer and Swinkels [1997], Harstad and Rothkopf [2000], and Maskin and Riley [2000].

[19] Note that there exist Nash equilibria in which seller selects an otherwise inferior $M'$ because, for example, all $N$ potential bidders respond to any $M \neq M'$ by not participating, or otherwise punishing seller. These equilibria fail to be subgame-perfect; in ignoring them, I follow a standard but usually implicit practice.

$\pi(M)$ the symmetric equilibrium probability of participating given mechanism $M$, and $n$ and $a$ numbers of participants and of actual bidders. Throughout, the usual binomial formula for the probability of $z$ successes in $Z$ trials, each with independent success probability $\varsigma$ is denoted

$$\beta(z, Z, \varsigma) = \binom{Z}{z} \varsigma^z (1-\varsigma)^{Z-z}.$$

Thus, if $N$ potential bidders each participate with probability $\pi$, the probability of $n$ participants is $\beta(n, N, \pi)$. A potential bidder analyzing the consequences of proceeding to the next step of the game (participating or actually bidding) rationally evaluates the likelihood of different numbers of rival participants according to $\beta(n-1, N-1, \pi)$, which accounts for the presumption that he (the analyzing potential bidder) proceeds–even if this is not a certainty, all behavior is otherwise payoff-irrelevant.[20] When context makes clear, I will shorten $\beta(n, N, \pi)$ to $\beta_n$ and $\beta(n-1, N-1, \pi)$ to $\beta_{n-1}$. When an arbitrary potential bidder $i$ becomes a participant, I will harmlessly treat the renumbering function $ren(i, n, N)$ that would provide his position in the numerically ordered set of participants as if it were the identity function, and refer to the continuing roles of the player who begins as potential bidder $i$ as if he becomes participant $i$ if he participates, and actual bidder $i$ if he pays the entry fee. Symmetry attained through $\mathcal{A}.1$ and $\mathcal{A}.6$ allow a focus throughout on potential bidder 1, participant 1, and actual bidder 1.

At the point that the decision to become an actual bidder (to incur the bid-preparation cost $b$ and pay the entry fee $\varphi$) is made, participant 1 has observed signal $X_1 = x$. Two cases must be developed. The first arises when the auction form $m$ specifies that participants know (perhaps because the seller informed them, perhaps because the seller could not prevent their knowing) the number of participants, $n$, before deciding whether to pay the entry fee. This case is identified with $M \in \mathbb{M}^K$. Let $\xi^K(M, n, x)$ be the expected profitability (gross of bid-preparation cost and entry fee, but net of participation cost) of actually bidding in auction $M$, when there are $n$ participants, and participant 1 observes $X_1 = x$. Define

$$\begin{aligned} \Xi^K(M, n, \varphi) &= \left\{x \in \mathcal{X} | \xi^K(M, n, x) \geq \varphi + b\right\}, \text{ and} \qquad (1) \\ \alpha^K(M, n) &= \int_{\Xi^K(M, n, \varphi)} d\mathfrak{B}_1(x). \end{aligned}$$

[20]This insight is originally due to Matthews [1987] (who is credited in McAfee and McMillan [1987a]), and is employed in Harstad, Kagel and Levin [1990]. In all three papers, the uncertain number of bidders follows an exogenous distribution.

Here, $\Xi^K(M,n,\varphi)$ is the subset of $\mathcal{X}$ consisting of those signals with a sufficiently high interim expected profitability to justify continuing to compete in auction $M$ with $n-1$ rival participants, and $\alpha^K(M,n)$ is the ex-ante probability that a potential bidder whose action is to participate will end up becoming an actual bidder.

Continuing with the first case, $M \in \mathbb{M}^K$, define, for $n = 1,\ldots,N, i = 1,\ldots,n$, the event that participants $1,\ldots,i$ observe signals leading them to continue, while participants $i+1,\ldots,n$ observe signals leading them to cease competing:

$$\Gamma^K(M,i,n) = \left[\left\{X_j \in \Xi^K(M,n,\varphi)\right\} \Leftrightarrow \{j \leq i\}, \, j = 1,\ldots,n\right].$$

Let

$$\begin{aligned} \mu^K(M,i,n) &= \Pr\left[\mathbf{X}_n \in \Gamma^K(M,i,n)\right], \text{ and} \\ \lambda^K(M,i,n) &= \Pr\left[\mathbf{X}_n \in \Gamma^K(M,i,n) \,|X_1 \in \Xi^K(M,n,\varphi)\right] \end{aligned}$$

denote the probability of this event, and its probability conditional on participant 1 observing a signal leading to continued competition. By Bayes' Formula, for $n = 1,\ldots,N, i = 1,\ldots,n$,

$$\mu^K(M,i,n) = \lambda^K(M,i,n)\,\alpha^K(M,n). \tag{2}$$

The other case, when the auction form $m$ specifies that the number of participants is unknown when deciding whether to pay the entry fee, is identified with $M \in \mathbb{M}^U = \mathbb{M}\backslash\mathbb{M}^K$. Let $\xi^U(M,\pi,x)$ be the expected profitability (again, gross of bid-preparation cost and entry fee, but net of participation cost) of actually bidding in auction $M$, when $N$ potential bidders each participate with probability $\pi$, and participant 1 observed $X_1 = x$. Define, correspondingly,

$$\begin{aligned} \Xi^U(M,\pi,\varphi) &= \left\{x \in \mathcal{X} | \xi^U(M,\pi,x) \geq \varphi + b\right\}, \text{ and} \\ \alpha^U(M,\pi) &= \int_{\Xi^U(M,\pi,\varphi)} d\mathfrak{B}_1(x). \end{aligned}$$

Continuing as in the first case, define the event that participants $1,\ldots,i$ actually bid, and $i+1,\ldots,n$ cease competing:

$$\Gamma^U(M,\pi,i,n) = \left[\left\{X_j \in \Xi^U(M,\pi,\varphi)\right\} \Leftrightarrow \{j \leq i\}, \, j = 1,\ldots,n\right],$$

which is well-defined although the participants are unaware that they number $n$. The probability of the first $i$ participants becoming the only actual bidders, given that $N$ potential bidders each become a participant with probability $\pi$, must take the probabilities of events $\Gamma^U(M,\pi,i,n)$ and weight them according to their likelihood:

$$\begin{aligned}
\mu^U(M,\pi,i) &= \sum_{n=1}^{N} \beta(n,N,\pi) \Pr\left[\mathbf{X}_n \in \Gamma^U(M,\pi,i,n)\right], \text{ and} \\
\lambda^U(M,\pi,i) &= \sum_{n=1}^{N} \beta(n,N,\pi) \Pr\left[\mathbf{X}_n \in \Gamma^U(M,\pi,i,n) \,|X_1 \in \Xi^U(M,\pi,\varphi)\right]
\end{aligned}$$

is the conditional probability given that participant 1 observes a signal leading to continued competition. As before,

$$\mu^U(M,\pi,i) = \lambda^U(M,\pi,i)\,\alpha^U(M,\pi). \tag{3}$$

To combine the two cases, define

$$\begin{aligned}
\alpha(M,\pi,n) &= \begin{cases} \alpha^K(M,n), & M \in \mathbb{M}^K, \\ \alpha^U(M,\pi), & M \in \mathbb{M}^U, \end{cases}, \\
\mu_i(M,\pi,n) &= \begin{cases} \mu^K(M,i,n), & M \in \mathbb{M}^K, \\ \mu^U(M,\pi,i), & M \in \mathbb{M}^U, \end{cases}, \text{ and} \\
\lambda_i(M,\pi,n) &= \begin{cases} \lambda^K(M,i,n), & M \in \mathbb{M}^K, \\ \lambda^U(M,\pi,i), & M \in \mathbb{M}^U, \end{cases},
\end{aligned}$$

with, for any $M$, each of $\alpha, \mu_i, \lambda_i$ degenerate in one of its last two variables. Thus, $\alpha(M,\pi,n)$ takes an *ex-ante* view, from the viewpoint of a potential bidder: it is the probability, should he participate, that he will go on to become an actual bidder, evaluated *before* the signal $x$ is observed. Similarly, $\lambda_a(M,\pi,n)$ is the *ex-ante* probability, should he become an actual bidder, that a potential bidder will find himself to be one of the set $\{1,\dots,a\}$ actual bidders, and $\mu_a(M,\pi,n)$ is the (unconditional) *ex-ante* probability of $\{1,\dots,a\}$ being the set of actual bidders.

Note that prior models of endogenous participation have, explicitly or implicitly, assumed a *screening level*: some $\widetilde{x}(M,n,\varphi) \in \mathcal{X}$ such that $\left\{x \in \Xi^K(M,n,\varphi)\right\} \Leftrightarrow \{x \geq \widetilde{x}(M,n,\varphi)\}$. Landsberger and Tsirelson [2002] demonstrate that this is impossible in a common-value auction, for large numbers of potential bidders, under mild assumptions, satisfied by this and most prior models. This paper is careful to allow for the fact that $\Xi^K(M,n,\varphi)$ and $\Xi^U(M,\pi,\varphi)$ may not be upper contours of $\mathcal{X}$.

Two cases are also distinguished with respect to actual bidders. An auction form $m \in \mathbb{M}^{K'}$ if the number $a$ of actual bidders becomes known before bidding strategies are selected; let the probability of a sale be $s^{K'}(M,a)$, which is the probability that at least one of $a$ actual bidders is willing to pay the reserve price $r$. For $m \in \mathbb{M}^{U'} = \mathbb{M} \backslash \mathbb{M}^{K'}$, the number of actual bidders is unknown when bidding; let $s^{U'}(M,\pi,n)$ be the probability that at least one actual bidder is willing to pay the reserve price $r$ when either [a] each of $n$ participants becomes an actual bidder iff $X_j \in \Xi^K(M,n,\varphi)$, if $m \in \mathbb{M}^K$ (degenerate in $\pi$), or [b] if $m \in \mathbb{M}^U$, each of $N$ potential bidders becomes a participant with probability $\pi$, and if a participant, becomes an actual bidder iff $X_j \in \Xi^U(M,\pi,\varphi)$ (degenerate in $n$). Again, combine these cases via

$$s(M,a,\pi,n) = \begin{cases} s^{K'}(M,a), & M \in \mathbb{M}^{K'}, \\ s^{U'}(M,\pi,n), & M \in \mathbb{M}^{U'}. \end{cases} \tag{4}$$

Notation will be slightly abused when context makes clear by representing this probability as $s_r$ (the reserve price $r$ is the principal component of $M$ affecting this probability).

