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Preferences and the Dynamic Representative Consumer

Abstract This paper provides families of time-separable, twice continuously differentiable, and strictly concave utility functions of a group of consumers that are both sufficient and necessary in order to have linear aggregation in a single-commodity-type deterministic dynamic environment, in the presence of consumer wealth-, labor-productivity, and preference heterogeneity, for alternative settings where the rates of time preference can be the same or different across consumers. The employed concept of linear aggregation pertains the existence of a representative consumer with a time-separable utility function. It is proved that when the rates of time preference are choice-independent and heterogeneous across consumers, a representative consumer exists if, and only if, the momentary utility functions of all consumers are exponential. Results are also provided for, (i) common across consumers choice-independent rates of time preference, and, (ii) heterogeneous choice-dependent rates of time preference, and compared with previously identified sufficient conditions for aggregation in the existing literature.

Keywords: heterogeneity, linear aggregation, representative consumer

JEL classification: D11, D31, D91

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1. Introduction

The findings of Gorman (1953) on the necessary and sufficient conditions for the existence of social indifference curves in static frameworks, have provided us with a pure message for demand aggregation: social preferences (represented by a 'social' utility function over aggregated equilibrium demands) exhibit indifference curves under all price domains and under all domains of income- and preference heterogeneity if, and only if, consumer preferences imply that all consumer Engel curves are parallel at the same prices.[1] In that case, social preferences can be seen as the preferences of a single, not necessarily existing consumer, the 'representative consumer,' whose choice of goods, by construction, always coincides with aggregate demand. Later on, Pollak (1971), fully characterized the family of additively separable utility functions of consumers over many commodities that deliver the Gorman (1953)-type of aggregation result in a static framework.

This paper shares a similar goal to this of Pollak (1971), but in a dynamic environment. In particular, the goal of the present study is to identify the comprehensive family of time-separable, twice continuously differentiable, and strictly concave utility functions that lead to a representative consumer with a time-separable utility function in deterministic dynamic environments with one commodity type. In such an environment, at each point in time, Engel curves depend on both wealth- and labor income, and also on future savings and consumption paths, a key complication that necessitates a different treatment.

Gorman's (1953) study was motivated by the desire to build static demand systems with tractable features in order to facilitate the applied analysis of consumer choice. Yet, in modern theory of the dynamic consumer who has savings options and forms optimal consumption plans over time, the need for aggregation results is more essential. Static

[1] This type of aggregation is usually called "exact linear aggregation."

models stress wealth heterogeneity as an exogenous economic attribute. On the contrary, in dynamic analysis, the evolution of the wealth distribution is endogenous, and, above all, it depends on individual demands. The distribution of individual demands determine the distribution of savings that maps to a new distribution of future consumption possibilities for each household.

A key determinant of demands in a dynamic environment is also the consumers' time preferences, i.e. their preferences for waiting until the consumption of a quantity of a good in the future. Our insights for exact linear aggregation in static environments that are provided by the existing literature on the subject, can fail to be generalized for dynamic environments, especially when the rates of time preference vary across consumers. The present study pays special attention to examining the potential for aggregation in dynamic economies with heterogeneous rates of time preference.

In an infinite-horizon dynamic world, the commodity space for each individual is infinite-dimensional. It is understandable that the existence of Gorman (1953)-type social indifference curves in an infinite-dimensional commodity space requires a special mathematical treatment. The scope of this paper is not up to that. Instead, the goal is to show necessary and sufficient conditions on the community preference profile so that there exist social preferences consistent with the independence axiom of Koopmans (1960), namely that if two different intertemporal paths have a common outcome at a certain point in time, preferences over these two paths should always, and solely, be determined by comparing them with remaining outcomes at that particular date that differ. In other words, the focus of this paper is on characterizing community preference profiles where social preferences are time-separable and, at each point in time, social indifference curves exist.

This focus has a strong motive. The fact that the implied (and identified) utility function of the representative consumer is time-separable, allows the extension to general-equilibrium models, where the computation of future prices becomes tractable through solving a simple dynamic-programming problem of the representative consumer using the aggregate-economy consumer wealth and productivity. This can be a great facility, if it is applicable to an economic question of distribution dynamics, especially through paper-and-pencil methods.

Several authors, starting with the contribution of Chatterjee (1994), which was followed by Atkeson and Ogaki (1996), Caselli and Ventura (2000), Maliar and Maliar (2001) and (2003), provide dynamic setups and suggest specific parametric forms for individual utility functions that are sufficient for the existence of a dynamic representative consumer. All these studies provide insights about the qualitative and quantitative relationship between the specific utility functions they employ and wealth distribution dynamics. In light of the powerful and directly testable analytics of these applications, a complete characterization of the family of preferences that lead to a representative consumer seems crucial. Such a characterization can define the scope of further questions on heterogeneity that can be asked, while retaining the analytical facility provided by the linear aggregation property.

The first part of the analysis of the present paper pertains deterministic economies with a single type of commodity, and time-invariant momentary utility functions, similarly to Chatterjee (1994). Under the restrictions of no preference heterogeneity and of time-invariant momentary utility functions, Chatterjee (1994) has provided the complete family of utility functions that lead to a representative consumer, although he presents these utility functions only as sufficient, but not as necessary. In Pollak (1971), these utility functions were also proved to be the only ones to deliver the aggregation result in a static framework with time-separable preferences over multiple goods.

But the scope of the necessary and sufficient community preference profiles for a representative consumer is more restricted while considering another issue that has been rather unexplored. Since Becker's (1980) result that heterogeneity in (constant) choice-independent rates of time preference leads asymptotically to a degenerate wealth distribution, the primitives that may lead to a representative consumer have not been studied. It is proved that, with heterogeneity in choice-independent rates of time preference, a representative consumer exists if, and only if, the momentary utility functions of all consumers are exponential.

Moreover, a more general setting is studied, with time-variant momentary utility functions when individual rates of time preference have a consumption-choice-independent part wich is common-across agents, and a consumption–choice-dependent part implied by their momentary utility function. The case of Caselli and Ventura (2000) arises as a special case. Yet, it is proved that one cannot generalize much more than in Caselli and Ventura (2000): the only parameter that can be time-variant is the subsistence levels (or bliss points- if any), with additional restrictions for the case where consumer utility functions are all exponential.

The starting point of the paper, Section 2, deals with the economy where consumers have choice-independent rates of time preference and time-invariant momentary utility functions. Section 2 is split into two subsections, one studying the case where all agents have the same rate of time preference but heterogeneous momentary utility functions, and one subsection where rates of time preference can be, in addition, heterogeneous. Section 3 deals with the case where agents have time-variant momentary utility functions. Section 4 summarizes the results.

2. Choice-independent rates of time preference (time-invariant momentary utility functions)

Time is continuous and the time horizon is infinite, $t \in [0, \infty)$. Consumers are all infinitely-lived and comprise a constant set $\mathcal{I}$ of different types, with generic element i. The set of consumer types can be countable, finite, or a continuum. It can also be that all consumers are of the same type, but in all cases there is a "large" number of households, making each of them having negligible impact on the aggregate economy, or else, all consumers are price-takers. Assume a measure $\mu : \mathcal{I} \to [0,1]$, which has a density, $d\mu$, with,

$$\inf \{d\mu(i) \mid i \in \mathcal{I}\} > 0 . \tag{1}$$

So, if $\mathcal{I}$ is finite, $d\mu(i) > 0$ for all $i \in \mathcal{I}$, whereas if $\mathcal{I}$ is a compact interval, $d\mu(i)$ is continuous on $\mathcal{I}$ and bounded away from 0. Consumers of different types can differ with respect to their initial endowment of capital claims (assets) and also with respect to their labor productivity which is given by the exogenous function of time, $\theta^i : \mathbb{R}_+ \to \mathbb{R}_+$. Asset holdings for household $i \in \mathcal{I}$ at time 0 are denoted as a_0^i.

There is a single private consumable good. Consumer preferences of each $i \in \mathcal{I}$, are given by the general additively-separable utility function with the rate of time preference captured by the positively-valued function $\rho^i : \mathbb{R}_+ \to \mathbb{R}_{++}$, where

$$U^i\left(\left(c^i(t)\right)_{t\geq 0}, t\right) = \int_0^\infty e^{-\int_0^t \rho^i(\tau)d\tau} u^i\left(c^i(t)\right) dt . \tag{2}$$

Assumption 1 *For all $i \in \mathcal{I}$, $u^i : \mathbb{R}_+ \to \mathbb{R}$, is twice-continuously differentiable and such that $u_1^i(c) > 0$ and $u_{11}^i(c) < 0$ on some interval, $\mathbb{C}^i \subseteq \mathbb{R}_+$, with both $u_1^i(c) < \infty$ and $-\infty < u_{11}^i(c)$ for all $c \in \mathbb{C}^i \subseteq \mathbb{R}_+$,with $\underline{c}^i \equiv \inf(\mathbb{C}^i) < \sup(\mathbb{C}^i) \equiv \bar{c}^i$.*

Assumption 1 simply secures that, for all $i \in \mathcal{I}$, there is a choice domain, $\mathbb{C}^i \subseteq \mathbb{R}_+$, which is an interval, and where standard desirable properties of momentary utility functions are present.

Another restriction on preferences pertains the rates of time preference.

Assumption 2 $\int_0^\infty e^{-\int_0^t \rho^i(\tau)d\tau} dt < \infty$ *for all $i \in \mathcal{I}$.*

Apart from the technical facility provided by Assumption 2 in developing proofs of the theorems below, one of its implications is that any agent can choose a consumption path such that the consumption level is asymptotically non-decreasing.

All consumers are endowed with the same amount of time at each moment, that is supplied inelastically for labor. The momentary time endowment is normalized to one.

Regardless of the production process underlying the economy, for any given price vector $(r(t), w(t))_{t\geq 0} >> 0$, with $r(t)$ being the interest rate and $w(t)$ the labor wage at each moment, the budget constraint faced by household $i \in \mathcal{I}$ is,

$$\dot{a}^i(t) = r(t) a^i(t) + \theta^i(t) w(t) - c^i(t) \ , \tag{3}$$

for all $t \geq 0$, and the transversality condition is,

$$\lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} a^i(t) = 0 \ . \tag{4}$$

Last, to define the domains of wealth- and productivity heterogeneity at any given price vector, for which the existence of a representative consumer is conceptually relevant. That is the domain that guarantees interiority of solutions to each individual optimization problem. The following assumption states this formally.

