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Choice-Nash-Equilibria

August 2002

Working Paper No: 0209

# DEPARTMENT OF ECONOMICS

# UNIVERSITY OF VIENNA

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# Choice-Nash Equilibria*

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August 8, 2002

**Abstract**

We provide existence results for equilibria of games where players employ abstract (non-binary) choice rules. Such results are shown to encompass as a relevant instance that of games where players have (non-transitive) SSB (Skew-Symmetric Bilinear) preferences, as well as other well-known transitive (e.g. Nash's) and non-transitive (e.g. Shafer and Sonnenschein's) models in the literature. Further, our general model contains games where players display procedural rationality.

**Journal of Economic Literature** Classification Numbers: C60, C72, D83

**Keywords:** non-cooperative games, non-binary choice, SSB preferences, procedural rationality.

## 1 Introduction

Game Theory postulates fully rational agents who are able to employ their reasoning capabilities to coordinate in an equilibrium. The (empirically based and experimentally nourished) dissatisfaction of economists with such extreme assumptions has given rise to a large literature where agents are assumed to display "bounded rationality" instead (see e.g. Simon [29]). A large part of this literature is behavioral in nature, i.e. it conceives of agents as focusing on actions, not being necessarily able to predict the ultimate outcome. Technically, we can think of agents as being endowed with a *behavioral rule*, which is merely a mapping from the current situation (e.g. the profile of actions taken by other players) to (a probability distribution over) actions. One such rule is, naturally, best reply, which in a sense puts classical game theory back into the picture, since Nash equilibria are the fixed points of this particular rule. Other rules that come to mind are imitation, better reply, and procedural rationality (in the sense of Osborne and Rubinstein [23]).[1]

*We thank Manfred Nermuth for helpful comments. The first author acknowledges financial support from the Ministerio de Ciencia y Tecnologa under the Research Project BEC2002-02456, and the Junta de Castilla y León under the Research Project SA061/02. The second author acknowledges financial support from the European Science Foundation and the Austrian FWF, under project P15281.

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[1]The term procedural rationality refers to an explicitly modeled reasoning procedure on the part of players. See [30].

Bounded rationality, though, is not only a concern of Game Theory but of all Decision Theory. Within this realm, a different branch of the literature has built upon the recognition that bounded rationality might still be captured by decision-theoretic models, prompting a re-examination of the most basic assumptions of the agents' way of perceiving alternatives (preferences). For instance, the theory of Skew-Symmetric-Bilinear (SSB) preferences (see e.g. Fishburn [11]) provides a framework for preferences which might fail one of the most basic rationality tests, that of transitivity (see also Shafer [27]).

An alternative approach to account for non-rational factors would be to move away from preferences, focusing instead on choices. Hence, agents are endowed with an abstract way to make choices which might not be supported by any underlying binary relation whatsoever (see e.g. Nehring [20]). Technically, agents are now endowed with *choice rules*,[2] which specify the acceptable choices given the range of available actions.

If we now consider agents interacting in a strategic situation, it is obvious that there exists a strong analogy between behavioral rules and choice rules. Formally, both are mappings from situations to actions. Conceptually, both are attempts to encompass bounded rationality within a general formal model of agents' behavior.

In the present work we aim at illustrating the analogies and connections between both approaches. To this end, we consider general games where players employ abstract (non-binary) choice rules (or, alternatively, behavioral rules).

The existence of equilibria in games has been extensively justified on the basis of fixed point theorems, beyond Nash's original contribution [17, 18]. Debreu [8] showed that the linearity assumption can be weakened to quasiconcavity. Kreweras [15] and Fishburn and Rosenthal [12]) consider non-transitive preferences of the SSB form; another work accounting for lack of transitivity is Shafer and Sonnenschein [28] (see also Border [4, Corollary 19.4]). Dekel, Safra, and Segal [9] consider certain non-expected utility preferences (where violations of the reduction of compound lotteries assumption are allowed). Crawford [6] analyzes preferences not satisfying the independence axiom (and exhibits an example of non-existence when quasiconcavity fails). From a different perspective, Osborne and Rubinstein [23] consider equilibria where players do not necessarily act on the basis of preference maximization, but rather follow certain, specific reasoning processes (procedural rationality).

Adding to this literature, we investigate conditions for the existence of equilibria (*Choice-Nash equilibria*) of games where (some of) the agents are able to express their tastes on prospective outcomes only through (not-necessarily binary) choice rules.

As a first, almost trivial but necessary step, we translate some elementary fixed point results into straigtforward existence proofs for Choice-Nash equilibria. This gives rise to a number of different existence results of considerable generality. Later on the paper, we illustrate their reach deducing from them well-established existence theorems such as Shafer and Sonnenschein's [28] theorem on equilibria in normal-form games.

Relying on the recent literature on non-binary choice (Nehring [19], Alcantud [1]), we move on to provide a new existence result for Choice-Nash equilib-

[2]The term choice function is also used, but they are actually correspondences. In choosing the term choice rule, we follow Mas-Colell *et al* [16].

ria. Then we show that such a new existence result encompasses games where players have SSB preferences (which, in general, need neither be transitive nor satisfy the independence axiom), as well as Nash's theorem and the classical von Neumann-Morgenstern expected utility case.

Last, we go on to illustrate that our general model allows a treatment of games where players display "procedural rationality", which we define as a generalization of Osborne and Rubinstein's [23].

