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Trees and Decisions

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Abstract. The traditional model of sequential decision making, for instance, in extensive form games, is a tree. Most texts define a tree as a connected directed graph without loops and a distinguished node, called the root. But an abstract graph is not a domain for decision theory. Decision theory perceives of acts as functions from states to consequences. Sequential decisions, accordingly, get conceptualized by mappings from sets of states to sets of consequences. Thus, the question arises whether a natural definition of a tree can be given, where nodes are sets of states. We show that, indeed, trees can be defined as specific collections of sets. Without loss of generality the elements of these sets can be interpreted as representing plays. Therefore, the elements can serve as states and consequences at the same time.

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1. Introduction

Traditional decision theory under uncertainty is a genuinely static theory. The objects of choice are either lotteries over consequences (von Neumann and Morgenstern [17]), or functions from states to consequences, known as "acts" (Savage [21]), or functions from states to lotteries over consequences (Anscombe and Aumann [3]). When such a theory is used to model sequential decision making, the only adjustment is to restrict the possible states and update the probability assignments according to Bayes' rule. Otherwise, each consecutive decision is treated like a single static decision problem.

This traditional concept of "sequential" decision making has come under attack, because it rules out many features that one would think of as being relevant to sequential decisions: a preference for flexibility (Kreps [15]), temptation and self-control (Gül and Pesendorfer [10]), or unforeseen contingencies (Dekel, Lipman, and Rustichini [9]). (For alternative approaches see Blume, Brandenburger, and Dekel [6, 7].) These studies address such issues by introducing preferences defined over subsets, rather than elements, of the space of possible consequences.

The best-known example is that of making a reservation at a restaurant. "Imagine that the only way that restaurants vary is in the menu of meals which they will serve. The individual is assumed to know the menus at all restaurants that he might select. Eventually, the individual will choose a meal, but his initial choice is of a restaurant/menu from which he will later choose his meal." (Kreps [15], p. 565) Such a two-stage set-up, of course, is for simplicity, not realism. Realistically, one expects more "stages" and a long chain of consecutive decisions. But this requires a careful specification of what the structure of the domain for preferences is.

There is another traditional model, that was invented as a domain for sequential decision making: the *tree* of the extensive form representation of a game (Kuhn [16]). Trees serve as a transparent graphical model of how consecutive decisions refine the selection among possible outcomes. And they are closely related to collections of subsets of an underlying space of consequences or outcomes - as already highlighted by von Neumann and Morgenstern ([17], p. 65).

Yet, usually trees are defined as directed connected graphs without loops and with a distinguished node, the "root," that comes before any other node. Though this is an intuitive concept, it is formally at variance with specifying sequential decisions over (increasingly smaller) sets of outcomes. Thus, the issue arises whether arbitrary trees can be recast into collections of subsets of some underlying space, thereby making them an adequate domain for sequential decision theory.

The present paper addresses this issue in full generality. We start from the order-theoretic concept of a tree (that encompasses the graphical model traditionally used for extensive form games) and show that it can be represented as a set of sets with a particular structure which we characterize. Yet, to be able to interpret the elements of these sets as consequences/outcomes requires a bit more. In particular, (maximal) chains of sets (called *plays*) need to identify elements that all sets in the chain contain. In the language of the restaurant example, a menu needs to correspond to a collection of meals, because meals are what the decision maker will ultimately consume.

Accordingly, we show that every tree has a set representation which meets this requirement. Characterizing these particular set representations of trees generates a definition of set trees that lends itself naturally to a theory of sequential decision making. For these set trees a node can be thought of as an event, just like in probability theory, i.e. as a set of states. Moreover, when the elements of these sets/nodes represent plays (maximal chains of nodes), they correspond to outcomes/consequences, thereby providing the adequate framework for the modern versions of sequential decision theory mentioned above.

Essentially, the present paper represents the first step towards a general definition of an extensive form as a framework for the application of truly sequential decision theories of the aforementioned type. This is why we start with utmost generality, rather than restricting to simple cases. Specifically, most of the traditional definitions of trees use a discreteness property: for every node there is an immediate predecessor. For instance, in von Neumann games ([17], Chapter II) the number of predecessors of every node in an information set is required to be the same. In our set-up, however, immediate predecessors may not even exist, and the number of predecessors may not be finite. This allows us to consider examples as exotic as decision problems in continuous time ("differential games").

1.1. Overview. The investigation starts with the most general definition of a tree and maps this into a collection of subsets (of some underlying set) with a particular structure: its "set representation." (This operation is always possible; see Proposition 1.) But it turns out that in a set-theoretic environment this structure can be "cleaned" without affecting the properties of the trees. Thus, we "clean" the structure in three steps, where each step corresponds to adding structure that enables increasingly specific interpretations of the tree.

Section 2 is concerned with strengthening the characterizing properties of set representations (of trees) to something that can only be obtained in a set-theoretic framework. Section 2.1 begins by characterizing set representations of trees (Proposition 2(a)) and reveals that the set-theoretic analog of the defining order-theoretic structure can be modified such that *unordered* nodes correspond to *disjoint* sets, without affecting the properties of the tree. We show that every tree indeed has a set representation that satisfies such a stronger set-theoretic property, called "Trivial Intersection" (Proposition 2(b)). As a leading example, a differential game tree (i.e. the tree of a decision problem in continuous time) is presented.

Yet, general trees may contain trivial structures that serve no purpose for decision theory. Ruling those out leads to "decision trees," in Section 2.3. Again, characterizing set representations of decision trees (Proposition 3(a)) shows that, under Trivial Intersection, the set-theoretic analog of the defining order-theoretic property of decision trees gets strengthened, to "Separability" (Lemma 4). Every decision tree has a set representation that satisfies the two strong characterizing properties (Trivial Intersection and Separability; see Proposition 3(b)).

Moreover, it turns out, in Section 2.4, that every decision tree has a "canonical" set representation, where the elements of the underlying set are *plays* (maximal chains), as we show in Theorem 1. This set representation satisfies the strong versions of the characterizing properties (by Lemma 5 and Corollary 1). This concludes the first step of "cleaning" and yields the first milestone: "set trees."

Thus, we turn next to the class of set trees, that satisfy the two strong properties, Trivial Intersection and Separability. The goal of Section 3 is to give meaning to the elements of the underlying set in such set trees. In particular, when can the elements of the underlying set be perceived as representing plays, as suggested by Theorem 1? It turns out that this requires the underlying set to be neither too large nor too small.

First, we construct in Section 3.1 a "reduced form" (Proposition 4) in which redundancies in the underlying set are eliminated. In Section 3.2 we find that, in this reduced form, the elements of the underlying set indeed correspond injectively to plays (Proposition 5 and Lemma 8). This is, in fact, the first gain from using the stronger set-theoretic properties: Trivial Intersection is equivalent to the property that the elements of the underlying set in the reduced form map one-to-one into plays (Proposition 5). Hence, elements of the underlying set (in the set representation) could potentially serve as representatives of ultimate outcomes or states.

When is a set tree already in reduced form? In Section 3.3 we show that this is the case if, roughly, no element of the underlying set can be dropped, i.e. if the set tree is "irreducible." Irreducibility is equivalent to the elements of the underlying set in the reduced form being the singleton sets of the originally underlying set (Proposition 6).

This leads, in Section 3.4, to the notion of a "proper" order isomorphism; those are the order isomorphisms between collections of sets that preserve the "strong" properties (Lemma 12). This concept enables a characterization of set trees: a collection of sets is a set tree if and only if it is properly isomorphic to its reduced form and the latter is an irreducible set tree (Theorem 2). But Irreducibility of the reduced form implies Irreducibility of the original set tree only if the order isomorphism has a reflection in the underlying set, i.e. if the two are "doubly isomorphic." Irreducible set trees are *precisely* those that are doubly isomorphic to their reduced forms (Proposition 7). These considerations clarify when the underlying set (of a set tree) is not too large.

It may, however, still be too small. Section 3.5 aims at characterizing when the elements of the underlying set map also *surjectively* onto plays. The condition that ensures this is "boundedness." Achieving this involves, possibly, enlarging the underlying set. Due to the stronger properties, Trivial Intersection and Separability, irreducible set trees are *precisely* those, where elements can be added, so that every play is represented by a distinct element of the underlying set (Proposition 8). Hence, a set tree is bounded if and only if the elements of the underlying set in the reduced form represent *all* plays (Proposition 9).

This yields the second milestone: "game trees," defined as bounded irreducible set trees. In Section 4.1 it is shown that game trees are *precisely* those for which there is a bijection between the elements of the underlying set and the set of plays; equivalently, they are *precisely* those which are decision trees that are their own "canonical" set representation by plays (Theorem 3). Hence, our results combine to the insight that there is no loss of generality in assuming boundedness and Irreducibility when working with set trees. Yet, once a tree has been turned into a game tree, we have arrived at a representation that can serve as an "objective" description of a sequential decision problem: nodes are (represented as) set of plays.

Section 4.2, finally, takes a modelling decision by entering "terminal nodes." Boundedness of a set tree does not necessarily imply that the singletons from the underlying set belong to the set of nodes. But it is shown that adding the singletons (from the underlying set) does not change any essential features of the tree, provided it is bounded and irreducible (Proposition 10). This yields the third and last milestone: "complete game trees." A set tree is a complete game tree if and only if it is irreducible and every play has a minimum (Proposition 11). In the finite case complete game trees are indeed very simple objects: they are collections of subsets that contain all singletons and satisfy Trivial Intersection (Proposition 12).

In Section 5 we provide an application by showing that extensive forms can be defined with game trees. The familiar strategy notions translate smoothly to this general framework, and pure strategy combinations give rise to plays.

Some issues remain open, though. On the one hand, complete game trees are so general that they even capture decision problems in continuous time. On the other hand, this generality may be insufficient for important game theoretic structures. For instance, alternating moves by different players, as in perfect information games, cannot always be modelled in such a framework. Intuitively, this is because in such a game tree a play may never "build up" by consecutive decisions, since these general set trees may lack a discrete structure. Hence, for some purposes, this version of game trees may be too general. How more structure can be added and what this simplifies is, however, left for future research on discrete trees (Alós-Ferrer and Ritzberger [1], in preparation). Section 6 discusses such directions for further research.

Proofs of major results are included in the text; proofs of selected Lemmata are relegated to the Appendix. Straightforward proofs are omitted.

2. Set Representations

The following basic definitions are used throughout the paper.

Definition 1. *A* ***preordered set*** *is a pair* $(N, \geq)$ *consisting of a nonempty set* N *and a reflexive and transitive binary relation* $\geq$ *on* N*. A preordered set* $(N, \geq)$ *for which the relation* $\geq$ *is antisymmetric is a (partially)* ***ordered set*** *(or a* ***poset****).*

In particular, a V*-poset* is a poset $(M, \supseteq)$ where M is a collection of nonempty subsets of a given set V and $\supseteq$ is set inclusion.

Definition 2. *A nonempty subset* $c \subseteq N$ *of a preordered set* $(N, \geq)$ *is a* ***chain*** *if for all* $x, y \in c$ *either* $x \geq y$ *or* $y \geq x$ *(or both), i.e. if the induced preorder on* c *is complete.*

Given a preordered set $(N, \geq)$ and an element $x \in N$ define the *up-set* (or *order filter*) $\uparrow x$ and the *down-set* (or *order ideal*) $\downarrow x$ by[1]

$$\uparrow x = \{y \in N \,|\, y \geq x\} \text{ and } \downarrow x = \{y \in N \,|\, x \geq y\} \tag{1}$$

Let $\downarrow N = \{\downarrow x \,|\, x \in N\} \subseteq 2^N$ denote the set of all down-sets of $(N, \geq)$.

Definition 3. *An* ***order isomorphism*** *between two preordered sets* $(N_1, \geq_1)$ *and* $(N_2, \geq_2)$ *is a bijection* $\varphi : N_1 \to N_2$ *such that*

$$x \geq_1 y \text{ if and only if } \varphi(x) \geq_2 \varphi(y) \tag{2}$$

for all $x, y \in N_1$*. This last property is referred to as "order embedding."*

Remark 1. *If* $(N_1, \geq_1)$ *is a* poset, $(N_2, \geq_2)$ *a preordered set, and* $\varphi : N_1 \to N_2$ *an order embedding function, then* φ *is necessarily injective (one-to-one). For, given* $x, y \in N_1$ *such that* $\varphi(x) = \varphi(y)$*, reflexivity of* $\geq_2$ *implies* $\varphi(x) \geq_2 \varphi(y)$ *and* $\varphi(y) \geq_2 \varphi(x)$ *and hence* $x \geq_1 y$ *and* $y \geq_1 x$ *(by the "if"-part of (2)), together implying* $x = y$ *(by antisymmetry for* $\geq_1$*). In particular, any order-embedding* surjection *(onto function) between two posets is an order isomorphism.*

Order isomorphism is an equivalence relation on the class of all preordered sets. Two order-isomorphic preordered sets can be regarded as identical for all practical purposes.

Definition 4. *A preordered set* $(N, \geq)$ ***admits a set representation*** *if there is an order isomorphism between* $(N, \geq)$ *and a* V*-poset* $(M, \supseteq)$*.*

Proposition 1. *A preordered set* $(N, \geq)$ *admits a set representation if and only if it is a poset. In that case a possible set representation is* $(\downarrow N, \supseteq)$ *with order isomorphism given by* $\varphi(x) = \downarrow x$ *for all* $x \in N$*.*

Proof. "if:" Suppose $(N, \geq)$ is a poset. Then $\varphi : N \to \downarrow N$ as given in the statement is onto by construction. Let $x, y \in N$ and $y \geq x$. Consider any $z \in \downarrow x$. By transitivity $y \geq x \geq z$ implies $z \in \downarrow y$, so $\varphi(y) \supseteq \varphi(x)$. Conversely, let $x, y \in N$ and $\varphi(y) \supseteq \varphi(x)$. Then $x \in \downarrow x = \varphi(x) \subseteq \varphi(y) = \downarrow y$ implies $y \geq x$. Thus, $y \geq x \Leftrightarrow \varphi(y) \supseteq \varphi(x)$ shows that φ is order embedding. By Remark 1 an order embedding surjection is an order isomorphism.

"only if:" Let $(N, \geq)$ be a preordered set which admits a set representation. Let $(M, \supseteq)$ be the associated poset and $\psi : N \to M$ the order isomorphism. If both $x \geq y$ and $y \geq x$ hold for some $x, y \in N$, then by (2) $\psi(x) = \psi(y) \in M$ implies $x = y$, because ψ is one-to-one. Hence, $\geq$ is antisymmetric. ■

Proposition 1 identifies an order isomorphism between N and $\downarrow N$. The resulting set representation is referred to as the *set representation by principal (order) ideals.* Similar results are known, for instance for finite arbitrary ordered sets (Davey and Priestley [8], Theorem 8.19).

[1] In the context of sequential decision problems more appropriate names may be "past" (for the up-set) and "future" (for the down-set).

2.1. Trees and Subtrees. The following definition introduces the central object of this study.

Definition 5. *A* ***tree*** *is a poset* $(N, \geq)$ *such that* $\uparrow x$ *is a chain for all* $x \in N$*. In a tree the elements of* N *are called* ***nodes****. For nodes* $x, y \in N$ *say that* x ***precedes*** *(resp. follows)* y *if* $x \geq y$ *(resp.* $y \geq x$*) and* $x \neq y$*. A tree is* ***rooted*** *if there is a node* $x_o \in N$*, called the root, such that* $x_o \geq x$ *for all* $x \in N$*.*

The definition could, of course, also be stated dually, i.e. with an element that is not followed by other nodes (a "bottom" instead of a "top") and $\downarrow x$, but here the opposite convention is preferred.[2] With this caveat, this is the most general definition of trees in order theory.