Assumption 3 *Given a community preference profile captured by the collection of functions $(\rho^i, u^i)_{i\in\mathcal{I}}$, the domain of initial distribution of assets $(a_0^i)_{i\in\mathcal{I}}$, of the collection of labor-productivity functions $(\theta^i)_{i\in\mathcal{I}}$, and of prices $(r(t), w(t))_{t\geq 0}$, is restricted so that the problems of all consumers $i \in \mathcal{I}$ are well-defined and the solution to each individual problem is interior for all $t \geq 0$.*

Given Assumption 3, maximizing (2) subject to the constraints (3) and (4) for any given a_0^i is an optimal-control problem with necessary optimal conditions given by,

$$\dot{c}^i(t) = -\frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} \left[r(t) - \rho^i(t)\right] \ , \tag{5}$$

together with (3) and (4), that lead to decision rules of the form,

$$c^i(t) = C^i\left(a^i(t) \,\middle|\, (r(\tau), w(\tau), \theta^i(\tau))_{\tau \geq t}\right), \tag{6}$$

i.e., consumption rules at each moment are memoryless, depending only on current personal assets and current and future prices. Assumptions 1 and 4 have a particular connection, that is revealed from equation (5). The term $-\frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))}$ must be always well-defined in order to have interiority. Thus, in order to meet Assumption 3 (interior solutions), it is necessary that $c^i(t) \in \mathbb{C}^i$, for all $t \geq 0$, and all $i \in \mathcal{I}$.

The solution to the partial-equilibrium problem of the households yields all individual consumption demands and asset supply functions that can be combined with a large class of settings for the production side guaranteeing that market-clearing prices are such that both general-equilibrium decision rules and prices can be either memoryless or time-invariant, depending only on the distribution of household assets at each moment. Such supply-side settings are not discussed. The focus is on the basic partial-equilibrium household problem, and, in particular, on the conditions on the utility function that lead to a representative consumer with time-separable preferences.

Definition 1 *Given a community preference profile captured by the collection of functions $(\rho^i, u^i)_{i\in\mathcal{I}}$, complying with Assumptions 1 and 2, a representative consumer (denoted by "RC") is a (fictitious) consumer who has time-separable preferences, $\int_0^\infty v^{RC}(c(t), t)\, dt$, with $v_1^{RC}(c,t), v_{11}^{RC}(c,t)$ and $v_{12}^{RC}(c,t)$ existing, and with $v_1^{RC}(c,t) < \infty$ and $-\infty < v_{11}^{RC}(c,t), v_{12}^{RC}(c,t)$ for all consumption levels, $c \in \mathbb{C}^{RC} \equiv \left\{c \in \mathbb{R}_+ \,\middle|\, c = \int_{\mathcal{I}} c^i d\mu(i)\ ,\ c^i \in \mathbb{C}^i, i \in \mathcal{I}\right\}$, for all $t \geq 0$, and who possesses the economy-wide aggregate wealth and productivity at all times, and whose demand functions coincide with the aggregate demand functions of the* *economy at all times, namely,*

$$c^{RC}(t) = C^{RC}\left(\int_{\mathcal{I}} a^i(t)\, d\mu(i) \,\middle|\, \left(r(\tau), w(\tau), \int_{\mathcal{I}} \theta^i(\tau)\, d\mu(i)\right)_{\tau \geq t}\right) =$$

$$= \int_{\mathcal{I}} C^i\left(a_i(t) \,\middle|\, \left(r(\tau), w(\tau), \theta^i(\tau)\right)_{\tau \geq t}\right) d\mu(i) \ , \qquad (7)$$

for all $t \geq 0$, *for the complete domain of prices* $(r(t), w(t))_{t \geq 0}$, *initial distributions of assets,* $(a_0^i)_{i \in \mathcal{I}}$, *and functions* $\left(\theta^i : \mathbb{R}_+ \to \mathbb{R}\right)_{i \in \mathcal{I}}$ *that comply with Assumption 3.*

It is emphasized, again, that this is a strong concept of a representative consumer that aims at providing the facility of solving only the representative-cosumer's problem using standard optimal-control techniques, in order to derive aggregate demands at all times.

2.1 Common choice-independent rates of time preference among all consumers at all times

The case where $\rho^i : \mathbb{R}_+ \to \mathbb{R}_{++}$ is such that $\rho^i = \rho$ for all $i \in \mathcal{I}$ is a standard convention in most literature with dynamic consumers. The first theorem pertains this special case, providing a complete characterization of the collection of momentary utility functions, $(u^i)_{i \in \mathcal{I}}$, in order that a representative consumer exists. This is done under an additional assumption in this setting.

Assumption 4 $\bigcap_{i \in \mathcal{I}} \mathbb{C}^i$ *is non-empty and not a singleton.*

Assumption 4 places a constraint on the scope of preference heterogeneity. It says that nobody's bliss point (if any), should be lower than (or equal to) anyone else's subsistence level of consumption (if any), hence $\bigcap_{i \in \mathcal{I}} \mathbb{C}^i$ is an interval.

Theorem 1 *When $\rho^i(t) = \rho(t)$ for all $i \in \mathcal{I}$ and all $t \geq 0$, under Assumptions 1 through 4, a representative consumer exists iff*

$$u^i(c) = \begin{cases} \frac{(\alpha c + \beta_i)^{1-\frac{1}{\alpha}} - 1}{\alpha\left(1 - \frac{1}{\alpha}\right)} & \text{with } \alpha > 0 \text{ and } \beta_i \in \mathbb{R} \text{ or } \alpha < 0 \text{ and } \beta_i \in \mathbb{R}_{++} \\ \text{or} & \\ -e^{-\frac{1}{\beta_i}c} & \text{with } \beta_i > 0 \end{cases}, \tag{8}$$

for all $i \in \mathcal{I}$. The representative consumer has the common, across households, rate of time preference, $\rho(t)$, at all times, and momentary utility function given by,

$$u^{RC}(c) = \begin{cases} \frac{(\alpha c + \beta_{RC})^{1-\frac{1}{\alpha}} - 1}{\alpha\left(1 - \frac{1}{\alpha}\right)} & \text{for } \alpha \neq 0 \\ -e^{-\frac{1}{\beta_{RC}}c} & \text{else} \end{cases}, \tag{9}$$

with

$$\beta_{RC} = \int_{\mathcal{I}} \beta_i d\mu(i) \ .$$

Proof of Theorem 1

Part 1: Necessity

Fix any function $\rho : \mathbb{R}_+ \to \mathbb{R}_{++}$, and any collection $(u^i)_{i\in\mathcal{I}}$, with properties complying with Assumptions 1,2 and 4. Assume that a representative consumer exists with some momentary utility function $v^{RC} : \mathbb{C}^{RC} \times \mathbb{R}_+ \to \mathbb{R}$, of the form $v^{RC}(c(t), t)$, at each point in time. Under Assumption 3, from Definition 1 and (5) it must be that,

$$\frac{v_1^{RC}\left(\int_{\mathcal{I}} c^i(t) d\mu(i), t\right)}{v_{11}^{RC}\left(\int_{\mathcal{I}} c^i(t) d\mu(i), t\right)} \left[r(t) + \frac{v_{12}^{RC}\left(\int_{\mathcal{I}} c^i(t) d\mu(i), t\right)}{v_1^{RC}\left(\int_{\mathcal{I}} c^i(t) d\mu(i), t\right)}\right] = \int_{\mathcal{I}} \mu(i) \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} di \left[r(t) - \rho(t)\right] \ , \tag{10}$$

where the term

$$-\frac{v_{12}^{RC}\left(\int_{\mathcal{I}} c^{i}(t)\, d\mu(i), t\right)}{v_{1}^{RC}\left(\int_{\mathcal{I}} c^{i}(t)\, d\mu(i), t\right)}$$

is the temporal rate of time preference of the representative consumer.

(Necessity) Step 1: preliminary characterization of the function $\int_0^\infty v^{RC}(c(t), t)\, dt$.

According to Definition 1, the existence (and the implied preference primitives) of the representative consumer should be independent from any price regime. The case where $r(t) = \rho(t)$ for all $t \geq 0$, should always be included in the price domain. To see this, fix any moment in time, $t \in \mathbb{R}_+$, pick any consumer $i \in \mathcal{I}$, and multiply her budget constraint, (3), by the integrating factor $e^{-\int_t^\tau r(s)ds}$, integrate over all $\tau \in [t, \infty)$, and apply the transversality condition, to get,

$$\int_t^\infty e^{-\int_t^\tau r(s)ds} c^{i}(t)\, d\tau = a^{i}(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds} \theta^{i}(\tau)\, w(\tau)\, d\tau \ . \tag{11}$$

For the case $r(t) = \rho(t)$ for all $t \geq 0$, under Assumption 3, (5) implies that $\dot{c}^{i}(t) = 0$ for all $t \in \mathbb{R}_+$, and all $i \in \mathcal{I}$, so, (11) implies that

$$c^{i}(t) = \hat{c}^{i} = \frac{a^{i}(t) + \int_t^\infty e^{-\int_t^\tau \rho(s)ds} \theta^{i}(\tau)\, w(\tau)\, d\tau}{\int_t^\infty e^{-\int_t^\tau \rho(s)ds} d\tau} \text{ , for all } t \geq 0. \tag{12}$$

For the given $(u^{i})_{i \in \mathcal{I}}$, (12) implies that there are always $\left(a_0^{i}, \theta^{i}\right)_{i \in \mathcal{I}}$ and $(w(t))_{t \geq 0}$ securing that $\hat{c}^{i} \in \mathbb{C}^{i}$ for all $i \in \mathcal{I}$, and for all $t \geq 0$, so, Assumption 3, indeed, holds. Therefore, the case $r(t) = \rho(t)$ for all $t \geq 0$, is always part of the domain complying with Assumption 3, for any $(u^{i})_{i \in \mathcal{I}}$ that satisfies Assumptions 1 and 2.