Getting closer to a characterization: Relying on $\mathcal{A}.4$, let $p(M,a,\pi,n,v)$ be a function indicating the expected price paid by the winning bidder, given auction $M$, $a$ actual bidders, $\pi$ probability of participating, $n$ participants, and conditional on a realization $v$ of asset value $V$. Depending on which cases above apply, $p(\cdot)$ will typically be degenerate in at least one variable. It bears emphasis that $p(\cdot)$ is an *ex-ante* calculation, and thus is symmetric across potential bidders.

Momentarily assume a potential bidder is one of $n$ participants and one of $a \leq n$ actual bidders; his *ex-ante* expected payoff is

$$\frac{s(M,a,\pi,n)}{a} E\left\{V - E\left[p(M,a,\pi,n,\cdot)\,|V\right]\right\} - \varphi - b - c. \tag{5}$$

In essence, conditioning the price on asset value (the inner expectation) makes the outer expectation simply the expected difference between what the winner gets and what he pays for it. The probability that the winner obtains this difference is simply the probability of a sale ($s$). Ex ante, given a winner, the probability that any one of the $a$ actual bidders is the winner is $1/a$, by $\mathcal{A}.1$ and $\mathcal{A}.6$. For an actual bidder, the bid-preparation cost $b$, entry fee $\varphi$ and participation cost $c$ are subtracted with certainty. (Note that this calculation need not require that the actual bidder know the value of $n$ or $a$.)

Continuing to assume $n$ participants, the *ex-ante* probability of being an actual bidder is $\alpha(M, \pi, n)$, and of any particular formula (5) being the relevant calculation for an assumed actual bidder is $\binom{n-1}{a-1}\lambda_a(M, \pi, n)$, since there are $\binom{n-1}{a-1}$ ways in which actual bidder 1 could face $a-1$ remaining rivals. Now to step back, assume only that a potential bidder is one of $n$ participants. His expected profit, for $n = 1, \ldots, N$, is

$$w(M, n) = \alpha(M, \pi, n)\left(\sum_a \left[\frac{s_r}{a} E\{V - E[p(M, \cdot)|V]\} - \varphi - b - c\right]\binom{n-1}{a-1}\lambda_a(M, \pi, n)\right). \quad (6)$$

Throughout, $\sum_a$ and $\sum_n$ are to be taken as abbreviated forms of $\sum_{a=1}^n$ and $\sum_{n=1}^N$. Each formula (6), for different $n$, is relevant (assuming participation) with probability $\beta_{n-1} = \beta[n-1, N-1, \pi(M)]$. Thus,

**Conclusion 2** *Equilibrium participation is that* $\pi \in (0, 1)$ *characterized by*

$$\begin{aligned} 0 &= \sum_n \beta_{n-1} w(M, n) \\ &= \sum_n \beta_{n-1}\left\{\alpha(M, \pi, n)\sum_a \left[\frac{s_r}{a} E\{V - E[p(M, \cdot)|V]\} - \varphi - b - c\right]\binom{n-1}{a-1}\lambda_a(M, \pi, n)\right\} \quad (7) \end{aligned}$$

*equating the payoff from nonparticipation to the net expected benefits.*

The right-hand side of (7) can be lowered by increasing $\pi$. If $\pi = 1$ is allowed as an equilibrium possibility, $0 = r.h.s.(7)$ must be replaced by $[r.h.s.(7)] \leq 0 = (\pi - 1)[r.h.s.(7)]$. Equation (7), by implicitly defining the symmetric participation probability function $\pi(M)$, together with equilibrium continuation, provides a complete characterization of potential bidders' behavior.[21]

# 4 General Revenue Formulation

Begin, analogous to the participation analysis, with a specification of seller's expected revenue, conditional on assuming $a \geq 1$ actual bidders and $n \geq a$ participants:

$$\begin{aligned} & s_r E[p(M, a, \pi, n, \cdot)] + a\varphi \\ = & \left(s_r E\{E[p(M, a, \pi, n, \cdot)|V]\} + a\varphi\right), \end{aligned}$$

[21] A corresponding equation is asserted by French and McCormick [1984], and found in simpler models by Harstad [1990] and Levin and Smith [1994]. The current development is original in avoiding a monotonicity assumption and allowing for the full variety of information flows.

which simply sums the price paid by the winner (multiplied by the probability of a sale), and entry fees paid by all actual bidders. It is harmless to condition on the asset's expected value.

Stepping back by replacing an assumed number of actual bidders and then of participants with probabilities gives our specification of expected revenue:

$$
\begin{aligned}
\mathcal{R}(M,n) &= \sum_a \left(s_r E\left\{E\left[p(M,a,\pi,n,\cdot)\,|V\right]\right\} + a\varphi\right)\binom{n}{a}\mu_a(M,\pi,n). && (8)\\
R(M) &= \sum_n \mathcal{R}(M,n)\,\beta\left[n,N,\pi(M)\right] \\
&= \sum_n \left\{\sum_a \left(s_r E\left\{E\left[p(\cdot)\,|V\right]\right\} + a\varphi\right)\binom{n}{a}\mu_a(M,\pi,n)\right\}\beta_n. && (9)
\end{aligned}
$$

Equation (9) is still a simple sum of the price paid and entry fees, itself summed over the objective probabilities of $a$ actual bidders and $n$ participants (there are $\binom{n}{a}$ ways that $\mu_a(M,\pi,n)$ might correctly predict the number of actual bidders). The summation harmlessly ignores the events of 0 participants and 0 actual bidders, which contribute 0 revenue. To interpret expected revenue, natural definitions of the expected value transferred, expected number of participants, and expected number of actual bidders, are invoked:

$$
\begin{aligned}
\overline{V}(M) &= \sum_n \sum_a s(M,a,\pi(M),n)\,E[V]\binom{n}{a}\mu_a(M,\pi,n)\,\beta_n, \\
\overline{n}(M) &= \sum_n n\beta_n = N\pi(M) \\
\overline{a}(M) &= \sum_n \left\{\sum_a a\binom{n}{a}\mu_a(M,\pi,n)\right\}\beta_n.
\end{aligned}
$$

These summations also harmlessly ignore the cases $n = 0$, $a = 0$. Note that these definitions depend on the mechanism; in particular, $\overline{V}(M)$ treats as a zero transfer an asset that does not sell. (Expected value transferred will retain the same natural economic meaning in Appendix C when that value will also come to depend on private-values components of asset value, but will have a more complicated definition.)

**Theorem 3** *(The Fundamental Revenue Identity): In symmetric equilibrium continuation with endogenous bidder participation, for any $M \in \mathbb{M}$,*

$$
R(M) = \overline{V}(M) - b\overline{a}(M) - c\overline{n}(M). \qquad (10)
$$

Theorem 3 is proven simply by separating out terms in (9) that are zero by equilibrium participation (eq. (7)). The notationally cumbersome details have been moved to Appendix B, and extended to affiliated-values auctions in Appendix C.

In simple language, the Identity says that revenue in symmetric equilibrium continuation is equal in expectation to the expected value transferred less aggregate participation and bid-preparation costs.[22] It is particularly important that this identity provides a simple formula for revenue for all $M$; there is no need for separate formulas for first-price, second-price, and English auctions, or for different information-revealing policies (except to determine $\pi[M]$), and the entry fee does not directly enter the calculation. The reserve price enters only through the probability of a sale.

Viewing efficiency as the sum of expected surplus of seller and all $N$ potential bidders, Theorem 3 yields a general and striking contrast to prior optimal auctions models, in which revenue is maximized by enforcing allocative inefficiencies:

**Corollary 4** *The Bayesian equilibrium in which seller maximizes expected revenue is allocatively efficient. Indeed, seller's preferences over any set of auction mechanisms match those of an efficient social planner.*

**Proof.** The right-hand-side of (10) is an efficiency measure, and in equilibrium continuation is also seller's objective. ■

As presented here, inefficiencies arise solely through failures to sell: with a common-value asset, any successful bidder is an efficient recipient. However, in Appendix C, this result is extended to affiliated-values auctions. There, a seller optimally attains a non-zero probability of no sale, and a non-zero probability of selling to an actual bidder who values the asset less than some other participant; these accord exactly with the preferences of an efficient social planner.[23]

[22] No result approaching comparable generality is in the literature, but this Theorem has many antecedents. Relative to Levin and Smith [1994], for example, it is original in its allowance for costs incurred both before and after bidders observe private information, in allowance for numbers of participants and actual bidders to be either learned or inferred, in the number and variety of surplus-extracting devices allowed for, and in dealing with the impossibility of a screening level. Moreover, Appendix C obtains the corresponding revenue identity for affiliated-values settings combining common-value and private-values elements.

In special cases, a corresponding result is found by Samuelson [1985] and Hausch and Li [1990], can be calculated in the example of Theorem 5.2 in Milgrom [1981], and found as an asymptotic approximation in Matthews [1984] (where the number of bidders is not necessarily an equilibrium level, but the participation costs are). Theorem 3 verifies shortcuts taken, but not justified explicitly, in equations (2) and (3) in Harstad [1990]. French and McCormick [1984] discuss a similar heuristic feature of first-price, common-value auctions, but do not provide a complete model or equilibrium characterization. McAfee and McMillan [1987b] assert the corresponding equation for a nonstochastic but supposedly endogenous $n$, without justification either for the equation or the source of $n$, and proceed incorrectly to dismiss the possibility that seller could enhance expected revenue via a positive entry fee.