These developments account for the suitability of the model that we propose as a quite general and useful context where one can frame other different models.

# 2 The model

Consider an (abstract) set of players $I$. For each player $i \in I$, let $A_i$ denote his *action set*. Denote $A := \prod_{i \in I} A_i$ and $A_{-i} := \prod_{j \neq i} A_j$ henceforth. Besides, we decompose $a = (a_i, a_{-i})$ following the usual convention in game theory.

Let $\mathcal{D}_i \subseteq \mathcal{P}^*(A_i)$ be a collection of (nonempty) subsets of $A_i$ such that $A_i \in \mathcal{D}_i$. We call $\mathcal{D}_i$ the *choice domain* for player $i$.

A *choice rule* for player $i$ is a correspondence

$$C_i : \mathcal{D}_i \times A_{-i} \mapsto A_i$$

which depends on the actions of other players, and such that $C_i(D, a_{-i}) \subseteq D$ for every $D \in \mathcal{D}_i$ and $a_{-i} \in A_{-i}$.

Abusing notation, we denote $C_i(a_{-i}) = C_i(A_i, a_{-i})$, the evaluation of the choice rule $C_i(\cdot, a_{-i})$ on the full set $A_i$ (for a given vector of strategies $(a_{-i})$ of the other players).

A (normal-form) *game* is a tuple $(I, \{A_i, C_i\}_{i \in I})$ where $I$ is the set of players, $A_i$ are the action sets, and $C_i$ are the choice rules.

A *Choice-Nash equilibrium* of the game so defined is a profile $a^* \in A$ such that, for all $i \in I$, $a_i^* \in C_i(a_{-i}^*)$. In words: given the other players' strategies $(a_{-i}^*)$, player $i$ chooses $a_i^*$.

For the sake of the exposition, we proceed to argue how we can subsume some well-known models in our framework. We stick to the set $I$ of players, each having $A_i$ as her action set.

**Example 2.1.** *Consider the case of quasiconcave preferences (Debreu [8]) as exposed e.g. in [22, Chapter 2]; namely, each player has a preference (complete, transitive) relation $\succcurlyeq_i$ on $A$ (perhaps derived from payoffs or utilities) satisfying quasiconcavity (i.e. the set $\{a_i \in A_i | (a_i, a_{-i}^*) \succcurlyeq a^*\}$ is convex for all $a^* \in A$). Define, for every agent $i$, $\mathcal{D}_i = \mathcal{P}^*(A_i)$ and the choice rule $C_i : \mathcal{D}_i \times A_{-i} \mapsto A_i$ given by maximality, i.e. $C_i(D, a_{-i}) = \{x_i \in D : (x_i, a_{-i}) \succcurlyeq_i (y_i, a_{-i})$ for all $y_i \in D\}$ whenever $D \subseteq A_i$. Then, a Nash equilibrium in the sense of that work is a Choice-Nash equilibrium of $(I, \{A_i, C_i\}_{i \in I})$.*

**Example 2.2.** *We can put aside the transitivity assumption of the previous example by dealing e.g. with SSB games; see Kreweras [15], Fishburn and Rosenthal [12]. This model accounts for cycles in the players' preferences as well. In this case, each player expresses her preferences on $A$ by means of $\phi_i : A \times A \longrightarrow \mathbb{R}$ bilinear and skew-symmetric ($\phi_i(x, y) = -\phi_i(y, x)$); we should read that $i$'s preference for $x$ is at least as much as her preference for $y$ if and* *only if $\phi_i(x,y) \geqslant 0$ (see Section 3.1 below for more details). Define now, for every agent $i$, $\mathcal{D}_i = \mathcal{P}^*(A_i)$ and the choice rule $C_i : \mathcal{D}_i \times A_{-i} \mapsto A_i$ given by $C_i(D, a_{-i}) = \{x_i \in D : \phi_i((x_i, a_{-i}), (y_i, a_{-i})) \geqslant 0$ for all $y_i \in D\}$ whenever $D \subseteq A_i$. Then, a Nash equilibrium in the sense of [15] is a Choice-Nash equilibrium of $(I, \{A_i, C_i\}_{i \in I})$.*

## 2.1 Elementary existence results

There are conditions in the literature that grant the existence of Nash equilibria for models as in the examples above. The references we have cited provide sets of conditions that apply to each instance. Under continuity of the preferences, and quasi-concavity of each on the corresponding $A_i$, Proposition 20.3 of [22] ensures the existence of Nash equilibria when each $A_i$ is a nonempty, compact and convex subset of a Euclidean space. [15], and afterwards [12], prove that SSB games have a Nash equilibrium when $A_i$ is $i$'s simplex of mixed strategies on a finite set of pure strategies. Both papers use a proof exactly analogous to Nash's [18] proof for the existence of equilibria in finite noncooperative games.

So the question arises as to when can we ensure that games under our requirements have a Choice-Nash equilibrium. In order to apply the classical tools, the proverbial fixed-point characterization is available. We just have to define $C : A \mapsto A$ such that $C(a) = \prod_{i \in I} C_i(a_{-i})$ for every $a \in A$. Now, a Choice-Nash equilibrium is a fixed point of $C$.

Following Nash [17], Kakutani's fixed-point Theorem is typically used to yield the existence of a Nash equilibrium. We proceed now to make this quite trivial development explicit while introducing our basic requirements. Other possibilities are outlined as well.