Thus, if we set $r(t) = \rho(t)$ for all $t \geq 0$, pick an appropriate $\left(a_0^{i}, \theta^{i}\right)_{i \in \mathcal{I}}$ and $(w(t))_{t \geq 0}$ securing that $\hat{c}^{i} > \underline{c}^{i}$ for all $i \in \mathcal{I}$, and for all $t \geq 0$, and also set,

$$c \equiv \int_{\mathcal{I}} \mu(i)\, \hat{c}^{i} di \ ,$$

equations (10) and (12) imply that the necessary optimality conditions of the representative consumer are,

$$-\frac{v_{12}^{RC}(c,t)}{v_1^{RC}(c,t)} = \rho(t) \ ,$$

so, standard Riemann integration with respect to t over the time interval $[0,t]$ implies that,

$$v_1^{RC}(c,t) = e^{-\int_0^t \rho^i(\tau)d\tau} v_1^{RC}(c,0) \ ,$$

or,

$$v^{RC}(c,t) = e^{-\int_0^t \rho(\tau)d\tau} v^{RC}(c,0) \ ,$$

ignoring the constant, since this is a utility function. Setting,

$$u^{RC}(c) \equiv v^{RC}(c,0) \ ,$$

we conclude that the objective of the representative consumer must be of the form,

$$U^{RC}\left((c(t))_{t\geq 0}, t\right) = \int_0^\infty e^{-\int_0^t \rho(\tau)d\tau} u^{RC}(c(t))\, dt \ . \tag{13}$$

For notational ease, let, $f^{RC} : \mathbb{C}^{RC} \to \mathbb{R}_{++}$ and $\left(f^i : \mathbb{C}^i \to \mathbb{R}_{++}\right)_{i\in\mathcal{I}}$, with

$$f^{RC}(\cdot) = -\frac{u_1^{RC}(\cdot)}{u_{11}^{RC}(\cdot)} \qquad \text{and} \qquad f^i(\cdot) = -\frac{u_1^i(\cdot)}{u_{11}^i(\cdot)} \quad \text{for all } i \in \mathcal{I}.$$

Combining (13) with (10), it follows that,

$$f^{RC}\left(\int_{\mathcal{I}} c^i(t)\, d\mu(i)\right) = \int_{\mathcal{I}} f^i\left(c^i(t)\right) d\mu(i) \ , \tag{14}$$

for all $\left(c^i(t) \in \mathbb{C}^i\right)_{i\in\mathcal{I}}$, that are consumer-equilibrium choices, and $t \geq 0$.

(Necessity) Step 2: characterization of $f^{RC}:\mathbb{R}_+\rightarrow\mathbb{R}_{++}$ *and* $\left(f^i:\mathbb{R}_+\rightarrow\mathbb{R}_{++}\right)_{i\in\mathcal{I}}$. In this step it is shown that,

$$(14)\Leftrightarrow\left\{\begin{array}{c} f^i(c)=\alpha c+\beta_i\ ,\quad \text{and,} \\ f^{RC}(c)=\alpha c+\int_{\mathcal{I}}\beta_i d\mu(i)\ , \\ \text{for some } \alpha\in\mathbb{R} \text{ and some } \beta_i\in\mathbb{R}, \text{ for all } i\in\mathcal{I} \end{array}\right\}. \tag{15}$$

The sufficiency part of (15) is both straighforward and not needed for the proof of the theorem, so, the focus is on proving the necessity part of (15). So, let (14) hold, being the only information available about $f^{RC}:\mathbb{R}_+\rightarrow\mathbb{R}_{++}$ and the collection $\left(f^i:\mathbb{R}_+\rightarrow\mathbb{R}_{++}\right)_{i\in\mathcal{I}}$. Suppose, again, that $r(t)=\rho(t)$ for all $t\geq 0$, and, given (12), find a common distribution of $\left(a_0^i,\theta^i\right)_{i\in\mathcal{I}}$ and $(w(t))_{t\geq 0}$, where $a_0^i=a_0$ and $\theta^i=\theta$, so that $c^i(t)=\tilde{c}$ for all $i\in\mathcal{I}$, and all $t\geq 0$, also with $\tilde{c}\in\bigcap_{i\in\mathcal{I}}\mathbb{C}^i$.

Let,

$$\Phi^{RC}(c)\equiv f^{RC}(c)-f^{RC}(\tilde{c})\ , \tag{16}$$

and,

$$\Phi^i(c)\equiv f^i(c)-f^i(\tilde{c})\ ,\ \text{ for all } i\in\mathcal{I}\ . \tag{17}$$

For this distribution, (14) implies that,

$$f^{RC}(\tilde{c})=\int_{\mathcal{I}}\mu(i)\,f^i(\tilde{c})\,di\ . \tag{18}$$

Moreover, given (1), set $\underline{\mu}$ such that,

$$0<\underline{\mu}\leq\inf\left\{d\mu(i)\ \mid i\in\mathcal{I}\right\}\ . \tag{19}$$

Now pick any arbitrary consumer type $i\in\mathcal{I}$, keep prices as before and modify the previous distribution by just adding to $\underline{\mu}$ of this consumer type different wealth or productivity that yields $c^i(t)=(\tilde{c}+\Delta c)\in\bigcap_{i\in\mathcal{I}}\mathbb{C}^i$, for all $t\geq 0$. Since prices are the same, $c^j(t)=\tilde{c}$, for all $j \in \mathcal{I} \backslash \{i\}$ and for some consumers of type i with measure $\mu(i) - \underline{\mu}$, and for all $t \geq 0$. Combining (14), (18), (16) and (17), it is,

$$\Phi^{RC}\left(\underline{\mu}\Delta c + \tilde{c}\right) = \underline{\mu}\Phi^{i}\left(\Delta c + \tilde{c}\right) \ . \tag{20}$$

Since the choice of $i \in \mathcal{I}$, Δc, and $\tilde{c} \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i$, were arbitrary, and since we can construct the same distribution of consumption choices for all $i \in \mathcal{I}$, (20) holds for all $i \in \mathcal{I}$, so,

$$\Phi^{i}\left(c\right) = \Phi\ \left(c\right) \quad \text{for all } c \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i \text{ and for all } i \in \mathcal{I}. \tag{21}$$

Given (12), we are able to construct any interior optimal path with distribution of consumptions with $c^i(t) = c \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i$ for all $i \in \mathcal{I}$, and all $t \geq 0$. Therefore, (14), (18) and (21) imply that,

$$\Phi^{RC}\left(c\right) = \Phi^{i}\left(c\right) = \Phi\left(c\right) \quad \text{for all } c \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i \text{ and for all } i \in \mathcal{I}\ , \tag{22}$$

and most importantly,

$$\Phi\left(\int_{\mathcal{I}} c^i\left(t\right) d\mu\left(i\right)\right) = \int_{\mathcal{I}} \Phi\left(c^i\left(t\right)\right) d\mu\left(i\right)\ , \quad \text{for all } \left(c^i\left(t\right) \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i\right)_{i \in \mathcal{I}}, \text{ and } t \geq 0\ , \tag{23}$$

i.e. for the whole domain of wealth/labor-productivity heterogeneity and prices where consumer choices fall in $\underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i$ and are interior, as imposed by Assumption 4. Equation (23) gives us the chance to further characterize Φ. In particular,

$$(23) \Leftrightarrow \Phi \text{ is affine on } \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i. \tag{24}$$

The sufficiency part of (24) is straightforward, so for the necessity part of (24) let's set,

$$z^i \equiv c^i - \tilde{c}\ , \tag{25}$$

with $\tilde{c}$ defined as above for an arbitrary $\tilde{c} \in \underset{i \in \mathcal{I}}{\cap} \mathbb{C}^i$, in the case where $r(t) = \rho(t)$ for all $t \geq 0$. So, fix $\tilde{c}$ and set,

$$\Psi\left(z\right) \equiv \Phi\left(z\right) - \Phi\left(0\right)\ , \tag{26}$$

since we know that for the transformed variable, z, the choice of 0 falls in the class of interior solutions to a distribution in the domain of $(u^i)_{i\in\mathcal{I}}$, namely the case where all consumers choose $\tilde{c} \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^i$ at all times. It is now shown that Ψ is a linear functional. For any partition of consumers, irrespective of their consumer types, say, $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I}$, with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, and $\int_{\mathcal{I}_1} d\mu(i) = \mu$, retaining $r(t) = \rho(t)$ for all $t \geq 0$, provide the same a_0 and a labor-productivity function θ to all $i \in \mathcal{I}_1$, so that consumption is equal to $(\Delta c + \tilde{c}) \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^i$ for all $i \in \mathcal{I}_1$ at all times, provide to the remaining consumers $\tilde{a}_0$ and a labor-productivity $\tilde{\theta}$, so that their consumption is equal to $\tilde{c} \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^i$ for all $i \in \mathcal{I}_2$ at all times. Then, $z^i = \Delta c$ for all $i \in \mathcal{I}_1$, and $z^i = 0$ for all $i \in \mathcal{I}_2$, so,

$$\Phi(\mu\Delta c) = \Phi(\mu\Delta c + (1-\mu)\,0) \ ,$$

and (23) and (26) imply that,

$$\Phi(\mu\Delta c) = \mu\Phi(\Delta c) + (1-\mu)\,\Phi(0) \ ,$$

or,

$$\Psi(\mu\Delta c) = \mu\Psi(\Delta c) \ . \tag{27}$$

It is important to notice that the choices of Δc and μ were arbitrary. So, we can take any $\mu_1, \mu_2 \in (0,1)$ with $(\mu_1\Delta c + \tilde{c}), (\mu_1\Delta c + \tilde{c}) \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^i$ and $\frac{\mu_2}{\mu_1} = \xi \in \mathbb{R}_+$. Repeating the same steps, (27) yields, $\Psi(\mu_1\Delta c) = \mu_1\Psi(\Delta c)$ and $\Psi(\xi\mu_1\Delta c) = \xi\mu_1\Psi(\Delta c)$, or,

$$\Psi(\xi\mu_1\Delta c) = \xi\Psi(\mu_1\Delta c)\,, \quad \text{for all } \xi \in \mathbb{R}_+ \ . \tag{28}$$

Since Ψ is a univariate function, (28) is sufficient to prove that it is linear. So, let,

$$\Psi(z) = \alpha z \ , \quad \alpha \in \mathbb{R},$$

and, due to the linearity of Ψ, the transformation (25) can be ignored, having (26) and (22) implying that, $\Phi(c) = \alpha c + \Phi(0)$. But since (16) and (17) imply that $\Phi(\tilde{c}) = 0$, $\Phi(0) = -\alpha\tilde{c}$, so,

$$\Phi^{RC}(c) = \Phi^{i}(c) = \Phi(c) = \alpha c - \alpha\tilde{c} \ , \ \ \alpha \in \mathbb{R}, \text{ for all } c \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^{i} \text{ and for all } i \in \mathcal{I} \ . \tag{29}$$

With (29) at hand, it is easy to show that,

$$\Phi^{i}(c) = \Phi(c) = \alpha c - \alpha\tilde{c} \ , \ \ \alpha \in \mathbb{R}, \text{ for all } c \in \mathbb{C}^{i} \text{ and for all } i \in \mathcal{I} \ . \tag{30}$$

To prove (30), consider the case where an arbitrary $c^{j} \in \mathbb{C}^{j}$ is such that $c^{j} \leq \inf\left(\bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}\right)$ or $c^{j} \geq \sup\left(\bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}\right)$ for some $j \in \mathcal{I}$, whenever any of the two is possible (i.e. whenever $\inf\left(\bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}\right) > 0$, or $\sup\left(\bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}\right) < \infty$). There always exist some $\mu \in (0,1)$, with $\mu \leq \mu(j)$, such that $(\mu c^{j} + (1-\mu)\tilde{c}) \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}$. So, retaining $r(t) = \rho(t)$ for all $t \geq 0$, provide a level a_0 and a labor-productivity function θ to a mass of μ of type $j \in \mathcal{I}$, so that consumption is equal to c^{j} at all times, and also provide to the remaining consumers $\tilde{a}_0$ and a labor-productivity $\tilde{\theta}$, so that their consumption is equal to $\tilde{c} \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}$ at all times. Combining (14), (16), (17) and (18), it is,

$$\mu\Phi^{j}\left(c^{j}\right) = \Phi^{RC}\left(\mu c^{j} + (1-\mu)\tilde{c}\right) \ ,$$

but since $(\mu c^{j} + (1-\mu)\tilde{c}) \in \bigcap_{i\in\mathcal{I}} \mathbb{C}^{i}$, (29) implies that $\Phi^{RC}(\mu c^{j} + (1-\mu)\tilde{c}) = \alpha(\mu c^{j} + (1-\mu)\tilde{c}) - \alpha\tilde{c}$, or

$$\Phi^{j}\left(c^{j}\right) = \alpha c^{j} - \alpha\tilde{c} \ .$$

since the choice of $j \in \mathcal{I}$ and $c^{j} \in \mathbb{C}^{j}$ were arbitrary, (30) is proved.

