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Mai 2001

Working Paper No: 0107

# DEPARTMENT OF ECONOMICS

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# On Core-Walras (Non-) Equivalence for Economies with a Large Commodity Space*

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May 16, 2001

**Abstract**

Addressing a question raised by Tourky and Yannelis (1998), we show that given any non-separable Banach space $E$, and given any atomless measure space $(T, \mathcal{T}, \nu)$, there is an economy with $(T, \mathcal{T}, \nu)$ as space of traders and $E$ as commodity space fulfilling the usual standard assumptions but having a core allocation not supportable as a Walrasian equilibrium, and in fact, having no Walrasian equilibria at all. We shall also consider the framework of economies with weakly compact consumption sets as developed by Khan and Yannelis (1991). We prove that in this setting the core of an economy with a measure space of traders is non-empty, regardless of whether or not the commodity space is separable. On the other hand, we show that when the commodity space contains weakly compact subsets that are non-separable, then, again, there are atomless economies for which core-Walras equivalence fails. Thus, in particular, for very large commodity spaces the notion of the core seems to be more robust than that of a Walrasian equilibrium.

*Journal of Economic Literature Classification Numbers* C62, C71, D41, D50.

*Keywords*: Non-separable commodity space, measure space of traders, core, Walrasian equilibrium, core-Walras equivalence.

*Thanks to E. Dierker, M. Nermuth, R. Tourky, and N. C. Yannelis for helpful discussions and suggestions.

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# 1 Introduction

This note addresses the problems of non-emptiness of the core and core-Walras equivalence for an exchange economy with a measure space of traders, the commodity space being a general Banach space. Recently, Tourky and Yannelis (1998) showed that, when aggregation of individual commodity bundles is formalized in terms of the Bochner integral, given any atomless measure space $(T, \mathcal{T}, \nu)$ there is a non-separable Banach space $E$ such that an economy with $(T, \mathcal{T}, \nu)$ as space of traders and $E$ as commodity space can be constructed that has a non-empty core but—in spite of the fact that $(T, \mathcal{T}, \nu)$ is atomless, and although several strong conditions are met[1]—does not have a Walrasian equilibrium. Contrasting this result with the classical equilibrium existence and core-Walras equivalence theorems of Aumann (1966, 1964) and their extensions to commodity spaces being separable Banach spaces (e.g. Khan and Yannelis, 1991; Rustichini and Yannelis, 1991), Tourky and Yannelis (1998) convincingly argued that the crucial condition behind these latter results is not only that the space of traders is atomless but, in fact, that there are "many more agents than commodities."[2]

Actually, the commodity space in the counter examples to core-Walras equivalence presented by Tourky and Yannelis (1998) is a non-separable *Hilbert* space. At the end of their paper, however, Tourky and Yannelis raised the question as to whether one can view the construction in these counter examples "as a concrete version of the proof of a more general result. One that characterizes a class of Banach spaces as those spaces in which Bochner existence and Bochner core-Walras equivalence hold?"[3] In this note we will attack this problem.

In our first contribution we show that, indeed, a core-Walras non-equivalence result as in Tourky and Yannelis (1998) does not depend on the commodity space being (isomorphic to) a Hilbert space, but actually holds in every non-separable *Banach* space: We shall prove that given any such Banach space $E$, and given any atomless measure space $(T, \mathcal{T}, \nu)$, there exists an economy with $E$ as commodity space and $(T, \mathcal{T}, \nu)$ as space of traders such that, as in Tourky and Yannelis (1998), there is a core allocation not supportable as a Walrasian equilibrium (in fact there are no Walrasian equilibria at all) but such that all the conditions listed in Tourky and Yannelis (1998) under the term "desirable assumptions" are fulfilled: a) for some ordering on $E$, $E$ is an ordered Banach space so that $E_+$, the positive cone of $E$, has a non-empty interior; (b) endowments be-

[1] In particular, consumption sets have a non-empty interior.

[2] The paper of Tourky and Yannelis (1998) contains several refinements of this result and, in particular, a discussion of the meanings and implications of various notions of measurability with respect to allocations and preference profiles in this context.