[23] Indeed, with some additional notation, one can readily build an extension of this model (including affiliated-values

# 5 Inferiority of a Nontrivial Reserve Price

A seller can attain the entire interval of equilibrium participation probabilities, 0 through 1; this result is shown below for a second-price auction, chosen purely for convenience. The range of equilibrium values of $\pi$ is attained by varying only the entry fee $\varphi$ (including possibly $\varphi < 0$, reimbursing a fraction of participation and bid-preparation costs), while keeping the reserve price fixed at $r = 0$. Let $\overline{M}_{\varphi} = (\overline{m}, \varphi, 0)$, where $\overline{m}$ is a "vanilla" second-price auction with no disclosure of seller's information, and with $n$ and $a$ revealed to bidders; $\overline{M}_{\varphi}$ sells "without reserve."

**Theorem 5** *For any $\pi_0 \in [0,1]$, there exists an entry fee $\varphi_0$ such that $\pi\left(\overline{M}_{\varphi_0}\right) = \pi_0$.*

**Proof.** Setting $\varphi = E[V]$ generates $\pi = 0$; $\varphi = -c - b$ generates $\pi = 1$. Interim expected profitability $\xi\left(\overline{M_{\varphi}}, n, x\right)$ is degenerate in $\varphi$, so the mapping $\varphi \mapsto \Xi^K\left(\overline{M}_{\varphi}, n, \varphi\right)$ [(1)] is continuous. Smoothness of $\mathcal{B}_1$ (from $\mathcal{A}$.1) implies that $\alpha^K\left(\overline{M}_{\varphi}, n\right)$ and $\lambda^K\left(\overline{M}_{\varphi}, a, n\right)$ are continuous in $\varphi$. Since, for $\overline{m}$, the price function $p(\cdot)$ is degenerate in $\varphi$, it follows that the $\pi\left(\overline{M}_{\varphi}\right)$ function implicitly defined in (7) is continuous in $\varphi$. The Intermediate Value Theorem yields the conclusion. ■

A reserve price $r$ is *nontrivial* if at least one actual bidder does not guarantee a sale, that is, if there is an $a > 0$ such that $s(M, a, \pi, n) < 1$, $\mu_a(M, \pi, n) > 0$.

**Corollary 6** *Any auction mechanism $M$ with a nontrivial reserve price, yielding $\pi(M) \in (0,1)$, is an expected-revenue-inferior mechanism for seller to adopt.*

**Proof.** Consider any $M = (m, \varphi, r)$ such that $\pi(M) = \widehat{\pi} \in (0,1)$, with $r$ nontrivial. By Theorem 5, there exists $\overline{M}_{\widehat{\varphi}} = (\overline{m}, \widehat{\varphi}, 0)$ so that $\pi\left(\overline{M}_{\widehat{\varphi}}\right) = \widehat{\pi}$. Expected revenue, $R(M)$, with $r$ nontrivial, is

$$\begin{aligned} & \sum_n \left[ \sum_a (E[V] - ab) \left\{ s(M, a, \widehat{\pi}, n) \binom{n}{a} \mu_a(M, \pi, n) \right\} \right] \beta_n - c\widehat{\pi} N \\ < \; & \sum_n \left[ \sum_a (E[V] - ab) \left\{ s\left(\overline{M}_{\widehat{\varphi}}, a, \widehat{\pi}, n\right) \binom{n}{a} \mu_a\left(\overline{M}_{\widehat{\varphi}}, \pi, n\right) \right\} \right] \beta_n - c\widehat{\pi} N, \end{aligned} \tag{11}$$

aspects considered in Appendix C) featuring a number $N_L$ of potential bidders who face participation costs $c_L$ and $N_H$ who face costs $c_H > c_L$. For such a model, it is straightforward to show that a seller will prefer an auction mechanism $M_1$ for which the high-cost potential bidders participate with positive probability (in which case, all low-cost potential bidders strictly prefer to participate) to a mechanism $M_2$ for which the high-cost potential bidders do not participate iff an efficient social planner prefers $\mathrm{M}_1$ to $M_2$.

which is $R\left(\overline{M}_{\widehat{\varphi}}\right)$. Naturally, the probability that a given participant pays the reserve price is strictly less than the probability that he wins, while the probability that he pays the entry fee is strictly greater than the probability that he wins. Hence, with both mechanisms attaining participation probability $\widehat{\pi}$, $\widehat{\varphi} < \varphi + r$. The inequality then results from the terms in $\{\cdot\}$ on the left-hand side of (11) summing to less than the corresponding terms on the right-hand side. ■

Corollary 6 is quite intuitive. There is an unavoidable imperfection in this model, whenever $\pi < 1$. That is, with probability $(1-\pi)^N$, the independent mixed-strategy decisions lead to no potential bidder participating, which is a cost that both seller and a social planner would take into account. (Appendix A considers an alternative model avoiding this imperfection.)

Should a nontrivial reserve price be used, a further imperfection that is an inefficiency is introduced: not only is there no sale with probability $(1-\pi)^N$, there is also a probability

$$\sum_{n}\sum_{a=0}^{n}\left[1-s\left(M,a,\pi\left(M\right),n\right)\right]\binom{n}{a}\mu_a\left(M,\pi,n\right)\beta_n \tag{12}$$

that one or more potential bidders participate, but none of them are willing to pay the reserve price.

When the number of bidders responds endogenously to the profitability of competing, there is no counterbalance to make up for the loss of a sale due to a nontrivial reserve price. Occasionally, a reserve price would have, for example, fallen between the highest and second-highest bids in a second-price auction, or prevented a single participant from obtaining the asset for merely the entry fee, but the increased revenue such events create will have been taken into account in bidders' calculations of the probability with which to participate. Levin and Smith [1994] find that a nontrivial reserve price enhances revenue in common-value auctions with entry; their result is entirely due to disallowing entry fees, disclosure of seller's information, and other surplus-extracting devices that shed the revenue losses in (12).[24]

[24] Levin and Smith [1994] have a more primitive device in their model that they call an entry fee, but it is an information fee, in that it must be paid before bidders learn their signals; it in essence allows seller to employ a lump-sum tax on participants before they become privately informed. They obtain the result that a nontrivial reserve price is called for when they assume this information fee has to be set to zero. This paper follows a tradition in the literature, led by Cassady [1967], Milgrom and Weber [1982] and Samuelson [1985] in the normal usage of the term entry fee (as a fee incurred after participants become privately informed).

It also follows the tradition in the optimal auctions literature, and indeed in auction theory more generally (the only other exception I know of is McAfee and Reny [1992]), of assuming that any surplus-extracting device is potentially distortive, and thus ruling out devices that are in essence lump-sum taxes. I thank Jeroen Swinkels for emphasizing this issue, and for pointing out that a limit to the generality of this paper's results is that they apply only after a seller has exhausted usage of devices that are essentially lump-sum taxes.

Via Corollary 6, endogenizing bidder participation turns much of standard auction theory on its head. A reserve price (or bidder-specific reserve prices if bidders draw types asymmetrically) is the focus of Myerson's [1981] original "Optimal Auction Design" paper, and of much of the optimal auctions and mechanism design literature since (see, for examples, surveys in Klemperer [2000] and Krishna [2002]). Indeed, auction policy papers also focus on the reserve price as if it were a key variable (Klemperer [2002]). Yet when the number of bidders becomes an endogenous variable, the reserve price becomes a uniquely inferior tool for extracting surplus from bidders; a rational seller does not use it, and an efficient social planner is glad he doesn't.

## 6 Revenue and Participation

In view of Corollary 6, the remainder of the text limits mechanisms to $M \in \mathbb{M}^Z = \{M \subset \mathbb{M} \,|\, r = 0\}$, zero-reserve-price auctions. This section shows that seller's announcement of $M$ affects expected revenue solely through its effects on the participation probability $\pi$. To develop and understand this result, I begin with some natural comparative statics: two auction mechanisms with the same $\pi$ have the same expected revenue, and a change in mechanism which would lead to a higher expected revenue for any exogenously given number of bidders will lead to a lower $\pi$. Formally,

**Proposition 7** *For any* $\{M, M'\} \subset \mathbb{M}^Z$,
*[i]:* $\{\pi(M) = \pi(M')\} \Rightarrow \{R(M) = R(M')\}$;
*[ii]:* $\{\pi(M) = \pi(M')\} \Rightarrow \left\{\overline{V}(M) - b\overline{a}(M) = \overline{V}(M') - b\overline{a}(M')\right\}$;
*[iii]:* $\left\{\mathcal{R}(M, n) \gtreqless \mathcal{R}(M', n)\, \forall n \in \mathbb{N}\right\} \Rightarrow \left\{\pi(M) \lesseqgtr \pi(M')\right\}$.

The proof is in Appendix B.[25]

Proposition 7[iii] contrasts with an antecedent argument that the number of bidders may outweigh more direct aspects of auction design. Bulow and Klemperer [1996] find that an English auction with a zero reserve price and an exogenously given $N + 1$ bidders attains higher expected revenue than any standard auction with an optimal reserve price and $N$ bidders. While, once

Levin and Smith criticize Samuelson [1985] for considering the impact of entry fees in a model where the total expenditures on becoming privately informed are exogenous. The current model withstands that criticism.

Chakraborty and Kosmopoulou [2001] report a similar characterization to Corollary 6, that with entry, an auction with a nontrivial reserve price is revenue-inferior to some auction with a lower reserve and an entry fee. It is not clear what model of entry yields the nonstochastic number of participants in their paper, which depends on a screening level.