So, combining (17) with (30) it is,

$$f^{i}(c) = \alpha c - \alpha\tilde{c} + f^{i}(\tilde{c}) \quad \text{for all } c \in \mathbb{C}^{i} \text{ and all } i \in \mathcal{I} \ . \tag{31}$$

Now that all f^{i}'s are completely characterized over their domains, $\mathbb{C}^{i}$, we can consider the case of $c = 0$, irrespective from whether $0 \in \mathbb{C}^{i}$ or not, in order to set the intercepts of all

f^i's. Apparently, (31) implies that,

$$f^i(\tilde{c}) = \alpha\tilde{c} + f^i(0) \ , \tag{32}$$

and setting $f^i(0) = \beta_i$ for some $\beta_i \in \mathbb{R}$, for all $i \in \mathcal{I}$, a final combination of (31) with (32), while, consistently with (14), setting $\beta_{RC} = \int_{\mathcal{I}} \beta_i d\mu(i)$, complete the proof of (15).

(Necessity) Step 3: characterization of $(u^i : \mathbb{R}_+ \to \mathbb{R}_{++})_{i\in\mathcal{I}}$ *and* $u^{RC} : \mathbb{R}_+ \to \mathbb{R}_{++}$.

In light of (15), the derivation of the consumer primitives comes through simple Riemann integration. There are two general cases, these of $\alpha \neq 0$ and $\alpha = 0$.[2]

For the case that $\alpha \neq 0$, (15) implies that,

$$\frac{u_1^i(c)}{u_{11}^i(c)} = -\frac{1}{\alpha c + \beta_i} \ ,$$

and the indefinite Riemann integral of this expression with respect to c yields,

$$\ln\left[u_1^i(c)\right] = -\frac{1}{\alpha}\ln(\alpha c + \beta_i) + \kappa_i \ ,$$

where κ_i is some constant in $\mathbb{R}$, that can be consumer-specific, and integrating once more, it is,

$$u^i(c) = e^{\kappa_i}\frac{(\alpha c + \beta_i)^{1-\frac{1}{\alpha}}}{\alpha\left(1 - \frac{1}{\alpha}\right)} + \kappa \ ,$$

where κ is, again some constant. Setting $e^{\kappa_i} = 1$, without loss of generality, and κ accordingly, the result of (8) is obtained. The special case where $\alpha = 1$, is known to yield the result that $u^i(c) = \ln(\alpha c + \beta_i) + \kappa$, through computing the limit of the above expression for $\alpha \to 1$ using L'Hôpital's rule. The preferences of the representative consumer are derived in the same way.

[2] The case where $\alpha = 1$ is also of special interest, but the particular functional form of $(u^i)_{i\in\mathcal{I}}$ and u^{RC} that result in this case, can be derived from the more general functional forms that apply to $\alpha \neq 0$.

For the case that $\alpha = 0$,

$$\frac{u_1^i(c)}{u_{11}^i(c)} = -\frac{1}{\beta_i} \ ,$$

and in order for $u_1^i > 0$ and $u_{11}^i < 0$ to hold, it must be that $\beta_i > 0$. So,

$$\ln\left[u_1^i(c)\right] = -\frac{1}{\beta_i}c + \kappa_i \ ,$$

and,

$$u^i(c) = -\frac{e^{\kappa_i}}{\beta_i}e^{-\frac{1}{\beta_i}c} + \kappa \ ,$$

so, setting $\frac{e^{\kappa_i}}{\beta_i} = 1$ and $\kappa = 0$ yields the corresponding function in (8). With the same reasoning for the representative consumer, the proof of the necessity part is complete.

Part 2: Sufficiency

The particular functional forms given by (8) enable a thorough analytical characterization of the demand functions of all consumers at all times. Again, two cases must be examined separately, this of $\alpha \neq 0$, and also the case where $\alpha = 0$.

Under the assumption that $\alpha \neq 0$, (5), implies,

$$\dot{c}^i(t) = \left[\alpha c^i(t) + \beta_i\right]\left[r(t) - \rho(t)\right] \ ,$$

so, multiplying this expression by the integrating factor $e^{-\alpha\int_t^\tau [r(s)-\rho(s)]ds}$ and integrating over the interval $[t,\tau]$ for any $\tau \in [t,\infty)$, yields,

$$c^i(\tau) = c^i(t)\,e^{\alpha\int_t^\tau [r(s)-\rho(s)]ds} + \beta_i e^{\alpha\int_t^\tau [r(s)-\rho(s)]ds}\int_t^\tau e^{-\alpha\int_t^\tau [r(s)-\rho(s)]ds}\left[r(s) - \rho(s)\right]ds \ .$$

Multiplying this last expression by $e^{-\int_t^\tau r(s)ds}$, integrating over all $\tau \in [t,\infty)$, and combining the result with (11), gives,

$$c^i(t) = \frac{a^i(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds}\theta^i(\tau)\,w(\tau)\,d\tau}{\int_t^\infty e^{\int_t^\tau [(\alpha-1)r(s)-\alpha\rho(s)]ds}d\tau} -$$

$$-\frac{\beta_i \int_t^\infty e^{\int_t^\tau [(\alpha-1)r(s)-\alpha\rho(s)]ds} \int_t^\tau e^{-\alpha\int_t^\tau [r(s)-\rho(s)]ds} \left[r(s)-\rho(s)\right] ds d\tau}{\int_t^\infty e^{\int_t^\tau [(\alpha-1)r(s)-\alpha\rho(s)]ds} d\tau}, \quad (33)$$

which can be linearly aggregated across all a^i's, θ^i's and β_i's, proving that a representative consumer exists, as long as Assumption 1 holds, which keeps all individual demands taking the form of (33).

For the case where $\alpha = 0$, when all individual utilities fall in the class of $u^i(c) = -e^{-\frac{1}{\beta_i}c}$, (33) implies that,

$$c^i(t) = \frac{a^i(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds} \theta^i(\tau) w(\tau) d\tau - \beta_i \int_t^\infty e^{-\int_t^\tau r(s)ds} \int_t^\tau \left[r(s) - \rho(s)\right] ds d\tau}{\int_t^\infty e^{-\int_t^\tau r(s)ds} d\tau}, \quad (34)$$

which can also be linearly aggregated across all a^i's, θ^i's and β_i's, completing the proof of the theorem.□

Having read Pollak (1971), the result stated by Theorem 1 is not that surprising. Pollak (1971) shows that if utility functions over a finite number of goods are additively-separable, the only functional forms of utility per good that yield linear Engel curves are the same as these stated by Theorem 1. Yet, the difference in the dynamic setup studied here is the existence of an asset that stores values, and the feature that individuals have a time preference. These two features can lead to complex trajectories of assets over time that could influence the potential to derive linear Engel curves at all points in time. Most importantly, unlike the main argument developed by Pollak (1971) in the finite-good static setting, it is difficult to characterize indirect utility functions in dynamic setups out of the requirement that Engel curves are linear.[3]

The work by Chatterjee (1994), Atkeson and Ogaki (1996), and Caselli and Ventura (2000) has shown that Pollak's (1971) utility functions do lead to a representative consumer in such dynamic settings. But unlike the spirit of the work by Pollak (1971) who provides

[3] Pollak's (1971) main theorem relies upon a finding by Gorman (1961) for static models. Gorman (1961) proves that Engel curves are linear if, and only if, indirect utility functions possess a certain general form. Such an extension to an optimal-control setup is, at least, demanding.

the full set of utility functions that guarantee the existence of a representative consumer in a static setting, it has been an open question whether there is anything more to say about the momentary utility functions in a dynamic setting with an asset that stores value over time. In the spirit of the setup examined by Chatterjee (1994), for the case of time-invariant momentary utility functions (unlike the setup studied by Caselli and Ventura (2000) who examine a case of time-variant momentary utility functions), in a single-commodity deterministic environment, (unlike the case studied by Atkeson and Ogaki (1996), Maliar and Maliar (2001) and (2003), who look at multi-type commodity setups in stochastic environments), an answer is given by Theorem 1 of the present study: there is nothing more to say than in Pollak (1971) when all individuals have the same rate of time preference and nobody's bliss point (if any) is never lower than or equal to anyone else's subsistence level (if any). But is there anything more (or less) to say about time-invariant utility functions when individual rates of time preference differ? This question is examined in the section that follows.

2.2 Heterogeneous choice-independent rates of time preference

Now the general economic environment with heterogeneous functions $\rho^i : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ is considered. For developing the aggregation theorem of this section, Assumption 4 is not necessary and it is dropped. Yet, another assumption should be added.