[3] "Bochner" referring to the use of the Bochner integral to formalize aggregation of individual consumption bundles.

long to the interior of $E_+$; (c) consumption sets are equal to $E_+$; (d) preferences are complete preorderings and are continuous, convex and strictly monotone; (e) the preference mapping is measurable in the sense of Aumann (1964)[4]. Now by the result of Rustichini and Yannelis (1991) mentioned above, when the commodity space is separable then core-Walras equivalence holds for any atomless economy satisfying these assumptions.[5] Thus, combining this latter result with ours, we have established that the class $\{E\}$ of Banach spaces such that, under the "desirable assumptions," any atomless economy with commodity space $E$ exhibits core-Walras equivalence is exactly the class of Banach spaces that are separable.

Note, though, that when the commodity space is an abstract ordered Banach space and consumption sets coincide with the positive cone, non-emptiness of the core for an economy with a measure space of agents is in general impossible to guarantee—even if separability of the commodity space is assumed. The reason is that, with such consumption sets, the set of feasible allocations lacks suitable compactness properties; see Khan and Yannelis (1991) for a broader discussion of this point. Thus, in this setting, the problem of core-Walras equivalence tends to become empty in some sense. However, as shown in Khan and Yannelis (1991), when the consumption sets of an economy with a measure space of traders are assumed to be weakly compact, then in case of a separable commodity space, Walrasian equilibria and thus core allocations do exist.[6] The second result of our paper is now that under the hypothesis of weakly compact consumption sets, the core of an economy with a measure space of traders is non-empty, regardless of whether or not the commodity space is separable. On the other hand, we show that given a Banach space $E$ containing non-separable weakly compact subsets, and given any atomless measure space $(T, \mathcal{T}, \nu)$, there is an economy with $(T, \mathcal{T}, \nu)$ as space of agents and $E$ as commodity space that satisfies the assumptions of our core non-emptiness result (or even stronger ones) but has no Walrasian equilibrium; in particular, core-Walras equivalence does not hold. Thus, even in settings where core allocations in general exist, the property of an economy being atomless does not guarantee core-Walras equivalence when the commodity space is non-separable.

Finally, it will be shown that for an atomless economy where consumption sets are weakly compact but separable (and do not vary too much among

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[4]That is, given any two allocations the set of agents preferring what they get in the first allocation to what they get in the second is measurable, allocations being defined as Bochner integrable (hence strongly measurable) functions.

[5]Actually, the assumptions employed by Rustichini and Yannelis (1991, Theorem 4.1) are weaker than the "desirable assumptions;" in particular, preferences are not assumed to be convex or complete. Note that measurability of the preference mapping as postulated in Rustichini and Yannelis (1991, Theorem 4.1) can be derived from measurability of this mapping in the sense of Aumann (e.g. by using the lemma in the appendix of our paper).

[6]Provided, of course, that certain standard assumptions are in force.

agents) core-Walras equivalence holds. In particular, then, in view of our second result, there must be a Walrasian equilibrium. Incidentally, this latter core equivalence result shows that our non-equivalence result for economies with weakly compact consumption sets has nothing to do with the fact that such consumption sets may have an empty interior, but that it is indeed essential that they be non-separable.

To sum up, under the condition that consumption sets be weakly compact (and not too dispersed among agents), the class $\{E\}$ of Banach spaces such that for any atomless economy with commodity space $E$ core-Walras equivalence holds (and Walrasian equilibria do exist) consists of those Banach spaces having no non-separable weakly compact subsets; in particular, the separable Banach spaces belong to this class. On the other hand, this condition ensures that the core is always non-empty. Thus, for "very large" commodity spaces the notion of the core seems to be more robust than that of a Walrasian equilibrium.