First, we restrict the sets $A_i$ to be simplexes over finite strategy sets.

**Condition 1.** *Every $A_i$ is the simplex of mixed strategies on a finite set of pure strategies $S_i = \{s_1^i, ..., s_{m_i}^i\}$ for each $i$.*

The second requirement is merely that choices should be "continuous" in other agents' actions.

**Condition 2.** *$C_i(\cdot)$ is "continuous" with respect to $a_{-i}$, in the closed-graph sense: if $\{a^k\}_{k \in \mathbb{N}}$ is a sequence in $A$ converging to $a$ and $a_i^n \in C_i(a_{-i}^n)$ for all $n$, then $a_i \in C_i(a_{-i})$.*

Note that, under Condition 1, $C_i$ has a closed graph (as required by condition 2) if and only if it is closed-valued and u.h.c. (see e.g. [14, 7.1.15, 7.1.16] or [4, 11.9]).

Under Conditions 1 and 2, Kakutani's Theorem translates into the following proposition:

**Proposition 2.3.** *Suppose Conditions 1 and 2 hold, and that $C_i$ is convex- and nonempty-valued for all $i \in I$. Then $(I, \{A_i, C_i\}_{i \in I})$ has a Choice-Nash equilibrium. Furthermore, the set of Nash equilibria is closed and hence compact.*

*Proof.* Apply Kakutani's fixed point Theorem to the correspondence $C : A \mapsto A$ (each $A_i$ is clearly convex and compact). It follows from a standard result (see e.g. [4, 11.18(a)]) that the set of fixed points of $C$ is closed in $A$, and hence compact (since $A$ is compact). ■

Similarly, subsequent generalizations of Kakutani's result would permit to widen the reach of this general Proposition in a predictable manner. For instance, the convexity requirement can be substantially weakened through an appeal to the Eilenberg-Montgomery fixed point theorem:

**Proposition 2.4.** *Suppose that, in Proposition 2.3, $C$ was only star-shaped valued (or, more generally, that the images of $C$ are contractible sets or even acyclic sets).*[3] *Then the thesis of Proposition 2.3 follows as well.*

*Proof.* $C$ has acyclic compact values and $A$ is a product of simplexes (hence a contractible polyhedron). Hence, the result follows from the Eilenberg-Montgomery fixed point theorem (see e.g. [4, 15.9]). ■

This latter result can also be further generalized, weakening Condition 1. Park [24, Theorem A or B] reexamines prior fixed point theorems in the line of the Eilenberg-Montgomery fixed point theorem.[4] Applying these results, the previous theorem still holds if the set $A$ is not a product of simplexes but only a nonempty and convex set of a locally convex topological vector space (hence *admissible*, see [24, p. 84]).

Besides, different applications might demand alternative axiomatizations. We here offer yet another existence result for cases where Condition 2 is not granted; it will allow us to derive existence results in a well-known model with non-transitive agents (cf. Subsection 3.2).

**Proposition 2.5.** *Suppose that Condition 1 holds and that $C_i$ is convex- and nonempty-valued, for all $i \in I$. Suppose, further, that the graph of $C$ is open in $A \times A$. Then $(I, \{A_i, C_i\}_{i \in I})$ has a Choice-Nash equilibrium.*

*Proof.* The set $G_i = \{(a_i, a_{-i}) : a_i \in C_i(a_{-i})\}$ is open in $A$, which is normal, for each $i$. By either Uryshon's characterization of normality or Tietze's extension theorem, there is $f_i : A \longrightarrow [0,1]$ continuous that vanishes on the complement of $G_i$ and is strictly positive in $G_i$. Now, $m_i : A_{-i} \longrightarrow A_i$ given by $m_i(a_{-i}) = \arg\max_{a_i \in A_i} f_i(a_{-i}, \cdot)$ is a well-defined correspondence and has closed graph by continuity of each $f_i$. Observe that $f_i(a_i, a_{-i}) > 0$ if and only if $a_i \in C_i(a_{-i})$. Let $\mathrm{co}(m_i(a_{-i}))$ be the convex hull of $m_i(a_{-i})$. Since $C_i(a_{-i})$ is a convex set, it follows that $\mathrm{co}(m_i(a_{-i})) \subseteq C_i(a_{-i})$; here we use that $C_i(a_{-i}) \neq \varnothing$ and therefore $\max_{a_i \in A_i} f_i(a_{-i}) > 0$. We define $B : A \longrightarrow A$ by $B(a) = \prod \mathrm{co}(m_i(a_{-i}))$, nonempty valued and with convex values. Its graph is closed, by a routine check using Nikaido [21, Theorems 4.5 and 4.8]. By Kakutani's theorem, there is $a \in A$ with $a \in B(a) \subseteq C(a)$. ■

We leave the details of further enhancements to the reader, for utmost generality is not our main purpose in this contribution. For instance: Condition 1 in Proposition 2.5 could be replaced with convexity and compactness of each $A_i$ without substantial changes in the proof. Observe that this latter result could also be proven from Browder's fixed point theorem (see [5], Border [4, 15.6], or Klein and Thompson [14, 8.2.2]); just note that $C$ satisfies that

[3] A set $S$ is star-shaped if there exists a distinguished element $x_0$ such that all convex combinations of $x_0$ and any other element of $S$ are in $S$; all convex sets are star-shaped. A set is contractible if it can be continuously deformed into a single point; all star-shaped sets are contractible. A set is acyclic if it has the same homology group as a singleton; all contractible sets are acyclic. For more details, see Park [24] or [4, 15.8].