Remark 2. *Definition 5 (stated dually) is given as an example of a poset by Davey and Priestley ([8], p. 23). However, in order theory the word "tree" is usually reserved for posets such that, additionally, the sets* $\uparrow x$ *are (dually) well-ordered: all their subsets have a first element according to* $\geq$ *(see Koppelberg [13], Chapter 6). This implies that immediate successors of non-terminal nodes are well-defined, but is unrelated to the existence of immediate predecessors, except in finite cases (see Alós-Ferrer and Ritzberger [1]). Koppelberg and Monk [14] dropped the well-ordered requirement and called the resulting concept (which coincides with the dual of our definition) a* pseudotree*. Order-theoretic analysis of pseudotrees, though, has concentrated on the analysis of the various (set) Boolean algebras that they give rise to: Koppelberg and Monk [14] study the algebra of subtrees (down-sets, in our notation); Baur and Heindorf [4] study the initial chain algebra (up-sets). For an order-theoretic characterization of the concept of pseudotree, see [2].*

The property of being a tree is preserved by order isomorphism, i.e., if a poset is order isomorphic to a tree, then it is itself a tree. By Proposition 1, every tree $(N, \geq)$ has a set representation by principal (order) ideals, $(\downarrow N, \supseteq)$. This is called the tree's *set representation by subtrees.* The name is motivated by the fact that for any $x \in N$ the ordered set $(\downarrow x, \geq)$ is itself a tree.

The next Lemma, the proof of which is immediate, identifies an alternative definition of trees as abstract order-theoretic structures.

Lemma 1. *A poset* $(N, \geq)$ *is a tree if and only if, for all* $x, y, z \in N$

$$\text{if } y \geq x \text{ and } z \geq x \text{ then } y \geq z \text{ or } z \geq y \tag{3}$$

If this last property is translated into set-theoretic properties, two alternatives are naturally identified.

[2]This convention is chosen to be able to associate the order relation $\geq$ on an abstract tree with set inclusion $\supseteq$ on its set representation.

Definition 6. *A V-poset $(M, \supseteq)$ satisfies* ***Trivial Intersection*** *if, for all $a, b \in M$*

$$\text{if } a \cap b \neq \emptyset \text{ then } a \subset b \text{ or } b \subseteq a \tag{4}$$

A V-poset $(M, \supseteq)$ satisfies ***Weak Trivial Intersection*** *if for all $a, b, c \in M$*

$$\text{if } c \subseteq a \cap b \text{ then } a \subset b \text{ or } b \subseteq a \tag{5}$$

Of course, Trivial Intersection implies Weak Trivial Intersection. If Weak Trivial Intersection is written in terms of an abstract partial order $\geq$, property (3) is obtained. Trivial Intersection, on the other hand, cannot be translated back into arbitrary posets, since there is in general a difference between an intersection of two nodes being empty and not containing any other node.

The next result makes use of these properties to give a full characterization of set representations of trees.

Proposition 2. *(a) A V-poset $(M, \supseteq)$ is a tree if and only if it satisfies Weak Trivial Intersection.*

(b) A poset $(N, \geq)$ is a tree if and only if its set representation by principal ideals $(\downarrow N, \supseteq)$ satisfies Trivial Intersection.

Proof. (a) It suffices to notice that Weak Trivial Intersection is equivalent to property (3) and apply Lemma 1.

(b) "if:" If $(\downarrow N, \supseteq)$ satisfies Trivial Intersection, then it satisfies Weak Trivial Intersection and by part (a) it is a tree. By isomorphism $(N, \geq)$ is a tree.

"only if:" Let $(N, \geq)$ be a tree and let $x, y \in N$ such that $\downarrow x \cap \downarrow y \neq \emptyset$. Let $z \in \downarrow x \cap \downarrow y$. It follows that $\downarrow z \subseteq \downarrow x \cap \downarrow y$. By isomorphism and part (a), $(\downarrow N, \supseteq)$ satisfies Weak Trivial Intersection, and hence either $\downarrow x \subset \downarrow y$ or $\downarrow y \subseteq \downarrow x$. ■

Any set representation of a tree is necessarily a tree (by order isomorphism), and, hence, Proposition 2(a) characterizes *all* set representations of trees as the V-posets satisfying Weak Trivial Intersection. Still, Trivial Intersection is more appealing. (Intuitively, we think of unordered nodes as disjoint entities.) Proposition 2(b) establishes that trees can also be characterized as those posets whose set representations by principal ideals satisfy Trivial Intersection. The implication is rather natural if one observes that, for $(\downarrow N, \supseteq)$, there is no difference between an intersection of two nodes being empty and not containing any other node, i.e. Weak Trivial Intersection and Trivial Intersection are the same property for this particular set representation. Still, there may be set representations of a tree which satisfy Weak Trivial Intersection but not Trivial Intersection.

Example 1. *Let $(M, \supseteq)$ be the $\{1, 2, 3\}$-poset given by*

$$M = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}\}$$

Trivial Intersection fails, because $\{1,2\} \cap \{2,3\} \neq \emptyset$ *and neither of the nodes contains the other. But Weak Trivial Intersection holds, because its hypothesis is void (for* $\{1,2\}$ *and* $\{2,3\}$*). The set representation by subtrees is given by*

$$M' = \{\{s_1, s_2, s_3\}, \{s_2\}, \{s_3\}\}$$

where $s_1 = \{1,2,3\}$, $s_2 = \{1,2\}$, *and* $s_3 = \{2,3\}$. *Now* $\{s_2\} \cap \{s_3\} = \emptyset$, *because nodes are elements of* 2^M *rather than* M.

But this set representation is still not satisfactory. Intuitively, one would like to remove the redundant element 2 from V *and obtain a set representation in terms of a* $\{1,3\}$*-poset* $(M'', \supseteq)$ *with* $M'' = \{\{1,3\}, \{1\}, \{3\}\}$.

This points to a fundamental question. In M' *the primitives are the nodes, i.e.,* M' *is formed by subsets of an underlying set (of sets). By contrast, in* M'' *only ultimate "outcomes" are elements of an underlying set of which the elements of* M'' *are subsets.*

2.2. Example: Differential Game (tree). The various concepts of trees that will be considered, starting with Definition 5, are quite general. We will allow for all classical examples from game theory, from finite trees to the infinite ones underlying repeated games, or Rubinstein's [20] bargaining game. The purpose of the following example, that will repeatedly be referred to, is to illustrate that our concepts go even further. They include as examples the trees of so-called "differential games" (decision problems in continuous time).

To see this, let V be the set of functions $f : \mathbb{R}_+ \to A$, where A is some given set of "actions," containing at least two elements, and let

$$\begin{aligned} N &= \{x_t(g) \,|\, g \in V,\ t \in \mathbb{R}_+\} \text{ where} \\ x_t(g) &= \{f \in V \,|\, f(\tau) = g(\tau),\ \forall \tau \in [0,t)\} \end{aligned}$$

for any $g \in V$ and $t \in \mathbb{R}_+$.

Intuitively, at each point in time $t \in \mathbb{R}_+$ a decision $a_t \in A$ is taken. The "history" of all decisions taken in the past (up to, but exclusive of, time t) is a function $f : [0,t) \to A$, i.e. $f(\tau) = a_\tau$ for all $\tau \in [0,t)$. A node at "time" t is the set of all functions which coincide with f on $[0,t)$, all possibilities still open for their values thereafter.

We claim that $(N, \supseteq)$ is a V-poset satisfying Trivial Intersection, and hence a tree by Proposition 2(a). To verify this claim, let $x_t(g)$ and $x_\tau(h)$ be two arbitrary nodes, with $g, h \in V$ and $t, \tau \in \mathbb{R}_+$. If $x_t(g) \cap x_\tau(h) \neq \emptyset$, then there is some $f \in V$ such that $f(s) = g(s)$ for all $s \in [0,t)$ and $f(s) = h(s)$ for all $s \in [0,\tau)$. If, say, $\tau \leq t$, then $g(s) = f(s) = h(s)$ for all $s \in [0,\tau)$, implying that $x_t(g) \subseteq x_\tau(h)$ as required.

In this tree, there is no "point in time" where the decision between two distinct nodes $x_t(g)$ and $x_t(h)$ for which $g(\tau) = h(\tau)$ for all $\tau \in [0,t)$, but $g(t) \neq h(t)$, is actually "taken."[3] However, the definition is operational in the sense that, in each

[3]Suppose the convention in the specification of nodes would be changed such that for two functions to belong to the same node they would have to agree on the closed interval $[0,t]$. Then there would be no "point in time" when the decision actually "becomes effective."

node $x_t(g)$, the decision that an agent has to take is clear (his action at time t), and the history up to that point is clearly specified (by the function g on $[0, t)$). Ultimately, a function $f \in V$ becomes a complete description of all decisions taken from the beginning to the end. The classical name for such descriptions is "plays."

2.3. Decision Trees. For a poset $(N, \geq)$ a chain $c \in 2^N$ is *maximal* if there is no $x \in N \setminus c$ such that $c \cup \{x\}$ is a chain. To see that every chain is contained in a maximal chain, recall the Hausdorff Maximality Principle, which is an equivalent form of the Axiom of Choice and, hence, Zorn's Lemma (see Birkhoff [5], Chapter VIII, or Hewitt and Stromberg [11], Chapter 1).

Hausdorff Maximality Principle. Let $(N, \geq)$ be a poset and $c \subseteq N$ a chain in N. Then there exists a maximal chain w in N such that $c \subseteq w$.

Definition 7. *For a tree $(N, \geq)$ a* ***play*** *w is a maximal chain in N. Denote by W the set of all plays. Given a node $x \in N$, let $W(x) = \{w \in W \,|\, x \in w\}$ be the set of all plays* ***passing through*** *x.*

The next result identifies key properties of the mapping $W : N \to W$. Its proof is straightforward and omitted (notice, e.g., that part (a) follows directly from the Hausdorff Maximality Principle).

Lemma 2. *For any tree $(N, \geq)$ and all nodes $x, y \in N$:*
(a) The set $W(x)$ of plays passing through x is nonempty,
(b) if $x \geq y$ then $W(x) \supseteq W(y)$.

Any chain is a tree. But in a chain nodes that follow a given node do not represent alternatives, because there is only one play for the whole tree. To model decisions, a given node should be followed by several others which are not related by $\geq$. The idea is that $\geq$ expresses "history," while nodes not related by $\geq$ model decisions among *alternative* "histories." If every node represents a decision, the following definition is obtained.

Definition 8. *A* ***decision tree*** *is a tree $(N, \geq)$ such that for all $x, y \in N$*

$$\text{if } W(x) = W(y) \text{ then } x = y \tag{6}$$

A decision tree is a tree without irrelevant nodes, where a node is irrelevant if it is followed only by one other node. The presence of irrelevant nodes would make it impossible to recover nodes as sets of plays, since the plays passing through two different nodes may be identical. Since irrelevant nodes serve no purpose for decision theory, Definition 8 rules them out and demands that, every time a node is reached, there must have been another alternative. An alternative definition of decision trees, relying explicitly on this intuition, is given next.

Lemma 3. *A tree $(N, \geq)$ is a decision tree if and only if for all $x, y \in N$*

$$\begin{array}{c}\text{if } x \geq y \text{ and } y \not\geq x \text{ then there is } z \in N \\ \text{such that } x \geq z,\ y \not\geq z, \text{ and } z \not\geq y\end{array} \tag{7}$$

Since property (7) is given purely in terms of the partial order $\geq$, it is easy to conclude that the property of being a decision tree is preserved by order isomorphism. That is, if a poset is order isomorphic to a decision tree, it must itself be a decision tree. The translation of (7) into set-theoretic terms gives rise to the following two concepts.

Definition 9. *A V-poset $(M, \supseteq)$ satisfies* ***Separability*** *if, for all $a, b \in M$*

$$\text{if } b \subset a, \text{ then there is } c \in M \text{ such that } c \subseteq a \text{ and } b \cap c = \emptyset \tag{8}$$

A V-poset $(M, \supseteq)$ satisfies ***Weak Separability*** *if, for all $a, b \in M$,*

$$\text{if } b \subset a, \text{ there is } c \in M \text{ such that } c \subseteq a \text{ but } c \setminus b \neq \emptyset \text{ and } b \setminus c \neq \emptyset \tag{9}$$

Clearly, Separability implies Weak Separability. If Weak Separability is written in terms of an abstract partial order $\geq$, property (7) is obtained. Separability, on the other hand (analogously to Trivial Intersection), cannot be translated back into arbitrary posets. However, as the next (immediate) lemma shows, the difference only exists in the absence of Trivial Intersection.

Lemma 4. *Let $(M, \supseteq)$ be a V-poset satisfying Trivial Intersection, and let $b, c \in M$. Then,*

$$b \setminus c \neq \emptyset \text{ and } c \setminus b \neq \emptyset \text{ if and only if } b \cap c = \emptyset$$

In particular, under Trivial Intersection, Weak Separability holds if and only if Separability holds.

It follows that Separability and Weak Separability are equivalent for the set representation by subtrees, but not necessarily for arbitrary set representations of trees. The next result makes use of these properties to give a full characterization of set representations of decision trees.

Proposition 3. *(a) A V-poset $(M, \supseteq)$ is a decision tree if and only if it satisfies Weak Trivial Intersection and Weak Separability.*

(b) A poset $(N, \geq)$ is a decision tree if and only if its set representation by subtrees $(\downarrow N, \supseteq)$ satisfies Trivial Intersection and Separability.

Proof. (a) It suffices to notice that Weak Separability is equivalent to property (7) and apply Lemma 3 and Proposition 2(a).

(b) "if:" If $(\downarrow N, \supseteq)$ satisfies Trivial Intersection and Separability, then it satisfies Weak Trivial Intersection and Weak Separability and by part (a) it is a decision tree. By isomorphism $(N, \geq)$ is a decision tree.

"only if:" Let $(N, \geq)$ be a decision tree. By Proposition 2(a), $(\downarrow N, \supseteq)$ satisfies Trivial Intersection. But, by isomorphism, $(\downarrow N, \supseteq)$ is a decision tree and by part (a) satisfies Weak Separability. By Lemma 4, we have that $(\downarrow N, \supseteq)$ satisfies also Separability. ■

This result is the analogous to Proposition 2 for decision trees. Any set representation of a decision tree is necessarily a decision tree (by order isomorphism), and hence Proposition 3(a) characterizes *all* set representations of decision trees as the V-posets satisfying Weak Trivial Intersection and Weak Separability. Proposition 3(b) establishes that decision trees can also be characterized as those posets whose set representations by principal ideals satisfy Trivial Intersection and Separability.

Since the set (of sets) M in the set representation of a decision tree can be arbitrary, there may be set representations of a decision tree for which Separability fails, but then, by Lemma 4, Trivial Intersection must also fail.

Example 2. *Consider again the* $\{1,2,3\}$*-poset* $(M, \supseteq)$ *from Example 1. Separability (8) does not hold, because* $\{1,2\} \subset \{1,2,3\}$ *and yet the only other node contained in* $\{1,2,3\}$*, that is,* $\{2,3\}$*, has a nonempty intersection with* $\{1,2\}$*. However, Weak Separability (9) holds. Neither of the nodes* $\{1,2\}$ *and* $\{2,3\}$ *contains the other.*

Example 3. *The differential game tree example from Section 2.2 above is a decision tree. For, let* $x_t(g)$ *and* $x_\tau(h)$ *be two nodes, with* $g, h \in V$ *and* $t, \tau \in \mathbb{R}_+$*, such that* $x_t(g) \subset x_\tau(h)$*. Then,* $\tau < t$*. Choose any* $f \in V$ *such that* $f(s) = h(s)$ *for all* $s \in [0, \tau)$ *and* $f(\tau) \neq g(\tau)$*. Then, for any* s *with* $\tau < s < t$*, we have that* $x_s(f) \subseteq x_\tau(h)$ *but* $x_s(f) \cap x_t(g) = \emptyset$*, verifying Separability. Since this example also satisfies Trivial Intersection, it follows from Proposition 3(a) that this tree is a decision tree.*

2.4. Representation by Plays. The arbitrariness of the V-poset representing a tree makes it difficult to interpret the elements of V. In this subsection, it is shown that every decision tree $(N, \geq)$ admits a set representation $(M, \supseteq)$ where $M \subseteq 2^W$ is a collection of nonempty sets of plays, i.e., every decision tree can be represented by a W-poset.

Intuitively, one should be able to take plays and nodes alternatively as the primitives of a tree. If nodes are the primitives, plays are derived as maximal chains. If plays are the primitives, nodes are recovered as sets of plays sharing a common history.