Assumption 5 *Given a collection of functions* $(\rho^i)_{i\in\mathcal{I}}$, *that comply with Assumption 2, the collection of functions* $(u^i)_{i\in\mathcal{I}}$ *possesses structure such that for all interest rates,* $(r(t))_{t\geq 0}$, *satisfying,*

$$\int_0^\infty e^{-\int_0^t r(\tau)d\tau} dt < \infty \ , \tag{35}$$

the domain of initial distribution of assets $(a_0^i)_{i\in\mathcal{I}}$*, of the collection of labor-productivity functions* $\left(\theta^i\right)_{i\in\mathcal{I}}$*, and of wages* $(w(t))_{t\geq0}$*, so that the problems of all consumers* $i\in\mathcal{I}$ *are well-defined and the solution to each individual problem is interior for all* $t\geq0$*, is non-empty.*

Assumption 5 puts some restriction on the class of preferences, $(\rho^i, u^i)_{i\in\mathcal{I}}$, but not one that makes the issue under study uninteresting. For having a representative consumer, interiority of individual optimal paths is an essential feature of equilibrium. In the case of heterogeneous rates of time preference, the necessary optimality condition given by (5) can drive the consumption of consumers towards different directions even when consumers have the same momentary utility function. For such a setting, assumptions such as Inada conditions are quite usual in order to guarantee interiority. Assumption 5 does not restrict attention to utility functions that satisfy usual Inada conditions. It simply says that for a given $(\rho^i)_{i\in\mathcal{I}}$, for any path of interest rates that satisfies (35), $(u^i)_{i\in\mathcal{I}}$, should allow some of the remaining domain of prices and wealth/labor-productivity distributions to generate interior solutions.

Theorem 2 *Under Assumptions 1,2,3 and 5, a representative consumer exists iff*

$$u^i(c) = -e^{-\frac{1}{\beta_i}c}\ , \tag{36}$$

for $\beta_i > 0$ *for all* $i\in\mathcal{I}$*. The representative consumer's preferences are given by,*

$$U^{RC}\left((c(t))_{t\geq0}, t\right) = -\int_0^{\infty} e^{-\int_0^t \rho^{RC}(\tau)d\tau} e^{-\frac{1}{\beta_{RC}}c}dt\ , \tag{37}$$

with

$$\rho^{RC}(t) = \frac{\int_{\mathcal{I}} \rho^i(t)\,\beta_i d\mu(i)}{\int_{\mathcal{I}} \beta_i d\mu(i)}\ , \text{ for all } t \geq 0,$$

and

$$\beta_{RC} = \int_{\mathcal{I}} \beta_i d\mu(i)\ .$$

Proof of Theorem 2

Part 1: Necessity

Fix any collection $(\rho^i, u^i)_{i\in\mathcal{I}}$, with properties complying with Assumptions 1, 2, and 5. Assume that a representative consumer exists with some momentary utility function $v^{RC} : \mathbb{C}^{RC}\times\mathbb{R}_+ \to \mathbb{R}$, of the form $v^{RC}(c(t), t)$, at each point in time. Under Assumption 3, from Definition 1 and (5) it must be that,

$$\frac{v_1^{RC}\left(\int_{\mathcal{I}} c^i(t)\,d\mu(i), t\right)}{v_{11}^{RC}\left(\int_{\mathcal{I}} c^i(t)\,d\mu(i), t\right)}\left[r(t) + \frac{v_{12}^{RC}\left(\int_{\mathcal{I}} c^i(t)\,d\mu(i), t\right)}{v_1^{RC}\left(\int_{\mathcal{I}} c^i(t)\,d\mu(i), t\right)}\right] =$$

$$= r(t)\int_{\mathcal{I}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} d\mu(i) - \int_{\mathcal{I}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))}\rho^i(t)\,d\mu(i)\ . \tag{38}$$

(Necessity) Step 1: preliminary characterization of the function $\int_0^\infty v^{RC}(c(t), t)\,dt$.

Pick any arbitrary $j \in \mathcal{I}$ and assume that $r(t) = \rho^j(t)$ for all $t \geq 0$. Assumption 5 implies that there exists some initial-wealth and labor-productivity-function distributions together with some wage vector $(w(t))_{t\geq 0}$ guaranteeing an interior optimal path of consumption choices for $t \geq 0$. In that case, (5) implies that all consumer types j will choose a consumption path with constant consumption over time, given by,

$$c^j(t) = \hat{c}^j = \frac{a^j(t) + \int_t^\infty e^{-\int_t^\tau \rho^j(s)ds}\theta^j(\tau)\,w(\tau)\,d\tau}{\int_t^\infty e^{-\int_t^\tau \rho^j(s)ds}d\tau}\ , \text{ for all } t \geq 0. \tag{39}$$

With these settings, taking into account that $r(t) = \rho^j(t)$ for all $t \geq 0$, and also (39), rearranging (38) it is,

$$\frac{v_{11}^{RC}\left(\int_{\mathcal{I}\setminus\{j\}} c^i(t)\, d\mu(i) + \mu(j)\,\hat{c}^j, t\right)}{v_1^{RC}\left(\int_{\mathcal{I}\setminus\{j\}} c^i(t)\, d\mu(i) + \mu(j)\,\hat{c}^j, t\right)} \times$$

$$\times\left[\rho^j(t)\int_{\mathcal{I}\setminus\{j\}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} d\mu(i) - \int_{\mathcal{I}\setminus\{j\}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} \rho^i(t)\, d\mu(i)\right] +$$

$$+\frac{v_{12}^{RC}\left(\int_{\mathcal{I}\setminus\{j\}} c^i(t)\, d\mu(i) + \mu(j)\,\hat{c}^j, t\right)}{v_1^{RC}\left(\int_{\mathcal{I}\setminus\{j\}} c^i(t)\, d\mu(i) + \mu(j)\,\hat{c}^j, t\right)} = -\rho^j(t)\ . \tag{40}$$

Now fix any $t \in [0, \infty)$ and, for notational ease, set,

$$\int_{\mathcal{I}\setminus\{j\}} c^i(t)\, d\mu(i) = \omega(t)\ ,$$

$$\rho^j(t)\int_{\mathcal{I}\setminus\{j\}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} d\mu(i) - \int_{\mathcal{I}\setminus\{j\}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} \rho^i(t)\, d\mu(i) = \chi(t)\ .$$

So, (40) becomes,

$$\frac{v_{11}^{RC}(\omega(t) + \mu(j)\,\hat{c}^j, t)}{v_1^{RC}(\omega(t) + \mu(j)\,\hat{c}^j, t)}\chi(t) = -\rho^j(t) - \frac{v_{12}^{RC}(\omega(t) + \mu(j)\,\hat{c}^j, t)}{v_1^{RC}(\omega(t) + \mu(j)\,\hat{c}^j, t)}\ . \tag{41}$$

For the same point in time, we are able to consider alternative distributions, keeping the same wealth and productivity for all $i \in \mathcal{I}\setminus\{j\}$ and give to j all combinations of initial wealth or productivity that can fully span $\mathbb{C}^j$, given equation (39). So, we can integrate equation (41) over $\hat{c}^j$, as $\mathbb{C}^j$ is a continuum. Doing so, leads to,

$$v_1^{RC}\left(\omega(t) + \mu(j)\,\hat{c}^j, t\right) = e^{-\frac{\mu(j)\rho^j(t)}{\chi(t)}\hat{c}^j - \frac{\mu(j)}{\chi(t)}\int \frac{v_{12}^{RC}\left(\omega + \mu(j)\hat{c}^j, t\right)}{v_1^{RC}\left(\omega + \mu(j)\hat{c}^j, t\right)} d\hat{c}^j + \frac{\mu(j)}{\chi(t)}\kappa}\ , \tag{42}$$

where κ is the constant of integration. Viewing the right-hand side of (42) as the product of two functions,

$$g\left(\hat{c}^j, t\right) = e^{-\frac{\mu(j)\rho^j}{\chi(t)}\hat{c}^j + \frac{\mu(j)}{\chi(t)}\kappa}\ ,$$

and

$$h\left(\hat{c}^j,t\right)=e^{-\frac{\mu(j)}{\chi(t)}\int\frac{v_{12}^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}d\hat{c}^j} \ ,$$

and integrating again (42) with respect to $\hat{c}^j$, using integration by parts, it is,

$$\frac{1}{\mu(j)}v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)=-\frac{\chi(t)}{\mu(j)\rho^j(t)}e^{-\frac{\mu(j)\rho^j(t)}{\chi(t)}\hat{c}^j-\frac{\mu(j)}{\chi(t)}\int\frac{v_{12}^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}d\hat{c}^j+\frac{\mu(j)}{\chi(t)}\kappa}-$$

$$-\frac{1}{\rho^j(t)}\int\frac{v_{12}^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}e^{-\frac{\mu(j)\rho^j(t)}{\chi(t)}\hat{c}^j-\frac{\mu(j)}{\chi(t)}\int\frac{v_{12}^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}d\hat{c}^j+\frac{\mu(j)}{\chi(t)}\kappa}d\hat{c}^j \ ,$$

where the constant of integration is ignored, since this is a utility function. Using (42) again, this last equation gives,

$$v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)=-\frac{\chi(t)}{\rho^j(t)}v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)-\frac{\mu(j)}{\rho^j(t)}\int v_{12}^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)d\hat{c}^j \ ,$$

or,

$$v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)=-\frac{\chi(t)}{\rho^j(t)}v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)-\frac{\mu(j)}{\rho^j(t)}v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right) \ , \tag{43}$$

having, again, ignored the constant of integration. Rearranging terms in (43), it is,

$$\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}=-\frac{\chi(t)}{\mu(j)}\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}-\frac{\rho^j(t)}{\mu(j)} \ . \tag{44}$$

If the terms $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}$, on the left-hand side of (44) and $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}$ on the right-hand side of (44) depend on $\hat{c}^j$, it must be that both are downward sloping functions of $\hat{c}^j$. In order that the representative-consumer's problem be well-defined, the momentary function of the representative consumer must have $v_1^{RC}(c,t)>0$, $v_{11}^{RC}(c,t)<0$ on $\mathbb{C}^{RC}$, and also $v_{12}^{RC}(c,t)<0$: the rate of time preference of the representative consumer, $-\frac{v_{12}^{RC}(c,t)}{v_1^{RC}(c,t)}$ cannot be negative while all consumers in the economy always possess strictly positive rates of time preference and, at least asymptotically, $-\frac{v_{12}^{RC}(c,t)}{v_1^{RC}(c,t)}>0$, so that the objective function of the representative consumer is well-defined. Given these observations, let's examine, case by case, whether $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$, or $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$, or both, depend on $\hat{c}^j$, guided by (44).