[4] An overview of this branch of the fixed-point literature can be found in [25].

$C^{-1}(a) = \{\overline{a} \in A : a \in C(\overline{a})\}$ is open in $A$ by definition of the product topology in $A \times A$. The reader might also appreciate a comparison of Proposition 2.5 and the (fixed point) theorem in Gale and Mas-Colell [13]; despite the similarity in the statements, we employ a different proof technique that was inspired by the prospective application.

## 2.2 Conditions from non-binary choice

The aim of Subsection 2.1 was merely to translate well-known patterns of proof into (elementary) existence results within our framework. Of course, the appeal to explicit conditions on the best reply correspondence has clear formal advantages and, in consequence, has been frequently adopted in the literature. However, the motivation for doing so has been notational convenience more often than real gain in generality from moving to non-binary preferences. This statement is exemplified e.g. in the aforementioned motivation for the (fixed point) theorem in Gale and Mas-Colell [13]. The lack of a non-binary background is explicitly admitted in more recent texts, such as the authoritative [4, Chapter 19]. There, the author complains that despite being "convenient to describe preferences in terms of the good reply correspondence rather than the preference relation [...] we lose some information by doing this." This position contrasts with Nehring's [20]; this author argues that "the traditional assumption of binariness on preference relations or choice functions is [...] analytically unhelpful and normatively unfounded [...]" on the basis e.g. of unresolvedness of preferences. "Dropping binariness," he says, "may lead to a deeper understanding of abstract choice theory."

In this Subsection we aim at providing and discussing sensible conditions which will allow us to apply the arguments above, and by doing so derive non-trivial existence results of non-binary nature that incorporate classical theorems as particular cases. Relying on Nehring [19] and Alcantud [1], we present now two sets of elementary conditions on choice rules which we will then use to provide new general existence theorems.

It is convenient to introduce the following notation. For each $i = 1, ..., n$ and $a_{-i} \in A_{-i}$, let

$$C_{a_{-i}} : \mathcal{D}_i \mapsto A_i$$

be the correspondence defined by $C_{a_{-i}}(D_i) = C_i(D_i, a_{-i})$ on each $D_i \in \mathcal{D}_i$. We are hence generalizing the concept of best response to the other players' profile (the vector $a_{-i}$), under the restriction that only a part of the action set (the subset $D_i$) is available, and according to the primitive choice procedure determined by $C_i$. Note that $C_{a_{-i}}(A_i) = C_i(a_{-i})$ for each agent $i$.

**Condition 3.** *For each $i = 1, ..., n$, the choice domain $\mathcal{D}_i$ includes all finite subsets of $A_i$. Further, for all $a_{-i} \in \prod_{j \neq i} A_j$, $C_{a_{-i}}$ satisfies that:*

*(a) If $S \subseteq A_i$ is finite, then $C_{a_{-i}}(S) \neq \varnothing$ and also $M_S = \{a_i \in A_i : a_i \in C_{a_{-i}}(S \cup \{a_i\})\}$ is closed.*

*(b) For all $S, T \in \mathcal{D}_i : T \subseteq S$ implies $C_{a_{-i}}(F) \cap T \subseteq C_{a_{-i}}(T)$.*

*(c) For all $S \in \mathcal{D}_i$, if $a_i \in S$ satisfies that for all $T \subseteq S$ finite, $a_i \in T$ implies $a_i \in C_{a_{-i}}(T)$, then $a_i \in C_{a_{-i}}(S)$.*

(d) *For all $S \in \mathcal{D}_i$ finite, $M_S$ is convex.*

**Lemma 2.6.** *Suppose $A_i$ and $C_i$ satisfy Conditions 1 and 3 (a), (b) and (c). Then, for any $a_{-i} \in \prod_{j \neq i} A_j$, the set $C_i(a_{-i})$ is nonempty.*

*Proof.* $C_i(a_{-i}) = C_{a_{-i}}(A_i) \neq \varnothing$ by the theorem in [19]. ■

Observe that, for the last Lemma, compactness of each $A_i$ (rather than the full Condition 1) would suffice.

**Remark 2.7.** *Let us briefly explain Condition 3 (for a more detailed discussion, see Nehring [19]). Part (a) refers to the non-emptiness axiom A1 in [19] and the technical continuity axiom A3 there. Part (b) is contraction consistency, A2 in [19], sometimes called Chernoff condition and a very basic choice consistency requirement. Part (c) is finitariness, A4 in [19]; in some sense, it extends the range of $C_{a_{-i}}$ from finite to infinite sets. It is a weakening of binariness, A5 in [19], which in turn implies it under non-emptiness and contraction consistency (Remark 1 in [19]).*

We now offer an alternative set of conditions leading to the same conclusion. We denote by $\mathrm{co}(B)$ the convex hull of a set $B \subseteq A_i$ and $[x, y] = \mathrm{co}(\{x, y\})$.

