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A FRICTIONLESS ECONOMY WITH SUBOPTIMIZING AGENTS

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Abstract. The existence of short-term monetary equilibrium in a frictionless economy with suboptimal agents is proved for any (reasonable) given interest rate. Separability ideas (as defined in Decision Theory) are applied. Two financial markets are in operation: for bank contracts (deposits and credits) and for shares.

1. Introduction

We consider a sequential model where the decision-makers (consumers and firms) face a labour market, a market for commodities and two financial markets: for bank contracts (deposits and credits) and for shares. A "bounded rationality" approach is followed, i.e., it is supposed that the agents do not necessarily make "optimal" decisions, but rather "acceptable" or "suboptimal" ones. As in [6], what is meant by "suboptimal" is characterized by a set of behavioural assumptions.

Among the assumptions imposed on the preferences of the decision-makers figure separability and dual separability conditions, as defined in Decision Theory (v. [3] for a survey).

Long ago it was realized that, as money has only exchange value, it can be made to consist of convertible claims. In the words of Schumpeter ([12], p. 321), "you cannot ride on a claim to a horse, but you can pay with a claim to money"; in the older words (1571) of Tomás de Mercado ([9], book 2, ch. 17), "never men distinguish morally in their business money from the right to have it, if, as I say, is safe and liquid[1]". In our model, there is a clearing house, in charge of the settling of all payments in the economy, and a bank, representing all financial intermediation. The credit positions of the agents at the clearing house are *money.* Money has to be homogeneous, in order to be determined only by its quantity, in terms of standard units; the debit positions of the bank at the clearing house are formulated in terms of *ad hoc* units, and so money is formulated in terms of (convertible claims to) these units. The monetary units are taken as units of account for all prices in the period and serve as standard of deferred payment.

Key words and phrases. Separability in Decision Theory, bounded rationality, short-term equilibrium, monetary frictions.

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[1]Mercado did not compose [9] in Latin, but in Spanish: "nunca los hombres distinguen moralmente en sus negocios el dinero del derecho de haberlo, si, como digo, está seguro y líquido". The statement is in the context of a qualified disapproval of usury.

In the model, money is not a commodity or asset, and properly there is no (final) demand o supply of it. Nevertheless, there are prices of money (against labour and all commodities and assets). These prices of money give a further degree of freedom to the model (in comparison with historical commodity money).

We define monetary assets as those whose prices (in terms of money) are fixed by the bank (or the incumbent monetary authority). Gold has been in the past a monetary asset, and one unit of money has been set as a fixed quantity of gold. The equations relating supplies and demands of monetary assets in equilibrium may contribute to determining the general levels of prices. In our model, the relevant equation will be equivalent to a condition on aggregate demand (provided that the market for shares is in equilibrium too). Bank credits and deposits are the monetary assets: the rates of interest are supposed to be set by the bank. The quantities of monetary assets are endogenous, and the bank implements monetary policy by setting the interest rates.

It is supposed that there are not transaction costs. On the other hand, the possibility that agents have to maintain some amount of a particular monetary asset in order to facilitate payments is admitted. This is a liquidity friction. As it is the only possible transaction friction considered here, we call the model *frictionless* if it does not exist (cf the definition of monetary frictions in [16]). At any rate, this friction is not likely to have appreciable consequences on the workings of the model.

In static equilibrium, both the income of the agents and their expenses are determined simultaneously. In the real world, consumers make their expenditure decisions on the basis of the income that they have secured before. On the other hand, firms contract labour and invest in fixed capital before the resulting income arrives. There is an organizational ordering then, where income comes before expenditure in the case of consumers and conversely in the case of firms. In the model, the decisions determining income and expenditure are made in this sequence: earning-then-expending, for consumers, and investing-then-earning for firms. Thus decisions on salaries and dividends result in deferred payments.

We discuss temporary short-term competitive equilibria. In short-term equilibrium, the fixed capital endowment of firms is supposed to be given. This entails that investment in fixed capital has effects on commodity demand, but the adjustment of the production capacity does not take place within the period of the investment, and thus the equality of demand and supply in the labour market is not necessarily assured. The production system may not be able to employ all labour available, at least paying at the level of the subsistence wage, either because the fixed capital is not up to it (out of lack of capacity or lack of efficiency) or because the firms limit the employment of labour as the expectations on the commodity market are not optimistic enough.

The existence of short-term equilibrium is proved for any (reasonable) given interest rate. In particular, equilibrium exists in the frictionless case.

In models with an infinite horizon, no equilibrium is possible if consumers have an incentive to resort to credit because the successive postponement of the debt payment (using new credit to pay interests) is allowed (*Ponzi schemes*). Three procedures that have been considered in the literature to check debt-inclined consumers into equilibrium have been debt constraints, transversality conditions and exogenous collateral (v. [1]). In the model, the debt constraint approach is followed for consumers. For firms, a "Marx was wrong" assumption is imposed, according to which the aggregate fixed capital investment of firms does not swallow up aggregate expenditure, because the income of consumers acts as an anchor. The proof of existence of equilibrium relies on a closed graph argument.

The discussion on the existence of equilibrium turns round two necessary equilibrium conditions, which are orthogonality conditions.

The model is introduced in Section 2. The concept of short-term equilibrium, and some necessary conditions, are considered in Section 3. Money and monetary assets are discussed in Section 4. The existence of equilibrium is proved in Section 5. Some final remarks are included in Section 6.

Given two vectors $x, y \in \mathbb{R}^n$, $x \leq y$ means $x_i \leq y_i$ for $i = 1, ..., n$, and $x < y$ means $x_i < y_i$ for $i = 1, ..., n$; the scalar product of the two vectors is denoted by $\langle x, y\rangle$. We write $\mathbb{R}^n_+ \triangleq \{x \in \mathbb{R}^n : x \geq 0\}$, and $\mathbb{R}^n_{++} \triangleq \{x \in \mathbb{R}^n : x > 0\}$.

2. The model

We consider a sequential monetary closed economy where n consumers have available l commodities produced by n' firms, in each period t, with $t = 1, 2, ...$ Commodities are not necessarily perishable, but they can be sold only in the period when they are produced. They can be bought by the consumers and the firms (as investment in capital goods). Consumers and firms are *agents* of the economy. There is a (central) bank. We contemplate the situation in a certain period t, although this is not made explicit through subscripts or superscripts.

There are a labour market, a market for commodities and two financial markets: the bank market and the market for shares. Instead of through barter, the markets operate through a payments system. The market contracts involve the exchange of a good or asset for a payment. These contracts are not the only source of obligations leading to payments, as it will be considered below. The obligations incurred by the agents are always complied with.

Payments can be spot payments, when they are carried out exactly when they are agreed to, or deferred payments, if they are to be implemented after they are agreed to.

There is a clearing house centralizing the payments of the period. Each agent and the bank have an account there, whose balance is zero at the beginning of the period and has to be zero again at the end of it (v. (1), (2) and (3) below). Purchases are paid by debiting the account of the buyer and crediting that of the seller, for the same amount. The other payments are implemented analogously. All payments are settled simultaneously at the end of the period. The credit positions of the agents at the clearing house are *money.* The debits of the bank are formulated in terms of *ad hoc* units. Money is formulated in terms of these units, which are taken as units of account for all prices in the period and serve as standard of deferred payment.

The agents and the bank can issue loans. Both loans and shares are *assets.* Loans last one period and are formulated in terms of the unit of account in period $(t + 1)$. The possibility of default is excluded. On the one hand, the agents can issue loans to be bought by the bank, bearing interest at the rate ρ, resulting in *credit* contracts. On the other hand, the bank can also issue loans to be taken by the agents, resulting in *deposit* contracts. There are two sorts of deposits: *demand deposits* and *saving deposits*, the former with the rate of interest $\underline{\rho}'$ and the latter with the rate $\underline{\rho}$. We assume that $\underline{\rho}' \leq \underline{\rho} \leq \rho$. For the sake of simplicity, we assume that $\rho = \underline{\rho} + \varsigma$, for some constant $\varsigma \geq 0$, and that $\underline{\rho}' = 0$. Thus only ρ is to be determined. Now ρ can be freely set by the bank within I, where $I \subseteq \mathbb{R}_+$ is a nonvoid given interval. The bank imposes no rationing in its deposit and credit facilities. Following actual use, credits and deposits are measured by their cost (e.g., $1/(1+\rho)$ units of credit (or deposit) correspond to 1 unit of loan: the promise of payment in the period $(t+1)$ of 1 unit of account).