Suppose that $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ depends on $\hat{c}^j$, but $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ does not. Then, at a given point in time, (44) implies that $\hat{c}^j$ is a function of the primitives of the representative consumer, and of $\chi(t)$, driven by what the rest of the consumers in the economy choose. But $\hat{c}^j$ can be any, by assigning to consumer type j different initial wealth or labor productivity processes, a contradiction. Similarly, we can contradict that $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ depends on $\hat{c}^j$, but $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ does not. Now suppose that both $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ and $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ depend on $\hat{c}^j$. In that case, nothing could prevent the rest of the consumers to choose consumption so that $\chi(t)>0$ for some $t\geq 0$. In that case, given that both $\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ and $\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}$ could only be downward-sloping, $\hat{c}^j$ is, again, a function of the primitives of the representative consumer, and of $\chi(t)$. Therefore,

$$\frac{\partial\frac{v_1^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}}{\partial\hat{c}^j}=\frac{\partial\frac{v_2^{RC}\left(\omega(t)+\mu(j)\hat{c}^j,t\right)}{v^{RC}(\omega(t)+\mu(j)\hat{c}^j,t)}}{\partial\hat{c}^j}=\kappa\in R\ ,\quad t\geq 0. \tag{45}$$

Since the choice of $t\in[0,\infty)$ and, especially, the choice of $j\in\mathcal{I}$ were arbitrary, a condition similar to this given by (43) can be obtained for all $i\in\mathcal{I}$ and al $t\geq 0$, namely,

$$v^{RC}(c,t)=\gamma(t)\,v_1^{RC}(c,t)-\delta(t)\,v_2^{RC}(c,t)\ ,\quad\text{for some }\gamma(t)\in\mathbb{R},\ \delta(t)\in\mathbb{R}_{++},\ c\in\mathbb{C}^{RC},\ t\geq 0\ , \tag{46}$$

since we can span the whole range of $\mathbb{C}^{RC}$, by setting $r(t)=\rho^i(t)$ for all $i\in\mathcal{I}$, and repeating the same analysis. Now, set,

$$\upsilon(c,t)=\ln\left[\left|v^{RC}(c,t)\right|\right]\ , \tag{47}$$

and divide both sides of (46) by $v^{RC}(c,t)$, to get,

$$\gamma(t)\,\upsilon_1(c,t)-\delta(t)\,\upsilon_2(c,t)=1\ ,\ \text{for some }\gamma(t)\in\mathbb{R},\ \delta(t)\in\mathbb{R}_{++},\ c\in\mathbb{C}^{RC},\ t\geq 0\ , \tag{48}$$

Accordingly, condition (45) on the whole range of $\mathbb{C}^{RC}$, reads,

$$\upsilon_{11}(c,t) = \upsilon_{12}(c,t) = 0 \ , \quad c \in \mathbb{C}^{RC},\ t \geq 0. \tag{49}$$

Conditions (48) and (49) imply that $\upsilon(c,t)$ must be additively separable with respect to c and t. Specifically, $\upsilon(c,t)$ can only be the sum of a linear function of c and of a function of t, namely, $\upsilon(c,t) = -\phi c + \zeta(t)$. Since $\upsilon_2(c,t)$ is the negative of the representative consumer's rate of time prefrerence, setting, without loss of generality, $\zeta(t) = -\int_0^t \rho^{RC}(\tau)\,d\tau$, using (47) it is,

$$\upsilon^{RC}(c,t) = -e^{-\phi c - \int_0^t \rho^{RC}(\tau)d\tau} \text{ with } \gamma(t)\,\phi - \delta(t)\,\rho^{RC}(t) = 1 \tag{50}$$

$$\text{for some } \gamma(t) \in \mathbb{R},\ \phi, \rho^{RC}(t), \delta(t) > 0,\ c \in \mathbb{C}^{RC},\ t \geq 0\ .$$

Treating $\gamma(t), \phi, \delta(t)$ and $\rho^{RC}(t)$ as undetermined for the moment, returning to (38), after using (50) it becomes,

$$\int_{\mathcal{I}} \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))}\left[r(t) - \rho^i(t)\right] d\mu(i) = \frac{r(t) - \rho^{RC}(t)}{\phi}\ ,\quad t \geq 0\ . \tag{51}$$

(Necessity) Step 2: characterization of the functions $(u^i)_{i\in\mathcal{I}}$.

In this step, it is shown that,

$$(51) \Rightarrow \frac{u_1^i(c^i(t))}{u_{11}^i(c^i(t))} = -\beta_i\ ,\ \beta_i > 0 \text{ for all } c^i(t) \in \mathbb{C}^i \text{ and } i \in \mathcal{I},\ t \geq 0\ . \tag{52}$$

To prove (52), consider a constant path of interest rates, $r(t) = r > 0$, for all $t \geq 0$. Assumption 5 tells us that there always exists a distribution of initial-wealth, labor-productivity paths and a wage vector, $(w(t))_{t\geq 0}$, such that all individuals can have an interior optimal consumption path. Pick one of these, say $(a_0^i)_{i\in\mathcal{I}}$, $(\theta^i)_{i\in\mathcal{I}}$, and $(w(t))_{t\geq 0}$. Set $\tilde{\theta}^i(t) = \frac{\theta^i(t)w(t)}{\tilde{w}}$, where $\tilde{w} > 0$ is some constant. From the necessary conditions of the problem, it is transparent that for $r(t) = r$, $w(t) = \tilde{w}$, for all $t \geq 0$, and $(a_0^i)_{i\in\mathcal{I}}$, $\left(\tilde{\theta}^i\right)_{i\in\mathcal{I}}$, the consumption paths will remain unchanged. In this setting with constant prices, the consumption decision rule of each consumer type, $i \in \mathcal{I}$, is memoryless and depends only on the continuation of $\tilde{\theta}^i$ from the current moment of the decision and on. This means that, in period 0, one can start any consumer type from any point of its optimal path, *at any moment in time*, and let the consumer continue on the same path, by modifying its initial wealth so that it matches a choice of wealth at a future moment and by modifying $\tilde{\theta}^i$, giving the continuation of $\tilde{\theta}^i$ from that future moment at time 0. Let's call such a modification a *path-preserving modification*. Suppose that, contrary to (52), $\frac{u_1^i(c^i)}{u_{11}^i(c^i)}$ depends on c^i, for some $i \in \mathcal{I}$. If that consumer type, i, has an optimal path such that $c^i(t_1) \neq c^i(t_2)$ for some $t_1 \neq t_2$, $t_1, t_2 \geq 0$, an appropriate path-preserving modification (which is always possible and does not violate interiority), while keeping the features of all other consumer types the same, would violate (51). In the very extreme case that $c^i(t)$ is constant at all times, after solving the necessary conditions for $r(t) = r$, $w(t) = \tilde{w}$, for all $t \geq 0$, and a_0^i, $\tilde{\theta}^i$, it is,

$$c^i(t) = c^i = \frac{a^i(t) + \tilde{w} \int_t^\infty e^{-rt} \tilde{\theta}^i(\tau)\, d\tau}{\int_t^\infty e^{-rt} d\tau} \text{ , for all } t \geq 0 \text{ ,}$$

where $c^i \in \mathbb{C}^i$ is a constant, then we can simply change a_0^i to $\hat{a}_0^i = \xi a_0^i$ and $\tilde{\theta}^i(t)$ to $\hat{\theta}^i(t) = \xi \tilde{\theta}^i(t)$ so that, according to the equation above, $c^i(t) = \hat{c}^i \neq c^i$, with $\hat{c}^i \in \mathbb{C}^i$, contradicting (51) again. Since the choice of $i \in \mathcal{I}$ was arbitrary, $\frac{u_1^i(c^i)}{u_{11}^i(c^i)}$ is a constant for all $i \in \mathcal{I}$, and given Assumption 1, it must be a negative constant, so (52) is proved.

Integrating $\frac{u_1^i(c^i)}{u_{11}^i(c^i)} = -\beta_i$ with respect to c^i, leads to,

$$u^i(c) = -e^{-\frac{1}{\beta_i} c} \text{ ,} \quad \beta_i > 0 \text{, for all } i \in \mathcal{I} \text{ .} \tag{53}$$

(Necessity) Step 3: characterization of u^{RC} and ρ^{RC}.

Combining (51) with (52), it is,

$$\left[\int_{\mathcal{I}} \beta_i d\mu(i) - \frac{1}{\phi}\right] r(t) = \frac{\rho^{RC}(t)}{-\phi} + \int_{\mathcal{I}} \beta_i \rho^i(t)\, d\mu(i) \ . \tag{54}$$

But since (54) should hold independently from $r(t)$, it can only be that,

$$\phi = \frac{1}{\int_{\mathcal{I}} \beta_i d\mu(i)} \ , \quad t \geq 0 \ ,$$

and, accordingly,

$$\rho^{RC}(t) = \frac{\int_{\mathcal{I}} \beta_i \rho^i(t)\, d\mu(i)}{\int_{\mathcal{I}} \beta_i d\mu(i)} \ , \quad t \geq 0 \ .$$

A combination of these two last equations with (50) completes the necessity part of the theorem.

Part 2: Sufficiency

Employing the same algebra as in Part 2 of Theorem 1, the resulting demand for individual $i \in \mathcal{I}$ at any point in time, $t \geq 0$, is,

$$c^i(t) = \frac{a^i(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds} \theta^i(\tau)\, w(\tau)\, d\tau - \beta_i \int_t^\infty e^{-\int_t^\tau r(s)ds} \int_t^\tau \left[r(s) - \rho^i(s)\right] ds d\tau}{\int_t^\infty e^{-\int_t^\tau r(s)ds} d\tau} \ , \tag{55}$$

which can be linearly aggregated across all a^i's, θ^i's, β_i's, and $\rho^i(t)$ at all times, completing the proof of the theorem.□

Theorem 2 relates to a number of studies that deal with the issue of heterogeneity in rates of time preference. Becker's (1980) result in a deterministic world, that the most patient consumer accumulates asymptotically all the wealth of the economy as time approaches infinity, relies heavily on imposing Inada conditions on the momentary utility functions of the different agent types, that $\lim_{c \to 0} u_1^i(c) = \infty$ and $\lim_{c \to \infty} u_1^i(c) = 0$. If one needs facility in computing the dynamics of aggregate wealth in a Becker (1980)-type economy by modeling so that a representative consumer exists, according to Theorem 2, the only way is to use $u^i(c) = -e^{-\frac{1}{\beta_i}c}$, $\beta_i > 0$, for all $i \in \mathcal{I}$. However, in this case, a problem is that $\lim_{c \to 0} u_1^i(c) = \frac{1}{\beta_i}$ for all $i \in \mathcal{I}$. Thus, the issue of interiority of optimal paths in a Becker (1980)-type economy is at stake, unless one makes appropriate (although possibly ad-hoc) assumptions on the productivity processes of less patient consumer types.

Becker's (1980) observation on a general equilibrium model with heterogeneous rates of time preference has triggered numerous directions of research, especially along the lines of stochastic general-equilibrium environments with idiosyncratic shocks. Probably the most well-known application is this of Krusell and Smith (1998), who exploit the tendency of heterogeneous rates of time preference to generate wealth disparities. They use a random process of individual rates of time-preference in order to match U.S. wealth data that exhibit a high Gini coefficient. As this random process has a trivial probability that a person is always the most patient, the asymptotic distribution of wealth is not degenarate, but it simply leads to a high Gini coefficient.