One way to understand the reason underlying core-Walras non-equivalence in non-separable commodity spaces when aggregation of individual commodity bundles is formalized by means of the Bochner integral is to look at the connection between the fact that a Bochner integrable allocation must be essentially separably valued and the assumption that the preference mapping be Aumann measurable. The requirement that allocations be essentially separably valued implies that the property of an allocation being in the core is separably determined in the sense that a feasible allocation is a core allocation already when it is a core allocation relative to every separable subspace of the commodity space. On the other hand, it makes the requirement that the preference mapping be Aumann measurable weak, so that across the separable subspaces of the commodity space the profile of agents' preferences may be very dispersed. As a consequence, since the property of an allocation being Walrasian is determined relative to the entire commodity space, the core may be larger than the set of Walrasian allocations—even when the economy in question is atomless. More technically, this can be illustrated as follows. Suppose $f$ is a feasible allocation of some (atomless) economy, and suppose there is a price system $p$ such that relative to every fixed separable subspace $G$ of the commodity space almost all agents are optimizing at $p$. Then, since allocations have to be almost separably valued, $f$ is a core allocation.[7] However, for any price system with the above property, the exceptional set of measure zero of agents not optimizing relative to $G$ may vary with $G$—which reflects the fact that the preferences profile may be dispersed across the separable subspaces of the commodity space—so it might well happen that the set of agents not optimizing relative to the entire commodity space is a non-null set. That is, $f$ need not be Walrasian.

[7] It may be shown that, conversely, for a given core allocation a price system such as above exists (even when the commodity space is non-separable) provided the economy in question is atomless and, say, the "desirable assumptions" hold.

We finish up the introduction with a remark concerning the method of proof. In the Hilbert space setting of Tourky and Yannelis (1998), the core-Walras non-equivalence result is derived by these authors by using the fact that a non-separable Hilbert space has an orthonormal set with uncountably many members. Now an orthonormal set in a Hilbert space can be seen as a special case of a so called biorthogonal system, a concept that is defined for any Banach space. (Recall that given a Banach space $E$, a biorthogonal system is a family $(e_i, e_i^*)_{i\in I}$ in $E \times E^*$ such that $e_i^*(e_j) = \delta_{ij}$ for all $i, j \in I$, $E^*$ standing for the dual space of $E$.) However, it is known that there are non-separable Banach spaces for which every biorthogonal system is countable. For this reason, in the general Banach space setting one cannot proceed by using biorthogonal systems. Instead, we shall use a result by Juhász and Szentmiklóssy (1992) on transfinite sequences in compact spaces, applied to the unit ball in the dual of the commodity space, endowed with the weak* topology. (See Section 4.1.)

## 2 Notation and Terminology

Let $E$ be a real Banach space.

(a) The space $E$ is said to be an *ordered Banach space* if it is endowed with a vector ordering $\geq$ such that the positive cone $E_+ = \{x \in E\colon x \geq 0\}$ is closed in $E$. By "$\geq$ a vector ordering on $E$" we mean that $\geq$ is a reflexive, transitive and anti-symmetric binary relation on $E$ such that $x \geq y$ entails $x + z \geq y + z$ for any $z \in E$, and $\lambda x \geq \lambda y$ for any real number $\lambda \geq 0$. Thus, when $E$ is an ordered Banach space then $E_+$ is a closed convex cone satisfying $E_+ \cap -E_+ = \{0\}$. Recall that, conversely, any closed convex cone $C$ with $C \cap -C = \{0\}$ defines, by "$x \geq y$ if and only if $x - y \in C$," a vector ordering on $E$ under which $E$ is an ordered Banach space with positive cone $E_+ = C$.

(b) $E^*$ denotes the dual space of $E$, i.e. the space of all continuous linear functions from $E$ into $\mathbb{R}$. If $x \in E$ and $p \in E^*$, the value $p(x)$ of $p$ at $x$ will often be denoted by $\langle p, x\rangle$ for notational convenience. When $E$ is an ordered Banach space, we write $E_+^*$ for the set $\{p \in E^*\colon p(x) \geq 0 \text{ for all } x \in E_+\}$.

(c) $B_E$ denotes the closed unit ball of $E$, i.e. $B_E = \{x \in E\colon \|x\| \leq 1\}$.

(d) $(B_{E^*}, \text{weak}^*)$ denotes the closed unit ball of $E^*$, endowed with the weak* topology.

(e) Let $A$ be a subset of $E$. Then:

– $\operatorname{int} A$ denotes the (norm) interior of $A$;
– $c\ell\, A$ or $\bar{A}$ denote the (norm) closure of $A$;
– $\operatorname{co} A$ denotes the convex hull of $A$;
– $\operatorname{span} A$ denotes the linear span of $A$;
– $\langle p, A\rangle$, $p \in E^*$, denotes the set $\{p(x)\colon x \in A\}$.