Definition 10. *For a tree* $(N, \geq)$ *its* ***image in plays*** *is the tree* $(W(N), \supseteq)$*, where*

$$W(N) = \{W(x)\}_{x \in N} = \left\{ a \in 2^W \,|\, \exists x \in N : a = W(x) \right\}$$

and $\supseteq$ *is set inclusion.*

It is easy to see that a tree's image in plays satisfies Trivial Intersection, and, hence, is itself a tree (by Proposition 2(a)).

Lemma 5. *Let $(N, \geq)$ be a tree. Its image in plays $(W(N), \supseteq)$ satisfies Trivial Intersection.*

Definition 11. *A tree $(N, \geq)$ can be* ***represented by plays*** *if the mapping*[4] *$W : N \to W(N)$ is an order isomorphism between $(N, \geq)$ and its image in plays $(W(N), \supseteq)$. The latter is then called the tree's (set)* ***representation by plays****.*

The image in plays is the natural candidate for a "canonical" set representation. An arbitrary tree, though, need not be order isomorphic to its image in plays.

Theorem 1. *A tree $(N, \geq)$ can be represented by plays if and only if it is a decision tree.*

Proof. "if:" Let W be the set of plays. The set $W(N)$ and its elements are non-empty by Lemma 2(a). The mapping $W : N \to W(N)$ is one-to-one by (6) and onto by construction. Next, it is verified that the bijection W is order embedding.

Let $x, y \in N$. If $y \geq x$, then by Lemma 2(b) $W(x) \subseteq W(y)$. Conversely, suppose $W(x) \subseteq W(y)$. Choose $w \in W(x) \subseteq W(y)$. Since $x, y \in w$, either $x \geq y$ or $y \geq x$. In the first case, the previous argument would imply $W(x) = W(y)$ and, therefore, $x = y$, because W is one-to-one. Since $\geq$ is reflexive, in both cases $y \geq x$. Hence, $y \geq x \Leftrightarrow W(x) \subseteq W(y)$ for all $x, y \in N$, i.e. $W(\cdot)$ is an order isomorphism.

"only if:" Let $x, y \in N$. If $W(x) = W(y)$, then $x = y$ because the mapping W is one-to-one. ■

The set representation by plays of a decision tree is itself a decision tree (by isomorphism) which satisfies Trivial Intersection by Lemma 5 and Weak Separability by Proposition 3(a). Hence, it also satisfies Separability (by Lemma 4).

Corollary 1. *If $(N, \geq)$ is a decision tree then its image in plays $(W(N), \supseteq)$ satisfies Separability.*

Hence, the set representation by plays of a decision tree satisfies Trivial Intersection and Separability, and is order-isomorphic to the decision tree. These results can also be understood as follows. The properties that characterize set representations of decision trees, Weak Trivial Intersection and Weak Separability, have order-theoretic analogues, that are preserved by order isomorphisms. Trivial Intersection and Separability, on the other hand, make sense only for V-posets and, hence, are not preserved by order isomorphisms. However, both the set representation by subtrees and the set representation by plays of a decision tree satisfy Trivial Intersection and Separability

[4]No confusion should arise between the mapping $W(\cdot)$ assigning to each node x the set of plays passing through x and the set W of all plays.

and can be taken as "canonical." The former gives a particularly "bulky" representation, while the latter conforms to our intuition, according to which we should be able to take either nodes or plays as primitives. Besides, it will be seen that the representation by plays satisfies stronger properties.

Example 4. *Let $(M, \supseteq)$ be a decision tree with $M = \{\{1,2,3\}, \{1,2\}, \{3\}\}$. This tree satisfies Trivial Intersection and Separability, but a decision between* 1 *and* 2 *is never taken, i.e., there is a redundant element in the underlying set. Its set representation by subtrees is given by*

$$M' = \{\{s_1, s_2, s_3\}, \{s_2\}, \{s_3\}\}$$

where $s_1 = \{1,2,3\}$, $s_2 = \{1,2\}$, and $s_3 = \{3\}$. In this representation there is also an irrelevant element in the underlying set, because no decision is ever taken to select s_1. The representation by plays of $(M, \supseteq)$ is given by $M' = \{\{w_1, w_2\}, \{w_1\}, \{w_2\}\}$, where $w_1 = \{\{1,2,3\}, \{1,2\}\}$ and $w_2 = \{\{1,2,3\}, \{3\}\}$. In a sense, the redundant element 2 *has disappeared.*

This example shows that the representation by plays "reduces" the underlying set, eliminating irrelevant elements. What is still missing is a further separation property which guarantees that, given two elements of the underlying set, there is always a decision to distinguish between them.

Example 5. *Since the differential game tree example of Section 2.2 is a decision tree, it can be represented by plays. Given a node $x_t(g)$, the set of plays passing through it is given by*

$$W(x_t(g)) = \Big\{ \{x_\tau(f)\}_{\tau \in [0,\infty)} \,|\, f \in V \text{ with } f(\tau) = g(\tau)\, \forall\, \tau \in [0,t) \Big\}.$$

3. Set Trees

In this section we consider decision trees which are V-posets satisfying the two key properties of the set representations by plays and by subtrees.

Definition 12. *A V-poset $(M, \supseteq)$ is a V**-set tree** if it satisfies (4) and (8), i.e., for all $a, b \in M$*

(Trivial Intersection) *if $a \cap b \neq \emptyset$ then either $a \subset b$ or $b \subseteq a$, and*

(Separability) *if $b \subset a$ then $\exists\, c \in M$ such that $c \subseteq a$ and $b \cap c = \emptyset$.*

*A V-set tree is **rooted** if $V \in M$.*

As we have seen, an example of a V-set tree is given by the differential game (decision) tree from Section 2.2. All V-set trees are decision trees, but not all V-posets, that are decision trees, are also V-set trees, (see Proposition 3(a)). However, given a decision tree $(N, \geq)$, we can find two alternative set representations which turn out to be V-set trees. The first is the set representation by subtrees, which is a $\downarrow N$-set tree (where $\downarrow N$ is the set of subtrees of $(N, \geq)$) by Proposition 3(b). The second (by Theorem 1) is the image in plays, which is a W-set tree (where W is the set of plays of $(N, \geq)$) by Lemma 5 and Corollary 1.

3.1. Reduced-Form Posets. Intuitively, Separability for a V-set tree $(M, \supseteq)$ ensures that there are *no redundant nodes* in M. Yet, there may still be *redundant elements* in V. Roughly, an element $v \in V$ is redundant, if it can be deleted without affecting the structure of the tree. But there are two meanings for when an element of V is redundant.

Example 6. *Let* $V = \{1, 2, 3, 4, 5\}$ *and* $M = \{\{1, 2, 3, 4\}, \{1, 2\}, \{3\}\}$. *Then* $(M, \supseteq)$ *satisfies Trivial Intersection (4) and Separability (8). For, if* $b \subset a$ *then* $a = \{1, 2, 3, 4\}$, *so that there always is* $c \in M \setminus \{a, b\}$ *such that* $c \subset a$ *and* $b \cap c = \emptyset$. *On the other hand,* V *contains redundant elements for two reasons.*

First, $4 \notin \{1, 2\} \cup \{3\}$, *but* $4 \in a \in M$ *implies* $a = \{1, 2, 3, 4\}$ *so that* $\{1, 2\} \cup \{3\} \subset a$*; hence, there is no* $b \in M$ *with* $v \in b \setminus a$ *for* $v = 1, 2, 3$. *Intuitively, element* $4 \in V$ *is* not separable. *Similarly, since there is no* $a \in M$ *with* $5 \in a$, *there are no* $a, b \in M$ *such that* $5 \in a \setminus b$ *and* $v \in b \setminus a$ *for* $v = 1, 2, 3, 4$. *Second,* $1 \neq 2$, *but* $1 \in c \in M$ *if and only if* $2 \in c \in M$. *Intuitively, elements* $1, 2 \in V$ *are* duplicates.

In this example we attribute the first redundancy to the two elements $4, 5 \in V$ not being *separable*. The structure of the tree $(M, \supseteq)$ would not be affected by eliminating elements 4 and 5 from V. The second redundancy we attribute to elements $1, 2 \in V$ being *duplicates*. If one of them were eliminated (or they would be identified), the structure of the tree $(M, \supseteq)$ would not be affected.

To pin down these redundancies, extend the definition of the up-set to elements of the underlying set as follows. Let $(M, \supseteq)$ be a V-poset, $v \in V$, and define

$$\uparrow\{v\} = \{a \in M \mid v \in a\} \tag{10}$$

If $\{v\} \in M$, this coincides with the previously defined up-set. With this convention, the aforementioned redundancies can be tackled.

We start with duplicates. Define the equivalence relation $\sim$ on V by

$$v \sim v' \text{ if } \uparrow\{v\} = \uparrow\{v'\} \tag{11}$$

that is, if, for all $a \in M$, $v \in a \Leftrightarrow v' \in a$. Note that $v \in a \Leftrightarrow [v] \subseteq a$ for all $a \in M$ and all $v \in V$, where $[v]$ denotes the equivalence class (with respect to $\sim$) to which v belongs. In Example 6 we have $1 \sim 2$, so $[1] = [2] = \{1, 2\}$, but $[v] = \{v\}$ for $v = 3, 4, 5$. By definition, it is now justified to write $\uparrow[v] = \uparrow\{v\}$.

Obviously, any V-poset $(M, \supseteq)$ can be identified with a $(V/\sim)$-poset, where $V/\sim$ is the quotient set, and this representation will contain no duplicate elements.

Turning to separable elements, consider the subset $S(V)$ of the quotient space $V/\sim$ defined by

$$S(V) = \left\{[v] \in V/\sim \,\middle|\, \cap_{a \in \uparrow[v]} a = [v]\right\} \tag{12}$$

which will be referred to as the set of *separable* equivalence classes. In Example 6 we have $\uparrow[4] = \{\{1, 2, 3, 4\}\}$ and $\uparrow[5] = \emptyset$, so $[4], [5] \notin S(V)$, while $\cap_{a \in \uparrow[v]} a = [v]$ for $v = 1, 2, 3$. The following justifies the use of the word "separable" for these classes.

Lemma 6. *Let* $(M, \supseteq)$ *be a* V*-poset. The equivalence class* $[v] \in V/\sim$ *is separable, i.e.* $[v] \in S(V)$*, if and only if for all* $v' \in V \setminus [v]$ *there is* $a \in M$ *such that* $[v] \subseteq a$ *and* $v' \notin a$*, i.e.* $V \setminus [v] = V \setminus \cap_{a \in \Uparrow [v]} a = \cup_{a \in \Uparrow [v]} (V \setminus a)$

This result characterizes separable equivalence classes as those which can be "separated" from other classes by nodes. The next result shows that the intersection of any two elements from a V-poset $(M, \supseteq)$ contains at least one separable equivalence class.

Lemma 7. *Let* $(M, \supseteq)$ *be a* V*-poset. If* $a, b \in M$ *are such that* $a \cap b \neq \emptyset$ *(not necessarily* $a \neq b$*) then there is* $[v] \in S(V)$ *such that* $[v] \subseteq a \cap b$*.*

Definition 13. *For a* V*-poset* $(M, \supseteq)$ *its* ***reduced form*** *is the* $S(V)$*-poset* $(M^*, \supseteq)$ *given by*

$$M^* = \{a^* \subseteq S(V) \mid \exists\, a \in M : [v] \in a^* \Leftrightarrow [v] \subseteq a\}$$

For instance, $S(V) = \{[1], [3]\}$ and $M^* = \{\{[1], [3]\}, \{[1]\}, \{[3]\}\}$ in Example 6.

Proposition 4. *If the* V*-poset* $(M, \supseteq)$ *is a* V*-set tree, then it is order isomorphic to its reduced form with order isomorphism* $\varphi : M \to M^*$ *given by* $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$*.*

Proof. We first show that the mapping φ, as defined in the statement, is onto. Let $a^* \in M^*$. Then there is $a \in M$ such that $[v] \in a^*$ if and only if $[v] \subseteq a$, i.e. $a^* = \varphi(a)$ and φ is onto.

Let $a, b \in M$ be such that $a \subseteq b$. Then $a \supseteq [v] \in S(V)$ implies $b \supseteq [v] \in S(V)$, so $\varphi(a) \subseteq \varphi(b)$. Conversely, if $a, b \in M$ are such that $\varphi(a) \subseteq \varphi(b)$, then $[v] \in \varphi(a)$ implies $[v] \in \varphi(b)$, so $a \supseteq [v] \in S(V)$ implies $b \supseteq [v]$, hence, $a \cap b \neq \emptyset$. By Trivial Intersection, either $a \subseteq b$ or $b \subset a$. If $b \subset a$, then by Separability there is $c \in M$ such that $c \subseteq a$ and $b \cap c = \emptyset$. By Lemma 7, we can choose $[v'] \in S(V)$ such that $[v'] \subseteq c$. Then $\varphi(c) \subseteq \varphi(a) \subseteq \varphi(b)$ implies $[v'] \subseteq b$, in contradiction to $b \cap c = \emptyset$. Hence, $a \subseteq b$ must hold and φ is order embedding. By Remark 1, the statement is verified. ■

That the hypothesis of a V-set tree (rather than a V-poset) is *necessary* for Proposition 4 is illustrated by the following example.

Example 7. *Reconsider Example 1. There,* $\uparrow \{1\} = \{\{1, 2, 3\}, \{1, 2\}\}$*,* $\uparrow \{2\} = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}\} = M$*, and* $\uparrow \{3\} = \{\{1, 2, 3\}, \{2, 3\}\}$*, so all equivalence classes with respect to* $\sim$ *are singletons, but only* $2 \in V$ *is separable, i.e.* $S(V) = \{[2]\}$*, and* $[2] \subseteq a$ *for all* $a \in M$*. Therefore,* $M^* = \{\{2\}\}$ *cannot be order isomorphic to* $(M, \supseteq)$*. This is due to a violation of Trivial Intersection.*

Yet, this example does not mean that Proposition 4 can be strengthened to a characterization. The next example shows that there are V-posets (in fact, trees) that are order isomorphic to their reduced form, but are not V-set trees. The crucial point is the step from Weak Trivial Intersection to Trivial Intersection.

Example 8. *Let* $V = \{1,2,3\}$ *and* $M = \{\{1,2,3\},\{1,2\},\{2,3\},\{1,3\}\}$. *Then,*

$$\begin{aligned} \uparrow\{1\} &= \{\{1,2,3\},\{1,2\},\{1,3\}\} \text{ and } \cap_{a\in\uparrow\{1\}} a = \{1\} = [1], \\ \uparrow\{2\} &= \{\{1,2,3\},\{1,2\},\{2,3\}\} \text{ and } \cap_{a\in\uparrow\{2\}} a = \{2\} = [2], \\ \uparrow\{3\} &= \{\{1,2,3\},\{2,3\},\{1,3\}\} \text{ and } \cap_{a\in\uparrow\{3\}} a = \{3\} = [3], \end{aligned}$$

so all equivalence classes are singletons and all elements of V *are separable, i.e.,* $S(V) = \{[1],[2],[3]\}$. *Therefore,*

$$M^* = \{\{[1],[2],[3]\},\{[1],[2]\},\{[2],[3]\},\{[1],[3]\}\},$$

so $(M,\supseteq)$ *and* $(M^*,\supseteq)$ *are order isomorphic by* $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$.

This example also shows that without Trivial Intersection (but still with Separability) it may not be possible to find a subset $V' \subseteq V$ such that Trivial Intersection holds for $(M',\supseteq)$ with

$$M' = \{a' \subseteq V' \mid \exists\, a \in M : a' = a \cap V'\}$$

and $(M',\supseteq)$ is order isomorphic to $(M,\supseteq)$. For, if $1 \in V'$ then $1 \in \{1,2\} \cap V'$ and $1 \in \{1,3\} \cap V'$. So, if $(M',\supseteq)$ satisfies Trivial Intersection, then $1 \notin V'$. Therefore, if $\{1,2\} \cap V' \neq \emptyset$, it follows that $2 \in V'$ and $2 \in \{1,2\} \cap V'$. But then $2 \in \{2,3\} \cap V'$ contradicts Trivial Intersection on $(M',\supseteq)$. Hence, $(M',\supseteq)$ cannot satisfy Trivial Intersection and be order isomorphic to $(M,\supseteq)$ at the same time.