The capital of the firms is divided into dividend-providing equities, which can be bought by the agents or the bank. Equities last indefinitely (unless they are redeemed), and in each period new equities can be issued (or old ones redeemed) by firms. In period t, $q_k \in \mathbb{R}_+$ is the (unitary) price of the equities of firm k, and $q \triangleq (q_1, ..., q_{n'})$. At the conclusion of period $(t-1)$, firm k has fixed the unit dividend $\upsilon_k \geq 0$ attributable to the equities of the firm circulating at that moment, and payable in period t. The firm also decides then how many equities it intends to issue (or to redeem) in period t, leaving the total number of equities in circulation at σ_k. We write $\upsilon \triangleq (\upsilon_1, ..., \upsilon_{n'})$, $\sigma \triangleq (\sigma_1, ..., \sigma_{n'})$, and assume that $\sigma \neq 0$. Gold bullion can be thought of as shares of a particular firm (with zero dividend).

The possibility that the mere possession of an asset should produce satisfaction, apart from the yields and price expected in the future, is admitted.

Production takes time. All through the period, firms have in use the same fixed capital, resulting from the investments of the periods before. Thus production depends only on the labour employed; for the sake of simplicity, intermediate goods are not contemplated. The decisions on labour (supply and demand) are made at the beginning of the period, and it is then when the labour market takes place. All the other markets meet simultaneously later in the period, at the time when credits and deposits mature and all payments take place (including those of approved dividends and agreed labour contracts).

The purchases in the market for commodities and the financial markets lead to spot payments. On the other hand, the contracts in the labour market and the settlement of deposits and credits at maturity and that of approved dividends lead to deferred payments.

The intended supply of labour of consumer j in period t is $\beta_j(w)$, where w is the wage (for unit of labour). Thus the intended wage income is given by the function $\gamma_{0j} : \mathbb{R}_{++} \to \mathbb{R}_+$ defined by $\gamma_{0j}(w) \triangleq w\beta_j(w)$. The production function of firm k in period t is $\xi_k : \left[0, \overline{N}_k\right] \to \mathbb{R}^l_+$, where $\overline{N}_k > 0$ is the maximum labour capacity. For given wage $w \geq \underline{w}$ (where $\underline{w} > 0$ is the minimum wage, which prevails through custom or law), the intended demand for labour of the firm is $\alpha_{0k}(w) \leq \overline{N}_k$, and the vector of commodities that it intends to produce is $e_{0k}(w) \triangleq \xi_k(\alpha_{0k}(w))$. There can be different decision criteria to make this decision (by considering profit, sales, ...); at any rate the firm will have to make predictions on the behaviour of the commodity market and the financial markets in the period.

After the operation of the labour market, let the wage be in fact $\widetilde{w}$ and consumer j devote effectively $\widetilde{N}_{jk}$ units of labour to firm k. Then, denoting $\alpha_k \triangleq \sum_{j=1}^{n} \widetilde{N}_{jk}$, firm k produces in fact $e_k \triangleq \xi_k(\alpha_k)$, and consumer j receives effectively $\gamma_j \triangleq \widetilde{w} \sum_{k=1}^{n'} \widetilde{N}_{jk}$ as wage income. We do not assume that these values actually correspond to an equilibrium of supply and demand. They are taken as data when the agents face decisions for the other markets in period t, even if we do not reflect it explicitly in the notation. We write $e \triangleq \sum_{k=1}^{n'} e_k$, and assume that $e > 0$.

The bank announces the value of ρ after the labour market has ended (before that, the agents have to resort to estimations).

In period t, given equity prices q and commodity prices $p \triangleq (p_1, ..., p_l)$, consumer j makes estimations about the future and has to decide: how much he deposits in the bank (and how much of it in a demand or in a saving account), how much he borrows from the bank, the portfolio of shares, and the commodity bundle in the current period. The first three decisions involve the financial aspect (implying how much to spend), and the last one the spending aspect (i.e., how to spend it). The preferences of the consumer are given by a vector of functions $(d_{1j}, d_{2j}, c_j, f_j, z_j) : \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n'}_+ \times \mathbb{R}^l_+$ defined for pairs (p, q), where $d_{1j}(p, q)$ represents the desired demand deposits, $d_{2j}(p, q)$ the desired saving deposits, $c_j(p, q)$ the desired credit from the bank (corresponding to the desired supply of loans to be taken by the bank), $f_j(p, q)$ the desired portfolio of shares ($f_{jk}(p, q)$ corresponds to the number of shares of firm k), and $z_j(p, q)$ the desired commodity bundle when the amount to be spent is that determined (after the financial decisions) by (p, q). We denote $d_j \triangleq d_{1j} + d_{2j}$.

Firm k faces also decisions in the markets for deposits, credits and commodities. Now $d'_{1k}(p, q)$ represents the desired demand deposits in period t, $d'_{2k}(p, q)$ the desired saving deposits, $c'_k(p, q)$ the desired credit from the bank, $f'_k(p, q)$ the desired portfolio of shares, and $z'_k(p, q)$ the desired commodity bundle (produced by the other firms) to be used as capital goods. Thus the preferences of the firm are given by a vector of functions $(d'_{1k}, d'_{2k}, c'_k, f'_k, z'_k) : \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n'}_+ \times \mathbb{R}^l_+$. We denote $d'_k \triangleq d'_{1k} + d'_{2k}$.

As for the bank, $f''(p, q)$ is the desired portfolio of shares and $z''(p, q)$ the desired commodity bundle. The preferences of the bank are given by a vector of functions $(f'', z'') : \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \to \mathbb{R}^{n'}_+ \times \mathbb{R}^l_+$ and a correspondence $\kappa : \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \twoheadrightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, with $\kappa(p, q) \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, $\kappa(p, q) \neq \emptyset$. If $(\overline{d}_1, \overline{d}_2, \overline{c}) \in \kappa(p, q)$, then $\overline{d}_1$ represents a desired supply of demand deposits in period t, $\overline{d}_2$ a desired supply of saving deposits and $\overline{c}$ a desired credit (corresponding to a desired demand of the loans issued by the agents).

We write $D_1 \triangleq \sum_{j=1}^{n} d_{1j} + \sum_{k=1}^{n'} d'_{1k}$, $D_2 \triangleq \sum_{j=1}^{n} d_{2j} + \sum_{k=1}^{n'} d'_{2k}$, $D \triangleq D_1 + D_2$, $C \triangleq \sum_{j=1}^{n} c_j + \sum_{k=1}^{n'} c'_k$, $F \triangleq \sum_{j=1}^{n} f_j + \sum_{k=1}^{n'} f'_k + f''$ and $Z \triangleq \sum_{j=1}^{n} z_j + \sum_{k=1}^{n'} z'_k + z''$.

Given $(p, q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$, let $\psi_j(p, q) \triangleq \langle p, z_j(p, q)\rangle$, $\psi'_k(p, q) \triangleq \langle p, z'_k(p, q)\rangle$, $\psi''(p, q) \triangleq \langle p, z''(p, q)\rangle$, $\omega_j(p, q) \triangleq \langle q, f_j(p, q)\rangle$, $\omega'_k(p, q) \triangleq \langle q, f'_k(p, q)\rangle$, $\omega''(p, q) \triangleq \langle q, f''(p, q)\rangle$, for $j = 1, ..., n$, $k = 1, ..., n'$, and $\Psi(p, q) \triangleq \langle p, Z(p, q)\rangle$, $\Omega(p, q) \triangleq \langle q, F(p, q)\rangle$. Thus consumer j intends to spend ψ_j, firm k intends to invest ψ'_k in commodities (to be used as capital goods) of the other firms, etc. The bank has a budget, and we assume that $\psi''(p, q) = \overline{\psi}''$, with given $\overline{\psi}'' > 0$.

Predictions affect preferences; implicit in $(d_{1j}, d_{2j}, c_j, f_j, z_j)(p, q)$ is the fact that, for periods t', with $t' > t$, consumer j (considering (p, q)) assigns, in period t, probability distributions to the bank rate $\rho^{t'}$, the vector of share prices $q^{t'}$, the demand for labour $\sum_{k=1}^{n'} \alpha_{0k}^{t'}(w)$, etc. The same applies analogously to firm k (also, mutatis mutandis, to the bank), and to the preferences of both consumers and firms concerning production and labour.