(f) Let $p \in E^*$. Then:

– $\ker p$ denotes the kernel of $p$, i.e. $\ker p = \{x \in E \colon p(x) = 0\}$.

(g) $\mathcal{B}(E)$ denotes the Borel $\sigma$-algebra of $E$.

(h) Let $(T, \mathcal{T}, \nu)$ be a measure space. Then:

– Given a mapping $f \colon T \to E$, by "$f$ is integrable" we always mean $f$ is Bochner integrable.

– Given a correspondence $\varphi \colon T \to 2^E$, $\int_T \varphi(t)\, d\nu(t)$ means the set

$$\Big\{ z \in E \colon z = \int_T f(t)\, d\nu(t) \text{ for some (Bochner) integrable function } f \colon T \to E \text{ with } f(t) \in \varphi(t) \text{ for almost all } t \in T \Big\}.$$

# 3 The Model and the Results

## 3.1 The Basic Model

Let $E$ be an ordered Banach space. An *economy* $\mathcal{E}$ with commodity space $E$ is a pair $[(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ where

- $(T, \mathcal{T}, \nu)$ is a complete finite measure space of agents;[8]
- $X(t) \subset E_+$ is the consumption set of agent $t$;
- $\succcurlyeq_t \subset X(t) \times X(t)$ is the preference/indifference relation of agent $t$;
- $e(t) \in X(t)$ is the initial endowment of agent $t$;

and where the endowment mapping $e \colon T \to E$, given by $t \mapsto e(t)$, is assumed to be integrable[9].

An *allocation* (assignment) is an integrable function $f \colon T \to E$ such that $f(t) \in X(t)$ for almost all $t \in T$. An allocation $f$ is said to be *feasible* if

$$\int_T f(t)\, d\nu(t) \le \int_T e(t)\, d\nu(t) .$$

Thus our feasibility notion allows for free disposal. We follow the model of Khan and Yannelis (1991) in this respect. (See however Remarks 2 and 7.)

A *Walrasian equilibrium* of the economy $\mathcal{E}$ is a pair $(p, f)$ where $f$ is a feasible allocation and $p \in E^*_+ \setminus \{0\}$ is a price system such that for almost all $t \in T$:

(i) $\langle p, f(t) \rangle \le \langle p, e(t) \rangle$ and

(ii) if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x \rangle > \langle p, e(t) \rangle$.

[8]Whenever we speak of a measure space, the measure in question is meant to be positive and non-zero.

[9]As said in the previous section, throughout this paper "integrable" means "Bochner integrable." We do not discuss the implications of other notions of integrability in the context of the core equivalence problem.

(By requiring an equilibrium price system to be positive we follow the model of Khan and Yannelis (1991) again. Of course, the positivity requirement on equilibrium price systems can be seen as a consequence of the free disposal assumption embodied in the definition of feasibility. See also Remarks 3 and 8.)

A feasible allocation $f$ is said to be a *Walrasian allocation* if there is a $p \in E^*_+ \setminus \{0\}$ such that $(p, f)$ is a Walrasian equilibrium. An allocation $f$ is a *core allocation* if it is feasible and if there does not exist a coalition $S \in \mathcal{T}$ with $\nu(S) > 0$ and an integrable function $g\colon T \to E$ such that

(i) $\int_S g(t)\, d\nu(t) \le \int_S e(t)\, d\nu(t)$, i.e. $g$ is feasible for $S$, and

(ii) $g(t) \succ_t f(t)$ for almost all $t \in S$.

We denote by $C(\mathcal{E})$ the set of all core allocations of the economy $\mathcal{E}$ and by $\mathcal{W}(\mathcal{E})$ the set of Walrasian allocations.

We shall make use of the following standard assumptions:

(A1) $X(t)$ is closed and convex for every $t \in T$.

(Note that by definition of an economy, $e(t) \in X(t)$, i.e., $X(t)$ is non-empty.)

(A2) $\succcurlyeq_t$ is reflexive, transitive, and complete for every $t \in T$.