3.2. Reduced Form and Plays. In this subsection the relation between separable classes and plays (maximal chains of nodes) is explored.

Recall that, by Lemma 7, for a V-poset $(M,\supseteq)$ every element $a \in M$ contains at least one separable equivalence class. Reciprocally, if for a V-poset $(M,\supseteq)$ and $v \in V$ there exists some $a \in M$ such that $a \subseteq [v]$, then $[v] \in S(V)$. For, $a \subseteq [v]$ implies $a = [v]$ (because $v' \in [v]$ implies $v' \sim v$ and, therefore, $v' \in a$ and $[v] \subseteq a$) so that $a \in\uparrow[v]$ and $a = [v] \subseteq b$ for all $b \in\uparrow[v]$ imply that $a = \cap_{b\in\uparrow[v]} b = [v]$, as required.

That there is $a \in M$ such that $a \subseteq [v]$ is, therefore, sufficient for $[v] \in S(V)$. But it is not necessary, as the next example shows.

Example 9. *Let* $V = [0,1]$ *and* $M = \left\{(\{v\})_{v\in(0,1]}, (x_t)_{t=1}^{\infty}\right\}$, *where* $x_t = \left[0, \frac{1}{t}\right]$ *for all* $t = 1,2,\ldots$ *Then* $\cap_{a\in\uparrow[0]} a = \{0\} = [0]$, *but there is no* $a \in M$ *such that* $a = [0]$.

Hence, there are more separable equivalence classes than those which coincide with a node without a successor.[5] The significance of separable equivalence classes is revealed by the next result.

[5] Notice that a Cantor-set construction would allow to build an example with infinitely many such classes.

Proposition 5. *A V-poset $(M, \supseteq)$ satisfies Trivial Intersection if and only if $\uparrow[v] \in W$ for all $[v] \in S(V)$, where W denotes the set of plays (maximal chains) for $(M, \supseteq)$.*

Proof. "if:" Let W be the set of maximal chains for $(M, \supseteq)$ and assume that $\uparrow[v] \in W$ for all $[v] \in S(V)$. If $a, b \in M$ are such that $a \cap b \neq \emptyset$ then by Lemma 7 there is $[v] \in S(V)$ such that $[v] \subseteq a \cap b$, i.e., $a, b \in \uparrow[v]$. But then $\uparrow[v] \in W$ implies either $a \subset b$ or $b \subseteq a$, verifying Trivial Intersection.

"only if:" By Trivial Intersection $\uparrow[v]$ is a chain for all $[v] \in S(V)$. Suppose there is $a \in M \setminus \uparrow[v]$ such that $\uparrow[v] \cup \{a\}$ is a chain. If there would be some $b \in \uparrow[v]$ such that $b \subseteq a$, then $[v] \subseteq a$ in contradiction to $a \notin \uparrow[v]$. Thus, if $\uparrow[v] \cup \{a\}$ is a chain, then $a \subset b$ for all $b \in \uparrow[v]$. Then $a \subseteq \cap_{b \in \uparrow[v]} b = [v]$, i.e. $a = [v]$, again in contradiction to $a \notin \uparrow[v]$. ■

Even for arbitrary V-posets, on $S(V)$ the mapping $[v] \mapsto \uparrow[v]$ is one-to-one (injective). For, if $\uparrow[v] = \uparrow[v']$ then $[v] = \cap_{a \in \uparrow[v]} a = \cap_{a \in \uparrow[v']} = [v']$.

Lemma 8. *Let $(M, \supseteq)$ be a V-poset. For all $[v], [v'] \in S(V)$, if $\uparrow[v] = \uparrow[v']$ then $[v] = [v']$.*

Hence, we have seen that on the set of separable equivalence classes $S(V)$ for a V-poset satisfying Trivial Intersection the mapping $[v] \mapsto \uparrow[v]$ defines an injection into the set of plays W (by Lemma 8). And if $\uparrow[v] \in W$ for all $[v] \in S(V)$, then the V-poset $(M, \supseteq)$ satisfies Trivial Intersection, by Proposition 5.

Proposition 5 could be simply re-stated as follows: Trivial Intersection is equivalent to $\{\uparrow[v] \,|\, [v] \in S(V)\} \subseteq W$. The reverse inclusion, though, is not true, as a slight modification of the last example shows.

Example 10. *Let $V = (0, 1]$ and $M = \left\{(\{v\})_{v \in (0,1]}, (x_t)_{t=1}^{\infty}\right\}$, where $x_t = \left(0, \frac{1}{t}\right]$ for all $t = 1, 2, ...$ Then $w = \{x_t\}_{t=1}^{\infty}$ is a play that corresponds to no separable class.*

The problem in this example is that V itself is not large enough, since intuitively a play fails to lead to an ultimate outcome (even in the limit).

3.3. Irreducible Set Trees. Proposition 5 suggests that separable equivalence classes in $S(V)$ can be used to represent plays for a V-set tree. If the elements of $S(V)$ would correspond to singletons in V, this would yield an interpretation of the elements of V as representatives of plays. In this subsection V-set trees $(M, \supseteq)$ are identified for which all separable equivalence classes are singletons of V.

Definition 14. *A V-poset $(M, \supseteq)$ satisfies **Irreducibility**, or is **irreducible**, if, for all $v, v' \in V$*

$$\text{if } v \neq v' \text{ then } \exists\, a, b \in M \text{ such that } v \in a \setminus b \text{ and } v' \in b \setminus a \tag{13}$$

Remark 3. *By Lemma 4 it follows that, if a V-poset $(M, \supseteq)$ satisfies Trivial Intersection, then Irreducibility holds if and only if* ***Strong Irreducibility*** *holds:*[6] *for all* $v, v' \in V$

$$\text{if } v \neq v' \text{ then } \exists\, a, b \in M : v \in a,\ v' \in b, \text{ and } a \cap b = \emptyset \tag{14}$$

Under Trivial Intersection, Irreducibility implies Separability. The proof of this implication is straightforward and omitted.

Lemma 9. *If a V-poset $(M, \supseteq)$ satisfies Trivial Intersection and Irreducibility, then it satisfies Separability.*

The converse of Lemma 9 is not true. The trivial V-set tree $(\{V\}, \supseteq)$ satisfies Separability, because the hypothesis is void, but it fails Irreducibility, whenever V is not a singleton set.

The set representation by plays of a decision tree satisfies Trivial Intersection by Lemma 5. It is easy to show that it also satisfies Irreducibility and, therefore, is an irreducible set tree.[7] Hence, every decision tree $(N, \geq)$ is order isomorphic to an irreducible W-set tree. But the hypothesis of a *decision* tree is only required to make the image in plays a set representation.

Lemma 10. *Let $(N, \geq)$ be a tree. Its image in plays $(W(N), \supseteq)$ is an irreducible tree.*

Recall that, by Lemma 6, separable equivalence classes can be "separated" from other classes by choosing appropriate elements of M. This immediately implies the following:

Lemma 11. *Let $(M, \supseteq)$ be a V-poset. Its reduced form $(M^*, \supseteq)$ is irreducible.*

Finally, a V-poset is irreducible if and only if the elements of $S(V)$ are the singleton subsets of V.

Proposition 6. *A V-poset is irreducible if and only if $S(V) = \{\{v\}\}_{v \in V}$.*

Proof. "if:" If $S(V) = \{\{v\}\}_{v \in V}$ then by Lemma 11 Irreducibility holds for all $v, v' \in V$ (the set tree and its reduced form must then be identical).

"only if:" To see the converse, let $(M, \supseteq)$ be an irreducible V-poset and consider any $v \in V$. By Irreducibility for any $v' \in V \setminus \{v\}$ there are $a, b \in M$ such that $v \in a \setminus b$ and $v' \in b \setminus a$, implying that $[v] \neq [v']$ and, therefore, $[v] = \{v\}$ for all $v \in V$. That is, all equivalence classes are singletons. We still have to show that they are separable.

[6] Observe the formal analogy of Strong Irreducibility with the definition of a Hausdorff space in topology.

[7] The set representation by subtrees of a decision tree *cannot* satisfy Irreducibility. For, if $x \neq y$ and $x \geq y$ then $\downarrow y \subseteq \downarrow x$, so that $x \in \downarrow z$ and $y \in \downarrow z'$ for $z, z' \in N$ implies $\downarrow x \subseteq \downarrow z$ and $\downarrow y \subseteq \downarrow z'$ and, therefore, $y \notin \downarrow z' \setminus \downarrow z$.

Consider any class $[v] = \{v\}$. If $[v] \subset \cap_{a\in\uparrow[v]} a$ then there exists $v' \neq v$ such that $v' \in a$ for all $a \in\uparrow \{v\}$. This contradicts Irreducibility, because the latter requires the existence of some $a \in\uparrow \{v\}$ and $b \in M$ such that $v' \in b \setminus a$. Hence, $[v] = \cap_{a\in\uparrow[v]} a$, i.e. $[v]=\{v\} \in S(V)$.

This implies $\{\{v\}\}_{v\in V} \subseteq S(V)$. Since all equivalence classes are singletons, the reverse inclusion follows. ■

Hence, for irreducible V-set trees the set $S(V)$ of separable equivalence classes simply consists of the collection of all singleton subsets of V. Therefore, for an irreducible V-set tree the mapping $v \mapsto\uparrow \{v\}$ on V is a one-to-one function into the set W of plays (by Proposition 5 and Lemma 8).

Example 11. *The differential game tree of Section 2.2 satisfies (Strong) Irreducibility. For, if $g, f \in V$ and $g \neq f$, then there is some $t \in \mathbb{R}_+$ such that $f(t) \neq g(t)$. For any τ with $t < \tau$ it then follows that $x_\tau(f) \cap x_\tau(g) = \emptyset$ (because elements of V are functions), verifying (14). Moreover, every $g \in V$ uniquely induces the play $\uparrow\{g\} = \{x_t(g)\}_{t\in[0,\infty)}$.*

3.4. Proper Order Isomorphisms. Set trees are not only decision trees, but have more structure. In this subsection we study when the "strong" properties, Trivial Intersection and Separability, are preserved by order isomorphisms. This will shed more light on Irreducibility.

Definition 15. *Let $(M, \supseteq)$ be a V-poset, $(M', \supseteq)$ a V'-poset, and $\varphi : M \to M'$ an order isomorphism between the two. The order isomorphism φ is **proper** if*

$$\varphi(a) \cap \varphi(b) = \emptyset \text{ implies } a \cap b = \emptyset \text{ for all } a, b \in M \tag{15}$$

Note that properness need not be symmetric. That $(M, \supseteq)$ is properly isomorphic to $(M', \supseteq)$ does not necessarily imply that $(M', \supseteq)$ is properly isomorphic to $(M, \supseteq)$. Yet, a proper order isomorphism is necessary and sufficient to preserve the strong properties, as the next result states.

Lemma 12. *Let $(M, \supseteq)$ be a V-poset, $(M', \supseteq)$ a V'-poset, and $\varphi : M \to M'$ an order isomorphism between the two.*

(a) If $(M, \supseteq)$ satisfies Trivial Intersection, then φ is proper.

(b) If $(M', \supseteq)$ satisfies Trivial Intersection, then $(M, \supseteq)$ satisfies Trivial Intersection if and only if φ is proper.

(c) If $(M', \supseteq)$ is a V'-set tree, then $(M, \supseteq)$ is a V-set tree if and only if φ is proper.

Consider a decision tree $(M, \supseteq)$. Its image in plays, $(W(M), \supseteq)$, is order isomorphic to $(M, \supseteq)$ by Theorem 1. Still, the order isomorphism is not necessarily proper. To see why, recall that decision trees are characterized only by Weak Trivial Intersection and Weak Separability (Proposition 3(a)), while $(W(M), \supseteq)$ is a W-set tree by Lemma 5 and Corollary 1. Hence, the concept of proper order isomorphism adds further structure.

On the other hand, if both $(M, \supseteq)$ and $(M', \supseteq)$ satisfy Trivial Intersection, then all order isomorphism between them *and their inverses* are trivially proper. Thus, the following characterization says that set trees are precisely those posets (of sets) that are properly order isomorphic to their reduced forms, provided the latter are irreducible set trees.

Theorem 2. *A V-poset $(M, \supseteq)$ is a V-set tree if and only if its reduced form $(M^*, \supseteq)$ is an irreducible $S(V)$-set tree and $(M, \supseteq)$ is properly (order) isomorphic to $(M^*, \supseteq)$.*

Proof. The "if"-part is immediate from Lemma 12. For the "only if"-part, note that Irreducibility of $(M^*, \supseteq)$ follows from Lemma 11. That $(M, \supseteq)$ is order isomorphic to its reduced form follows from Proposition 4. That the order isomorphism $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$ from Proposition 4 is proper follows from Lemma 7. For, if $a \cap b \neq \emptyset$, for some $a, b \in M$, then by Lemma 7 there is $[v] \in S(V)$ such that $[v] \subseteq a \cap b$, hence $[v] \in \varphi(a) \cap \varphi(b)$ by the construction of φ. ■

Intuitively, Theorem 2 shows that any V-set tree is properly order-isomorphic to an irreducible set tree, obtained by appropriately shrinking the underlying set V (to the set of separable equivalence classes). Moreover, by Proposition 5 for any V-set tree the mapping $[v] \mapsto \uparrow[v]$ on $S(V)$ is an injection into the set W of plays (by Lemma 8). If this mapping were onto (surjective), W and $S(V)$ could be identified. (Recall, though, Example 10.) In particular, if this mapping were onto for an *irreducible* V-set tree, then there would be no distinction between W and V (rather than $S(V)$), due to Proposition 6. This discussion motivates the following.

Definition 16. *Let $(M, \supseteq)$ be a V-poset and $(M', \supseteq)$ a V'-poset. An order isomorphism $\varphi : M \to M'$ is an **isomorphic embedding** if there is an injection $f : V \to V'$ such that*

$$f(a) = \{v' \in V' \,|\, \exists\, v \in a : v' = f(v)\} \subseteq \varphi(a) \text{ for all } a \in M \tag{16}$$

*If $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$ and, moreover, f is also onto (surjective) and satisfies $f(a) = \varphi(a)$ for all $a \in M$, then $(M, \supseteq)$ and $(M', \supseteq)$ are **doubly** (order) **isomorphic**.*

It is trivially true that every isomorphic embedding is a proper order isomorphism. For, if $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$ and $a, b \in M$ are such that $v \in a \cap b$, then $f(v) \in \varphi(a) \cap \varphi(b)$ implies $\varphi(a) \cap \varphi(b) \neq \emptyset$. The converse is not true: By Theorem 2 a V-set tree is properly order isomorphic to its reduced form, the latter is irreducible, but the former may not be. But, as the next lemma establishes, only irreducible V-posets can be isomorphically embedded in irreducible V'-posets.

Lemma 13. *Let $(M, \supseteq)$ be a V-poset which is isomorphically embedded in a V'-poset $(M', \supseteq)$. If $(M', \supseteq)$ satisfies Irreducibility, then so does $(M, \supseteq)$.*

In other words, like other "strong" properties, Irreducibility (which refers to the underlying set and, hence, cannot be stated in purely order-theoretic terms) is inherited by V-posets isomorphically embedded in irreducible V'-posets. Therefore, irreducible set trees are precisely those that are doubly isomorphic to their reduced forms.

Proposition 7. *A V-set-tree is irreducible if and only if it is doubly isomorphic to its reduced form.*

Proof. The "if"-part follows from Lemmata 11 and 13. For the "only if"-part, let $(M, \supseteq)$ be an irreducible V-set tree. By Theorem 2 it is properly order isomorphic to its reduced form. By Proposition 6 $S(V) = \{\{v\}\}_{v \in V}$ and the mapping f, given by $f(v) = [v]$, is a bijection such that $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\} = f(a)$ for all $a \in M$. ■

This clarifies the status of Irreducibility. It is equivalent to the property that a set tree is not only properly order isomorphic to its reduced form, but the underlying sets, V for the set tree and $S(V)$ for its reduced form, also "look alike."