In period t, the values of parameters and magnitudes that obtained effectively in period $(t-1)$ are known. We shall simplify the notation by writing d_{1j}^0 for the amount effectively deposited in demand accounts by consumer j in period $(t-1)$, or $c_k'^0$ for the amount borrowed from the bank by firm k, etc. In this way, $c_k'^0$ does not represent a function, but a number. Note that, *ex post*, the realized values of supply and demand coincide (e.g., $\sigma^0 = F^0$).

We can consider price indices (approximately Paasche) for commodity prices and equity prices, with base period $t_0 < t$, where the price vectors at that period satisfy that $p^{t_0} > 0$ and $q^{t_0} > 0$. For a commodity price vector p we define the price index $P \triangleq \langle p, e\rangle / \langle p^{t_0}, e\rangle$, and for an equity price vector q we define the corresponding price index $Q \triangleq \langle q, \sigma\rangle / \langle q^{t_0}, \sigma\rangle$. Now $\langle p, e\rangle = \lambda_0 P$, with $\lambda_0 \triangleq \langle p^{t_0}, e\rangle$, and $\langle q, \sigma\rangle = \mu_0 Q$, with $\mu_0 \triangleq \langle q^{t_0}, \sigma\rangle$.

Given an equity price vector q, the value of the shares of consumer j passed on from period $(t-1)$ becomes $\langle q, f_j^0\rangle$ in period t. Thus his initial financial wealth in period t depends on the equity prices, and it is a function $v_j : \mathbb{R}_{++}^{n'} \to \mathbb{R}$ defined by $v_j(q) \triangleq d_j^0 - c_j^0 + \langle q, f_j^0\rangle$. Before considering the financial markets and the commodity market, the amount available to the consumer is $v_j + y_j$, where $y_j \triangleq \gamma_j + \underline{\rho}^0 d_{2j}^0 + \langle \upsilon, f_j^0\rangle - \rho^0 c_j^0$ is his income (after having settled interest and dividend payments). Let $a_j(p,q) \triangleq d_j(p,q) - c_j(p,q) + \omega_j(p,q)$ be his desired accumulated saving at the end of the period. We assume that the plans of the consumer are consistent, i.e.,

$$\psi_j = v_j + y_j - a_j \tag{1}$$

In other words, the balance of his account in the clearing house is planned to be zero at the end of the period. On the other hand, $u_{0j} \triangleq v_j - a_j^0$ is his speculative gain, and thus $a_j - a_j^0 = u_j + u_{0j}$, where $u_j \triangleq y_j - \psi_j$ is his intended financing capacity (for the rest of the economy), positive or negative. If it were supposed that consumer j cannot have directly capital goods, u_j would be his (ordinary, i.e., "non-speculative") intended saving during period t.

We reflect now on the situation of firm k. We define the function $v_k' : \mathbb{R}_{++}^{n'} \to \mathbb{R}$ by $v_k'(q) \triangleq d_k'^0 - c_k'^0 + \langle q, f_k'^0\rangle - q_k\sigma_k^0$. The (gross ordinary) profit of the firm is a function $\delta_k : \mathbb{R}_{++}^l \to \mathbb{R}$ depending on commodity prices, defined by $\delta_k(p) \triangleq \langle p, e_k\rangle - \widetilde{w}\alpha_k + \underline{\rho}^0 d_{2k}'^0 + \langle \upsilon, f_k'^0\rangle - \rho^0 c_k'^0$, and its "income" (also its gross saving) is $y_k' \triangleq \delta_k - \upsilon_k\sigma_k^0$. Let $a_k'(p,q) \triangleq d_k'(p,q) - c_k'(p,q) + \omega_k'(p,q) - q_k\sigma_k$. We assume that the plans of the firm are consistent, i.e.,

$$\psi_k' = v_k' + y_k' - a_k' \tag{2}$$

On the other hand, $u_k' \triangleq y_k' - \psi_k'$ is the intended financing capacity of the firm.

In the case of the bank, we define the function $v'' : \mathbb{R}_{++}^{n'} \to \mathbb{R}$ by $v''(q) \triangleq C^0 - D^0 + \langle q, f''^0\rangle$. For the sake of simplicity, we take as both profit and income the net interest income $y'' \triangleq \rho^0 C^0 + \langle \upsilon, f''^0\rangle - \underline{\rho}^0 D_2^0$. Similarly as above, let $a''(p,q) \triangleq \{\overline{c} - (\overline{d}_1 + \overline{d}_2) + \omega''(p,q) : (\overline{d}_1, \overline{d}_2, \overline{c}) \in \kappa(p,q)\}$. We assume that the plans of the bank are consistent, i.e.,

$$\psi'' = v'' + y'' - a'' \tag{3}$$

Thus $a'' : \mathbb{R}^{n'}_{++} \to \mathbb{R}$ is singleton-valued. In order to guarantee (3), we define $\kappa(p,q)$ as the largest set such that this assumption holds[2]:

$$\kappa(p,q) \triangleq \left\{(\overline{d}_1, \overline{d}_2, \overline{c}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : \overline{c} - (\overline{d}_1 + \overline{d}_2) = v''(q) + y'' - \omega''(p,q) - \psi''\right\}$$

for all $(p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$. We define $u'' \triangleq y'' - \psi''$; so the elements of $\kappa(p,q)$ can be characterized as those $(\overline{d}_1, \overline{d}_2, \overline{c}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$(\overline{c} - C^0) - (\overline{d}_1 + \overline{d}_2 - D^0) + \left\langle q, f''(p,q) - f''^0 \right\rangle = u'' \tag{4}$$

The bank is called "non-interventionist" when $u'' = 0$ and $\left\langle q, f''(p,q) - f''^0 \right\rangle = 0$.

The aggregate planned expenditure is $\Psi = \sum_{j=1}^{n} \psi_j + \sum_{k=1}^{n'} \psi'_k + \psi''$ and the aggregate planned accumulated saving is $A \triangleq \sum_{j=1}^{n} a_j + \sum_{k=1}^{n'} a'_k + a''$. From easy calculations,

$$\sum_{j=1}^{n} v_j + \sum_{k=1}^{n'} v'_k + v'' = 0 \tag{5}$$

$$\sum_{j=1}^{n} y_j + \sum_{k=1}^{n'} y'_k(p) + y'' = \langle p, e \rangle \ \text{, for all } p \in \mathbb{R}^l_{++} \tag{6}$$

A sort of Walras' law follows now from (1), (2) and (3):

$$\Psi(p,q) = \langle p, e \rangle - A(p,q) \ \text{, for all } (p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \tag{7}$$

or, equivalently,

$$\sum_{j=1}^{n} u_j + \sum_{k=1}^{n'} u'_k + u'' = A$$

Let us write $D'_1 \triangleq \sum_{k=1}^{n'} d'_{1k}$, $D'_2 \triangleq \sum_{k=1}^{n'} d'_{2k}$, $D' \triangleq D'_1 + D'_2$, $C' \triangleq \sum_{k=1}^{n'} c'_k$, $F' \triangleq \sum_{k=1}^{n'} f'_k$, $\Delta \triangleq \sum_{k=1}^{n'} \delta_k$ and $\Psi' \triangleq \sum_{k=1}^{n'} \psi'_k$. We have for the aggregate profit:

$$\Delta(p) = \langle p, e \rangle - \sum_{j=1}^{n} \gamma_j + \underline{\rho}^0 D'^0_2 + \left\langle \upsilon, F'^0 \right\rangle - \rho^0 C'^0 \tag{8}$$

(An alternative to aggregate profit could also have been defined: aggregate profit excluding dividends, i.e., $\Delta - \left\langle \upsilon, F'^0 \right\rangle$). Since $\langle p, e \rangle = \lambda_0 P$, it follows that the aggregate profit depends on p only through the corresponding index P. In fact,

[2] "It is the rule that the bank turns away a customer whom it considers deserving of credit only if it is compelled to do so" ([14], III.5). Of course, in the real world the bank acts not only through the rate of interest, but also by setting solvency conditions (and this affects the amount of credit). We suppose that the agents are blessed with moderation and restrain themselves up to the point of incorporating the bank's personal solvency conditions into their preferences.

the aggregate profit is a strictly increasing affine function of the commodity price index. Also, the trade-off between real profits and real wages can be made precise:

$$\frac{\Delta(p)}{P}+\frac{\sum_{j=1}^{n}\gamma_j}{P}=\lambda_0+\frac{\underline{\rho}^0 D_2'^0+\langle \upsilon, F'^0\rangle-\rho^0 C'^0}{P} \tag{9}$$

where the right-hand side of the equation, except for P, is already determined before period t begins. Most often (certainly, if $\underline{\rho}^0 D_2'^0+\langle \upsilon, F'^0\rangle \leq \rho^0 C'^0$), real profits will increase with (commodity) prices, and real wages will behave conversely.