**Condition 4.** *For each $i = 1, ..., n$, the choice domain $\mathcal{D}_i$ includes all convex hulls of finite subsets of $A_i$. Further, for all $a_{-i} \in \prod_{j \neq i} A_j$, $C_{a_{-i}}$ satisfies that, if $T = \{x_1, \ldots, x_m\} \subseteq A_i$ with $m \geqslant 2$,*

(a) *If $z \in \mathrm{co}(T)$, then there exists $k \in \{1, ..., m\}$ such that $z \in C_{a_{-i}}([x_k, z])$.*

(b) *For any $x_k \in T$, the set $M_T(x_k) = \{z \in \mathrm{co}(T) | z \in C_{a_{-i}}([x_k, z])\}$ is closed.*

(c) *If $x_k \notin co(T \setminus \{x_k\})$ $\forall k = 1, \ldots, m$, then $C_{a_{-i}}(\mathrm{co}(T)) = \bigcap_{k=1}^{m} M_T(x_k)$.*

(d) *For each $i \in I$ and $a_{-i} \in A_{-i}$, $M_{S_i}(s_j^i)$ is convex for all $j = 1, ..., m_i$.*

Note that we implicitly assume Condition 1 to be able to state part (d) above. This could be avoided requiring (d) to be fulfilled for all sets $M_T(x_k)$ as in (b).

The conditions (a), (b) and (c) above are conditions B1(a), B2(a), and B1(c) in Alcantud [1]. Applying the results there, we have:

**Lemma 2.8.** *Suppose $A_i$ and $C_i$ satisfy Conditions 1 and 4 (a), (b) and (c). Then, for any $a_{-i} \in \prod_{j \neq i} A_j$, the set $C_i(a_{-i})$ is nonempty.*

*Proof.* $C_i(a_{-i}) = C_{a_{-i}}(co((s_1^i, ..., s_{m_i}^i)) \neq \varnothing$ by Theorem 1 in [1]. Observe that, by construction, the domain of each $C_{a_{-i}}$ meets the restriction imposed in that work (the convex hull of every finite set should be in the domain). ■

**Remark 2.9.** *Let us now discuss Condition 4 (for more details, see [1]). Part (a) says that, when considering a finite number of options, the agent can always declare a mixture $z$ eligible when compared with obtaining for sure some of the possible outcomes of $z$. As long as the rationality of the agent allows him to dispose of at least one option out of a finite number of alternatives, this requirement is quite weak (see, however, Section 3.4).*

*Condition 4(b) and (d) are technical requirements. Part (b) captures a certain continuity of the choice rule with respect to changes in the set of available* *alternatives. Part (d) is a weak convexity requirement; notice that, under Condition 1, it is only required for $M_T(x_k)$ such that $T$ is the whole pure-strategy set and $x_k$ is a pure strategy.*

*While somewhat more involved, Condition 4(c) is quite sensible in the current framework. Basically, it says as follows. Agent $i$ considers the other players' profile $a_{-i}$ as given, and ponders what she could get from varying her own (mixed) strategy $a_i$. In the event that two profiles of these, namely $\overline{a} = (\overline{a_i}, a_{-i})$ and $\widetilde{a} = (\widetilde{a}_i, a_{-i})$, are each chosen against a fixed pure strategy $s_j$ (and every mixture of $s_j$ and it), then Condition 4(c) requests that any arbitrarily selected mixture of them must satisfy the same requirement. But that mixture of them consists of randomizing between $\overline{a_i}$ and $\widetilde{a}_i$, while the others' strategies are kept. We think that this behavior is quite a weak rationality assumption on the agent's behavior, that is met in most normal-form game models. We will return to this point later on in subsequent Sections.*

*Condition 4(c) is a strengthening of*

*(c′) if $z \in C_{a_{-i}}([x_k, z])$ for all $k = 1, \ldots, m$ then $z \in C_{a_{-i}}(\mathrm{co}(T))$*

*(which is condition B1 (b) in [1]), in the sense that the former not only accounts for the inclusion $\bigcap_{j=1,\ldots,n} M_T(a^j) \subseteq C_{a_{-i}}(co(T))$ for all $i = 1, .., n$ and $T = \{a^1, ..., a^n\} \in A_i$, but it rather states that both sets are equal. The reader should note that, for choice correspondences that derive from a binary relation, the non-trivial implication in Condition 4(c) is that given by (c′). Therefore, for that type of choice correspondences, requesting one or the other is equivalent.*

**Theorem 2.10.** *Suppose that $(I, \{A_i, C_i\}_{i\in I})$ satisfies Conditions 1,2 and either of 3 or 4. Then the set of Choice-Nash equilibria is nonempty and compact.*

*Proof.* Let us check that the requirements of Proposition 2.3 are met by the correspondence $C$, i.e. we should show that each $C_i$ is convex- and nonempty-valued.

Firstly, either Lemma 2.6 or 2.8 applies, ensuring $C_i(a_{-i}) \neq \varnothing$.

Secondly, we prove that any of Conditions 3 or 4 imply convex-valuedness of each $C_i$.

Under Condition 3, it follows by (b) plus (c) -see Remark 3 in Nehring [19]- that $C_i(a_{-i}) = \bigcap_{S\subseteq A_i \text{finite}} M_S$, and so it is convex due to (d).