Krusell and Smith (1998) employ homothetic preferences with a constant elasticity of intertemporal substitution, and a common-across-consumers random process of varying rates of time preference. The random process of the rates of time preference is persistent, so it implies different conditional means for each individual in each period, whereas all individuals have the same unconditional means of rates of time preference. The first feature triggers wide differences in the savings propensity across individuals, leading to a high Gini coefficient for wealth. According to Theorem 2 of the present study, since preferences are not exponential, one could first think that there should not be a representative consumer in the Krusell-Smith (1998) setting, since the conditional means of the rates of time preference are different. Yet, although the economic environment has, in addition to its other features, incomplete markets, Krusell and Smith (1998) find that equilibrium prices can be computed very well using only the law of motion of the mean wealth, i.e. there is trivial improvement if one computes prices from the evolution of many moments of the wealth distribution that should match wealth-distribution dynamics very well.

On the contrary, Carroll (2000) uses exactly the same framework as in Krusell and Smith (1998), and the same stochastic processes (even numerically, in some of his applications) for both idiosyncratic labor productivity and aggregate shocks, with the sole difference that he employs two types of consumers, one always patient and another always impatient, both having constant rates of time preference. The fact that, unlike in Becker (1980), the patient consumers do not end up accumulating the economy's wealth in the model of Carroll (2000), rests upon the fact that all consumers receive a common process of idiosyncratic stochastic labor earnings. Thus, the impatient ones do not save as (relatively) low as in Becker's (1980) model, for precautionary reasons. Carroll (2000) finds totally different results, namely, the approximate aggregation result of Krusell and Smith (2000) disappears. According to Theorem 2, Carroll's (2000) findings should not be a surprise, because the employed preferences (the preferences of Theorem 1 with $\beta_i = 0$ and $\alpha > 0$) are not exponential. Even if no consumer has a corner solution, aggregation should fail. At the same time, equation (33) shows that if ρ increases, the marginal propensity to consume increases as well, if $\alpha > 0$, as Carroll (2000) intuitively emphasizes.[4] As the exponential function implies zero quasi

[4] In order to match the utility function employed by Carroll (2000) and Krusell and Smith (1998), one should set $\beta_i = 0$ and $\alpha > 0$ in equation (33), which reconfirms Carroll's (2000) intuition, at least in the deterministic continuous-time version of the model.

elasticity of intertemporal substitution ($\alpha = 0$), changes in impatience leave the individual marginal propensities to consume unaffected, as equation (55) shows.

However, the results by Krusell and Smith (1998) are still somewhat puzzling. But equation (33), although it refers to a deterministic continuous-time environment, can offer a *conjecture*. Extensions to adding the continuous-time analogue of the Krusell-Smith (1998) persistent Markov processes to the model of the present study, are likely to retain most of the structure of demands conveyed by equation (33). If we set $\alpha > 0$ and $\beta_i = 0$, and also replace ρ with ρ^i in (33), at a first glance, we can think that aggregation should fail, as marginal propensities to consume (with respect to asset holdings) would change due to temporal persistent shocks in ρ^i's. But it is important to notice that the aggregation question in the Krusell-Smith (1998) framework pertains decision rules where the accumulable state variable is asset holdings. For forming the wealth-distribution law of motion, each agent is identified by this sole attribute, asset holdings. So, identifying an agent by $a(t)$ at any point in time, the fact that there are infinite agents in the Krusell-Smith (1998) model (a continuum), makes the law of large numbers to apply. The latter means that the *average-across-θ^i's-and-ρ^i's agent holding* $a(t)$, should be aggregated over the *unconditional distribution* of θ^i's and ρ^i's, both today and in the future. But local aggregation of consumers (at a certain level of wealth) over the domain and probabilities of all θ^i's and ρ^i's, should imply similar attributes for the demands, as these stochastic processes are *common across agents* in the Krusell-Smith (1998) setting. Thus, the *average-across-θ^i's-and-ρ^i's demand of an agent holding* $a(t)$ should have the same Engel-curve slope (as implied by the unconditional means of θ^i's and all ρ^i's) as this of *any average-across-θ^i's-and-ρ^i's agent holding* $\tilde{a}(t) \neq a(t)$, at any point in time. These parallel Engel curves among *average-across-θ^i's-and-ρ^i's* agents holding some level of wealth must be the basis for the Krusell-Smith (1998) aggregation result for the law of motion of the wealth distribution.

Fixing $a(t)$ and a temporal $\theta^i(t)$, as it is the case in Figure 2 of Krusell and Smith (1998, p. 879), still leaves the Engel curves across wealth and temporal-productivity types parallel, since the demands of each of these types are *unconditionally average across* ρ^i*'s*, as it is conveyed by equation (33) of the present study. To the extent that a very small fraction of agents might hit corner solutions, as Krusell and Smith (1998, p. 880) point out, gives their "approximate aggregation" result across agents of $a(t)$ and temporal $\theta^i(t)$'s who are *average across* ρ^i*'s.*

At the same time, the fact that agents holding $a(t)$ receive all kinds of temporal and persistent shocks of θ^i's and all ρ^i's and, thus, *agents distinguished both by* θ^i*'s and all* ρ^i*'s, have non-parallel temporal Engel curves*, adds to having a higher long-run dispersion in wealth. Thus, the Krusell-Smith (1998, Table 1, p. 884) setting with stochastic rates of time preference delivers a higher Gini coefficient on the one hand, but does not ruin the "approximate aggregation" result on the other. On the contrary, in light of Theorem 2 and equation (33), the Carroll (2000) setting with some consumers having permanently lower rates of time preference than others, even if no consumers face a high risk to hit a corner solution, there is no way that aggregation would hold. If consumers face tight borrowing constraints and high probabilities of receiving very low incomes, and thus, consumers have precautionary motives to lower their marginal propensity to consume as their wealth approaches zero, then aggregation in the Carroll(2000) model becomes even more unlikely.

3. Choice-dependent rates of time preference (time-variant momentary utility functions)

Unlike the class of preferences examined by Gollier and Zeckhauser (2005), in the present paper the extension to time-variant momentary utility functions pertains the case where individual rates of time preference have a consumption-choice-independent part wich is common-across agents, and a consumption-choice-dependent part implied by their momentary utility function. In particular, consumer preferences of each $i \in \mathcal{I}$, are given by the general additively-separable utility function,

$$U^i\left(\left(c^i(t)\right)_{t\geq 0}, t\right) = \int_0^\infty e^{-\int_0^t \rho(\tau)d\tau} u^i\left(c^i(t), t\right) dt \ . \tag{56}$$

with $\rho : \mathbb{R}_+ \to \mathbb{R}_{++}$. A sequence of assumptions are important for the analysis that follows.

Assumption 6 *For all $i \in \mathcal{I}$, and all $t \geq 0$, $u^i : \mathbb{R}^2_+ \to \mathbb{R}$, is twice-continuously differentiable with respect to c, and once continuously differentiable with respect to t, and such that $u^i_1(c,t) > 0$ and $u^i_{11}(c) < 0$ on some interval, $\mathbb{C}^i(t) \subseteq \mathbb{R}_+$, with $u^i_1(c) < \infty$, $-\infty < u^i_{11}(c)$, $-\infty < u^i_{12}(c) < \infty$ for all $c \in \mathbb{C}^i(t) \subseteq \mathbb{R}_+$,with $\underline{c}^i(t) \equiv \inf\left(\mathbb{C}^i(t)\right) < \sup\left(\mathbb{C}^i(t)\right) \equiv \bar{c}^i(t)$.*

Assumption 7 *For all $i \in \mathcal{I}$, $\bigcap_{t\geq 0} u^i_1\left(\mathbb{C}^i(t), t\right)$ is non-empty and not a singleton.*

Assumption 8 *For any $i \in \mathcal{I}$, let,*

$$\mathfrak{C}^i \equiv \left\{ c \in \bigcap_{t\geq 0} \mathbb{C}^i(t) \;\middle|\; \bigcap_{t\geq 0} u^i_1\left(\mathbb{C}^i(t), t\right) \text{ is non-empty and not a singleton} \right\} .$$

Then, $\bigcap_{i\in\mathcal{I}} \mathfrak{C}^i$ is non-empty and not a singleton.

Based on these assumptions, the following theorem is provided.

Theorem 3 *Under Assumptions 2, 3, and 6 through 8, a representative consumer exists iff*

$$u^i(c,t) = \begin{cases} \frac{\left(\alpha c+\beta^i(t)\right)^{1-\frac{1}{\alpha}}-1}{\alpha\left(1-\frac{1}{\alpha}\right)} & \text{with } \alpha>0 \text{ and } \beta^i(t)\in\mathbb{R} \text{ or } \alpha<0 \text{ and } \beta^i(t)\in\mathbb{R}_{++} \\ \text{or} & \\ -e^{-\frac{1}{\beta_i G(t)}c} & \text{with } \beta_i>0 \end{cases}, \tag{57}$$

for all $i\in\mathcal{I}$, with functions $\beta^i(t)$ such that Assumptions 7 and 8 are met.

The representative consumer has

$$U^{RC}\left(\left(c^i(t)\right)_{t\geq 0},t\right) = \int_0^\infty e^{-\int_0^t \rho(\tau)d\tau} u^{RC}\left(c^i(t),t\right)dt\ , \tag{58}$$

with,

$$u^{RC}(c) = \begin{cases} \frac{\left(\alpha c+\beta^{RC}(t)\right)^{1-\frac{1}{\alpha}}-1}{\alpha\left(1-\frac{1}{\alpha}\right)} & \text{for } \alpha\neq 0\ ,\ \ \beta^{RC}(t)=\int_{\mathcal{I}}\beta^i(t)\,d\mu(i) \\ -e^{-\frac{1}{\beta_{RC}G(t)}c} & \text{else, }\ \beta_{RC}=\int_{\mathcal{I}}\beta_i d\mu(i) \end{cases}. \tag{59}$$

Proof of Theorem 3

Part 1: Necessity

Fix any function $\rho:\mathbb{R}_+\to\mathbb{R}_{++}$, and any collection $\left(u^i\right)_{i\in\mathcal{I}}$, with properties complying with Assumptions 2,3 and 6 through 8. Assume that a representative consumer exists with some momentary utility function $v^{RC}:\mathbb{C}^{RC}\times\mathbb{R}_+\to\mathbb{R}$, of the form $v^{RC}(c(t),t)$, at each point in time.