[25] Proposition 7 applies only when $\pi(M)$ is a function. The only auction mechanism this appears to rule out is the McAfee, McMillan and Reny [1989] mechanism that extracts full surplus if $n \geq 2$ bidders are exogenously given. I cover the details of comparison with that mechanism in footnote 27.

present, the $(N+1)^{\text{st}}$ bidder is assumed to behave rationally, no rational reason is given for his being there. They advise seller, rather than worrying about how to design the auction for the first $N$ bidders, to "somehow" find another bidder; how is unaddressed. Indeed, since their methods depend upon the potential bidder with the highest signal becoming the winning bidder, a model of costly entry with many potential bidders that could be consistent with their results faces serious problems.

Instead, Proposition 7 implies that revenue consequences of an increased number of bidders are necessarily less rosy when the extra bidders arrive via a rational participation calculation:

**Remark 1** *Suppose a seller can switch to an auction mechanism that increases equilibrium participation. Then each participant has a lower chance of winning, and so in equilibrium requires a higher expected profit in the event of winning. The winner's higher expected profit means an expected revenue* further below *expected value transferred.*

A host of econometric studies of auction markets are not sensible in this context, cases where revenue, the high bid, or some similar variable is estimated using the number of bidders as an exogenous explanatory variable. If ten potential bidders decided to participate expecting about three participants, but mixed strategy participation decisions happened to lead to six showing up, no wonder the extra bidders led to higher revenue: the auction rules were sufficiently extractive of bidders' surplus that no one wanted to be a fourth bidder. It would be interesting to discover the circumstances under which a higher *expected* number of bidders was associated with a higher expected price, but the historical record of auctions (outside carefully designed laboratory experiments) does not include data on the expected number of bidders. The actual number of bidders is no substitute. *That revenue is higher when the* realized *number of bidders is higher does not imply that a seller prefers to take steps to increase the* expected *number of bidders.*

If a historical series of auctions arguably results from the same equilibrium for each auction, then the binomial distribution that is the number of participants can be estimated from the series. Many empirical auction databases, however, arise from situations far enough from ex-ante symmetry to warn against direct application of this model. Nonetheless, observed data on the number of bidders may incorporate entry decisions based on which rivals were expected to take part with what probabilities. If the identities of participants are recorded in the database, a separate binomial distribution representing participation of each potential bidder could be estimated.

For many empirical studies, especially merger-and-acquisition studies, it remains a problem that the record does not indicate the number of participants, but at most the number of actual bidders. Hence, the size of a winner's curse adjustment a bidder ought rationally to make depends on a variable (or inferences about that variable) unavailable to the empirical analyst.

Proposition 7.$[ii]$ finds the difference $\overline{V}(M) - b\overline{a}(M)$ takes on the same value for any mechanisms $M, M'$ that attain the same $\pi$; let $W(\pi)$ denote this difference. Then let a simple function $R$ on the entire unit interval be defined by

$$R(\pi) = W(\pi) - c\pi N. \tag{13}$$

**Corollary 8** *$R(M) = R[\pi(M)]$ for all $M \in M^Z$; that is, $R(M)$ can be projected onto $[0,1]$ to yield $R(\pi)$.*

***Proof.*** *Proposition 7 implies that $M$ influences $R(M)$ only through its influence on $\pi(M)$. Theorem 5 shows that the entire interval $[0,1]$ can be reached.* ■

**Remark 2** *Corollary 8 finally changes auction theory's view of the comparative roles of the various surplus-extracting devices available to a seller, from complements (their role in exogenous-number-of-bidders models in the tradition of Milgrom and Weber [1982]) into their common-sense role of substitutes. With exogenous $n$, a seller who had introduced some subset of: switching to an English auction, releasing public information, setting an entry fee, and adding a nontrival reserve price, would still gain by incorporating the remaining surplus-extracting devices. With endogenous participation, it is (solely) the equilibrium probability $\pi$ that interests seller, and alternative methods of accomplishing an improvement in this variable are substitutes for each other.*

# 7 The Content of Optimal Auctions

Theorem 5 and Corollary 8 imply that $R(\pi)$ in (13) is continuous. As its range is obviously bounded, it attains a maximum. Let $R^* > 0$ be the maximum attainable level of revenue. Since a screening level is impossible, it is unsurprising that I have not been able to demonstrate strict concavity of $R(\pi)$. Accordingly, there may be multiple values of $\pi$ attaining $R^*$; let $A = \{\pi \in [0,1] \mid R(\pi) = R^*\}$. By continuity, $A$ contains a minimal and a maximal element, $\pi_0^*$ and $\pi_1^*$ (not necessarily different). As $R(0) = 0$, $\pi_0^* > 0$.

**Proposition 9** $\pi_1^* < 1$.

**Proof.** Evaluation of (13) as $\pi$ decreases from 1 to $1-\triangle\pi$ shows a revenue gain from decreased participation costs that is linear in $\triangle\pi$, and a revenue loss from a probability of 0 participants that is of the order $(\triangle\pi)^N$. ■

Proposition 9 heightens the contrast with Bulow and Klemperer [1996]. They recommend obtaining an $(N+1)^{\text{st}}$ bidder (in a setting where there are otherwise $N$ exogenously specified bidders) as the most important revenue-enhancing decision a seller can make. Far from seeking another bidder, when any bidder, including the $(N+1)^{\text{st}}$, arrives via a rational, expect-profit-seeking decision, a seller is instead always attempting at least some degree of potential bidder discouragement. Stated differently, a seller might benefit from extra competition, but does not wish to give an extra potential competitor an expectation of being able to enter the fray profitably.

Indeed, Proposition 9 (like all other results) applies to $N=2$ potential bidders. In that case, it is revenue-inferior to adopt an auction that leads to both bidders participating with probability one. The seller will have some revenue-superior alternative which will lead potential bidder 1 to be indifferent over participating even when he infers that there will be at least a $(1-\pi_1^*)>0$ probability of facing no competition.

A little structure enables characterizing the size of the set of optimal auctions. Let $\mathcal{M}^C$ be the set of auction forms $m$ for which some variable (explicit, or implicit in $m$) continuously alters $\pi$ and spans $[0,1]$. Theorem 5 above shows that the entry fee $\varphi$ makes the second-price auction form an element of $\mathcal{M}^C$; the continuity used in that proof is known to hold for the English and first-price auctions. Moreover, there are other modeling options (examples are below in this section) that may render many auction forms elements of $\mathcal{M}^C$. Next, define a set of auction mechanisms $\underline{\mathbb{M}}=\left\{M\in\mathbb{M}\,|m\in\mathcal{M}^C,\, r=0\right\}$; this is the set of mechanisms selling without reserve for which the auction form exhibits continuity and spanning in some variable. I treat $\underline{\mathbb{M}}$ as the domain of choice for seller, in light of Corollary 6; let $d$ denote the (nontrivial) dimensionality of $\underline{\mathbb{M}}$.[26] Without loss of generality, $\underline{\mathbb{M}}$ can be treated as embedded in $\Re^d$, and the coordinates of $\Re^d$ can be ordered so that coordinate $i\in\Delta=\{1,\ldots,d^*\}$ $(0<d^*<d)$ denotes a spanning, continuous variable. Let $\mathbb{M}^*=\{M\subset\underline{\mathbb{M}}\,|\,\pi(M)\in A\}$, a collection of optimal auctions. Then,[27]

[26] The exact dimensionality depends on modeling choices (as to what constitute the variables) that otherwise distract from the paper. The entry fee is one dimension. Whether $n$ and $a$ are revealed generates two more. At least one dimension could be generated by whether the auction is dynamic (if the degree of information dispersal during the course of the auction is an issue, more than one dimension), and still first-price and second-price auctions have not been distinguished. Mares and Harstad [2002] show that seller's information disclosure options cannot be summed up in a single dimension. Note that there is no problem to having dimensions consisting of discrete elements.

[27] Some readers may question how these optimal auction mechanisms compare to the mechanism which extracts full surplus in McAfee, McMillan and Reny [1989]. For their mechanism, call it $M^{MMR}$, the unique equilibrium

**Theorem 10** *Any auction in* $\underline{\mathbb{M}}$ *can be converted into an optimal auction in* $\mathbb{M}^*$ *merely by adjusting* any one *parameter in* $\Delta$.

**Proof.** Select an arbitrary $\pi^* \in A$, and an arbitrary $M_0 \in (\underline{\mathbb{M}} \backslash \mathbb{M}^*)$, so $\pi(M_0) \notin A$. Select an arbitrary coordinate $i \in \Delta$. Construct $\widehat{M} \in \underline{\mathbb{M}}$ by changing $M_0$ solely in coordinate $i$, as follows. Take an arbitrary $\pi^* \in A$; if $\pi(M_0) < \pi^*$, set $\pi(\widehat{M}) = 1$ (by $\varphi = -c - b$, for example), else if $\pi(M_0) > \pi^*$, set $\pi(\widehat{M}) = 0$ (by $\varphi = E[V]$). By spanning, this is always possible. As in Theorem 5, continuity implies the existence of a value for this $i^{\text{th}}$ coordinate yielding $M^* \in \mathbb{M}^*$, with $M^*$ differing from $M_0$ only in this $i^{\text{th}}$ coordinate. ■

**Corollary 11** $\mathbb{M}^*$ *contains a subset of dimension* $d-1$ *spanning* $\underline{\mathbb{M}}$.[28]

Thus, prior optimal-auction characterizations depend critically on the implicit assumption that a seller has a captive audience: there will be exactly $n$ bidders no matter how the seller changes auction rules to extract more surplus. When the number of bidders responds endogenously to the profitability of competing, the content of optimal common-value auctions is merely this: choose *any* auction form, commit to sell without reserve, and adjust *any* continuous parameter to avoid overly encouraging or overly discouraging bidder participation.