Assume now Condition 4. Due to (c) and (d),

$$C_{a_{-i}}(A_i) = C_{a_{-i}}(co((s_1^i, ..., s_{m_i}^i)) = \bigcap_{j=1,\ldots,m_i} M_{S_i}(s_j^i)$$

is convex for all $i = 1, .., n$ and $a_{-i} \in \prod_{j\neq i} A_j$ by (d). ∎

# 3 Particular results in specific frameworks

Along this Section, some known results in different frameworks will be obtained as Corollaries to Theorem 2.10 (Subsections 3.1 and 3.3). Through the use of the general existence results in Propositions 2.5 and 2.3 we shall derive: (a) the game-theoretical model contained in Shafer and Sonnenschein [28] (Subsection 3.2), and (b) a new result on existence of S($k$)-equilibria in games with procedurally rational players, plus a discussion of its relationship with the model at hand (Subsection 3.4).

### 3.1 SSB games

We turn our attention back to Example 2.2. SSB preferences (see e.g. [11]) are those which can be represented by a skew-symmetric bilinear function $\phi$ in the following way: an alternative $x$ is (weakly) SSB-better than another one, $y$, if and only if $\phi(x, y) \geqslant 0$. Such preferences (see [27] for a straightforward representation theorem) need not be transitive or satisfy the independence axiom, although of course the classical von Neumann-Morgenstern framework is encompassed as a particular case.[5]

We call an *SSB game* any game where players have a finite number of strategies (and hence every agent's action set is the simplex of mixed strategies) and players have SSB preferences. As commented above, Kreweras [15] already obtained existence of Nash equilibria in SSB games. In this subsection we first proceed to obtain such existence result as a Corollary to Theorem 2.10, and then we digress to give yet another possible alternative proof that does not appeal to the model at hand.

In this context, we should proceed with the choice correspondences given by $C_{a_{-i}}(D_i) = \{x_i \in D_i : \phi_i((x_i, a_{-i}), (y_i, a_{-i})) \geqslant 0 \ \forall y \in D_i\}$, for each $D_i \subseteq A_i$ and $a_{-i}$.

The argument of the Example in [1] shows that the choice correspondence $C_{a_{-i}}$ in fact satisfies Condition 4(a), (b), and (c$'$). In order to check that 4(c) does hold, we need to ensure that $a_i \in C_{a_{-i}}(co(a^1, ..., a^n))$ yields $a_i \in C_{a_{-i}}([a^j, a])$ for all $j = 1, ..., n$, which is quite immediate by binariness, irrespectively of $\{a^1, ..., a^n\} \subseteq A_i$ (see Remark 2.9).

The fact that each $M_{S_i}(s_j)$ is convex follows easily from linearity of $\phi$ in the first component.

Condition 2 is satisfied quite naturally here.

The argument above justifies, on the ground of Theorem 2.10:

**Corollary 3.1.** *[Kreweras [15], Fishburn and Rosenthal [12]] Any SSB game has a Nash equilibrium. Furthermore, the set of Nash equilibria is compact.*

Needless to say, the reasoning above could be adapted to yield a direct proof of Corollary 3.1. We proceed to give a short exposé of the proof arising, for it is different from that given by Kreweras and by Fishburn and Rosenthal.

Under the requirements of the Corollary, the best replies correspondence is nonempty due to Theorem 3 in Fishburn [10] and convex-valued due to linearity in the first component of each $\phi_i$. The fact that $C$ has a closed graph follows easily from linearity (therefore continuity) in each component of every $\phi_i$. By Kakutani's fixed point Theorem, the game has a Nash equilibrium.

### 3.2 The non-transitive model by Shafer and Sonnenschein

Another classical result on equilibrium existence when agents display non-transitive preferences is Shafer and Sonnenschein [28]. Although their theorem refers to abstract economies or generalized games, the version for games is available e.g. in Border [4, Corollary 19.4]. We here show how it can be derived from Proposition 2.5. As in Subsection 3.1, player $i$ is assumed to express his tastes on

[5] See [7] and [2] for a discussion of the SSB model as a framework for dynamic models of selection of preferences.

possible outcomes by a binary relation $\succ_i$, for each $i \in I$; then, a correspondence $P_i : A \mapsto A_i$ is defined by $P_i(a) = \{\overline{a_i} \in A_i : (\overline{a_i}, a_{-i}) \succ_i a\}$. Accordingly, thus, $a \in A$ is a Nash equilibrium of the defined game if and only if $P_i(a) = \varnothing$ for all $i \in I$. The only restrictions on the relations are those in the following statement:

**Corollary 3.2.** *[Shafer and Sonnenschein (1975)] Let $(I, \{A_i, \succ_i\}_{i\in I})$ be a game such that Condition 1 holds and each $P_i$ has open graph. Assume, further, that for each $a \in A$ there is a player $i$ for which $a_i \notin co(P_i(a))$. Then, the game has a Nash equilibrium, i.e. there is $a \in A$ with $P_i(a) = \varnothing$ for all $i = 1, ..., n$.*

*Proof.* Define the correspondence $C : A \mapsto A$ by $C(a) = \prod_{i=1,...,n} co(P_i(a))$. It has open graph, and, obviously, convex values. Observe that $C$ can not have a fixed point, since that would mean $a_i \notin co(P_i(a))$ for a fixed $a \in A$, irrespectively of $i$. Proposition 2.5 forcefully yields $P_i(a) = \varnothing$ for some $a \in A$ and all $i$. ■

Note that the restriction on the binary relations that model the players' preferences is yet weaker than that in the original statement. The original requirement that each $A_i$ is convex and compact has been replaced by Condition 1 by convenience, and could be restored without altering the proof, as noted before.