(A3) For every $t \in T$, $\succcurlyeq_t$ is continuous, i.e. for each $x \in X(t)$ the sets $\{y \in X(t)\colon y \succcurlyeq_t x\}$ and $\{y \in X(t)\colon x \succcurlyeq_t y\}$ are closed in $X(t)$.

(A4) For every $t \in T$, $\succcurlyeq_t$ is convex, i.e. for each $x \in X(t)$ the set $\{y \in X(t)\colon y \succcurlyeq_t x\}$ is convex.

(A5) If $f$ and $g$ are any two allocations then $\{t \in T\colon f(t) \succ_t g(t)\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$. ("Aumann measurability" of the profile of agents' preferences.)

### 3.2 Economies where consumption sets are equal to the positive cone of the commodity space

Let $E$ be an ordered Banach space whose positive cone has a non-empty interior, and let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ be an economy with commodity space $E$. In addition to the conditions listed in the previous section, we take the following ones into consideration in this section.

(B1) $e(t) \in \operatorname{int} E_+$ for every $t \in T$.

(B2) $X(t) = E_+$ for every $t \in T$.

(B3) For every $t \in T$, $\succcurlyeq_t$ is strictly monotone, i.e. whenever $x, x' \in X(t)$ with $x \ge x'$ but $x \ne x'$ then $x \succ_t x'$.

Note that (B2) makes (A1) redundant, and recall from the introduction that (A2) to (A5) and (B1) to (B3) together yield what is called the "desirable assumptions" in Tourky and Yannelis (1998).

We are ready to formulate our first result.

**Theorem 1.** *Let $(T, \mathcal{T}, \nu)$ be an arbitrary atomless complete finite measure space, and let $E$ be any non-separable Banach space. Assume the continuum hypothesis. Then there is an ordering $\geq$ on $E$ under which $E$ is an ordered Banach space with* $\operatorname{int} E_+ \neq \emptyset$ *and an economy $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t\in T}]$ with commodity space $E$ such that* (A1) *to* (A5) *and* (B1) *to* (B3) *are satisfied but*

$$C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E}); \text{ moreover, } \mathcal{W}(\mathcal{E}) = \emptyset.$$

**Remark 1.** As was noted in the introduction, when the commodity space $E$ is separable then for an atomless economy fulfilling the "desirable assumptions" core-Walras equivalence holds. Therefore, Theorem 1 implies that (under the continuum hypothesis) the class of Banach spaces for which (taken as commodity spaces) every atomless economy fulfilling the "desirable assumptions" exhibits core-Walras equivalence is exactly the class of Banach spaces that are separable.

**Remark 2.** Evidently, under assumptions (B2) and (B3), a core allocation $f$ of an economy must satisfy the feasibility condition with equality, necessarily; that is, $\int_T f(t)\, d\nu(t)$ and $\int_t e(t)\, d\nu(t)$ must be equal. Thus, the core-Walras non-equivalence result in Theorem 1 cannot be attributed to the fact that our feasibility definition allows for free disposal.

**Remark 3.** We have required an equilibrium price system to belong to $E_+^*$, i.e. to be positive and continuous. However, the result of Theorem 1 is not dependent on these requirements. Indeed, under (B2) and (B3) an equilibrium price system is automatically positive, and when $\operatorname{int} E_+ \neq \emptyset$ then every positive linear functional on $E$ is continuous.

**Remark 4.** If the economy satisfies (B2) and (B3), an equilibrium price system $p$ must in fact be strictly positive, i.e. $p(x) > 0$ must hold for all $x \in E_+ \setminus \{0\}$. In the proof of Theorem 1, $E_+$ will be constructed so that strictly positive price systems exist. Thus the result of Theorem 1 cannot be blamed on a missing of such price systems. We also remark here that $E_+$ will have the property that every continuous linear functional on $E$ is the difference of two continuous positive linear functionals. Thus $E^*$ will actually be rich in positive elements.