3.5. Bounded Set Trees. Proposition 5 says that certain plays for a V-set tree $(M, \supseteq)$ can be represented by elements of $S(V)$. According to Proposition 6 equivalence classes in $S(V)$ have to be used to represent plays, because V may be "too large." If the V-set tree were irreducible, V could be used directly.

But even Irreducibility does not ensure that *all* plays for $(M, \supseteq)$ can be represented by elements of V. The problem is that for some play $w \in W$ the set $\{v \in V \,|\, v \in a, \forall a \in w\}$ may be empty, so that not every play is represented by some $v \in V$, i.e., the given set V may be "too small." This was the case in Example 10 and is so in the following.

Example 12. *(Infinite Centipede) Let $V = \{1, 2, ...\}$ be the set of natural numbers, define $a_t = \{\tau \in V \,|\, t \leq \tau\}$ for all $t = 1, 2, ...$, and let $M = \{(\{t\})_{t=1}^{\infty}, (a_t)_{t=1}^{\infty}\}$. Since $a_\tau \subseteq a_t \Leftrightarrow \tau \geq t$ and $\{\tau\} \subseteq a_t \Leftrightarrow t \leq \tau$, for all $\tau, t = 1, 2, ...$, Trivial Intersection holds. Moreover, $\tau \neq t$, say, $t < \tau$, implies that $t \in \{t\}$, $\tau \in a_\tau$, and $\{t\} \cap a_\tau = \emptyset$, so Irreducibility also holds. Therefore, $(M, \supseteq)$ is an irreducible V-set tree.*

The set W of plays for $(M, \supseteq)$ consists of sets of the form $\left\{\{a_\tau\}_{\tau=1}^{t}, \{t\}\right\}$ for all $t = 1, 2, ...$ plus the play $\{a_t\}_{t=1}^{\infty}$. Every play of the form $\left\{\{a_\tau\}_{\tau=1}^{t}, \{t\}\right\}$ can be represented by the natural number t, for all $t = 1, 2, ...$, but the play $\{a_t\}_{t=1}^{\infty}$ cannot be represented by a natural number. Yet, if the element "∞" is added to V, the latter play can *be represented by this added element.*

This suggests that for an irreducible V-set tree $(M, \supseteq)$ the underlying set V could be used to represent *all* plays, *provided* V is "large enough." If this holds, then for any V-set tree with "large enough" V the separable equivalence classes $S(V)$ could be used to represent *all* plays. The next criterion makes precise what "large enough" V means.

Definition 17. *A V-poset $(M, \supseteq)$ is* ***bounded*** *(from below) if every chain in M has a lower bound in V, i.e., if for all chains $c \in 2^M$ there is $v \in V$ such that $v \in a$ for all $a \in c$.*

The image in plays of a tree $(N, \geq)$ is bounded. For, if $c \subseteq W(N)$ is a chain, then there is a chain $c' \subseteq N$ such that $x \in c'$ implies $W(x) \in c$. By the Hausdorff Maximality Principle there is a play $w \in W$ for $(N, \geq)$ such that $c' \subseteq w$. Therefore, $w \in W(x)$ for all $W(x) \in c$. It follows from Theorem 1 that the set representation by plays of any decision tree is bounded (and irreducible by Lemma 10). Another example of a bounded set tree is as follows.

Example 13. *The differential game tree from Section 2.2 is bounded. For, consider any chain $c \in 2^N$ and let $x_t(g), x_\tau(h) \in c$. If $\tau \leq t$, then (since c is a chain) $x_t(g) \subseteq x_\tau(h)$, and it follows that $g(s) = h(s)$ for all $s \in [0, \tau)$. Hence, the mapping $f_c : \mathbb{R}_+ \to A$ given by*

$$f_c(t) = \begin{cases} f(t) & \text{if there exists } x_\tau(f) \in c \text{ with } \tau > t \\ a_o \in A & \text{otherwise} \end{cases} \tag{17}$$

is a well defined function. By construction, $f_c \in x_\tau(f)$ for all $x_\tau(f) \in c$, which proves the claim.

For a V-poset that satisfies Trivial Intersection, to be bounded (from below) can be expressed by a useful double implication encompassing both Trivial Intersection and Boundedness.

Lemma 14. *A V-poset $(M, \supseteq)$ satisfies Trivial Intersection and is bounded (from below) if and only if*

$$c \in 2^M \text{ is a chain if and only if } \exists\, v \in V : v \in a, \forall a \in c \tag{18}$$

Condition (18) implies that $\cap_{a \in c} a \neq \emptyset$ for any chain c, thus preventing the situation in Examples 10 and 12. It will now be shown that there is no loss of generality in assuming that an irreducible V-set tree is bounded. To do this, the underlying set V gets enlarged to an appropriately constructed superset V_B such that a new V_B-set tree is obtained, that is bounded and properly order isomorphic to the original tree. The idea of enlarging the underlying set is captured by an isomorphic embedding, where the mapping $f : V \to V_B$ is simply the identity.

Since by Theorem 2 every V-set tree is order isomorphic to an irreducible set tree, any V-set tree is order isomorphic to a bounded and irreducible set tree.

Proposition 8. *A V-poset $(M, \supseteq)$ is an irreducible V-set tree if and only if it is isomorphically embedded in some bounded irreducible V_B-set tree $(M_B, \supseteq)$.*

Proof. Since the "if"-part follows from Lemmata 12 and 13, it suffices to demonstrate the "only if"-part. Let $(M, \supseteq)$ be an irreducible V-set tree. By Proposition 6 $S(V) = \{\{v\}\}_{v \in V}$ and by Proposition 5 the mapping $v \mapsto \uparrow \{v\}$ on V is an injection into the set W of plays. Let $W^* = \{\uparrow \{v\} \,| v \in V\}$ and define the superset V_B of V by $V_B = V \cup (W \setminus W^*)$. By construction, this union is disjoint.

For any $a \in M$ let $\phi(a) = a \cup (W(a) \setminus W^*) \subseteq V_B$. By construction $\phi(a) \cap V = a$ (and, therefore, $a \subseteq \phi(a)$). Let

$$M_B = \phi(M) = \{b \subseteq V_B \,| \exists\, a \in M : b = \phi(a)\}$$

It follows that $\phi : M \to M_B$ is onto and also one-to-one, because if $a, b \in M$ are such that $\phi(a) = \phi(b)$, then $a = \phi(a) \cap V = \phi(b) \cap V = b$. Moreover, $\phi(a) \subseteq \phi(b)$ implies $a = \phi(a) \cap V \subseteq \phi(b) \cap V = b$ for all $\phi(a), \phi(b) \in M_B$. Conversely, if $a, b \in M$ are such that $a \subseteq b$ then by Lemma 2(b) $W(a) \subseteq W(b)$ which implies (by the construction of ϕ) that $\phi(a) \subseteq \phi(b)$. Hence, ϕ is an order isomorphism.[8] To verify that ϕ is an isomorphic embedding, let $f : V \to V_B$ be given by the identity $f(v) = v$ for all $v \in V$, so that (16) holds trivially.

To establish that $(M_B, \supseteq)$ is a bounded irreducible V_B-set tree, the following is needed.

Claim. If $a \cap b = \emptyset$ then $\phi(a) \cap \phi(b) = \emptyset$ for all $a, b \in M$.

To see this, note that, because $V_B = V \cup (W \setminus W^*)$ and the union is disjoint, if $\phi(a) \cap \phi(b) \neq \emptyset$ and $a \cap b = \emptyset$, then there exists $w \in W(a) \cap W(b)$, in contradiction to $a \cap b = \emptyset$.

To verify Trivial Intersection for $(M_B, \supseteq)$, let $a', b' \in M_B$ be such that $a' \cap b' \neq \emptyset$ and $a, b \in M$ such that $a' = \phi(a)$ and $b' = \phi(b)$. By the Claim $a \cap b \neq \emptyset$. By Trivial Intersection for $(M, \supseteq)$ it follows that either $b \subset a$ or $a \subseteq b$. Therefore, because ϕ is order embedding, either $b' = \phi(b) \subset a' = \phi(a)$ or $a' = \phi(a) \subseteq b' = \phi(b)$.

Next, we verify Irreducibility on V_B. (By Lemma 9 this will imply that $(M_B, \supseteq)$ is both a V_B-set tree and irreducible.) If $v', w' \in V_B$ are such that $v' \neq w'$, there are three possibilities.

If $v', w' \in V$, Irreducibility for $(M, \supseteq)$ implies that there are $a, b \in M$ such that $v' \in a \setminus b$ and $w' \in b \setminus a$. It follows that $v' \in \phi(a) \setminus \phi(b)$ and $w' \in \phi(b) \setminus \phi(a)$.

If $v' \in V$ and $w' \in V_B \backslash V$ (and analogously for the reciprocal case), then w' is a play for $(M, \supseteq)$. If $v' \in a$ for all $a \in w'$, then $w' = \uparrow \{v'\} \in \cup_{v \in V} \uparrow \{v\}$, a contradiction. Hence, there exists $a \in w'$ such that $v' \notin a$. Therefore, $w' \neq \uparrow \{v'\}$, i.e., they are two different plays in $(M, \supseteq)$. Let $a, b \in M$ such that $a \in \uparrow \{v'\} \backslash w'$ and $b \in w' \backslash \uparrow \{v'\}$. Then $v' \in a$, because $a \in \uparrow \{v'\}$. And $v' \notin b$, because $b \in w' \backslash \uparrow \{v'\}$, i.e., $v' \in a \backslash b = a \backslash \phi(b) \subset \phi(a) \setminus \phi(b)$. On the other hand, because $b \in w' \in V_B \backslash V$, it follows that $w' \in W(b) \setminus (\cup_{v \in V} \uparrow \{v\}) \subset \phi(b)$. Since $a \notin w'$, we have that $w' \notin \phi(a)$ and hence $w' \in \phi(b) \setminus \phi(a)$.

Finally, if $v', w' \in V_B \backslash V = W \setminus (\cup_{v \in V} \uparrow \{v\})$, then v' and w' are two different plays in $(M, \supseteq)$. Let $a, b \in M$ such that $a \in v' \backslash w'$ and $b \in w' \backslash v'$. It follows that $v' \in \phi(a) \setminus \phi(b)$ and $w' \in \phi(b) \setminus \phi(a)$.

[8] Clearly, ϕ has inverse $\phi^{-1} : M_B \to M$ given by $\phi^{-1}(a) = a \cap V$ for all $a \in M_B$.

This shows that Irreducibility holds for all $v', w' \in V_B$. In other words, the modified V_B-poset $(M_B, \supseteq)$ continues to satisfy Trivial Intersection and Irreducibility and, therefore, is an irreducible V_B-set tree.

To establish that the irreducible V_B-set tree $(M_B, \supseteq)$ is bounded from below, consider any chain c' in M_B. By Irreducibility and the Claim above,

$$c = \phi^{-1}(c') \equiv \{a \in M \,|\, \exists\, a' \in c' : a' = \phi(a)\}$$

is a chain in M. By the Hausdorff Maximality Principle, there exists a play w in M such that $c \subseteq w$. If $w \in \cup_{v \in V} \uparrow \{v\}$, let $v \in V \subset V_B$ be such that $w = \uparrow \{v\}$. Then v is a lower bound for c and, therefore, for c'. If $w \in W \setminus \cup_{v \in V} \uparrow \{v\} \subset V_B$, then $w \in W(a) \setminus (\cup_{v \in V} \uparrow \{v\}) = \phi(a)$ for all $a \in c$ and, hence, is a lower bound for c'. ■

Combining this with Theorem 2 it follows that any V-set tree is properly order isomorphic to an irreducible set tree which itself is isomorphically embedded in a bounded irreducible set tree. In this sense, there is no loss of generality in assuming Irreducibility and Boundedness when working with set trees.

What Boundedness achieves is that the set $S(V)$ of separable equivalence classes maps surjectively onto the set W of plays.

Proposition 9. *A V-set tree $(M, \supseteq)$ is bounded if and only if the mapping $[v] \mapsto \uparrow [v]$ on $S(V)$ is onto W, where W is the set of plays for $(M, \supseteq)$.*

Proof. "if:" Suppose the mapping $[v] \mapsto \uparrow [v]$ on $S(V)$ is onto W. Let $c \in 2^M$ be a chain in M. By the Hausdorff Maximality Principle there is a play $w \in W$ such that $c \subseteq w$. By hypothesis there is $[v] \in S(V)$ such that $w = \uparrow [v]$. Therefore, $v \in a$ for all $a \in c \subseteq w$.

"only if:" Let $(M, \supseteq)$ be a bounded V-set tree and $w \in W$ a play. By hypothesis there is $v \in V$ such that $v \in a$ for all $a \in w$. By Trivial Intersection and the fact that w is a maximal chain, $\uparrow [v] = w$ and $[v] = \cap \{a \,|a \in w\}$, and, hence, $[v] \in S(V)$. ■

Combining the last result with Proposition 6 it follows that for a bounded irreducible V-set tree the mapping $v \mapsto \uparrow \{v\}$ is a bijection from V onto the set W of plays. This result is particularly transparent in the differential game example from Section 2.2.

4. GAME TREES

Theorem 2 and Proposition 8 show that every set tree can be modified into a bounded irreducible set tree by appropriately shrinking and enlarging the underlying set. The advantage of assuming a bounded and irreducible V-set tree is that the underlying set V and the set W of plays can be identified (by Propositions 5 and 9 and Lemma 8). Then, the underlying set can be taken to be the set of plays, and the mapping W from nodes to (sets of) plays passing through them becomes the identity.

Definition 18. *A **game tree** is a W-poset $(N, \supseteq)$ that satisfies*

$$c \in 2^N \text{ is a chain if and only if } \exists\, w \in W : w \in x,\ \forall x \in c \tag{19}$$

and for all $w, w' \in W$

$$\text{if } w \neq w' \text{ then } \exists\, x, y \in N \text{ such that } w \in x \setminus y \text{ and } w' \in y \setminus x \tag{20}$$

By Lemma 14 condition (19) holds if and only if Trivial Intersection holds and every chain in N has a lower bound in W. Since condition (20) is simply Irreducibility, a W-poset $(N, \supseteq)$ is a game tree if and only if it is a bounded irreducible W-set tree.

4.1. A Characterization. As pointed out earlier, the underlying set W in a bounded irreducible set-tree (i.e. a game tree) can then be taken to be the set of plays, and a bounded irreducible tree becomes *its own set representation by plays.* Yet, when precisely is a set tree its own set representation by plays? The idea is that, not only the set of plays W and the underlying set V are bijective, but, additionally, this bijection can be used to reconstruct the plays passing through a node from the elements which the node contains. This can be formalized as a double isomorphism.

Definition 19. *A V-poset $(M, \supseteq)$ is **its own set representation by plays** if there exists a bijection $\psi : V \to W$ such that $\psi(a) \equiv \{\psi(v)\}_{v \in a} = W(a)$ for all $a \in M$, where $W : M \to W(M)$ is the mapping assigning plays passing through a node, given by $W(a) = \{w \in W \,|a \in w\}$.*

Remark 4. *Let $2^\psi : 2^V \to 2^W$ be the trivial extension of ψ to the power set given by $2^\psi(a) = \psi(a)$ for all $a \subseteq V$. Let $i_M : M \to 2^V$ be the immersion of M into the power set of V (given by $i_M(x) = x$ for all $x \in M$) and $i_W : W(M) \to 2^W$ the analogous immersion of $W(M)$ into the set of sets of plays. Then, the previous definition amounts to the following diagram being commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\quad W(\cdot) \quad} & W(M) \\ \Big\downarrow {\scriptstyle i_M} & & \Big\downarrow {\scriptstyle i_W} \\ 2^V & \xrightarrow{\quad 2^\psi \quad} & 2^W \end{array}$$

A first, straightforward consequence of the definition is the following.