3. Equilibrium and its necessary conditions

We recall that $\widetilde{w}$ does not bring about necessarily the equality of demand and supply in the labour market. At any rate, the adjustment of the value of the parameters of the model may lead to the equilibrium of the other markets (the two financial markets and the commodity market): a *short-term equilibrium.* The parameters to contemplate are either endogenous parameters (p, q) or the policy parameter (ρ). Given $\rho \in I$, we call the model defined so far the *short-term model for* ρ. Obviously the value of ρ affects, e.g., the function $c_j(p,q)$; when we want to reflect explicitly that c_j changes with ρ, we write $c_j(\rho;p,q)$; the same notation is applicable to the other functions.

Definition 3.1. *Consider the short-term model for ρ. A pair $(\widetilde{p},\widetilde{q}) \in \mathbb{R}^l_{++}\times\mathbb{R}^{n'}_{++}$ is called a short-term equilibrium pair if $Z(\widetilde{p},\widetilde{q})=e$, $F(\widetilde{p},\widetilde{q})=\sigma$ and $(D_1,D_2,C)(\widetilde{p},\widetilde{q}) \in \kappa(\widetilde{p},\widetilde{q})$.*

The bank equilibrium condition $(D_1,D_2,C)(\widetilde{p},\widetilde{q}) \in \kappa(\widetilde{p},\widetilde{q})$ is obviously equivalent to the following:

$$(C(\widetilde{p},\widetilde{q})-C^0)-(D(\widetilde{p},\widetilde{q})-D^0)=u''-\langle \widetilde{q}, f''(\widetilde{p},\widetilde{q})-f''^0\rangle \tag{10}$$

Observe that the proportion of saving deposits and demand deposits in total deposits is irrelevant in the definition of equilibrium.

A necessary condition for $(p,q) \in \mathbb{R}^l_{++}\times\mathbb{R}^{n'}_{++}$ to be an equilibrium is obviously:

$$\Psi(p,q)=\langle p,e\rangle \tag{11}$$

In other words, $P=(1/\lambda_0)\Psi(p,q)$, where P is the index corresponding to p. Considering (7), (11) is equivalent to

$$A(p,q)=0 \tag{12}$$

Besides (11), another necessary condition for equilibrium is:

$$\Omega(p,q)=\langle q,\sigma\rangle \tag{13}$$

A third necessary condition for equilibrium is provided by (10):

$$(C(p,q)-C^0)-(D(p,q)-D^0)=u''-\langle q, f''(p,q)-f''^0\rangle \tag{14}$$

Considering that loans are supposed to be intermediated by the bank, (14) is a *loanable funds* equilibrium equation (although in the model shares are an alternative to loans for raising and placing funds). In fact, the equation becomes, if the bank is non-interventionist (as defined above),

$$C(p,q)-C^0=D(p,q)-D^0$$

Equation (11), the *expenditure equilibrium* equation, concerns the level of aggregate expenditure (alternatively, of saving (v. (12))); (13) concerns how much of the aggregate saving (or dissaving) goes on shares and (14) how much goes on loans. The three equations (11), (13) and (14) are equivalent to any two of them:

Proposition 3.1. *Consider the short-term model for ρ. Then the following statements are equivalent for $(p, q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$:*
(i) $\Psi(p, q) = \langle p, e\rangle$ *and* $\Omega(p, q) = \langle q, \sigma\rangle$
(ii) $(C(p, q) - C^0) - (D(p, q) - D^0) = u'' - \left\langle q, f''(p, q) - f''^0\right\rangle$
and $\Omega(p, q) = \langle q, \sigma\rangle$
(iii) $(C(p, q) - C^0) - (D(p, q) - D^0) = u'' - \left\langle q, f''(p, q) - f''^0\right\rangle$
and $\Psi(p, q) = \langle p, e\rangle$

Proof. Obviously $A(p, q) = D(p, q) - C(p, q) + \Omega(p, q) - \omega''(p, q) - \langle q, \sigma\rangle + a''(q)$. Hence, considering that $a'' = v'' + u''$, we have from (7) that

$$(C(p, q) - C^0) - (D(p, q) - D^0) = (\Psi(p, q) - \langle p, e\rangle) + (\Omega(p, q) - \langle q, \sigma\rangle) + u'' - \left\langle q, f''(p, q) - f''^0\right\rangle$$

The result is now immediate. ∎

Corollary 3.2. *Consider the short-term model for ρ. A pair $(\widetilde{p}, \widetilde{q}) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$ is a short-term equilibrium pair if and only if $Z(\widetilde{p}, \widetilde{q}) = e$, $F(\widetilde{p}, \widetilde{q}) = \sigma$.*

If (11) holds, the value of the aggregate profit is, by (8),

$$\Delta(p) = \Psi'(p, q) + \psi'' - \sum_{j=1}^{n} u_j(p, q) + \underline{\rho}^0 D_2^0 + \left\langle \upsilon, \sigma^0 - f''^0\right\rangle - \rho^0 C^0 \tag{15}$$

In the case of a non-interventionist bank, this boils down to

$$\Delta(p) = \left\langle \upsilon, \sigma^0\right\rangle + \Psi'(p, q) - \sum_{j=1}^{n} u_j(p, q) \tag{16}$$

Roughly, the aggregate profit equals distributed dividends (corresponding to the previous period) plus (fixed capital) investment of firms minus the financing capacity of consumers. The right-hand side of (15) or (16) depends on the value of ρ set by the bank, and, beyond the short term, the ability to wield this power (including the threat to use it) may allow the bank to exert influence on the agents acting in the labour market (v. also (9)), in order to steer the level of wages, and also on the firms deciding the amount of dividends.

4. Money

As said in Section 2, money provides the unit of account and the standard of deferred payment. Altogether, money provides the standard of payment. Money is not a commodity or asset, and properly there is no (final) demand o supply of it. On the other hand, there are prices of money, against labour and all commodities and assets. The prices of money are affected by the amount and structure of deferred payments (e.g., those of salaries).

The prices of the credit and deposit loans are fixed by the bank; thus we call them *monetary assets*. Credit loans are issued by the agents and deposit loans by the bank. The equations relating supplies and demands of monetary assets in equilibrium may contribute to determining the general levels of prices[3]. Equation (14) gives a necessary condition for equilibrium, relating the demands for monetary assets:

$$(C(p,q) - C^0) + \left\langle q, f''(p,q) - f''^0 \right\rangle - u'' = (D(p,q) - D^0)$$

This equality can be interpreted in terms of supply and demand of money, if a suitable meaning (not our own) is assigned to these terms. In fact, $C(p,q)$ corresponds to a supply of loans by the agents in exchange of money, and it can be said that represents a supply of money by the bank (or a demand of money by the agents). In the same way, $\left\langle q, f''(p,q) - f''^0 \right\rangle - u''$ can also be said to represent a supply of money by the bank. Analogously, $D(p,q)$ corresponds to a demand of loans by the agents in exchange of money, and it can be said that represents a demand of money by the bank (or a supply of money by the agents). Altogether, (14) stipulates that the planned inflow and the planned outflow of money between the bank, on the one hand, and the agents, on the other hand, are to be equal.

Note that, in double contrast with the Walrasian *numéraire,* here money is standard of deferred payment, and there are monetary assets.

Why are the agents willing to hold demand deposits without receiving interest when they can opt for interest-yielding saving deposits? It was noted in Section 2 that the possibility that the mere possession of an asset should produce satisfaction, apart from the yields and price expected in the future, is admitted. In fact, demand deposits may provide a benefit, *liquidity*, which is not contemplated explicitly in the model. We say that the model is *frictionless* if no such benefit exists, and thus the actual amount of demand deposits is zero for all agents. At any rate, a change on the part of the agents of the desired proportion between demand deposits and saving deposits, provided that the total amount of deposits is kept, is not likely to have much impact on the economy described by the model, apart from the effect on the profit of the bank.