## 3.3 An "extreme" case: players with continuous preferences

Consider Example 2.1, and the comment following it. In the event that the assumptions in Proposition 20.3 of [22] hold (continuity of the preferences, quasi-concavity of each on the corresponding $A_i$, that is assumed nonempty, compact and convex subset of a Euclidean space), the argument in the proof in [22] says that $C$ is nonempty-valued (by a well-known theorem by Bergstrom [3] and Walker [31]) and convex-valued (by quasi-concavity). Because the graph was also closed, Proposition 2.3 applies. Also, the reader can observe that since $C$ had closed values the set of Nash equilibria was closed. In fact, it is quite simple to check that this case is contained in the setting of Theorem 2.10, version Condition 3. This comes without surprise, since Nehring's [19] result extended the maximun theorem of Bergstrom and Walker, which is the key for Kakutani's argument to apply. In order to do that, he stated (Remark 1) that in conditions weaker than our current requirements -basically, because of binariness- Conditions 3 (a) -except for closedness of the $M_S$'s- (b) and (c) must hold. Now, closedness of each $M_S$ arises from continuity of the preferences. Also, (d) follows easily from quasi-concavity on the corresponding $A_i$, for if $S = \{u^1, ..., u^m\}$ then $M_S = \{a_i \in A_i : (a_i, a_{-i}) \succcurlyeq_i (s^k, a_{-i})\ k = 1, ..., m\} = \bigcap_{k=1,...,m}\{a_i \in A_i : (a_i, a_{-i}) \succcurlyeq_i (s^k, a_{-i})\}$ is an intersection of convex subsets.

Some further comments on this digression are in order.

Firstly: in fact, under Condition 1 all the assumptions in Theorem 2.10, version Condition 4, are fulfilled too, except Condition 4(c). Still, a relevant particular instance of that model does fit into the framework considered in this other version of Theorem 2.10. It suffices to restrict ourselves to the case where not only $\{a_i \in A_i : (a_i, a^*_{-i}) \succcurlyeq_i a^*\}$, but also $\{a_i \in A_i : a^* \succcurlyeq_i (a_i, a^*_{-i})\}$, are convex for each $a^* \in A$. The comment on the definition of B1 in [1] puts forward a number of preferences that satisfy that further requirement. Besides, that latter particular model accounts for the classical von Neumann-Morgenstern expected utility case.

Secondly: not only the original statement of Nash's equilibrium theorem is, therefore, encompassed by Proposition 2.3 (and Theorem 2.10). Also, generalizations such as Park's [24, Theorem 7] can be suitably subsumed in our model. An appeal to Proposition 2.4 would explain this latter generalization in the same manner as we have embodied Nash's result through Proposition 2.3.

### 3.4 Procedural rationality

In [23], Osborne and Rubinstein put forward a model of bounded rationality where players try each available action (strategy) *once* before deciding what to play. In the symmetric, two-player case, the corresponding solution concept is an $S(1)$-equilibrium: a probability distribution $\alpha^*$ over available actions such that, for each action $x$, $\alpha^*(x)$ gives the probability that a player finds $x$ to be the best (breaking ties equiprobably) when sampling each action once, given that the opponent plays according to $\alpha^*$.[6]

This "procedural rationality" and the corresponding equilibrium concept can obviously be generalized to encompass asymmetric $N$-player games. Assume Condition 1, and let $\pi_i : A \mapsto \mathbb{R}$ denote player $i$'s payoff function. Given a vector of probability distributions $a_{-i} \in A_{-i}$, and a pure strategy $s_j^i$, consider the random variables $\pi_i(s_j^i, s_j^{-i})$ where the opponents' strategies $s_j^{-i}$ are drawn according to $a_{-i}$. Let $C(a_{-i})(s_j^i)$ be the probability that strategy $s_j^i$ yields the highest payoff when each random variable $\pi_i(s_j^i, s_j^{-i})$ is drawn independently, breaking ties equiprobably.[7]

Further, the assumption of breaking ties in a pre-specified manner can be disposed with if we are willing to allow for choice correspondences in the sense of the present work. Given $a_{-i} \in A_{-i}$, interpreted as a vector of probability distributions over other agents' actions, define

$$\begin{aligned} C^+(a_{-i})(s_k^i) &= Prob(s_k^i \in \arg\max_j \pi_i(s_j^i, s_j^{-i})) = \\ &= Prob(\pi_i(s_k^i, s_k^{-i}) \geqslant \pi_i(s_j^i, s_j^{-i}) \ \forall\, j) \\ C^-(a_{-i})(s_k^i) &= Prob(\{s_k^i\} = \arg\max_j \pi_i(s_j^i, s_j^{-i})) = \\ &= Prob(\pi_i(s_k^i, s_k^{-i}) > \pi_i(s_j^i, s_j^{-i}) \ \forall\, j \neq i) \end{aligned}$$

where the $s_j^{-i}$ are drawn independently according to $a_{-i}$

In general, neither the $C_i^+$ nor the $C_i^-$ need to add up to 1, even generically, due to non-zero probability ties given $a_{-i}$. [23] postulate a particular form of breaking ties (equiprobably) in order to obtain a single-valued function. An alternative generating a choice correspondence is to allow for all *admissible* probabilistic choices, i.e. all vectors $\widetilde{C}_i(a_{-i})$ satisfying

(i) $C_i^-(a_{-i})(s_k^i) \leqslant \widetilde{C}_i(a_{-i})(s_k^i) \leqslant C_i^+(a_{-i})(s_k^i) \ \ \forall k = 1, ..., m_i$

[6] See Sethi [26] for a dynamic interpretation of this model.