Lemma 15. *If a tree $(M, \supseteq)$ is its own representation by plays, then it is a decision tree.*

By Theorem 1, a tree can be represented by plays if and only if it is a decision tree. Combining this fact with the previous Lemma shows that *a tree is its own representation by plays if and only if it is doubly isomorphic to its image in plays via the natural order isomorphism.*

The Infinite Centipede from Example 12 shows that Trivial Intersection and Irreducibility are *not* sufficient for $(M, \supseteq)$ to be its own set representation by plays. This is purely due to the fact that the underlying set V is given, and is unrelated to properties of the decision tree. That the underlying set V is large enough to contain a lower bound for every chain in M is expressed by adding the converse to the implication in Trivial Intersection, as in condition (19).

The "only if"-part of condition (19) in Definition 18 is purely a convention on how the set representation is chosen. That is, by contrast to its "if"-part (viz. Trivial Intersection) and Irreducibility, (20), it has no impact on the corresponding decision tree $(N, \geq)$, by Proposition 8.

The next Theorem shows that game trees are *precisely* those decision trees which are their own set representation by plays.

Theorem 3. *For any V-poset $(M, \supseteq)$ the following statements are equivalent:*

(a) $(M, \supseteq)$ is a game tree.

(b) $(M, \supseteq)$ is a tree and its own representation by plays.

(c) $\psi(v) = \uparrow\{v\}$ defines a bijection from the set V to the set W of maximal chains in M.

Proof. "(a) implies (b):" By Proposition 6 $S(V) = \{\{v\}\}_{v\in V}$, and by Proposition 5, for every $v \in V$, the set $\uparrow\{v\} \equiv \{a \in M \,|\, v \in a\} \subseteq M$ is a play (a maximal chain with respect to set inclusion).

Define $\psi : V \to W$ by $\psi(v) = \uparrow\{v\}$ for all $v \in V$. By Lemma 8 this function is one-to-one and by Proposition 9 it is onto. Hence, it is a bijection. Moreover, it follows that $\psi^{-1}(w)$ is given by the only element in $\cap_{a\in w} a = \{v \in V \,|\, v \in a, \forall a \in w\}$.

Next, it is verified that $\psi(a) \equiv \{\psi(v)\}_{v\in a} = W(a)$ for all $a \in M$. For $a \in M$ the plays passing through are $W(a) = \{w \in W \,|\, a \in w\}$, as usual. Let $w' \in W(a)$ and $v' = \psi^{-1}(w')$. Since $a \in w' = \psi(v') = \{a' \in M \,|\, v' \in a'\}$, it follows that $v' \in a$. Hence $w' \in \psi(a)$. Conversely, let $w' \in \psi(a)$. Then there is $v' \in a$ such that $\psi(v') = w'$. Since $w' = \psi(v') = \{a' \in M \,|\, v' \in a'\}$, it follows that $a \in w'$. Hence, $w' \in W(a)$. In summary, $\psi(a) = W(a)$ for all $a \in M$, as required by Definition 19.

"(b) implies (c):" Suppose that $(M, \supseteq)$ is a decision tree and that there exists some bijection $\widetilde{\psi} : V \to W$ such that $\widetilde{\psi}(a) \equiv \left\{\widetilde{\psi}(v)\right\}_{v\in a} = W(a)$ for all $a \in M$. Since $(W(M), \supseteq)$ is a decision tree by Lemma 10, it follows from Lemmata 12 and 13 that $(M, \supseteq)$ is an irreducible V-set tree.

Consequently, by Propositions 5 and 6, $\psi(v) = \uparrow\{v\}$ is a play for all $v \in V$. It now suffices to show that $\psi(v) = \widetilde{\psi}(v)$ for all $v \in V$. Let $v \in V$ and consider any

$a \in \psi(v) = \uparrow\{v\}$. Then $v \in a$ implies that $\widetilde{\psi}(v) \in \widetilde{\psi}(a) = W(a)$ and, hence, $a \in \widetilde{\psi}(v)$. It follows that $\psi(v) \subseteq \widetilde{\psi}(v)$ and, since both $\psi(v)$ and $\widetilde{\psi}(v)$ are plays, $\psi(v) = \widetilde{\psi}(v)$ by maximality.

"(c) implies (a):" Suppose there is $v \in V$ such that $v \in a$ for all $a \in c$ for some $c \in 2^M$. Then $c \subseteq \psi(v) = \uparrow\{v\} \in W$ implies that c is a chain. This verifies the "if"-part of (19).

If $c \in 2^M$ is chain, by the Hausdorff Maximality Principle there is a maximal chain $c' \in W$ such that $c \subseteq c'$. Since ψ is onto, there is $v \in V$ such that $\psi(v) = c' \in W$. Then $v \in b$ for all $b \in c'$ implies $v \in a$ for all $a \in c$. This verifies the "only if"-part of (19).

Let $v, v' \in V$ be such that $v \neq v'$. Then $\psi(v)$ and $\psi(v')$ are distinct plays, because ψ is one-to-one. If for every $a \in \psi(v)$ also $v' \in a$ would hold, then $\psi(v) \subseteq \psi(v')$ would imply the contradiction $\psi(v) = \psi(v')$ by maximality. Hence, there is $a \in \psi(v)$ such that $v' \notin a$. A symmetric argument shows that there is $b \in \psi(v')$ such that $v \notin b$. Thus $v \in a \setminus b$ and $w \in b \setminus a$ verifies (20). ■

Along the way it has been shown that it is justified to call the mapping $\psi : V \to W$ defined by $\psi(v) = \uparrow\{v\}$ *the canonical mapping*. After all, the proof of "(b) implies (c)" shows that the bijection from the underlying set to the set of plays is *unique*.

Corollary 2. *For any game tree $(N, \supseteq)$ the bijection in Definition 19 from V to the set W of plays is unique and given by $\psi(v) = \uparrow\{v\}$ for all $v \in V$.*

In a game tree, by Theorem 3, the sets V and W can be identified. The bijection $\psi : V \to W$ becomes the identity on all of $V = W$, and then $\psi(a) = W(a)$ for all $a \in M$, i.e., a node can be identified with the set of plays that pass through it. Game trees are those trees for which it is inconsequential whether nodes or plays are taken to be the primitives.

4.2. Complete Game Trees. As the Infinite Centipede from Example 12 shows, even for a game tree some plays may not end at nodes. As will be seen, Theorem 3 implies that one can always add terminal nodes (singletons) without altering the structure of the tree. Hence, having all plays ending at nodes or not becomes a modelling decision, and we choose to include terminal nodes in the following definition.

Definition 20. *A game tree $(N, \supseteq)$ is* ***complete*** *if $\{w\} \in N$ for all $w \in W$ (where W is the underlying set). A complete game tree is* ***rooted*** *if $W \in N$.*

A W-poset $(N, \supseteq)$ is a complete game tree if and only if (19) holds and $\{w\} \in N$ for all $w \in W$, because then Irreducibility, (20), holds trivially for all $w, w' \in W$. Theorem 3 implies that every game tree can be *completed* to a complete game tree by adding all singletons $\{w\}$ for $w \in W$ to N without affecting the set of plays:

Proposition 10. *If the V-poset $(N, \supseteq)$ is a game tree and $Z = \{\{v\} \mid v \in V\}$ is the collection of singleton sets, then $(N \cup Z, \supseteq)$ is a complete game tree. Moreover, $\phi(w) = w \cup \left\{\left\{\psi^{-1}(w)\right\} \cap Z\right\}$ defines a bijection between the set of plays for $(N, \supseteq)$ and the set of plays for $(N \cup Z, \supseteq)$.*

Proof. Let W be the set of plays for the game tree $(N, \supseteq)$ and W' the set of plays for $(N', \supseteq)$ where $N' = N \cup Z$. Let $\psi : V \to W$ be the canonical mapping, defined by $\psi(v) = \{x \in N \,|v \in x\} = \uparrow\{v\}$. Define $\phi : W \to W'$ by $\phi(w) = w \cup \{\{\psi^{-1}(w)\} \cap Z\}$. We show that ϕ is a bijection.

Let $w' \in W'$ and $w = w' \cap N$. Then $w \in W$, $\psi^{-1}(w) \in V$, and $\phi(w) = w' \in W'$. Hence, ϕ is onto (surjective). If $\phi(w) = \phi(\hat{w})$ for $w, \hat{w} \in W$ then

$$w = \phi(w) \cap N = \phi(\hat{w}) \cap N = \hat{w}$$

Thus, ϕ is one-to-one (injective). It follows that W and W' are set-isomorphic (bijective). By Theorem 3 (applied to $(N, \supseteq)$), ψ is bijective and hence $\psi' \equiv \phi \circ \psi : V \to W'$ is also bijective. But

$$\psi'(v) = \{x \in N \,|v \in x\} \cup \{\{v' \in V \,|v' \in x, \forall x \in \psi(v)\} \cap Z\} = \{x' \in N' \,|v \in x'\}$$

because $\{v' \in V \,|v' \in x, \forall x \in \psi(v)\} = \{\psi^{-1}(\psi(v))\} = \{v\}$ by Theorem 3. Therefore, Theorem 3 now implies that $(N', \supseteq)$ is a complete game tree. ■

In Proposition 10, all singletons are added to N. But some of the singletons may already have to belong to N, due to Irreducibility, (20). The following example illustrates this.

Example 14. *(Twins) Let $V = [0, 1]$ and $N = \{(\{v\})_{v \in V}, (x_t)_{t=1}^{\infty}\}$, where*

$$x_t = \left[0, \frac{1}{t+1}\right] \cup \left[\frac{t}{t+1}, 1\right] \text{ for all } t = 1, 2, ...$$

Because $\{v\} \in N$ for all $v \in V$, the poset $(N, \supseteq)$ is a complete game tree. The set $c_\infty = \{x_t \in N \,|t = 1, 2, ...\}$ is not a play (because $c_\infty \cup \{0\}$ is a chain), nor is any set $c_t = \{x_\tau \in N \,|\tau = 1, ..., t\}$ (for the same reason). The set of plays is given by

$$W = \left\{ \left((c_t \cup \{v\})_{v \in \left(\frac{1}{t+2}, \frac{1}{t+1}\right]}, (c_t \cup \{v\})_{v \in \left[\frac{t}{t+1}, \frac{t+1}{t+2}\right)} \right)_{t=1}^{\infty}, c_\infty \cup \{0\}, c_\infty \cup \{1\} \right\}$$

Hence, V and W are naturally isomorphic by the bijection $\psi : V \to W$, where $\psi(v) = c_{t(v)} \cup \{v\}$ and $t(v)$ is the largest integer such that $t(v) \leq (1 - v)/v$ for all $v \in (0, 1/2]$, $\psi(v) = c_{t(v)} \cup \{v\}$ and $t(v)$ is the largest integer such that $t(v) \leq v/(1 - v)$ for all $v \in (1/2, 1)$, $\psi(0) = c_\infty \cup \{0\}$, and $\psi(1) = c_\infty \cup \{1\}$.

The singletons $\{0\}$ and $\{1\}$ could not *have been added, as in Proposition 10, to a game tree. For, if originally, say $\{0\}$ would not be a node, then the original tree fails Irreducibility, (20). This is, because, without $\{0\}$, there would not be any node that separates $0 \in V$ from $1 \in V$ (i.e. that $0 \in V$ belongs to a node would imply that $1 \in V$ belongs to this node). Only if the underlying set is modified to become $V' = (0, 1]$, the resulting ordered set $(N \setminus \{\{0\}\}, \supseteq)$ would be a game tree; but then $\{0\}$ could not be added. Hence, the class of singletons that can truly be added (i.e. without already being there), as in Proposition 10, forms a particular subset of the set of all singletons. To unveil what special subset that is, however, takes extra concepts that are deferred to a separate paper (Alós-Ferrer and Ritzberger [1]).*

Since every game tree can be completed by adding all singletons without changing any essential features of the tree, by Proposition 10, the transition from game trees to complete game trees is a pure modelling decision.

By Lemma 5 the representation by plays of a decision tree is an irreducible set tree. By Proposition 8 every irreducible set tree can be modified - by adding elements to the underlying set V - to become a bounded irreducible tree, i.e. a game tree. (Hence, every decision tree is order isomorphic to a game tree.) By Proposition 10 every game tree can be modified - by adding nodes - to become a complete game tree. Neither of these modifications changes any essential features of the tree.

The next result gives a characterization of complete game trees in terms of minima of plays.

Proposition 11. *Let $(N, \supseteq)$ be a V-set tree. Then, $(N, \supseteq)$ is a complete game tree if and only if it is irreducible and every play has a minimum.*

Proof. "if:" Let $(N, \supseteq)$ be an irreducible V-set tree for which every play has a minimum. Then $(N, \supseteq)$ is a game tree by Theorem 3. For any $v \in V$ let $\psi(v) = \uparrow\{v\} = w \in W$ be the play associated to v by the canonical mapping. By hypothesis w has a lower bound $z \in w$ and, by the definition of w, we have that $v \in z$.

If there is $v' \in z$ with $v' \neq v$, then by Irreducibility there are $x, x' \in N$ such that $v \in x \setminus x'$ and $v' \in x' \setminus x$. Since $v \in x$, we have $x \in w$. Since $v' \in z$ and $z \subseteq x$ (because $z \subseteq y$ for all $y \in w$), it follows that $v' \in x$, a contradiction. Therefore, $z = \{v\} \in N$.

"only if:" If $(N, \supseteq)$ is a game tree, then it is irreducible, because $\{v\} \in N$ for all $v \in V$. If $w \in W$ then by (19) there is $v \in V$ such that $v \in x$ for all $x \in w$. By definition $x \supseteq \{v\} \in N$ for all $x \in w$, so $w \subseteq \uparrow\{v\}$. Since $\uparrow\{v\} \in W$ by Theorem 3(c), maximality implies $w = \uparrow\{v\}$. Therefore, if $z \in N$ is such that $z \subseteq x$ for all $x \in w$, then, in particular, $z \subseteq \{v\}$ implies $z = \{v\}$. Since, moreover, $\{v\} \in w$ the play $w = \uparrow\{v\}$ has the minimum $\{v\} \in N$. ■

In one important special case a complete game tree is characterized by a much simpler condition than the combination of (19) and (20).

Proposition 12. *If $(N, \supseteq)$ is a V-poset such that all chains in N are finite, then $(N, \supseteq)$ is a complete game tree if and only if it satisfies Trivial Intersection and $\{v\} \in N$ for all $v \in V$.*

Proof. "if:" If $v, v' \in V$ are such that $v \neq v'$ then, because $\{\hat{v}\} \in N$ for all $\hat{v} \in V$, $v \in \{v\} \setminus \{v'\}$ and $v' \in \{v'\} \setminus \{v\}$ verifies (20). If $c \in 2^N$ is a chain, then by hypothesis it is finite, so that $\cap_{x \in c} x \neq \emptyset$ implies that there is $v \in \cap_{x \in c} x \subseteq V$. Since the chain c is arbitrary, is bounded. By Lemma 14, (19) holds, and $(N, \supseteq)$ is a game tree.

"only if:" If $(N, \supseteq)$ is a game tree, then Trivial Intersection follows from (19) and Lemma 14. For any $v \in V$ the chain $\uparrow\{v\}$ is finite by hypothesis and, therefore, contains a smallest node $x^* \in N$ such that $v \in x^*$ (i.e. $v \in y \in N \Rightarrow x^* \subseteq y$). If there is $v' \in V \setminus \{v\}$ such that $v' \in x^*$, then by (20) there are $x, y \in N$ such that $v' \in x \setminus y$ and $v \in y \setminus x$. But

$v \in y$ implies $v \in x^* \subseteq y$ in contradiction to $v' \notin y$. Hence, there is no $v' \in V \setminus \{v\}$ such that $v' \in x^*$ and, therefore, $\{v\} = x^* \in N$. ■

Note also that all game trees with underlying *finite* set V are necessarily complete game trees. Without finiteness, however, a game tree as in Definition 20 is quite general. It obviously includes, as we intended, all classical examples from game theory, e.g. finite trees, the trees of infinitely repeated games, Rubinstein's [20] bargaining game, and the discrete trees defined by Osborne and Rubinstein [18], provided those are decision trees. Moreover, it includes the trees of decision problems in continuous time, like the differential game example from Section 2.2.