Figure 2 may help to visualize Theorem 10. It simplifies by imagining that $\underline{\mathbb{M}}$ has three dimensions: $\varphi$ on the vertical axis, plus one dimension in which $\underline{\mathbb{M}}$ can take on one of three discrete values (e.g., English, second-price, or first-price auction form, for a seller we imagine to be constrained to those three choices), and one dimension in which a variable can be chosen over an interval, but may not necessarily span the range of $\pi$. The set $\underline{\mathbb{M}}$ is then the union of three rectangles in parallel vertical planes, outlined in Figure 2 by dashed lines. The set of optimal auctions $\mathbb{M}^*$ is represented by a collection of curves shown lying in the three rectangles. In this illustration, the space of auctions

continuation is $\pi(M^{MMR}) = 0$, hence $R(M^{MMR}) = 0$. To arrive at a sensible comparison, consider mechanisms $M_\varphi^{MMR}$ with negative entry fees appended. Setting $\varphi < -c - b$ necessarily generates a revenue-inferior auction; the reverse inequality suffers the same $R(M_\varphi^{MMR}) = 0$ problem as $M^{MMR}$. So consider $\varphi = -c - b$: for $M_{-c-b}^{MMR}$, every $\pi \in [0,1]$ is an equilibrium continuation. Selecting $\pi \in (\pi_1^*, 1]$ yields excessive incurrence of participation costs with no compensation; selecting $\pi \in [0, \pi_1^*]$ runs into the same problem as a reserve price: there is an excess probability of no sale (happening anytime $a < 2$), with no compensation. So any equilibrium selection $\pi$ is revenue-inferior to the second-price auction $\overline{M}_\varphi$ of Theorem 5 that attains the same $\pi$, hence suboptimal. If a mechanism similar to Crémer and McLean [1985], [1988] were to apply to a common-value auction, it would suffer the same problems. So would the mechanism of McAfee and Reny [1992], which also depends on using information fees, ruled out here (cf. footnote 24).

[28] To see that a set of dimension $d-1$ has been obtained, construct the projection mapping $proj_\varphi : \Re^d \to \Re^{d-1}$ by deleting the $1^{st}$ component from any $d$-vector $M \in \underline{\mathbb{M}}$. Then the proof of Theorem 10 has shown for every $M \in \underline{\mathbb{M}}$ an element $M_M^* \in M^*$ such that $proj_\varphi(M) = proj_\varphi(M_M^*)$. Note that $M^*$ is typically a strict superset of the set thus obtained.

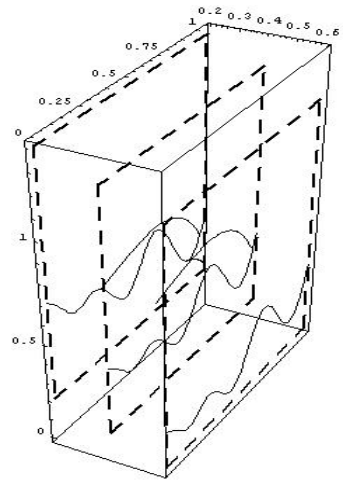

Figure 2: The Set of Optimal Auctions

with a 0 reserve price is 3-dimensional, and the set of optimal auctions (the union of the curves shown) a 2-dimensional subset. The optimal auctions $\mathbb{M}^*$ span $\underline{\mathbb{M}}$ in that, from any point in one of the three rectangles, it is possible to reach one of the curves by moving only vertically, that is, by adjusting only the entry fee. (To be consistent with the Theorem, the curves must, collectively, contain continuous paths from the left to the right edges of all rectangles.)

A variety of surplus-extracting devices might exhibit sufficient continuity to apply this logic. For example, suppose seller observes $X_{N+1}$ (which is affiliated with asset value $V$), and consider mechanisms $M_y^1$, all first-price auctions with $\varphi = \varphi_0$ (arbitrary), $r = 0$, and with seller making a public announcement of $Z_y = X_{N+1} + y\zeta$, where $\zeta$ is an independent standard normal (white noise), and $y$ a scalar parameter of the noisiness of this public announcement. Then (by Theorem 17 in Milgrom and Weber [1982] and Proposition 6.[*iii*] above), $\pi\left(M_y^1\right)$ is nondecreasing in $y$; assume (naturally) that it is continuous. Suppose $\pi\left(M_0^1\right) < \pi_0^* < \pi\left(M_Y^1\right)$ for sufficiently large $Y$. That is, full and honest public announcement of seller's information is overly extractive of surplus, but not a public announcement where the signal-to-noise ratio is very small (this is, in essence, a scanning supposition). Then the argument of Theorem 10 can be applied to derive the existence of a $y^*$ such that $M_{y^*}^1$ is an optimal auction.[29]

[29] A similar example: some sellers categorize assets being sold, so each asset in a category has an appraisal in a given range (e.g., $30K-$40K). For any auction form with the property that a broad enough range is insufficiently extractive and an exact announcement of the appraisal overly extractive, there exists a range width yielding an optimal auction. Thus, from an arbitrary auction form, only the range width of this categorization need be altered to obtain optimality.

In a world where a wide variety of auction mechanisms are employed by experienced and apparently successful auctioneers and frequent auction sellers or bid-taking procurers, Theorem 10 has the comforting conclusion that the variety is not *per se* unambiguous evidence that some of these sellers and auctioneers must be choosing suboptimally. If some aspect of a particular situation falling outside the model creates a preference for one auction form over another, nothing in the model surmounts that preference, so long as some variable remains sufficiently adjustable.

How do revenue comparisons across auction choices fare? To formulate a partial answer, note that the parameters of the model are $P = \{\mathfrak{B}, c, b, N\}$, the underlying probability measure, the levels of participation and bid-preparation costs, and the number of potential bidders. For simplicity, assume that $\mathfrak{B}$ can be specified by a real vector in $\Re_+^{D-2}$. Then $P \in \mathfrak{P} \subset \Re_+^D \times I$, where $I = \{2, 3, \ldots, N^U\}$, with $N^U < \infty$ an arbitrary upper bound on possible values of $N$, allows $\mathfrak{P}$ to represent the economic environments to which the model might be applied.

**Corollary 12** *In the special case where $R(\pi)$ is concave, every revenue comparison of two auction mechanisms with zero reserve prices for an exogenous number of bidders is extended with endogenous participation to a half-space of $\mathfrak{P}$ and is reversed in another half-space.*

**Proof.** Any such revenue comparison takes the form $\mathcal{R}(M_s, n) \geq \mathcal{R}(M_i, n) \forall n \in \mathbb{N}$, for some superior mechanism $M_s$ and inferior mechanism $M_i$.[30] Any $P \in \mathfrak{P}$ determines a mapping $\pi_P(M)$ on $\underline{\mathbb{M}}$, specified by (7). By Proposition 7, each $(M_s, M_i)$ comparison can, for any given $P$, be represented in $\mathfrak{P} \times [0,1]$ by a vector $g_P$ pointing from $(P, \pi_P[M_i])$ to $(P, \pi_P[M_s])$, where $\pi_P(M_s) < \pi_P(M_i)$. Of course, $\pi_0^*$ and $\pi_1^*$ depend on $P$ as well, but for any $P$, the collection $\{g_P, (P, \pi_0^*[P]), (P, \pi_1^*[P])\}$ are collinear in $\mathfrak{P} \times [0,1]$. By definition, $\pi_0^*[P] \leq \pi_1^*[P] \forall P \in \mathfrak{P}$. The direction of $g_P$ implies that a sufficient condition for $R(M_s) \geq R(M_i)$, that is, for extending the revenue comparison to endogenous participation, is $\pi_P(M_s) \geq \pi_1^*[P]$. Correspondingly, a sufficient condition for $R(M_s) \leq R(M_i)$, that is, for reversing the revenue comparison with endogenous participation, is $\pi_P(M_i) \leq \pi_0^*[P]$.[31] ■

As mentioned at the beginning of this section, concavity is not likely to hold in this general a model. However, most revenue comparisons in the literature[32] presume $b = 0$, and treat only the

[30] Examples would include: second-price versus first-price auctions in Milgrom and Weber [1982], alternating recognition versus second-price auctions in Harstad and Rothkopf [2000], cases of private revelation of seller's appraisal versus public announcement of seller's appraisal in Mares and Harstad [2002].

[31] If concavity of $R(\pi)$ is not strict, $A = [\pi_0^*, \pi_1^*]$ and $\pi_0^*$ and $\pi_1^*$ may be distinct. If so, these two half-spaces may not contain all possible parameters.

[32] These include all comparisons in Milgrom and Weber [1982] that do not involve a reserve price.

case $r = \varphi = 0$; for these assumptions, it is straightforward, though cumbersome, to show that the points of comparison can lie on a concave $R(\pi)$.

The sharpness of these results stems partly from the exactness attained via a mixed-strategy participation decision arrived at simultaneously by ex-ante symmetric (and thus not yet privately informed) potential bidders. Appendix A finds the bulk of these results attainable, if granted a sufficiently useful equilibrium selection, when potential bidders sequentially decide whether to participate. The intuition is this: if behavior once participating is symmetric, and one potential bidder who is indifferent over whether to participate does take part, then that one participant's indifference drives revenue whether his participation decision was made simultaneously or sequentially.

## 8 Concluding Remarks on Generality

The contrast is striking: Many papers calculate an "optimal auction," having innocuously (or at least without comment) assumed there are $n$ bidders. When this number remains fixed as the role of being a bidder is made far less profitable, these authors are in essence assuming irrational behavior, for most situations where they would have us apply their results. Those results typically find a particular auction form to be optimal, and it typically revolves around strategically setting a reserve price which has a significant chance of preventing a sale.

When bidders are also rational in deciding whether to bid, and the number of bidders is explicitly recognized as an endogenous variable, these results are completely overturned: the only aspect of an auction design that, *per se*, characterizes it as suboptimal is a nontrivial reserve price. Selling without reserve is the full content of optimal auctions when participation is endogenous.

As contemplated in auction theory, a nontrivial reserve price is almost never seen in practice (Cassady [1967]). The contemplated reserve price is a credible binding commitment that, if no bid exceeds it, the asset will not now and never in the future be available to the potential bidders. In some situations, such a commitment may stretch credibility, but in many, I suspect a tool so impacting yet so blunt is not used because it would pointlessly introduce inefficiency. What is common, and in the industry usually called a reserve price, is a price below which the current auction will end without a sale, but the same asset will be put up for sale again later (often, there may be negotiations between the seller and the high bidder to buy between the final bid and the reserve price). Such a policy, of course, limits potential inefficiencies to a wholly lower order of magnitude; introducing it would bring complications of dynamic negotiations into the model, which

I have avoided.