Recall the income version of the equation of exchange ([11]):

$$MV = PY \tag{17}$$

where M is the quantity of money, V the income velocity of circulation, P the implicit price deflator and Y national income. In the context of our model, the right-hand side of this equation of exchange becomes the right-hand side of (11):

$$\Psi(p,q) = \langle p, e \rangle$$

If behavioural assumptions are made about the variables in (17), the equation of exchange goes from identity to theory. The Cambridge Cash Balance Approach views (17) as stating the equilibrium between the supply and demand for money. On the

[3] In general, we define monetary assets as those whose prices (in terms of money) are fixed by the central bank (or whatever monetary authority). Gold has been historically a typical monetary asset, and one unit of money has been set as a fixed quantity of gold. Then the demand for gold has a "monetary" component (coined gold), a "pure asset" component (gold bullion) and a "commodity" component (for jewellery, etc). Assuming full convertibility (including free coinage) and certain hypotheses, a suitable equation relating the demand and the supply of gold could arguably determine the level of prices. If only convertibility into coined gold (and no free coinage) is considered, a similar equation for metallic money (coined gold) can be contemplated; Wicksell's quantity equation may be interpreted in this way.

In contrast with the case of the "value" of gold, the variations in the general price level will necessarily have a parallel effect (often, an almost inversely proportional effect) on the "value" of bank assets in relation to that of commodities.

other hand, Schumpeter [12] considers the "Income Approach" to (17), where MV is interpreted as aggregate demand, like the left-hand side in (11). An alternative decomposition of aggregate demand as the product of an "anchor" and a factor of adjustment is provided in (21) below. In the next section we shall see that (11) and (13) (where one of them can be replaced by (14), according to Proposition 3.1) play the role that the equation of exchange plays in the Quantity Theory: to determine the equilibrium price indices, while relative prices are determined elsewhere. Quite naturally, separability conditions (in the words of Decision Theory) will have to be assumed.

5. Existence of equilibrium

From Corollary 3.2, the necessary conditions for equilibrium considered in Proposition 3.1,

$$\Psi(p,q) = \langle p, e\rangle \ , \ \Omega(p,q) = \langle q, \sigma\rangle \tag{18}$$

become sufficient in the "macroeconomic" case, i.e., when there is only one commodity and one stock. Thus these necessary conditions define a sort of "macroeconomic equilibrium". The gist of our approach to prove the existence of equilibrium will be to show first (18) and then, considering separability (in the sense of Decision Theory), rely on the market for commodities and the market for shares being internally well-behaved.

Given two vectors of prices, $\overline{p} \in \mathbb{R}^l_{++}$ and $\overline{q} \in \mathbb{R}^{n'}_{++}$, we intend first to conclude, under suitable assumptions, that (18) holds for some prices $\lambda\overline{p}$ and $\mu\overline{q}$, with $\lambda, \mu > 0$.

The wealth and income $v_j(q) + y_j$ of consumer j do not depend on commodity prices. Recall that $v_j : \mathbb{R}^{n'}_{++} \to \mathbb{R}$ is continuous. Considering that good banking practices are internalized by the prudent consumer, we may assume that his intended debtor position at the end of the period (i.e., $-a_j(p,q)$) is bounded from above (the bound depending on q):

(A.1) There is a continuous function $\underline{a}_j : \mathbb{R}^{n'}_{++} \to \mathbb{R}$ such that $a_j(p,q) \geq \underline{a}_j(q)$ for all $(p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$

This is the sort of standard assumption to avoid *Ponzi schemes.* An hypothesis for firms analogous to (A.1) is not very plausible, as the existence of fixed capital alters solvency conditions (we do not contemplate consumers as having, as such, a substantial amount of fixed capital; they can always set up their own real estate firm). However, it is reasonable to assume for firms that their aggregate intended fixed capital investment does not swallow up the aggregate intended total expenditure (this is a "Marx was wrong" proposition):

(A.2) There is $\overline{\psi}' < 1$ such that $(\Psi'/\Psi)(p,q) \leq \overline{\psi}'$ for all $(p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$

The next assumption is a monotonicity property of the real aggregate intended expenditure when commodity prices change homothetically. Recall that P is the price index corresponding to p.

(A.3) $\frac{\Psi(\lambda p,q)}{\lambda P}$ is strictly decreasing as a function of $\lambda > 0$, for all $(p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$

Lemma 5.1. *Consider the short-term model for ρ, and assume (A.1)-(A.3) and that Ψ is continuous. Let $\overline{p} \in \mathbb{R}^l_{++}$. Then, for every $q \in \mathbb{R}^{n'}_{++}$, there exists one, and only one, $\pi(q) > 0$ such that $\Psi(\pi(q)\overline{p}, q) = \langle \pi(q)\overline{p}, e\rangle$. Moreover, the function $\pi : \mathbb{R}^{n'}_{++} \twoheadrightarrow \mathbb{R}_{++}$ so defined is continuous.*

Proof. Let $p \in \mathbb{R}^l_{++}$ and $q \in \mathbb{R}^{n'}_{++}$. From (A.2), $\Psi(p,q) \leq (1/(1-\overline{\psi}'))(\Psi - \Psi')(p,q) = (1/(1-\overline{\psi}'))(\sum_{j=1}^{n} \psi_j(p,q) + \psi'')$, and thus, by (1) and (A.1),

$$\Psi(p,q) \leq (1/(1-\overline{\psi}'))(\sum_{j=1}^{n} v_j(q) + \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} \underline{a}_j(q) + \psi'') \tag{19}$$

Also,

$$\Psi(p,q) \geq \psi'' > 0 \tag{20}$$

Consider now the function $\eta_q : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined by

$$\eta_q(\lambda) \triangleq \frac{\Psi(\lambda\overline{p},q)}{\lambda\overline{P}}$$

where $\overline{P} \triangleq \langle \overline{p}, e\rangle/\lambda_0$. We have that η_q is continuous. In view of (19) and (20), and applying the Intermediate Value Theorem to η_q, it follows that there exists $\lambda^* \in \mathbb{R}_{++}$ such that $\eta_q(\lambda^*) = \lambda_0$. By (A.3), η_q is injective, and so λ^* is unique in satisfying $\Psi(\lambda^*\overline{p},q) = \langle \lambda^*\overline{p}, e\rangle$. Hence we can define $\pi(q) \triangleq \lambda^*$.

Now we show that π has closed graph. Let $(q^h) \to q$ be a convergent sequence in $\mathbb{R}^{n'}_{++}$ and $(\pi(q^h)) \to \overline{\lambda}$. Then $\Psi(\pi(q^h)\overline{p}, q^h) = \langle \pi(q^h)\overline{p}, e\rangle$. By continuity of Ψ, we have that $\Psi(\overline{\lambda}\overline{p},q) = \langle \overline{\lambda}\overline{p}, e\rangle$, and $\pi(q) = \overline{\lambda}$. Thus π has closed graph. Now, $\pi(q) = \Psi(\pi(q)\overline{p},q)/\langle \overline{p}, e\rangle$. Considering (19) and (20), a compact co-domain can be taken for the restriction of π to any compact domain, and therefore π is continuous on compact domains. As $\mathbb{R}^{n'}_{++}$ is locally compact, we conclude that π is continuous. ∎

Note that π varies with $\overline{p}$, even if this is not made explicit in the notation. From Lemma 5.1, given $\overline{p} \in \mathbb{R}^l_{++}$ and $q \in \mathbb{R}^{n'}_{++}$, the relevant vector of commodity prices is $\pi(q)\overline{p}$. We define the function $\Upsilon : \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++} \to \mathbb{R}$ by $\Upsilon(p,q) \triangleq \Omega(\pi(q)p,q)$, representing the aggregate intended outlay on shares keeping expenditure equilibrium, when the equity price vector is q and the relative commodity prices are given by p.

The "Income Approach" to (17) considers the decomposition of aggregate demand as the product of an "anchor" (M) and a factor of adjustment (V). Assuming the hypotheses of Lemma 5.1, an alternative "anchor" is suggested by (19), namely $Y_0 \triangleq \sum_{j=1}^{n} y_j$, the income of consumers[4]. Provided that $Y_0 > 0$, and putting $V'(p,q) \triangleq \Psi(p,q)/Y_0$, (11) becomes:

$$Y_0 V'(p,q) = \langle p, e\rangle \tag{21}$$

In contrast with (17), here Y_0 is not a stock magnitude, and V' cannot be interpreted as a "velocity".