Example 15. *As shown before, this differential game tree is irreducible and bounded, and hence a game tree, where the plays $w \in W$ can be identified with the underlying elements (functions) $f \in V$. To turn this game tree it into a complete game tree, it is enough to add the singletons as nodes (applying Proposition 10), e.g. simply allowing t to be infinite in the definition of $x_t(g)$ and N.*

Since $V = x_o(g)$ for any $g \in V$, V belongs to N and every chain for a differential game tree has an upper bound in N (which is true in any rooted tree). Note, however, that no play for these trees has a lower bound in N, but only a lower bound in V, unless the singletons have been added.

Infima of chains, on the other hand, are easy to identify. Suppose that the tree is completed by adding the nodes $\{f\}$ for all $f \in V$. Let c be a chain. If, for all t, there exists $x_\tau(f) \in c$ with $\tau > t$ (which is true for instance for plays), then $\{f_c\}$, where f_c is defined in (17), is an infimum for c. If there exists t such that the chain contains no node $x_\tau(f)$ with $\tau > t$, then let t^* be the infimum over all such t. The node $x_{t^*}(f_c)$ is an infimum for the chain. To see this, take any node $x_\tau(f) \in c$; necessarily, $\tau \leq t^*$ and $x_{t^*}(f_c) \subseteq x_\tau(f)$.

However, $x_{t^*}(f_c) \notin c$ in general, i.e., the chain does not have a minimum. Consider a fixed $f \in V$ and take the chain $\{x_t(f) \,|t < t^*\}$ for a given t^*. Obviously, $x_{t^*}(f_c)$ is the infimum of the chain, but is not an element of the chain. This means that this tree fails a condition called "down-discreteness" (see Alós-Ferrer and Ritzberger [1]) and that all nodes (but the root) are "infinite." Intuitively, this says that no node other than the root can be reached in a finite number of "steps" from the root.

The last observation raises the issue whether or not a game tree has enough structure to serve as the "objective" description of what may happen in the course of an extensive form game. The next section tackles this issue.

5. EXTENSIVE FORMS

In this Section it will be shown that game trees can be used as part of a definition of an extensive form. Such a definition requires a specification of the players' choices on top of the specification of the tree. This is also true in the traditional definition of an extensive form: choices of players (and, thereby, the information structures of players) have to be specified separately. The concept of an "information set" (the set of nodes, where a certain menu of choices is available), however, need not make sense in the general setting: information sets may not exist.

5.1. A Definition. Let $T = (N, \supseteq)$ be a game tree with set of plays W and $X = \{x \in N \,|\, \exists\, z \in N : z \subset x\}$ the set of *moves*. For a set $a \subseteq W$ of plays let $\downarrow a = \{x \in N \,|\, x \subseteq a\}$ be its *down-set* and define the set of (immediate) *predecessors* of a as

$$P(a) = \{x \in N \,|\, \exists\, y \in \downarrow a : \uparrow x = \uparrow y \setminus \downarrow a\} \tag{21}$$

Since every node $x \in N$ is a set of plays, nodes too may, but need not have immediate predecessors. Say that a set a of plays is *available at* the move $x \in X$ if $x \in P(a)$.

Definition 21. *An* ***extensive form*** *with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a game tree with set of plays W and $C = \left(C_0, (C_i)_{i \in I}\right)$ is a system consisting of a collection C_0 (the set of chance's choices) and collections C_i (the sets of personal players' choices) of nonempty unions of nodes (hence, sets of plays) for all $i \in I$, such that*

(i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$ then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ and all $i \in I$;

(ii) $x \cap \left[\bigcap_{i \in J(x)} c_i\right] \neq \emptyset$ for all $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ and all $x \in X$;

(iii) if $y, y' \in N$ satisfy $y \cap y' = \emptyset$ then there are $i \in I \cup \{0\}$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$;

(iv) if $x \supset y \in N$ then for every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \subseteq c$, for all $x \in X$;

where $A_i(x) = \{c \in C_i \,|\, x \in P(c)\}$ are the choices available to i at x for all $i \in I \cup \{0\}$ and the set of decision makers at x, $J(x) = \{i \in I \cup \{0\} \,|\, A_i(x) \neq \emptyset\}$, is required to be nonempty for all $x \in X$.

An extensive form is ***determined by common probabilistic beliefs*** *if a function p (the belief function) is given, that maps the set of moves X into the set $\Delta(C_0)$ of probability measures (on a σ-algebra containing C_0) that have supports in C_0, such that, for all $x \in X$ and all $c \in C_0$,*

$$x \in P(c) \text{ if and only if } c \in \operatorname{supp}(p(x)) \tag{22}$$

What is added to the tree to obtain an extensive form are collections of "choices" $c \in C_i$ (i.e. collections of sets of plays) for all personal players $i \in I$ and for "chance" $i = 0$ (for events that are not under the control of personal players). These sets of choices C_i have to satisfy four constraints.

First, property (i) stands in for information sets. If two distinct choices are available at a common move, then their immediate predecessors are identical and the choices are disjoint. Thus, the player cannot infer from the available menu of choices at which move (in the common set of predecessors, i.e. information set) she chooses. And, two choices that are simultaneously available cannot overlap.

Second, property (ii) ensures that any combination of available choices yields something nonempty. If a combination of choices (one for each decision maker) is available at a common move, then the combination has a nonempty intersection that is contained in the move.

Third, property (iii) deals with the "residual" that remains after personal players have "made their choices." If two nodes differ, then at some point someone (possibly chance) takes a decision that separates them. It will be shown below that this implies that the intersection of all choices, belonging to personal players *or to chance*, that contain a particular play, yields precisely this play. Hence, "in the end" whatever is not decided by personal players will be decided by chance.

Fourth, property (iv) implies the traditional exclusion of absent-mindedness (Kuhn [16]). In the absence of such a condition a play may cross an information set more than once (Piccione and Rubinstein [19]) or, in the present formalism, the same choice may be available more than once along the same play. It will be shown below (Proposition 13) that (iv) implies "no-absent-mindedness."

The following examples serve to verify that conditions (i)-(iv) are independent.

Example 16. *(Two-sided absent-minded driver paradox) Let* $W = \{w_1, ..., w_4\}$, $N = \{W, \{w_3, w_4\}, (\{w\})_{w \in W}\}$, $I = \{1\}$, $C_0 = \{\{w_1, w_2\}, \{w_3, w_4\}\}$, *and* $C_1 = \{\{w_1, w_3\}, \{w_2, w_4\}\}$ *(see Figure 1). That* $P(\{w_1, w_3\}) = \{W, \{w_3, w_4\}\} = P(\{w_2, w_4\})$ *verifies (i). Because* $J(W) = \{0, 1\}$ *and* $J(\{w_3, w_4\}) = \{1\}$ *and* $c_0 \cap c_1 \neq \emptyset$ *for all* $(c_0, c_1) \in C_0 \times C_1$*, property (ii) also holds. Since for two nodes to be disjoint requires at least one of them not to be a move and none of them to be the root, the hypothesis of (iii) applies only if either* $y = \{w_3, w_4\}$ *and* $y' = \{w\}$ *for some* $w \in \{w_1, w_2\}$ *or both* y *and* y' *are singletons. For pairs of singletons* $y, y' \in N$ *there clearly is always a disjoint pair of choices of the same decision maker that separates them. Furthermore, the singletons* $\{w_1\}$ *and* $\{w_2\}$ *are separated from* $\{w_3, w_4\}$ *by the two choices of chance. Hence, (iii) also holds true. But (iv) fails, as* $1 \in J(W)$ *and* $x = W \supset y = \{w_3, w_4\}$*, but there is no* $c \in A_1(W) = C_1$ *such that* $\{w_3, w_4\} \subseteq c$*.*

Example 17. *Let* $W = \{w_1, w_2, w_3\}$, $N = \{W, (\{w\})_{w \in W}\}$, $I = \{1\}$, $C_0 = \{W\}$, *and* $C_1 = \{\{w_1, w_2\}, \{w_3\}\}$*. As* $P(\{w_1, w_2\}) = P(\{w_3\}) = W$*, property (i) holds true. As* $X = \{W\}$ *and* $J(W) = \{1\}$*, condition (ii) is also satisfied. Since for every singleton* $\{w\} \in N \setminus X$ *there is* $c \in C_1$ *such that* $\{w\} \subseteq c$*, condition (iv) also holds. But (iii) fails, because* $\{w_1\} \cap \{w_2\} = \emptyset$*, but either* $\{w_1, w_2\} \subseteq c$ *or* $\{w_1, w_2\} \cap c = \emptyset$ *for all* $c \in C_0 \cup C_1$*.*

Example 18. *Let* $W = \{w_1, w_2, w_3\}$, $N = \{W, (\{w\})_{w \in W}\}$, $I = \{1, 2\}$, $C_0 = \{W\}$, $C_1 = \{\{w_1, w_2\}, \{w_3\}\}$*, and* $C_2 = \{\{w_1\}, \{w_2, w_3\}\}$*. Then* $P(c) = \{W\}$ *for all* $c \in C_1 \cup C_2$ *verifies (i). If* $y, y' \in N$ *are such that* $y \cap y' = \emptyset$*, both* y *and* y' *must be singletons. But for every pair of singletons there exists a disjoint pair of choices of*

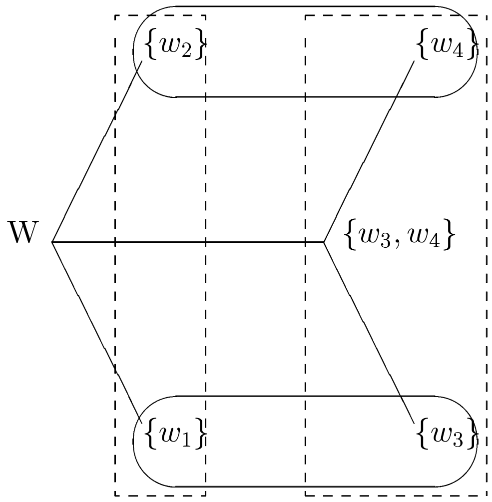

Figure 1: Rounded boxes represent personal player's choices, dashed boxes chance's choices. As choices are sets of plays, the choice $\{w_3, w_4\}$ includes the node $\{w_3, w_4\}$.

the same personal player that separates the two singletons. Hence, (iii) holds true. Since $X = \{W\}$ *and* $J(W) = \{1, 2\}$, *condition (iv) is also fulfilled. But property (ii) fails, as* $W \cap \{w_1\} \cap \{w_3\} = \emptyset$.

Example 19. *Let* $W = \{w_1, ..., w_5\}$,

$$N = \left\{W, \{w_1, w_2, w_3\}, \{w_4, w_5\}, (\{w\})_{w \in W}\right\},$$

$I = \{1\}$, $C_0 = \{\{w_1, w_2, w_3\}, \{w_4, w_5\}\}$, *and* $C_1 = \{\{w_1, w_4\}, \{w_2\}, \{w_3, w_5\}\}$. *Then* $J(W) = \{0\}$, $J(\{w_1, w_2, w_3\}) = J(\{w_4, w_5\}) = \{1\}$, $A_1(\{w_1, w_2, w_3\}) = C_1$, *and* $A_1(\{w_4, w_5\}) = C_1 \setminus \{\{w_2\}\}$ *show that (ii) holds. Verification of (iii) and (iv) is direct (though tedious). Yet, the first part of (i) fails, as* $\{w_1, w_2, w_3\} \in P(\{w_2\}) \cap P(\{w_1, w_4\})$, *but* $\{w_4, w_5\} \in P(\{w_1, w_4\}) \setminus P(\{w_2\})$.

Example 20. *Let* $W = \{w_1, ..., w_4\}$, $N = \left\{W, (\{w\})_{w \in W}\right\}$, $I = \{1\}$, $C_0 = \{W\}$, *and*

$$C_1 = \{\{w_1, w_2\}, \{w_3, w_4\}, \{w_2, w_3\}, \{w_1, w_4\}\}$$

Then $X = \{W\}$, $J(W) = \{1\}$, *and* $A_1(W) = C_1$. *Properties (ii), (iii), and (iv) are trivially satisfied. But the second part of (i) fails, as, say,* $\{w_1, w_2\} \cap \{w_2, w_3\} \neq \emptyset$.

Remark 5. *If every chain in* N *has a lower bound in* N, *then condition (ii) can be written as follows: For all* $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$,

$$\exists y \in \downarrow x \setminus \{x\} : y \subseteq x \cap \left[\cap_{i \in J(x)} c_i\right], \tag{23}$$

For, let $w \in x \cap \left[\cap_{i \in J(x)} c_i\right]$. *As choices are unions of nodes, there are* $z_i \in N$ *such that* $w \in z_i \in \downarrow c_i$ *for all* $i \in J(x)$. *Because* T *is a game tree, (the "if"-part of) condition*

(19) implies that the collection $\{z_i \mid i \in J(x)\}$ is a chain. By hypothesis this chain has a lower bound $y \in \downarrow x$. If $y = x$ would hold, then $y = x \subseteq c_i$ would contradict $x \in P(c_i)$ for all $i \in J(x)$. Therefore, $x \supset y \in N$, as desired.

The condition that every chain in N has a lower bound in N is, in fact, without loss of generality. Since every game tree can be completed by adding the singletons without changing the tree (by Proposition 10), and in a complete game tree every chain has a lower bound (the singleton in its intersection), condition (ii) is equivalent to (23) for all practical purposes.

Definition 21 captures the following quasi-operational specification of an extensive form. At every move $x \in X$ each personal player $i \in I$ is told (by an "umpire") which choices $c \in C^i$ she has available (in the sense that $x \in P(c)$, $c \in C^i$) and asked to select one of those. No other information is released to players. Given the decisions by all personal players, taking the intersection gives a nonempty set of plays, by (iii). From this a "chance move" selects how the game will continue. As will be seen below (Theorem 4), by (iii) this process will ultimately select a particular play.

What drives chance moves is, however, left open at first. In a classical probabilistic set-up one may want to "spin roulette wheels." That is, chance moves may be determined by a function p that assigns probability measures over chance's choices: a *common probabilistic belief*. Such a function p has to satisfy that chance only "chooses" *available* choices - the "if"-part of (22) - and that *all* available choices have positive probability - the "only if"-part of (22).[9] Note that chance is free to condition on the move at which it chooses, i.e., (ii) does not apply to chance.

5.2. Implications. The next result shows that Definition 21 yields an extensive form, in which the players' decisions (together with chance) ultimately lead to the realization of a play, owing to (iii).

Theorem 4. *Let (T, C) be an extensive form with player set I. Then,*

$$\cap \{c \in C_0 \cup (\cup_{i \in I} C_i) \mid w \in c\} = \{w\}$$

for all plays $w \in W$.

Proof. If $N = \{W\}$, i.e. the tree is trivial, there is nothing to prove. In nontrivial cases all plays pass through at least two nodes.

Let $C(w) = \{c \in C_0 \cup (\cup_{i \in I} C_i) \mid w \in c\}$ and note that this set is nonempty by (iii). For, let $x, y \in N$ be such that $w \in x \cap y$ and $x \supset y$. Since a game tree satisfies Separability, there is a third node $z \in N$ such that $x \supset z$ and $y \cap z = \emptyset$. By (iii) there is $i \in I \cup \{0\}$ and disjoint choices $c, c' \in C_i$ such that $y \subseteq c$ and $z \subseteq c'$. Since $w \in y \subseteq c$, $c \in C(w)$ verifies that $C(w)$ is nonempty.

[9]The latter is the assumption that zero-probability branches of the tree are pruned. If this is not desired, the "only if"-part of (22) can be dropped.

That $w \in \cap_{c \in C(w)} c$ follows from the definition. Suppose there is $w' \in W \setminus \{w\}$ such that $w' \in \cap_{c \in C(w)} c$. Because T is a game tree, there are $x, x' \in N$ such that $w \in x$, $w' \in x'$, and $x \cap x' = \emptyset$ (by Irreducibility). By (iii) there are $i \in I \cup \{0\}$ and $c, c' \in C_i$ such that $x \subseteq c$, $x' \subseteq c'$, and $c \cap c' = \emptyset$. Since $w \in x \subseteq c$, the choice c belongs to $C(w)$, so that by hypothesis $w' \in c$. But $w' \in x' \subseteq c'$ contradicts $c \cap c' = \emptyset$. ■

This result shows that plays "build up" from consecutive decisions by players (and/or chance) on sets of plays (or ultimate outcomes). Hence, the framework achieves what it was designed for.