The principal result of this paper is that while selling without reserve, the set of optimal auctions is large, consisting of single-parameter adjustments of all auctions. While the demonstration has hopefully been as straightforward as possible, it is clear that the characterizations allow several natural extensions:

- The results have been demonstrated for common-value auctions, wherein inefficiencies can only take the form of a failure to sell. However, complications introduced by adding private-value aspects to yield affiliated-value auctions change little. In particular, a seller has the same preferences across auction forms as a social planner, and the set of optimal auctions continues to be as large and dense as above. Arguments in favor of a nontrivial reserve price remain elusive, though a fully general characterization as strong as revenue-inferiority cannot quite be reached. Details are provided in Appendix C.

- The oft-seen formulation of an auction problem as an abstract mechanism design problem contemplates payments to or from losing bidders. Such payments are easily incorporated here, although Theorems 5 and 7 render them pointless.

- The bid-preparation cost above was exogenous, and independent of the form of the auction. If strategic issues were to make it more costly to prepare a bid in a first-price auction than in a second-price or an English auction (due to some variant on incentive compatibility), a more complicated twist on the tools provided here would be needed.[33]

- The assumption of a single asset for sale does not seem critical to the qualitative results. However, the ease with which extension to the modal multiple-unit auction model (where each bidder can acquire but one asset) arises in Milgrom [1981] and Pesendorfer and Swinkels

[33] Engelbrecht-Wiggans [2001] argues that the strategic simplicity of English auctions, in some simple settings, yields lower bid-preparation costs than first-price sealed-bid auctions, and so English auctions might attract more bidders and attain higher expected revenue. He demonstrates the possibility in a simple example with independent, uniformly distributed private values and infinitely many potential bidders. Unfortunately, the notion of strategic simplicity does not admit nearly as facile a sensible definition when going beyond such a simple setting. In particular, it may be that an English auction becomes far less simple strategically when a significant entry fee is prepended. (Engelbrecht-Wiggans does not write as if he is comfortable with the notion that an English auction necessarily remains strategically simpler once common-value elements enter the model.) Since in the equilibrium above, aggregate expected bid-preparation costs fall on the seller (Theorem 3), this by itself gives an incentive to favor devices which extract surplus while facing bidders with lower bid-preparation costs. For a given $\pi$, the mechanisms with lower bid-preparation costs yield higher revenue. For there to be a sufficiently large set of such lower-bid-preparation-cost mechanisms to span continuously the range of participation probabilities, and thus render all higher-bid-preparation-cost mechanisms inferior, is likely to depend on some controversial assumptions about strategic simplicity.

[1997] would be somewhat misleading. If $k$ identical assets are sold, the probabilities of $0, \ldots, k-1$ actual bidders would significantly clutter up the expected revenue formula.

- The model has been designed so that adding other sellers who are auctioning related assets is virtually automatic. (No problems are created if one seller's auction exhibits a participation cost of $c$, while a possibly more distant seller has a cost $c' > c$.)

A nontrivial dynamic structure would, however, introduce concerns not yet addressed. Among them, both sellers and potential bidders may have incentives to invest in building reputations. Nonetheless, a stride in this direction is made here: if an analysis of such reputational issues is to be applicable to markets where a subset of firms in an industry appear as bidders, reputational investments need to be viewed in terms of their discounted expected profitability when responses of other players include an endogenous decision as to whether and when to play.

This model assumes rational behavior consistent with a symmetric equilibrium. Asymmetric equilibria at the bidding stage are certainly not going to be unique, so it is unclear how to prepend an entry stage, without a unique expected profitability calculation. Asymmetric participation decisions are presumably rife for signaling a preferred asymmetric equilibrium. A "symmetric sequential" entry model which then assumed symmetric behavior following sequential entry decisions can be built; it yields similar but less sharp results. An outline is provided in Appendix A.

Laboratory evidence suggests the winner's curse is not easily overcome in common-value auctions (Kagel, Levin and Harstad [1995]); however, it is far from clear how to model participation decisions of potential bidders who will not follow up by bidding rationally. Nor can I envision how to model usefully the participation decision of a potential bidder who will himself behave rationally, but who cannot predict even the number of irrationally-behaving rivals who will participate.

# 9 Appendix A: An Alternative Sequential-Entry Model

Consider the following "symmetric sequential" participation model. First, the seller announces $M$, as above. An exogenous randomization assigns to the $N$ potential bidders a relabeling of their indices, with a potential bidder's realization that he is number $i$ in this relabeling his own private information, and all reorderings equally likely. Then potential bidders are in order given the opportunity to participate (at cost $c$, as above). As soon as a potential bidder declines to participate (an action that may be the result of a mixed strategy), no other potential bidder is given the opportunity.

As a potential bidder knows the step in this order in which he makes his decision, he knows how many potential bidders have already chosen to participate. As a participant's stage in the order does not get revealed, an opportunity to signal a favorite asymmetric equilibrium via becoming participant 1, for example, is unavailable. After participation decisions have been made, one of the participants knows privately that he is the marginal participant, but none knows the order in which rivals became participants.[34] For symmetric behavior to be possible, the private information of the last participant, as to the equilibrium number of participants, must become public; denote this number $n_e(M)$. Hence only mechanisms in $\mathbb{M}^K$ above can be considered.

The ex-ante probabilities of a participant becoming an actual bidder, and of participants $1, \ldots, a$ becoming the actual bidders (unconditional and conditional), are exactly the same as $\alpha^K[M, n_e(M)]$, $\mu^K[M, a, n_e(M)]$, and $\lambda^K[M, a, n_e(M)]$ above, and $s(\cdot)$ is unchanged from (4) except that it no longer can depend on $\pi$. Lack of dependence on $\pi$ is also the only change in $p(\cdot)$ above, so the expected profitability of being the $n_e(M)^{\text{th}}$ participant is still (6) above. Hence, $n_e(M)$ is determined by the equilibrium participation constraint

$$w[M, n_e(M)] \geq 0 > w[M, n_e(M) + 1]. \tag{14}$$

For equality in (14), participation by $n_e(M) - 1$ potential bidders with probability 1 and by the $n_e(M)^{\text{th}}$ potential bidder with probability $\pi \in [0, 1]$ are equilibria for all values of $\pi$. Selection of the $\pi = 1$ equilibrium can be based on it being the unique element of this set which is the limit of equilibria for mechanisms differing from $M$ by having infinitesimally smaller entry fees. Of course, virtually as strong a selection argument can be made for the $\pi = 0$ equilibrium, as the unique limit of equilibria for mechanisms differing from $M$ by having infinitesimally larger entry fees. However, usual problems with limits of open sets prevent existence of optimal auctions if the $\pi = 0$ equilibrium is selected. I will just consider the self-servingness of the $\pi = 1$ selection be a weakness of the alternative model, and proceed with it.

Revenue is now $R(M) = \mathcal{R}[M, n_e(M)]$, from (8). Define

$$\delta(M, n) = \begin{cases} 1, & n = n_e(M), \\ 0, & \text{otherwise.} \end{cases}$$

[34] It is solely for this reason that the model builds a counterfactual where a potential bidder's sequence order is his own private infomation.

This substitutes for the binomial coefficients $\beta(\cdot)$ in the formulas for expected value transferred and the expected number of actual bidders:

$$\begin{aligned} \overline{V}(M) &= \sum_n \sum_a s(M,a,n) E[V] \binom{n}{a} \mu^K (M,a,n) \delta(M,n), \\ \overline{a}(M) &= \sum_n \left\{ \sum_a a \binom{n}{a} \mu^K (M,a,n) \right\} \delta(M,n). \end{aligned}$$

Then expected revenue satisfies

$$R(M) \leq \overline{V}(M) - b\overline{a}(M) - cn_e(M), \tag{15}$$

with equality for and only for the selected equilibria attaining equality in (14); let $\mathbb{M}^=$ be the subset of $\mathbb{M}^K$ consisting of those mechanisms $M$ for which equality in (14) and (15) can be attained. Note that $\left(\mathbb{M}^K \backslash \mathbb{M}^=\right)$ contains an open and dense subset of $\mathbb{M}^K$. Define $\mathbb{M}_{\|n\|} = \left\{M \in \mathbb{M}^K \mid n_e(M) = n\right\}$, for $n = 1, \ldots, N$, and $\mathbb{M}^Z = \left\{M \in \mathbb{M}^K \,| r = 0\right\}$.

The following results can be obtained for such a model. [$i$]. $\{M_0, M_1\} \subset \mathbb{M}^=$ and $n_e(M_0) = n_e(M_1)$ implies $R(M_0) = R(M_1)$. This corresponds to a comparative static of the simultaneous entry model.

[$ii$]. Suppose a mechanism $M_0 \in \mathbb{M}^=$ with $n_e(M_0) = n_0$ participants in equilibrium. Then there exists $M_1 \in \left[\mathbb{M}_{\|n_0\|} \cap \mathbb{M}^= \cap \mathbb{M}^Z\right]$ (i.e., $M_1$ does not use a positive reserve price). This $M_1$ is revenue-maximal in the set $\mathbb{M}_{\|n_0\|}$; revenue comparisons across auction forms for an exogenous number of bidders apply within $\mathbb{M}_{\|n_0\|}$, and surplus-extracting devices are substitutes within $\mathbb{M}_{\|n_0\|}$, with the exception that nontrivial reserve prices are revenue-inferior.