[7] That is, for each strategy $s_j^i$, one extraction of $s^{-i}$, denoted $s_j^{-i}$, is made according to $a_{-i}$, all these extractions being independent. This way, the payoffs of the strategies $s_j^i$ are random variables and the probabilities of these payoffs to be the highest ones observed through this procedure are well-defined.

(ii) $\sum_k \widetilde{C}_i(a_{-i})(s^i_k) = 1$

Such choice vectors correspond to all possible forms of allocating the probability of payoff ties among the involved strategies. In summary, we view procedural rationality as the choice correspondence

$$C_i(a_{-i}) = A_i \bigcap \prod_k \left[C_i^-(a_{-i})(s^i_k), C_i^+(a_{-i})(s^i_k)\right]$$

The correspondence $C_i$ captures procedural rationality. An $S(1)$-equilibrium in the sense of [23] corresponds now to a Choice-Nash equilibrium of $(I, \{A_i, C_i\}_{i\in I})$.

Note that Condition 1 holds by construction, and Condition 2 follows from the continuity (actually, bilinearity) of the payoff functions $\pi_i$. Moreover, $C_i(a_{-i})$ is obviously nonempty (e.g. it contains the rule given by [23]), convex, and closed by construction (being the intersection of a product of intervals and a simplex).

Proposition 2.3 now implies:

**Corollary 3.3.** *Any game such that every agent's action set is the simplex of mixed strategies on a finite set of pure strategies has an $S(1)$-equilibrium. Moreover, the set of $S(1)$-equilibria is closed.*

Osborne and Rubinstein [23] provide an existence result for the case of symmetric, two-player games, using that, under equiprobable breaking of ties, the associated, single-valued choice rule is continuous and hence Brouwer's Theorem applies. Our result differs in that we consider a multi-valued choice rule by allowing for any allocation of the probabilities of ties among agents, and hence we ultimately resort to Kakutani's Theorem. [23] also define the concept of $S(k)$-equilibrium, where agents try each strategy not once but $k$ times before actually taking a decision. Of course, the arguments above can also be applied to this concept.

**Example 3.4.** *We conclude this section showing that Corollary 3.3 covers a case not covered in Theorem 2.10. For that, we show that procedural rationality does not fulfill Condition 4(a).*

*For this, we must first extend the choice correspondence to any set of the form $\mathrm{co}(T)$ with $T$ a finite subset of $A_i$. This can be done as follows. Given a "prospect" (or mixed strategy) $x \in T$, define the (expected) payoff $\tilde{\pi}(x, s_{-i}) = \sum_{s_i\in S_i} x(s_i)\cdot \pi(s_i, s_{-i})$, and let $\tilde{C}(T, a_{-i})(x)$ be defined as above (where the prospects in $T$ now play the role of pure strategies). This is a probability distribution over the elements of $T$ (which are mixed strategies), which in turn induces a probability distribution over pure strategies through $C(\mathrm{co}(T), a_{-i})(s_i) = \{\sum_{x\in T} x(s_i)\cdot z(x) \mid z \in \tilde{C}(T, a_{-i})\}$. Obviously, $C(\mathrm{co}(T), a_{-i})(s_i) \subseteq \mathrm{co}(T)$.*

*Consider the following "Rock-Scissors-Paper" symmetric, two-player game.*

| | *R* | *S* | *P* |
|---|---|---|---|
| *R* | *1* | *2* | *0* |
| *S* | *0* | *1* | *2* |
| *P* | *2* | *0* | *1* |

*The unique Nash equilibrium is symmetric and given by $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It can be easily checked that this profile also constitutes an $S(1)$-equilibrium, i.e. a Choice-Nash equilibrium under procedural rationality.*

*Let $a_{-i} = \sigma$. Since $\sigma$ is a convex combination of the three pure strategies, Condition 4(a) would require the existence of one pure strategy $s \in \{R, S, P\}$ such that $\sigma \in C_\sigma([s, \sigma])$. However, note that $\pi_i(\sigma, s^{-i}) = 1$ for any $s^{-i}$, and hence, given $T = \{\sigma, s\}$ with $s \in \{R, S, P\}$ and $D = \mathrm{co}(T)$,*

$$\begin{aligned} \tilde{C}^+(D,\sigma)(\sigma) &= Prob(\pi_i(\sigma, s^{-i}) \geqslant \pi_i(s, s'^{-i})) = \frac{2}{3} \\ \tilde{C}^-(D,\sigma)(\sigma) &= Prob(\pi_i(\sigma, s^{-i}) > \pi_i(s, s'^{-i})) = \frac{1}{3} \end{aligned}$$

*where we use the notation $s^{-i}$ and $s'^{-i}$ to emphasize that they correspond to independent random realizations of the opponents' strategies, according to $a_{-i}$. However, $\sigma \in C(D, \sigma)$ if and only if $(0, 1) \in \tilde{C}(D, \sigma)$, and hence in this case the mixture $\sigma$ is not preferred over any pure strategy, even though $(\sigma, \sigma)$ is both a Nash equilibrium and a Choice-Nash equilibrium of the game under Procedural Rationality.*

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