An important feature of the present definition of an extensive form is that information sets need not exist. If they do, they are given by the set of immediate predecessors of available choices. But at this level of generality nothing ensures that all choices have immediate predecessors. Still, in the following version of the differential game example information sets do exist and are merely singleton sets of moves.

Example 21. *Turn the tree of the differential game from Section 2.2 into an extensive form, say, with two personal players, as follows. Let the set of actions A be a product set $A = A_1 \times A_2$. Given any function $f \in V$, denote $f = (f_1, f_2)$. The interpretation is as follows. At any point in time the two players $i = 1, 2$ simultaneously decide on an action $a_i \in A_i$ for $i = 1, 2$. Up to that moment, they know the entire history, but cannot anticipate the decision taken by the other player at t. Choices are of the form $c = c_{it}(f) = \{g \in V \,|\, g \in x_t(f), \, g_i(t) = f_i(t)\}$ for some $f \in V$, some $t \in \mathbb{R}_+$, and $i = 1, 2$. (If player i were not to observe previous decisions by the other player, but recalls her own, choices would be defined only by the property that $g_i(\tau) = f_i(\tau)$ for all $\tau \in [0, t]$, rather than $g \in x_t(f)$.) When two players decide their choices at t by picking, say, $c_{it}(f^i) \in C^i$ for $i = 1, 2$, their intersection, $c_{1t}(f^1) \cap c_{2t}(f^2) = \{g \in V \,|\, (g_1(\tau), g_2(\tau)) = (f^1(\tau), f^2(\tau)), \, \forall \tau \in [0, t]\}$, keeps track of both decisions while leaving all possibilities open for the future. That choices are unions of nodes follows from $c_{it}(f) = \cup_{\tau > t} \cup_{g \in c_{it}(f)} x_\tau(g)$.*

The current definition of an extensive form encompasses most traditional ones plus some exotic cases—but *not* cases of absent-mindedness (see Piccione and Rubinstein [19]), where a play crosses an information set more than once (the same choice is available more than once along a play). Because in the present framework players choose among sets of ultimate outcomes, condition (iv) rules out that they "choose not to choose," that is, pick a choice that will become available once more, later on.

The argument relies on the definition of immediate predecessors, (21) and conditions (i) and (iv). It demonstrates that an extensive form as in Definition 21 satisfies "no-absent-mindedness", i.e., if a choice is available at two distinct moves, then these moves cannot be ordered (by set inclusion).

Proposition 13. *Let (T, C) be an extensive form with player set I as in Definition 21. Then, for all $x, y \in X$,*

$$A_i(x) \cap A_i(y) \neq \emptyset \text{ and } y \subseteq x \text{ imply } y = x \text{ for all } i \in I \tag{24}$$

Proof. Suppose for some $i \in I$ there are $c \in C_i$ and $x, y \in N$ such that $x, y \in P(c)$, i.e. $c \in A_i(x) \cap A_i(y)$, and $y \subseteq x$. By $y \in P(c)$ there is $y' \in \downarrow c$ such that $\uparrow y = \uparrow y' \setminus \downarrow c$. Hence, $y' \subset y \subseteq x$ and $y \setminus c \neq \emptyset$. If $y \subset x$ would hold, then by (iv) there would be $c' \in A_i(x)$ such that $y \subseteq c'$, implying that $c \neq c'$ from $y \setminus c \neq \emptyset$. But then $x \in P(c) \cap P(c')$ would imply that $c \cap c' = \emptyset$ by (i), in contradiction to $y' \subseteq c \cap c'$. Hence, $y \subseteq x$ must imply $y = x$, as desired. ■

Remark 6. *In the presence of (i) condition (iv) implies the following "partition" property:*

(iv') for all $x \in X$ the collection $\{x \cap c \,|\, c \in A_i(x)\}$ is a partition of x, for all $i \in J(x)$.

For, if two choices are available at x, then by (i) they must be disjoint. But, for any play $w \in x$, because $x \in X$, there is a node $y \in N$ such that $w \in y \subset x$. By (iv) y (and hence w) must be contained in a choice $c \in A_i(x)$ available at x. If (iv) is weakened to (iv'), though, some examples of absent-mindedness would be feasible, as Example 16 shows. There, conditions (i)-(iii) hold, while (iv) fails. But (iv') holds, as the choices of the personal player partition W (see Figure 1). This illustrates the implications of (iv): by imposing that all successors of a given node (where a player has choices available) be contained in some available choice, players cannot "jump ahead" in the tree and select a node skipping an intermediate step. As it turns out, this simple intuition rules out absent-mindedness.

5.3. Strategies. What remains is whether the usual strategy notions can be defined in the present framework. To fix this, let $X_i = \{x \in N \,|\, A_i(x) \neq \emptyset\}$ denote player i's decision points and define a *pure strategy* for player $i \in I$ as a function $s_i : X_i \to C_i$ such that, for all $c \in s_i(X_i) = \cup_{x \in X_i} s_i(x)$,

$$s_i(x) = c \text{ if and only if } x \in P(c) \tag{25}$$

The "if"-part of (25) say that, if a choice $c \in C_i$ is selected at all (i.e. if $c \in s_i(X_i)$), then at every move x, where c is available, strategy s_i picks this choice c. Hence, s_i picks the *same* choice at all moves, where this choice is available. The "only if"-part of (25) says that, if $c \in C_i$ is chosen by s_i, then it is chosen only where it is available.

Similarly, a *behavior strategy* for player $i \in I$ is a function ρ_i from X_i to the set of probability distributions on (a σ-algebra containing) C_i such that, for all $b \in \rho_i(X_i) = \cup_{x \in X_i} \rho_i(x)$,

$$\rho_i(x) = b \text{ if and only if } x \in \cap_{c \in \mathrm{supp}(b)} P(c) \tag{26}$$

Again, the "if"-part of (26) says that the *same* probability distribution b over choices is selected by ρ_i at all moves, where choices in the support of b are available. For, if b is a probability distribution over choices for which two choices c and c' in its support do *not* have precisely the same predecessor set, then by (i) their predecessor sets are disjoint; hence, there is no move $x \in X$ such that $x \in P(c) \cap P(c')$. Thus, ρ_i can only assign a distribution for which all choices in its support have the same predecessor sets, i.e., ρ_i assigns distributions supported on choices that are available at the same moves. If b is such a distribution, $x, x' \in \cap_{c \in \text{supp}(b)} P(c)$, $\rho_i(x) = b$, and $\rho_i(x') = b'$, then the "if"-part of (26) implies $b = b'$. Likewise, by the "only if"-part of (26), if $\rho_i(x) = b$, then $x \in P(c)$ for all choices $c \in C_i$ in the support of b, i.e., ρ_i assigns to moves $x \in X_i$ only distributions supported on choices that are available at x.

These specifications illustrate that the familiar strategy notions can be defined naturally for the present concept of an extensive form. There remains, of course, a measurability issue whether (pure or behavior) strategy combinations (one strategy for each personal player) do induce (together with the belief function p) well defined probability distributions on plays.[10] For the purpose of completing an extensive form to a full-fledged game, this can be bypassed by defining payoff functions directly on the space of strategy combinations. In any case, this is beyond the scope of the present paper.

There remains, though, a more important point regarding the use of game trees in extensive forms. In particular, with this generality nodes need not have immediate predecessors (even if choices do), as it is the case, for instance, in the differential game example (Section 2.2). This poses certain problems. For instance, how to define alternating moves, as they appear in games of perfect information. In the differential game example with two players, one could let A_2 be the set of functions from A_1 to an ultimate action space for player 2, modelling that player 2's decision conditions on what player 1 has chosen. But, in the tree, the two players' decision would formally be taken simultaneously. Ideally, one would like to let player 1 move at the immediate predecessor of player 2's decision points, thus separating the two decisions. In the absence of immediate predecessors this is precluded.

There are two possible reactions to this observation. One is to insist on a discrete structure imposed on the tree - this is what we study in the companion paper (Alós-Ferrer and Ritzberger [1]). The other is to view the presence of alternating moves as a property of the situation that is to be modelled by the tree, so that the tree inherits a discrete structure from what it models. That extensive forms can be defined without recourse to immediate predecessors shows that both possibilities do exist.

6. Discussion

This paper studies how arbitrary trees can be represented by a collection of sets. The purpose of such a representation is to provide a domain for sequential decision theory. To do this requires two things: First, a node should be an event in the sense of probability theory, i.e. a set of states. Second, the elements of the nodes/sets should have meaning as representatives of ultimate outcomes. We show that both desiderata can be met without any substantial loss of generality by the current definition of a game tree: a collection of subsets of an underlying set (of plays) such that (i) a

[10] For instance, consider the well-known problem with the law of large numbers arising if a player tosses a coin repeatedly in continuous time (e.g. Judd [12]).

family of those subsets is a chain if and only if all its elements (nodes) contain a common element (play), and (ii) for any two distinct elements (plays) threre are two sets (nodes) such that the first set (node) contains the first element (play), but not the second, and the second set (node) contains the second element (play), but not the first. (A game tree is rooted if the union of all its nodes belongs to it.)

This is the most general definition of a (set) tree available up to date. It is essentially equivalent to the order-theoretic notion of a decision tree, i.e., all requirements that enter on top of the property of being a set representation of an (order-theoretic) decision tree are purely modelling conventions. The definition is so general that it even encompasses "differential games" (decision problems in continuous time).

As an application we show that game trees are sufficient to define extensive forms by adding sets of choices for all players. The traditional strategy notions can then be translated into this general framework.

A problem remains open for further research, though. Some strategic situations, even as simple ones as games of perfect information, may require more structure on the tree. In particular, to model how players alternate in deciding, requires for each node an identification of its immediate predecessor. In a companion paper (Alós-Ferrer and Ritzberger [1]) we show that this can be tackled by adding a discreteness property. This property characterizes set trees for which every node (but the root) has an immediate predecessor. The latter then greatly simplifies the formalism required to define extensive form games and reintroduces familiar objects, that are potentially missing from the general framework, like information sets.

A. APPENDIX

Proof of Lemma 3. "if:" Let $(N, \geq)$ be a tree and $x, y \in N$ such that $W(x) = W(y)$. Then, for any $w \in W(x) = W(y)$, that $x, y \in w$ implies either $x \geq y$ or $y \geq x$ (or both), because $w \in W$ is a chain. Assume, without loss of generality, that $x \geq y$. Suppose $y \ngeq x$. Then, by (7) there exists $z \in \downarrow x$ such that $z \ngeq y$ and $y \ngeq z$. Since $W(z) \subseteq W(x)$ by Lemma 2(b), $y \notin w$ for all $w \in W(z)$. Hence, $W(y) \subseteq W(x) \setminus W(z)$ contradicts $W(x) = W(y)$. Therefore, also $y \geq x$ must hold, so that $x = y$ (by antisymmetry) verifies (6).

"only if:" Let $(N, \geq)$ be a decision tree, and let $x, y \in N$ such that $x \geq y$ and $y \ngeq x$. By Lemma 2(b), $W(x) \supseteq W(y)$. By (6), $W(x) \supset W(y)$, i.e. there exists $w \in W(x) \setminus W(y)$. For any $z \in w$, either $z \geq x$ or $x \geq z$. If $z \geq x$, transitivity implies $z \geq y$. Hence, there must be some $z \in w$ such that $x \geq z$ and both $z \ngeq y$ and $y \ngeq z$ hold. For, otherwise for all $z \in w$ either $z \geq y$ or $y \geq z$, which implies that $w \cup \{y\}$ is a chain. By maximality of $w \in W$, it follows that $y \in w$ and $w \in W(y)$, a contradiction. ■

Proof of Lemma 6. First, let $[v] \in V/\sim$ be such that $\uparrow[v] = \emptyset$. Then, $\cap_{a \in \uparrow[v]} a = \emptyset$, i.e. $[v]$ is not separable. Since there is no $a \in M$ such that $[v] \subseteq a$, the property is false, verifying the equivalence in this case.

Let now $[v] \in V/\sim$ be such that $\uparrow[v] \neq \emptyset$. Note that $[v] \subseteq \cap_{a \in \uparrow[v]} a$ whenever $\uparrow[v] \neq \emptyset$. Then, $[v]$ is separable if and only if $[v] = \cap_{a \in \uparrow[v]} a$, or, equivalently, $V \setminus [v] = V \setminus \cap_{a \in \uparrow[v]} a = \cup_{a \in \uparrow[v]} (V \setminus a)$, which proves the claim. ■

Proof of Lemma 7. For all $v \in a \cap b$ we have $[v] \subseteq \cap_{c \in \uparrow[v]} c \neq \emptyset$. If there is $v \in a \cap b$ such that $\cap_{c \in \uparrow[v]} c \subseteq [v]$ the statement is verified. Hence, suppose that $[v] \subset \cap_{c \in \uparrow[v]} c$ for all $v \in a \cap b$. But then $a \cap b = \cup_{v \in a \cap b} [v] \subset \cup_{v \in a \cap b} \left(\cap_{c \in \uparrow[v]} c \right) \subseteq a \cap b$ yields a contradiction. ■

Proof of Lemma 12. (a) Assume that $(M, \supseteq)$ satisfies Trivial Intersection and let $a, b \in M$ be such that $a \cap b \neq \emptyset$. Then Trivial Intersection implies either $a \subseteq b$ or $b \subset a$; hence, by order isomorphism, (2), either $\varphi(a) \subseteq \varphi(b)$ or $\varphi(b) \subset \varphi(a)$, i.e. $\varphi(a) \cap \varphi(b) \neq \emptyset$.

(b) Suppose $(M', \supseteq)$ satisfies Trivial Intersection. If is φ proper and $a \cap b \neq \emptyset$, for $a, b \in M$, then by (15) $\varphi(a) \cap \varphi(b) \neq \emptyset$. Therefore, either $\varphi(a) \subseteq \varphi(b)$ or $\varphi(b) \subset \varphi(a)$; hence, by order isomorphism, (2), either $a \subseteq b$ or $b \subset a$. The converse implication follows from (a).

(c) By (b) and Lemma 4 it is enough to establish Weak Separability, under the hypothesis that $(M', \supseteq)$ is a V'-set tree. But Weak Separability is preserved by order isomorphism. Hence, this is immediate. ■

Proof of Lemma 13. Assume that $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$. Let $v, w \in V$ such that $v \neq w$. Since the mapping f is one-to-one, $f(v) \neq f(w)$. By Irreducibility for $(M', \supseteq)$, there are $a', b' \in M'$ such that $f(v) \in a' \backslash b'$ and $f(w) \in b' \backslash a'$. Let $a, b \in M$ such that $a' = \varphi(a)$ and $b' = \varphi(b)$, where φ is the order isomorphism. Since $f(v) \in a'$ and f is one-to-one, it follows that $v \in a$. Since $f(v) \notin b'$, it follows that $v \notin b$. Hence, $v \in a \backslash b$ and, analogously, $w \in b \backslash a$. ■

Proof of Lemma 14. "if:" Suppose that $(M, \supseteq)$ satisfies that $c \in 2^M$ is a chain if and only if there is $v \in V$ such that $v \in a$ for all $a \in c$. Then, for any chain $c \in 2^M$ the "only if" part implies that there is $v \in V$ which forms a lower bound on c. Furthermore, if $a, a' \in M$ are such that $a \cap a' \neq \emptyset$, then that there is $v \in a \cap a'$ implies from the "if" part that $\{a, a'\} \in 2^M$ is a chain, i.e., either $a' \subset a$ or $a \subseteq a'$, verifying Trivial Intersection (4).

"only if:" Suppose $(M, \supseteq)$ satisfies Trivial Intersection (4) and every chain in M has a lower bound in V. Then if $c \in 2^M$ is a chain, there is $v \in V$ such that $v \in a$ for all $a \in c$. On the other hand, if $v \in a$ for all $a \in c$ for some arbitrary $c \in 2^M$, then if $a, a' \in c$ that $v \in a \cap a'$ implies from Trivial Intersection (4) either $a' \subset a$ or $a \subseteq a'$, i.e., $c \in 2^M$ is a chain. ■

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