[$iii$]. Suppose there exist $M_0 \in \mathbb{M}^=$, $M_1 \in \mathbb{M}^K$ such that $1 \leq n_e(M_0) < n_e(M_1) \leq N$, and $R(M_1) > R(M_0)$. Then there exists $n^* > n_e(M_0)$ such that [$a$] $\mathbb{M}^* = \left[\mathbb{M}_{\|n^*\|} \cap \mathbb{M}^= \cap \mathbb{M}^Z\right] \neq \emptyset$ (these are all zero-reserve-price auctions attaining equality in (14) for $n^*$ participants), and [$b$] every auction in $\mathbb{M}^*$ is an optimal auction. Moreover, for an arbitrary auction form $m'$ for which expected profitability is continuous in the entry fee $\varphi$, if there exists $M' = (m', \varphi', 0)$ such that $n_e(M') < n^*$, then there exists $\varphi^*$ such that $(m', \varphi^*, 0) \in \mathbb{M}^*$. In this sense, an arbitrary auction can be made optimal by the change of a single parameter, attaining a quite similar characterization to the principal result of the simultaneous entry model.

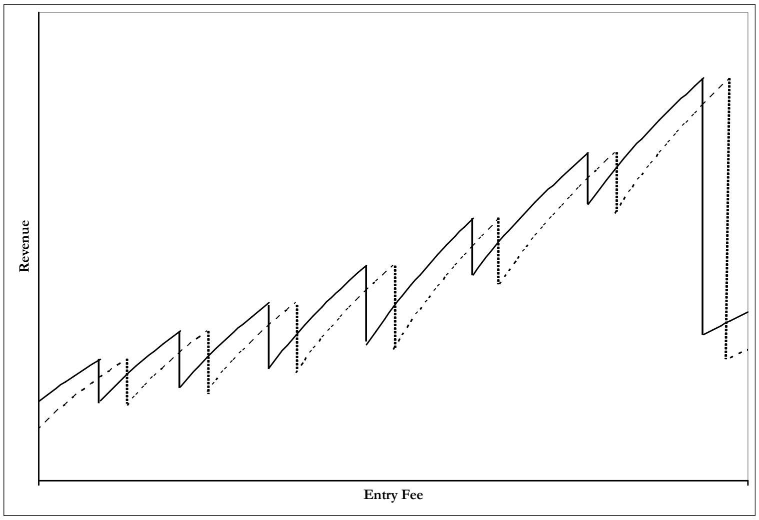


Figure 3: The Sequential Participation Model

$[iv]$. Let $M_0, M_1 \in \mathbb{M}^K$ be such that $n_e(M_0) < n_e(M_1)$. Suppose

$$\begin{aligned} & \sum_a \left\{ s[M_0, a, n_e(M_0)] \binom{n_e(M_0)}{a} \mu_a [M, \pi, n_e(M_0)] \right\} \\ \geq & \sum_a \left\{ s[M_1, a, n_e(M_1)] \binom{n_e(M_1)}{a} \mu_a [M, \pi, n_e(M_1)] \right\}, \end{aligned}$$

that is, suppose a sale is at least as likely ex ante under $M_0$ as under $M_1$. Then $R(M_0) > R(M_1)$, a sense in which the bidder-discouragement flavor of the simultaneous entry model extends to this model. Note that the sale-probability supposition is critical to result $[iv]$. (Proofs of these results correspond closely to methods used in the main text and Appendix B.)

Figure 3 illustrates these results, for auctions that use a 0 reserve price. The entry fee $\varphi$ is shown horizontally, expected revenue $R$ vertically. The solid curve illustrates one type of auction, the dotted curve a second type which extracts less surplus for a given number of bidders. For concreteness, we may call the solid curve English auction revenue, and the dotted curve first-price auction revenue. The vertical line segments on each correspond to values of $\varphi$ for which the specified auction mechanism lies in $\mathbb{M}^=$. In particular, each point in a vertical line segment is revenue associated with one of the multiple equilibria: the lower endpoint is associated with the marginal participant selecting to enter with probability 0, the upper endpoint associated with probability 1.

The rightmost vertical segments are where one participant in an English (solid vertical segment) and in a first-price (dotted) auction is enough to make a second potential bidder indifferent over participating. Along the sloped segment of each curve to the left of its rightmost vertical segment, the second potential bidder strictly prefers to participate, while staying out is the third potential bidder's strict preference. Then each curve reaches another vertical segment where the third potential bidder's indifference yields multiple equilibria, followed further left by a sloped segment along which there are three participants.

Each pair of vertical segments corresponding to the multiple equilibria where the $i^{\text{th}}$ potential bidder is indifferent over participating peak at exactly the same height. This is a result of equality in (14) and (15). A curve like those shown could be drawn for any auction form; for example, the curve for a second-price auction would have vertical segments that lie between the paired vertical segments shown. The vertical segments shown for the English auction would be shifted to the left if seller's information were publicly disclosed. All such curves for auction forms with 0 reserve prices would reach identical heights at the peaks of vertical segments.

The case illustrated will have as an optimal auction (given the self-serving equilibrium selection mentioned above) any auction without a reserve price where the entry fee is set so that the second potential bidder is indifferent over participating. Any auction form which is sufficiently extractive to strictly discourage the second potential bidder via a high enough entry fee will have an entry fee which makes that auction form (with $r = 0$) optimal.

In the general affiliated-values setting (Appendix C), it is possible that the rightmost pair of vertical line segments in Figure 4 do not attain the height of the pair to the left of them. If so, then optimal auctions are those where potential bidder $i^*$ is indifferent over participating, for some $i^* > 2$ (surely still a small number).

Consider, for an arbitrary auction form $m$, beginning with $n_0$ participants in equilibrium, impacts of increasing $\varphi$. Increasing from small enough $\varphi$, revenue is monotonically increasing, and $w$, the expected profitability of participating (the l.h.s. of (14)), is monotonically decreasing, while $s$, the probability of a sale (here, with $r = 0$, the probability that $a > 0$), holds nearly constant. As $\varphi$ continues to increase, past some level $s$ starts to decrease nonnegligibly. There will be some threshold $\widehat{\varphi}$ at which revenue from $n_0$ participants hits a local maximum and starts to decrease. Figure 4 is drawn assuming potential bidder $n_0$ is driven down to indifference over participating before $\varphi$ reaches $\widehat{\varphi}$.

I know of no assumption on the primitives of the model guaranteeing this will always be the case (this is why the results above in this appendix are stated with such specific conditions). In general, little is known about the behavior of auction mechanisms above $\widehat{\varphi}$. Revenue need not be monotonic in $\varphi$ above $\widehat{\varphi}$, nor need $w$ be monotonic. It is the case that, for $(m, \varphi', 0) \in \mathbb{M}^{=}$, revenue approaches from below as $\varphi$ approaches $\varphi'$ from below. Also, $(m, \varphi, 0) \in \mathbb{M}_{\|n_0\|} \Rightarrow \exists \varphi' |\, (m, \varphi', 0) \in \mathbb{M}^{=} \cap \mathbb{M}_{\|n_0\|}$. However, $\varphi' > \widehat{\varphi}$ will mean multiple local maxima of revenue in $\varphi$ for given $m$, across the set of $\varphi$ for which an equilibrium with $n_0$ participants is selectable. In the presence of such multiple local maxima, I know of no argument from primitives that implies the global maximum revenue must lie in $\mathbb{M}^{=}$. Should it not, in essence the theory of auctions with an exogenous number of bidders applies.

Several seminar attendees have insistently pursued the following assertion: a seller who (somehow) had a choice between selling via an auction following the "sequential symmetric" entry of this Appendix and via an auction following the "simultaneous symmetric" entry in the main text above would always prefer the former. I first provide a counterexample, and then discuss why the assertion appears to be so appealing.

Example: Let there be $N = 2$ potential bidders; denote $\overline{M} = (\overline{m}, 0, 0)$, a second-price auction. Fix $\mathfrak{B}$; then $E[V]$ and $\mathcal{R}\left(\overline{M}, 2\right)$ are fixed. Then choosing participation cost $c = \left[E[V] - \mathcal{R}\left(\overline{M}, 2\right)\right]/2$ and bid preparation cost $b = 0$ yields an environment for which $\overline{M}$ is an optimal auction in the sequential entry model, if the equilibrium is selected in which the second potential bidder is indifferent over participating but participates with probability 1. The expected revenue attained is $\mathcal{R}\left(\overline{M}, 2\right)$. In the simultaneous entry model, $\pi\left(\overline{M}\right) = 1$ and $R\left(\overline{M}\right) = \mathcal{R}\left(\overline{M}, 2\right)$. However, by Proposition 9, an increase in $\varphi$ from 0 to $d\varphi$ increases revenue, to a level unattainable in the sequential entry model.

The assertion pays attention to an obvious difficulty in the main model, the probability $(1-\pi)^N$ that no potential bidder participates, and thus no gains from trade occur. It neglects a more subtle advantage: for optimal mechanisms, the probability that a participant faces a smaller-than-average number of rival participants is far larger than $(1-\pi)^N$. In the example, a potential bidder making a sequential participation decision knows for sure that he faces one rival bidder, and is indifferent over participating when $\varphi = 0$; an entry fee of $d\varphi > 0$ will lead to his nonparticipation and a plunge in revenue (to $d\varphi$). However, a potential bidder making a simultaneous participation decision will face one rival bidder with probability $\pi^*$ slightly less than 1. If he faces one rival bidder, his net expected profitability is $-d\varphi$. Countering this loss is the $(1-\pi^*)$ probability that he faces no opposition and obtains the asset for a price of $d\varphi$. The seller gains because the resource costs have been reduced from $2c$ to $2\pi^* c$, and in each case there is a participant who is indifferent.

For those auction forms where the relationship between a bidder's expected profitability and an exogenously specified number of bidders is known, this relationship is strictly convex. Hence, a seller can sometimes attain a sizable $\pi$, even though the mechanism is strongly surplus-extractive, because a bidder is weighing in the chances of being the only participant or one of very few participants. With "sequential symmetric" entry, an optimal auction never faces a participant with fewer than $n_e(M) - 1$ rival participants. On average, the seller may be able to gain from this difference.