Observing at period t, and given commodity prices p and equity prices q, the global rate of return of shares (considering dividends and capital gains or losses) from period t to period $t+1$ is for consumer j a random variable ${}_j\widehat{F}$:

$${}_j\widehat{F} \triangleq \frac{\langle {}_j\widehat{q} + {}_j\widehat{v}, \sigma\rangle}{\langle q, \sigma\rangle} - 1 \tag{22}$$

[4]In the real world, the budget of the public sector provides also an anchor.

where ${}_j\widehat{q}$ and ${}_j\widehat{v}$ are the estimators of equity prices and dividends for period $t+1$ (p and q are not made explicit in the notations of the random variables).

The agents will tend to buy shares if ${}_j\widehat{F}$ is (in probabilistic terms) high enough in relation to $\underline{\rho}$ (moreover, the agents may consider advantageous to borrow from the bank if necessary to buy stock if ${}_j\widehat{F}$ is high enough in relation to ρ). Roughly, a reduction in Q (recall that $Q = \langle q, \sigma\rangle/\mu_0$) will produce this outcome on ${}_j\widehat{F}$ provided that it does not bring about an overcoming counter-effect (in the extreme, a panic) on the probability distributions of ${}_j\widehat{q}$ and ${}_j\widehat{v}$. A parallel argument is relevant for low values of ${}_j\widehat{F}$. The following macroeconomic assumption (large numbers help in probabilistic situations) is a boundary condition, considering the intended outlay on shares in real terms. Here the hypotheses of Lemma 5.1 are supposed to hold.

(A.4) For all $(p,q) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$,

$$\lim_{\mu\to 0} \frac{\Upsilon(p,\mu q)}{\mu Q} \to +\infty \ , \ \lim_{\mu\to +\infty} \frac{\Upsilon(p,\mu q)}{\mu Q} \to 0$$

Proposition 5.2. *Consider the short-term model for ρ, and assume (A.1)-(A.4) and that Ψ and Ω are continuous. Let $\overline{p} \in \mathbb{R}^l_{++}$ and $\overline{q} \in \mathbb{R}^{n'}_{++}$. Then (18) holds for some prices $p = \lambda\overline{p}$ and $q = \mu\overline{q}$, with $\lambda, \mu > 0$.*

Proof. By Lemma 5.1, $\Upsilon(\overline{p}, .)$ is continuous. From (A.4) and the Intermediate Value Theorem, there exists $\mu^* \in \mathbb{R}_{++}$ such that $\Upsilon(\overline{p}, \mu^*\overline{q})/\mu^*\overline{Q} = \mu_0$, where $\overline{Q} = \langle \overline{q}, \sigma\rangle/\mu_0$. Thus $\Upsilon(\overline{p}, \mu^*\overline{q}) = \langle \mu^*\overline{q}, \sigma\rangle$. Hence $\Omega(\pi(\mu^*\overline{q})\overline{p}, \mu^*\overline{q}) = \langle \mu^*\overline{q}, \sigma\rangle$, and the second equation in (18) holds for $p = \pi(\mu^*\overline{q})\overline{p}$ and $q = \mu^*\overline{q}$. Now, by Lemma 5.1, $\Psi(\pi(q)\overline{p}, q) = \langle \pi(q)\overline{p}, e\rangle$, and the first equation in (18) is also satisfied for p and q. ∎

We turn now to the workings of the commodity market and the stock market.

We may assume that the desired commodity bundle $z_j(p,q)$ of consumer j in the current period depends on q only through the planned expenditure $\psi_j(p,q)$, and analogously for the firms and the bank. This is a separability condition, in the sense of Decision Theory. We may also suppose for the resulting functions the usual hypotheses for demand functions. A similar assumption may be considered for the stock market. Formally:

(A.5) (i) There exist functions $\zeta_j, \zeta'_k, \zeta'' : \mathbb{R}_+ \times \mathbb{R}^l_{++} \to \mathbb{R}^l_+$, $j = 1,...,n$, $k = 1,...,n'$, continuous and homogeneous of degree zero, such that

$$z_j(p,q) = \zeta_j(\psi_j(p,q),p),\ z'_k(p,q) = \zeta'_k(\psi'_k(p,q),p),\ z''(p,q) = \zeta''(\psi'',p)$$

$$\langle p, \zeta_j(x,p)\rangle = x,\ \langle p, \zeta'_k(x,p)\rangle = x,\ \langle p, \zeta''(x,p)\rangle = x,\ \text{, for all } x \in \mathbb{R}_+$$

(ii) There exist functions $\varphi_j, \varphi'_k, \varphi'' : \mathbb{R}_+ \times \mathbb{R}^{n'}_{++} \to \mathbb{R}^{n'}_+$, $j = 1,...,n$, $k = 1,...,n'$, continuous and homogeneous of degree zero, such that

$$f_j(p,q) = \varphi_j(\omega_j(p,q),q),\ f'_k(p,q) = \varphi'_k(\omega'_k(p,q),q),\ f''(p,q) = \varphi''(\omega''(p,q),q)$$

$$\langle q, \varphi_j(x,q)\rangle = x,\ \langle q, \varphi'_k(x,q)\rangle = x,\ \langle q, \varphi''(x,q)\rangle = x \text{ , for all } x \in \mathbb{R}_+$$

We consider two boundary conditions, one of them for the commodity market and the other for the stock market:

(A.6) (i) If $(x_j(p) \in \psi_j(\mathbb{R}_+ \times \mathbb{R}^l_{++})$, $x'_k(p) \in \psi'_k(\mathbb{R}_+ \times \mathbb{R}^l_{++})$, $j = 1, ..., n$, $k = 1, ..., n'$, then

$$\lim_{p \to \overline{p}} \left\| \sum_{j=1}^{n} \zeta_j(x_j(p), p) + \sum_{k=1}^{n'} \zeta'_k(x'_k(p), p) + \zeta''(\psi'', p) \right\| \to +\infty$$

for all $\overline{p} \geq 0$ with $\overline{p}_i = 0$ for some i. (ii) If $x_j(q) \in \omega_j(\mathbb{R}_+ \times \mathbb{R}^l_{++})$, $x'_k(q) \in \omega'_k(\mathbb{R}_+ \times \mathbb{R}^l_{++})$, $x''(q) \in \omega''(\mathbb{R}_+ \times \mathbb{R}^l_{++})$, $j = 1, ..., n$, $k = 1, ..., n'$, then

$$\lim_{q \to \overline{q}} \left\| \sum_{j=1}^{n} \varphi_j(x_j(q), q) + \sum_{k=1}^{n'} \varphi'_k(x'_k(q), q) + \varphi''(x''(q), q) \right\| \to +\infty$$

for all $\overline{q} \geq 0$ with $\overline{q}_k = 0$ for some k.

In the two following assumptions, P denotes the price index corresponding to the price vector p (i.e., $P \triangleq \langle p, e \rangle / \lambda_0$), and analogously $\overline{P}$ for $\overline{p}$, Q for q (i.e., $Q \triangleq \langle q, \sigma \rangle / \mu_0$) and $\overline{Q}$ for $\overline{q}$. We may hypothesize that the aggregate planned expenditure Ψ depends on the index of prices and not on the particular commodity prices, and analogously for the stock market:

(A.7) Let $p, \overline{p} \in \mathbb{R}^l_{++}$ and $q, \overline{q} \in \mathbb{R}^{n'}_{++}$. If $P = \overline{P}$, then $\Psi(p, \overline{q}) = \Psi(\overline{p}, \overline{q})$. If $Q = \overline{Q}$, then $\Omega(\overline{p}, q) = \Omega(\overline{p}, \overline{q})$.[5]

It may be assumed that the aggregate demand for commodities depends on stock prices only through their price index. In the case of the stock market, a similar condition can only reasonably be assumed for the aggregate planned expenditure:

(A.8) Let $p, \overline{p} \in \mathbb{R}^l_{++}$ and $q, \overline{q} \in \mathbb{R}^{n'}_{++}$. If $Q = \overline{Q}$, then $Z(p, q) = Z(p, \overline{q})$. If $P = \overline{P}$, then $\Omega(p, q) = \Omega(\overline{p}, q)$.

A continuity assumption may also be contemplated:

(A.9) $\psi_j, \psi'_k, \omega_j, \omega'_k, \omega''$ are continuous, $j = 1, ..., n$, $k = 1, ..., n'$.