Considering any $i\in\mathcal{I}$, its optimality conditions imply that,

$$-\frac{u^i_{11}\left(c^i(t),t\right)}{u^i_1\left(c^i(t),t\right)}\dot{c}^i(t)-\frac{u^i_{12}\left(c^i(t),t\right)}{u^i_1\left(c^i(t),t\right)} = r(t)-\rho(t)\ \ ,\quad t\geq 0\ . \tag{60}$$

Now pick $r(t) = \rho(t)$ for all $t \geq 0$, substitute it to (60) and take the indefinite inegral with respect to time to get,

$$u_1^i\left(c^i(t), t\right) = \kappa\ , \quad t \geq 0\ . \tag{61}$$

where κ is some constant. Due to the fact that $u_{11}^i\left(c^i(t), t\right) < 0$, and due to Assumptions 7 and 9, there is always a $\kappa > 0$ such that $c^i(t) \in \mathbb{C}^i(t)$ for all $t \geq 0$, satisfying (61). For $r(t) = \rho(t)$, (60) implies that,

$$\dot{c}^i(t) = -\frac{u_{12}^i\left(c^i(t), t\right)}{u_{11}^i\left(c^i(t), t\right)}\ . \tag{62}$$

The level of κ in (61) will be uniquely identified by setting $u_1^i\left(c^i(0), 0\right) = \kappa$ and applying (11) at time 0, combined with the dynamics of $c^i(t)$ implied by (62). Due to Assumption 7, such an interior path exists on $\mathfrak{C}^i$, as $\mathfrak{C}^i$ is defined in assumption 8. This means that with the right choices of initial wealth and labor productivity, we can construct interior paths that span $\mathfrak{C}^i$. Moreover, always for the case where $r(t) = \rho(t)$ for all $t \geq 0$, due to Assumption 8, for any $i \in \mathcal{I}$, we can generate any choice of $c \in \bigcap_{i\in\mathcal{I}} \mathfrak{C}^i$ at any point in time, picking the appropriate initial wealth and labor productivity, since the dynamics of consumption are solely driven by (62).

With this facility at hand, we can look at the problem of the representative consumer, whose optimal Euler equation gives,

$$-\frac{v_{11}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)}{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)} \int_{i\in\mathcal{I}} \mu(i)\, \dot{c}^i(t)\, di - \frac{v_{12}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)}{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)} = r(t)\ , \quad t \geq 0\ , \tag{63}$$

and combining it with (60), it is,

$$\frac{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)}{v_{11}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)} r(t) + \frac{v_{12}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)}{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\, d\mu(i), t\right)} =$$

$$= \left[r(t) - \rho(t)\right] \int_{i\in\mathcal{I}} \frac{u_1^i\left(c^i(t), t\right)}{u_{11}^i\left(c^i(t), t\right)} d\mu(i) + \int_{i\in\mathcal{I}} \frac{u_{12}^i\left(c^i(t), t\right)}{u_{11}^i\left(c^i(t), t\right)} d\mu(i)\ . \tag{64}$$

Setting $r(t) = \rho(t)$ for all $t \geq 0$, (64) becomes,

$$\frac{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)}{v_{11}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)}\rho(t) + \frac{v_{12}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)}{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)} = \int_{i\in\mathcal{I}} \frac{u_{12}^i\left(c^i(t),t\right)}{u_{11}^i\left(c^i(t),t\right)}d\mu(i)\ . \quad (65)$$

But since, as explained above, for the case where $r(t) = \rho(t)$ for all $t \geq 0$, one can generate any distribution of consumption choices, (65) holds for the whole domain implied by Assumption 3. So, substituting (65) into (64), it is,

$$\frac{v_1^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)}{v_{11}^{RC}\left(\int_{i\in\mathcal{I}} c^i(t)\,d\mu(i),t\right)} = \int_{i\in\mathcal{I}} \frac{u_1^i\left(c^i(t),t\right)}{u_{11}^i\left(c^i(t),t\right)}d\mu(i)\ , \quad (66)$$

for the whole domain implied by Assumption 3, including the case where $r(t) = \rho(t)$ for all $t \geq 0$. But then, for any $t \geq 0$, the same argument that was developed in step 2 of the necessity part of the proof of Theorem 1, to get,

$$\begin{aligned} \frac{u_1^i(c,t)}{u_{11}^i(c,t)} &= \alpha(t)\,c + \beta^i(t)\ , \quad \text{and,} \\ \frac{v_1^{RC}(c,t)}{v_{11}^{RC}(c,t)} &= \alpha(t)\,c + \int_{\mathcal{I}} \beta^i(t)\,d\mu(i)\ , \end{aligned} \quad (67)$$
$$\text{for some } \alpha(t) \in \mathbb{R} \text{ and some } \beta^i(t) \in \mathbb{R}, \text{ for all } i \in \mathcal{I},\ t \geq 0$$

Using (67), with the same procedure as in step 3 of the necessity part of Theorem 1, candidate utility functions arise. Deriving individual demands, one can verify that this is possible only if

$$\alpha(t) = \alpha \neq 0\ , \text{ and } \beta^i(t) \text{ meeting Assumptions 7, 8, } t \geq 0\ ,$$

and

$$\alpha = 0, \quad \beta^i(t) = \beta_i G(t)\ ,$$

that match the utility functions of the theorem. In particular, for the case where $\alpha \neq 0$, demands are,

$$c^i(t) = \frac{a^i(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds}\theta^i(\tau)\,w(\tau)\,d\tau + \frac{1}{\alpha}\int_t^\infty e^{\int_t^\tau[(\alpha-1)r(s)-\alpha\rho(s)]ds}\beta^i(\tau)\,d\tau}{\int_t^\infty e^{\int_t^\tau[(\alpha-1)r(s)-\alpha\rho(s)]ds}d\tau} - \frac{\beta^i(t)}{\alpha}\ , \quad (68)$$

which are linear with respect to β^i's. On the contrary, the demands for the utility function,

$$u^i(c,t) = -e^{-\frac{1}{\beta^i(t)}c},$$

are,

$$c^i(t) = \frac{a^i(t) + \int_t^\infty e^{-\int_t^\tau r(s)ds}\theta^i(\tau)\, w(\tau)\, d\tau - \int_t^\infty e^{-\int_t^\tau r(s)ds}\beta^i(\tau)\int_t^\tau [r(s) - \rho(s)]\, ds d\tau}{\int_t^\infty e^{-\int_t^\tau r(s)ds}\frac{\beta^i(\tau)}{\beta^i(t)}d\tau}, \tag{69}$$

which can be linearly aggregated only if $\frac{\beta^i(t)}{\beta^i(0)} = \frac{\beta^j(t)}{\beta^j(0)}$ for all $i, j \in \mathcal{I}$, i.e. only when $\beta^i(t) = \beta_i G(t)$, $\beta_i > 0$ for all $i \in \mathcal{I}$, completing the necessity part.

Part 2: Sufficiency

Follows by (68) and (69), observing that, under the statement of the theorem, they are linear with respect to a^i's, θ^i's and β^i's.□

4. Summary and conclusions

A representative consumer is a fictitious consumer who always chooses the aggregate-demand level optimally, as a result of maximizing her own utility function subject to the aggregate-economy constraint, irrespective from the underlying distribution of wealth, income and taste heterogeneity across consumers. Her existence depends on the structure of individual utility functions in an economy. Gorman (1953) pointed out that, in a static framework, this is possible if, and only if, the utility functions imply that the Engel curves of all consumers are linear and parallel.

Compared to static worlds, it is more tempting to impose such a structure on individual utilities in dynamic general-equilibrium environments of heterogeneous agents. It is often useful, especially for paper-and-pencil research, to be able to compute current and future prices out of a sole optimal-control problem, the representative-consumer's problem, and to obtain insightful analyses of the dynamics of wealth distributions. The goal of this study has been to point out necessary and sufficient conditions on the structure of utility functions under which such an analytical strategy is possible. The focus has been on an environment where all individuals solve standard optimal-control problems of consumption/savings choice and a representative consumer with time-separable preferences exists.

Consumption choices depend on future-income processes and on the ability to accumulate an asset that stores value over time. Like in the static models, aggregation rests, again, upon having linear and parallel Engel curves, but this time with respect to both income and asset holdings. Identifying problems and questions of heterogeneity where such an analytical strategy pertaining linear Engel curves is appropriate (or empirically plausible), has been outside the scope of the analysis.

In particular, the family of individual utility functions that lead to the existence of a representative consumer has been characterized for three settings: (i) the typical case of time-invariant momentary utility functions where all agents share the same choice-independent rates of time preference, (ii) time-invariant momentary utility functions where agents have different choice-independent rates of time preference, and, (iii) time-variant momentary utility functions when individual rates of time preference have a choice-independent part wich is common-across agents, and a choice-dependent part implied by their momentary utility function.

The results pertaining case (i) are the same as these of Pollak (1971) for time-separable utility functions in static environments, and come as no surprise after Chatterjee's (1994) results in a dynamic setting. Yet, the main message of the present paper about case (i), given by Theorem 1, is that, in the dynamic world, there are no other utility functions that could deliver a representative consumer. Thus, the work by Chatterjee (1994) when momentary utility functions are common across agents is comprehensive. Yet, Theorem 1 states the only way that momentary utility functions can be heterogeneous in Chatterjee's (1994) setting for a continuous-time environment: only subsistence levels or bliss points can vary across consumer types, but their (quasi) elasticity of intertemporal substitution must be common.

About case (ii), there is a stronger result: the only utility functions that can accommodate a representative consumer are exponential (Theorem 2). Some implications of this result for recent research findings (Krusell and Smith (1998) and Carroll (2000)) about stochastic general-equilibrium models with rate-of-time preference heterogeneity were discussed.

In case (iii), it is proved that one can generalize the Caselli and Ventura (2000) setting very little. The only parameter that can be time-variant is the subsistence levels (or bliss points - if any) of consumers. In particular, for the exponential functions case, the corresponding parameters that make individual utilities to differ, must follow the same dynamic process over time (but the initial conditions of this process can be different for each individual). The only possible departure from the setting of Caselli and Ventura (2000) is that the subsistence levels (or bliss points - if any) can be general, under certain restrictions stated by Theorem 3. These conditions stated by Theorem 3, essentially allow for the existence of a steady state if the environment is generalized to standard general-equilibrium setups with a representative-firm production function, as all cases examined by Caselli and Ventura (2000). The result of Theorem 3 might prove useful for researchers studying the impact of changes in the demographic characteristics of population over time, under the assumption that each family type has to face some minimum household setup costs (demographically-dependent subsistence levels), such as housing and nutrition needs.

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