# 10 Appendix B: Proofs

**Proof of Theorem 3**: For the proof, shorten $\pi(M)$ to $\pi$, $\alpha(M, \pi, n)$ to $\alpha$, $\mu_a(M, \pi, n)$ to $\mu_a$, $\lambda_a(M, \pi, n)$ to $\lambda_a$, and continue to use $\beta_n$ for $\beta(n, N, \pi)$, $\beta_{n-1}$ for $\beta(n-1, N-1, \pi)$. Begin by adding 0 in useful forms at two locations in (9):

$$
\begin{aligned}
& R(M) \\
= & \sum_n \left\{ \sum_a \left( s_r E\left\{ E\left[p(\cdot)|V\right] - V - ab + V + ab \right\} + a\varphi \right) \binom{n}{a} \mu_a - cn + cn \right\} \beta_n \\
= & \sum_n \left\{ s_r E[V] - b \sum_a a \binom{n}{a} \mu_a(M, \pi, n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a \left( s_r E\left\{ E\left[p(\cdot)|V\right] - V \right\} + a[\varphi + b] \right) \binom{n}{a} \mu_a + cn \right\} \beta_n \\
= & \sum_n \left\{ s_r E[V] - b \sum_a a \binom{n}{a} \mu_a(M, \pi, n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a \left( s_r E\left\{ E\left[p(\cdot)|V\right] - V \right\} + a[\varphi + b] \right) \frac{n}{a} \binom{n-1}{a-1} \lambda_a \alpha + cn \right\} \beta_n,
\end{aligned}
$$

where the last equality uses the Bayes' formulas [(2) or (3)].

$$
\begin{aligned}
& R(M) \\
= & \sum_n \left\{ s_r E[V] - b \sum_a a \binom{n}{a} \mu_a(M,\pi,n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a \left( s_r E\{E[p(\cdot)|V] - V\} + [\varphi + b] \right) \binom{n-1}{a-1} \lambda_a \alpha + c \right\} n\beta_n \frac{\sum_i i\beta(i,N,\pi)}{\sum_i i\beta(i,N,\pi)} \\
= & \overline{V}(M) - b\overline{a}(M) - c\overline{n}(M) \\
& + \left[ \sum_n \left\{ \alpha \sum_a \left( s_r E\{E[p(\cdot)|V] - V\} + [\varphi + b] \right) \binom{n-1}{a-1} \lambda_a + c \right\} \beta_{n-1} \right] N\pi,
\end{aligned}
$$

where the first equality sorts $n$ out of $\sum_a$ and multiplies by 1 in a useful form, and the final equality simplifies the numerator and combines the denominator with $n\beta_n$. The term in large $[\cdot]$ is 0 by (7).∎

**Proof of Proposition 7**: [$i$]: The same $\pi$ implies that (r.h.s.) of (7) attains the same value. Now reversing the substitutions used in the proof of Theorem 3 demonstrates revenue equality. [$ii$]: $\{\pi(M) = \pi(M')\} \Rightarrow \{\overline{n}(M) = \overline{n}(M')\}$, so [$ii$] follows from [$i$]. [$iii$]: The proof for the equality has already been shown. Suppose $\mathcal{R}(M,n) > \mathcal{R}(M',n)\ \forall n \in \mathbb{N}$. Shorten $\pi(M')$ to $\pi'$. From (9), $\forall n \in \mathbb{N}$,

$$\begin{aligned} & \sum_a \left(E\{E[p(M,\cdot)|V]\} + a[\varphi+b]\right)\binom{n}{a}\mu_a(M,\pi(M),n) \\ > & \sum_a \left(E\left\{E\left[p\left(M',\cdot\right)|V\right]\right\} + a\left[\varphi'+b\right]\right)\binom{n}{a}\mu_a\left(M',\pi',n\right) \end{aligned}$$

implies, using Bayes' formula as in the previous proof:

$$\sum_n \beta_{n-1}\left\{\alpha\left(M',\pi',n\right)\sum_a\left[\frac{1}{a}E\left\{V - E\left[p\left(M',\cdot\right)|V\right]\right\} - \varphi' - b\right]\binom{n-1}{a-1}\lambda_a\left(M',\pi',n\right)\right\} > c,$$

implying $\pi(M) < \pi(M')$. The reverse inequality is identical.∎

## 11 Appendix C: Affiliated Values

This Appendix extends almost all results to the general affiliated-values case (Milgrom and Weber [1982]). For this case, let $V$ as used above be the *underlying asset value*, with asset value to a particular participant observing signal $X_i$ a continuous function $t(V, X_i)$, increasing in both variables (common across participants, in that $t$ does not have a subscript). Define

$$T(M,\pi,n,\varphi,v) = \begin{cases} \int_{\Xi^K(M,n,\varphi)} G^K(M,n,\varphi,x,v)\,t(v,x)\,d\mathfrak{B}_1(x|v), & M \in \mathbb{M}^K, \\ \int_{\Xi^U(M,\pi,\varphi)} G^U(M,\pi,\varphi,x,v)\,t(v,x)\,d\mathfrak{B}_1(x|v), & M \in \mathbb{M}^U, \end{cases} \tag{16}$$

where $G^K(M,n,\varphi,x,v)$ [resp., $G^U(M,\pi,\varphi,x,v)$] is the probability of becoming the winning bidder for a potential bidder who will participate in auction $M$, when there are $n-1$ other participants [when $N-1$ other potential bidders participate with probability $\pi$], the entry fee is $\varphi$, he will observe signal $x$, and underlying asset value is $v$. Then $T(M,\pi,n,\varphi,v)$ is the expected asset value to a potential bidder who will participate, conditional on his winning in the circumstances specified by its arguments.

Assumption $\mathcal{A}.2$ above needs to be adjusted to specify that the expectation of $T(\cdot)$ exceeds $c+b$ yet remains finite. Let $p(\cdot)$ now denote the price in an affiliated-values auction, for the same arguments. The ex-ante expected payoff for a potential bidder who will be one of $n$ participants and $a$ actual bidders [(5) above] becomes

$$\frac{s\left(M,\alpha,a,\pi,n\right)}{a}E\left\{E\left[T\left(M,\pi,n,\varphi,\cdot\right)-p\left(M,\alpha,a,\pi,n,\cdot\right)|V\right]\right\}-\varphi-b-c.$$

The revenue formulation is unchanged.

For the affiliated-values case, expected value transferred becomes

$$\overline{V}\left(M\right)=\sum_{n}\sum_{a}s_{r}E\left\{T\left[M,\pi\left(M\right),n,\varphi,V\right]\right\}\binom{n}{a}\mu_{a}\left(M,\pi,n\right)\beta_{n}. \tag{17}$$

**Proof of Corresponding Theorem 3**: As before, begin by adding 0 in useful forms at two locations in (9):

$$\begin{aligned}
& R\left(M\right)\\
= & \sum_{n}\left\{\sum_{a}\left(s_{r}E\left\{E\left[p\left(\cdot\right)-T\left(\cdot\right)-ab+T\left(\cdot\right)+ab|V\right]\right\}+a\varphi\right)\binom{n}{a}\mu_{a}-cn+cn\right\}\beta_{n}\\
= & \sum_{n}\sum_{a}\left\{s_{r}\sigma_{a}E\left[T\left(\cdot\right)\right]-ab-cn\right\}\mu_{a}\beta_{n}\\
& +\sum_{n}\left\{\sum_{a}\left(s_{r}E\left\{E\left[p\left(\cdot\right)-T\left(\cdot\right)|V\right]\right\}+a\left[\varphi+b\right]\right)\binom{n}{a}\mu_{a}+cn\right\}\beta_{n}\\
= & \overline{V}\left(M\right)-b\overline{a}\left(M\right)-c\overline{n}\left(M\right)\\
& +\left[\sum_{n}\left\{\alpha\sum_{a}\left(\frac{s_{r}}{a}E\left\{E\left[p\left(\cdot\right)-T\left(\cdot\right)|V\right]\right\}+\varphi+b\right)\binom{n-1}{a-1}\lambda_{a}+c\right\}\beta_{n-1}\right]N\pi,
\end{aligned}$$

by the same substitutions as in the previous proof. The term in large $[\cdot]$ is 0 by an equilibrium participation equation corresponding to (7) for the affiliated-values case.∎

Corollary 4 and Theorem 5 can be extended to the affiliated-values case by identical proofs. A corresponding proof of Corollary 6, the inferiority of a nontrivial reserve price, now depends on the existence of a screening level. It is unknown whether a screening level in affiliated-values auctions (where the private-values element is not degenerate) may be internally consistent (the impossibility result of Landsberger and Tsirelson [2002] no longer applies). It is also unknown whether some quite different method of proof could establish Corollary 6 without relying on a screening level. The intuition underlying Corollary 6 seems not to depend on the absence of private-values elements. Note that no screening-level assumption is needed or used to extend all other results to general affiliated values.

The last equation above in the proof for the Corresponding Theorem 3 can be used to extend Proposition 7, the comparative statics, to the affiliated-values case. Corollary 8 remains trivial. The claims of Proposition 9 (that $\pi = 1$ is always suboptimal), Theorem 10 (an arbitrary auction can be made optimal by altering any one continuous, spanning variable), and Corollaries 11 (a $(d-1)$-dimensional subset of optimal auctions) and 12 (that, given concavity of $R(\pi)$, antecedent revenue comparisons are extended, and are reversed, each in a half-space of environments) can be extended via identical proofs.

Finally, notice that the extension of Proposition 9 and Theorem 10 has made Corollary 4 (efficiency of revenue maximization with endogenous bidders) a much more powerful result. In this Appendix, inefficiency can result from either a failure to sell or the sale to a bidder other than the highest-valuing participant (possibly because that participant decides not to pay the entry fee). Here, optimal auctions often accept a positive probability of selling to a bidder other than the highest-valuing participant. Nonetheless, the seller and a social planner evaluate these possibilities identically.

## 12 References

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