Lemma 5.3. *Consider the short-term model for ρ, and assume (A.5)-(A.9). Then there exists a short-term equilibrium pair if and only if there exists a solution of (18).*

Proof. We have already seen that (18) is a necessary condition for equilibrium. Conversely, let $\Psi(\overline{p}, \overline{q}) = \langle \overline{p}, e \rangle$, $\Omega(\overline{p}, \overline{q}) = \langle \overline{q}, \sigma \rangle$. From Corollary 3.2, we have only to prove that $Z(\widetilde{p}, \widetilde{q}) = e$, $F(\widetilde{p}, \widetilde{q}) = \sigma$, for some $(\widetilde{p}, \widetilde{q}) \in \mathbb{R}^l_{++} \times \mathbb{R}^{n'}_{++}$. We shall show first that $Z(\widetilde{p}, \overline{q}) = e$ for some $\widetilde{p} \in \mathbb{R}^l_{++}$. Let $\theta_0 : \mathbb{R}^{1+l}_{++} \to \mathbb{R}$ and $\overline{\theta} : \mathbb{R}^{1+l}_{++} \to \mathbb{R}^l$ be defined by $\theta_0(p_0, p) \triangleq \langle p, e \rangle / p_0 - \Psi(\overline{p}, \overline{q})$ and $\overline{\theta}(p_0, p) \triangleq \sum_{j=1}^{n} \zeta_j(p_0 \psi_j((\overline{P}/P)p, \overline{q}), p) + \sum_{k=1}^{n'} \zeta'_k(p_0 \psi'_k((\overline{P}/P)p, \overline{q}), p) + \zeta''(p_0 \psi'', p) - e$, where $P \triangleq \langle p, e \rangle / \lambda_0$ and $\overline{P} \triangleq \langle \overline{p}, e \rangle / \lambda_0$. Thus $\theta \triangleq (\theta_0, \overline{\theta})$ is continuous and homogeneous of degree zero, and $\theta(p_0, p) \geq (-\Psi(\overline{p}, \overline{q}), -e)$ for all (p_0, p). Also $\langle (p_0, p), \theta(p_0, p) \rangle = \langle p, e \rangle - p_0 \Psi(\overline{p}, \overline{q}) + p_0 (\sum_{j=1}^{n} \psi_j((\overline{P}/P)p, \overline{q}) + \sum_{k=1}^{n'} \psi'_k((\overline{P}/P)p, \overline{q}) + \psi'') - \langle p, e \rangle = 0$, considering (A.5) and (A.7). Moreover, if $(\widehat{p}_0, \widehat{p}) \geq 0$, $(\widehat{p}_0, \widehat{p}) \neq 0$, and either $\widehat{p}_i = 0$ for some i or $\widehat{p}_0 = 0$, then

$$\lim_{(p_0, p) \to (\widehat{p}_0, \widehat{p})} \|\theta(p_0, p)\| \to +\infty \tag{23}$$

[5] A weakened version of (A.7) would be sufficient for our purposes, namely: If $\Psi(\overline{p}, \overline{q}) = \langle \overline{p}, e \rangle$ and $P = \overline{P}$, then $\Psi(p, \overline{q}) = \Psi(\overline{p}, \overline{q})$; if $\Omega(\overline{p}, \overline{q}) = \langle \overline{q}, \sigma \rangle$ and $Q = \overline{Q}$, then $\Omega(\overline{p}, q) = \Omega(\overline{p}, \overline{q})$.

In fact, if $\widehat{p}_0 > 0$, then (23) results from (A.6); on the other hand, if $\widehat{p}_0 = 0$, then

$$\lim_{(p_0,p)\to(\widehat{p}_0,\widehat{p})} \theta_0(p_0, p) \to +\infty$$

since $e > 0$ and $\widehat{p} \neq 0$, and (23) also follows. From these properties of θ, we conclude (by a well known result; v., e.g., 17.C.1 in [8]) that there exists $(p_0^*, p^*) > 0$ such that $\theta(p_0^*, p^*) = 0$. Thus $\langle p^*, e\rangle / p_0^* = \Psi(\overline{p}, \overline{q})$ and $\sum_{j=1}^{n} \zeta_j(p_0^*\psi_j((\overline{P}/P^*)p^*, \overline{q}), p^*) + \sum_{k=1}^{n'} \zeta'_k(p_0^*\psi'_k((\overline{P}/P^*)p^*, \overline{q}), p^*) + \zeta''(p_0^*\psi'', p^*) = e$, where $P^* \triangleq \langle p^*, e\rangle/\lambda_0$. Let $\widetilde{p} \triangleq p^*/p_0^*$; it follows that $\langle \widetilde{p}, e\rangle = \Psi(\overline{p}, \overline{q})$ and $\sum_{j=1}^{n} \zeta_j(\psi_j((\overline{P}/\widetilde{P})\widetilde{p}, \overline{q}), \widetilde{p}) + \sum_{k=1}^{n'} \zeta'_k(\psi'_k((\overline{P}/\widetilde{P})\widetilde{p}, \overline{q}), \widetilde{p}) + \zeta''(\psi'', \widetilde{p}) = e$, where $\widetilde{P} \triangleq \langle \widetilde{p}, e\rangle/\lambda_0$. We have that $\widetilde{P} = \Psi(\overline{p}, \overline{q})/\lambda_0 = \langle \overline{p}, e\rangle/\lambda_0 \triangleq \overline{P}$, and so $\sum_{j=1}^{n} \zeta_j(\psi_j(\widetilde{p}, \overline{q}), \widetilde{p}) + \sum_{k=1}^{n'} \zeta'_k(\psi'_k(\widetilde{p}, \overline{q}), \widetilde{p}) + \zeta''(\psi'', \widetilde{p}) = e$. Hence $Z(\widetilde{p}, \overline{q}) = \sum_{j=1}^{n} z_j(\widetilde{p}, \overline{q}) + \sum_{k=1}^{n'} z'_k(\widetilde{p}, \overline{q}) + z''(\widetilde{p}, \overline{q}) = e$.

Next we shall see that $F(\widetilde{p}, \widetilde{q}) = \sigma$ for some $\widetilde{q} \in \mathbb{R}^{n'}_{++}$. Now let $\theta'_0 : \mathbb{R}^{1+n'}_{++} \to \mathbb{R}$ and $\overline{\theta}' : \mathbb{R}^{1+n'}_{++} \to \mathbb{R}^{n'}$ be defined by $\theta'_0(q_0, q) \triangleq \langle q, \sigma\rangle / q_0 - \Omega(\widetilde{p}, \overline{q})$ and $\overline{\theta}'(q_0, q) \triangleq \sum_{j=1}^{n} \varphi_j(q_0\omega_j(\widetilde{p}, (\overline{Q}/Q)q), q) + \sum_{k=1}^{n'} \varphi'_k(q_0\omega'_k(\widetilde{p}, (\overline{Q}/Q)q), q) + \varphi''(q_0\omega''(\widetilde{p}, (\overline{Q}/Q)q), q) - \sigma$, where $Q \triangleq \langle q, \sigma\rangle/\mu_0$ and $\overline{Q} \triangleq \langle \overline{q}, \sigma\rangle/\mu_0$. Thus $\theta' \triangleq (\theta'_0, \overline{\theta}')$ is continuous and homogeneous of degree zero, and $\theta'(q_0, q) \geq (-\Omega(\widetilde{p}, \overline{q}), -\sigma)$ for all (q_0, q). Similarly as above (considering now also the second statement in (A.8)), $\langle (q_0, q), \theta'(q_0, q)\rangle = \langle q, \sigma\rangle - q_0\Omega(\widetilde{p}, \overline{q}) + q_0[\sum_{j=1}^{n} \omega_j(\widetilde{p}, (\overline{Q}/Q)q) + \sum_{k=1}^{n'} \omega'_k(\widetilde{p}, (\overline{Q}/Q)q) + \omega''(\widetilde{p}, (\overline{Q}/Q)q)] - \langle q, \sigma\rangle = 0$. Besides, if $(\widehat{q}_0, \widehat{q}) \geq 0$, $(\widehat{q}_0, \widehat{q}) \neq 0$, and either $\widehat{q}_k = 0$ for some k or $\widehat{q}_0 = 0$, then

$$\lim_{(q_0,q)\to(\widehat{q}_0,\widehat{q})} \left\|\theta'(q_0, q)\right\| \to +\infty$$

From these properties of θ', we conclude now that there exists $(q_0^*, q^*) > 0$ such that $\theta'(q_0^*, q^*) = 0$. Thus $\langle q^*, \sigma\rangle / q_0^* = \Omega(\widetilde{p}, \overline{q})$ and $\sum_{j=1}^{n} \varphi_j(q_0^*\omega_j(\widetilde{p}, (\overline{Q}/Q^*)q^*), q^*) + \sum_{k=1}^{n'} \varphi'_k(q_0^*\omega'_k(\widetilde{p}, (\overline{Q}/Q^*)q^*), q^*) + \varphi''(q_0^*\omega''(\widetilde{p}, (\overline{Q}/Q^*)q^*), q^*) = \sigma$, where $Q^* \triangleq \langle q^*, \sigma\rangle/\mu_0$. Let $\widetilde{q} \triangleq q^*/q_0^*$; it follows that $\langle \widetilde{q}, \sigma\rangle = \Omega(\widetilde{p}, \overline{q})$ and $\sum_{j=1}^{n} \varphi_j(\omega_j(\widetilde{p}, (\overline{Q}/\widetilde{Q})\widetilde{q}), \widetilde{q}) + \sum_{k=1}^{n'} \varphi'_k(\omega'_k(\widetilde{p}, (\overline{Q}/\widetilde{Q})\widetilde{q}), \widetilde{q}) + \varphi''(\omega''(\widetilde{p}, (\overline{Q}/\widetilde{Q})\widetilde{q}), \widetilde{q}) = \sigma$, where $\widetilde{Q} \triangleq \langle \widetilde{q}, \sigma\rangle/\mu_0$. We have that $\widetilde{Q} = \Omega(\widetilde{p}, \overline{q})/\mu_0 = \Omega(\overline{p}, \overline{q})/\mu_0 = \langle \overline{q}, \sigma\rangle/\mu_0 \triangleq \overline{Q}$, and so $\sum_{j=1}^{n} \varphi_j(\omega_j(\widetilde{p}, \widetilde{q}), \widetilde{q}) + \sum_{k=1}^{n'} \varphi'_k(\omega'_k(\widetilde{p}, \widetilde{q}), \widetilde{q}) + \varphi''(\omega''(\widetilde{p}, \widetilde{q}), \widetilde{q}) = \sigma$. Hence $F(\widetilde{p}, \widetilde{q}) = \sum_{j=1}^{n} f_j(\widetilde{p}, \widetilde{q}) + \sum_{k=1}^{n'} f'_k(\widetilde{p}, \widetilde{q}) + f''(\widetilde{p}, \widetilde{q}) = \sigma$. Finally, by (A.8), $Z(\widetilde{p}, \widetilde{q}) = Z(\widetilde{p}, \overline{q}) = e$. ∎

Note that the respective price indices of the solution pair of the necessary condition (18) and the short-term equilibrium pair found in the proof are the same, i.e., $\overline{P} = \widetilde{P}$ and $\overline{Q} = \widetilde{Q}$. On the other hand, if (A.7) and (A.8) are assumed, Ψ and Ω depend on commodity prices and equity prices only through the corresponding price indices. Formally: $\Psi(p,q) = \Psi_0(P,Q)$, where $\Psi_0 : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}_{+}$, and analogously for Ω. For the sake of simplicity, we denote Ψ_0 also by Ψ, and Ω_0 by Ω, if no misunderstanding can arise. If also it is assumed that $\omega''(p,q)$ depends on p and q only through the corresponding price indices, then, considering the proof of Proposition 3.1, the same convention applies to $(C-D)$, and we can write $(C-D)(P,Q)$. Now (11), (13) and (14) become:

$$\Psi(P,Q) = \lambda_0 P$$

$$\Omega(P,Q) = \mu_0 Q$$

$$(C - D + \omega'')(P,Q) - (C^0 - D^0 + \langle q, f''^0 \rangle) = u''$$

Recall that, by Proposition 3.1, these three equations are equivalent to any two of them. The next corollary of Lemma 5.3 results from the preceding discussion.

Corollary 5.4. *Consider the short-term model for ρ, and assume (A.5)-(A.9). Then the solutions (P,Q) of*

$$\Psi(P,Q) = \lambda_0 P \ , \ \Omega(P,Q) = \mu_0 Q \tag{24}$$

are precisely the price indices corresponding to the short-term equilibrium pairs (p,q).

Obviously (24) can be formulated as a fixed point condition.

From Proposition 5.2 and Lemma 5.3, the existence of short-term equilibrium follows immediately.

Proposition 5.5. *Consider the short-term model for ρ, and assume (A.1)-(A.9). Then there exists a short-term equilibrium pair.*

Note that all the results are valid in the particular case that the model is frictionless.

A modification of the model can be obtained eliminating the possibility of borrowing and lending (but keeping the market for shares). We leave the alterations to the reader (e.g., (A.1) and (A.4) drop). In the resulting model (11) is equivalent to (13), and short-term equilibrium exists corresponding to every equity price level Q. Also, there is no bank, with the consequences already indicated in Section 1. In this alternative model, fiat money leads to indetermination.

6. Concluding remarks

Already in the 16th century Luis de Molina became aware that clearing could perform the function of cash as a medium of payment. Speaking of fairs like that of Medina ([10], Disputatio 409), he observed: "However, I believe that usually, the custom is to pay at the end of the fairs, when there is a fixed time to pay off those debts, formalizing through signed documents [*chirographis*] the majority of the transactions that are previously carried out; since money is not so abundant that it allows buying in cash [*pecunia numerata*] the enormous amount of merchandise that is taken there to be sold if the payment is to be carried out in cash, nor that it allows carrying out so many transactions".[6] In a modern economy, banks have two different functions. On the one hand, they are financial intermediaries. On the other hand, they provide the payment system of the economy. It is convenient to separate both functions. In this paper we consider two centralized institutions: the *clearing house*, in charge of the settling of all payments in the economy, and the *bank*, representing all financial intermediation.

The monetary units are taken as units of account for all prices in the period and serve as standard of deferred payment. Altogether, money provides the standard of payment. "If a serious monetary theory comes to be written, the fact that contracts are made in terms of money will be of considerable importance" ([2], p. 357).

Apart from the transaction liquidity friction considered in the model, there is in the real world another liquidity friction that we do not contemplate because we assume that there is only one bank, to be taken as the central bank. If a complex hierarchical banking system is assumed instead, with a central bank at the top and private banks below, the credit positions in accounts at the central bank (e.g., banknotes) are considered more liquid than those at the private banks. By managing this superiority, the central bank is supposed to be able to control, to a large extent, the behaviour of private banks, providing the banking system with an adequate level of unity[7].

The question has been asked whether financial innovations, particularly those related with information and communication technology, may lead to the demise of central banking (v., e.g., [4], [5], [15]). In the frictionless case, could the bank be disposed with? In this case, the bank serves two functions in the model. Firstly, by setting the units in which its debit positions are formulated, the bank fixes the standard of payment. Secondly, all borrowing and lending is intermediated through the bank. In the first function, the bank could be replaced, at least in theory, by any solvent agent (and all agents are supposed to be solvent in the model). In the second function, it could be thought, *prima facie*, that no intermediation is necessary, and that the interplay of demand and supply of private loans would do. However, the absence of bank means chaos. Given any (reasonable) interest rate, short-term equilibrium exists for some general commodity price level P and equity price level Q. There is nothing like one equilibrium interest rate. Through the intermediation of borrowing and lending, the bank is able to set the interest rate and thus to affect the level of prices. Moreover, all this happens in the short term, where the income of the consumers is known before prices are determined. Beyond the short term, the ability to wield the power of setting the interest rate (including the threat to use it), may allow the bank to exert influence on the agents acting in the labour market, in order to steer the level of wages (v. (9) and (15)), and also on the firms deciding the amount of dividends.

[6]"Arbitror autem regulariter esse in more positum, ut in fine nundinarum soluantur, praescriptumque esse certum tempus finitis, aut fere finitis nundinis, eiusmodi solutionibus, atque ante id tempus quam plurima chirographis peragi, eo quod non sit tanta copia pecuniae, ut tam ingens mercium multitudo, quae eo deferuntur, ac venduntur, pecunia statim numerata emi possit, atque ut tam multa negotia pecunia ultro citroque in eis expediantur". The English translation above is in [7].

[7]How this control is implemented, and with what success, is an institutional issue that has changed along history. About the present situation, we agree with [13]: "In an open and efficient financial system, the central bank can determine the market rate of interest by standing in the market at its own rate, and rely on interest rate arbitrage to transmit that rate to the market".

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