**Remark 5.** In the presence of (B1), (B2) and (A3), any quasi-equilibrium of an economy is in fact a Walrasian equilibrium (regardless of positivity or continuity requirements on price systems). Hence, denoting by $\mathcal{Q}(\mathcal{E})$ the set of all quasi-equilibrium allocations of an economy $\mathcal{E}$, Theorem 1 continues to be true if in its statement "$\mathcal{W}(\mathcal{E})$" is replaced by "$\mathcal{Q}(\mathcal{E})$."

### 3.3 Economies with weakly compact consumption sets

The treatment in this section is based on the framework of economies with weakly compact consumption sets, introduced by Khan and Yannelis (1991). Let $E$ be an ordered Banach space whose positive cone has a non-empty interior and let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ be an economy with commodity space $E$. Following Khan and Yannelis (1991), we will take the following assumptions into consideration:

(C1) $X(t)$ is weakly compact for every $t \in T$.

(C2) The correspondence $X(\cdot)\colon T \to 2^E$ is integrably bounded; that is to say, there is an integrable function $\rho\colon T \to \mathbb{R}_+$ such that for each $t \in T$, $\sup\{\|x\| : x \in X(t)\} \leq \rho(t)$.

(C3) $\{(t, x) \in T \times E \colon x \in X(t)\} \in \mathcal{T} \otimes \mathcal{B}(E)$; that is, the graph of the consumption sets correspondence $X(\cdot)\colon T \to 2^E$ belongs to $\mathcal{T} \otimes \mathcal{B}(E)$.

(C4) There is a separable subset $J$ of $E_+$ such that for every $t \in T$ there is an $x_t^\circ \in J \cap X(t)$ such that $e(t) - x_t^\circ$ is an interior point of $E_+$.[10]

To make a remark concerning the role of these assumptions in the proof of the result presented below about non-emptiness of the core of an economy, (C1) and (C2) (together with convexity of the $X(t)$'s) will ensure that the set of feasible allocations has suitable compactness properties. (C3) and (C4) will, in particular, allow us to apply some known equilibrium existence theorems to the restrictions of the economy to certain separable subspaces of the commodity space. In the context of those theorems, (C3) is used to obtain non-empty aggregate demand sets, and (C4) to overcome minimum wealth problems.

Note that under (A1), (A3) and (A4), preferences are weakly upper semicontinuous. Hence, if (A2) and (C1) too hold then preferences have satiation points. With respect to this, we shall consider the following conditions:

(C5) For every $t \in T$, if $x \in X(t)$ is a satiation point for $\succcurlyeq_t$ then $x \geq e(t)$.

(C6) For every $t \in T$, if $x \in X(t)$ is not a satiation point for $\succcurlyeq_t$ then $x$ belongs to the closure of $\{y \in X(t) \colon y \succ_t x\}$. (Local non-satiation at non-satiation points).

Finally, we introduce a condition requiring that consumption sets do not vary too much among agents. Observe that this condition automatically holds when the commodity space $E$ is separable.

(C7) There is a countable set $T' \subset T$ such that $\bigcup_{t \in T'} X(t)$ is dense in $\bigcup_{t \in T} X(t)$.

[10] Of course, if the commodity space is separable, then the word "separable" in the statement of (C4) is superfluous.

Now when the commodity space of an economy is separable, then assumptions (A1) to (A5) together with (C1) to (C6) ensure that Walrasian equilibria (and hence core allocations) do exist. (See Khan and Yannelis, 1991, Main Theorem, or Podczeck, 1997, Theorem 5.1.[11]) Our next theorem says that these assumptions ensure non-emptiness of the core even without separability.

**Theorem 2.** *Let $E$ be any ordered Banach space with* $\operatorname{int} E_+ \neq \emptyset$ *and let $\mathcal{E}$ be an economy with commodity space $E$ which satisfies assumptions* (A1) *to* (A5) *and* (C1) *to* (C4). *Then*

$$C(\mathcal{E}) \neq \emptyset.$$

See Section 4.1 for the proof. Note that assumptions (C5) to (C7) are actually not needed in Theorem 2.

**Remark 6.** We have not investigated the question of whether in the setting of weakly compact consumption sets there are conditions sufficient to guarantee the existence of core allocations without fee disposal, i.e. of core allocations where feasibility holds with equality. We leave this as an open problem.

We will finally present two theorems which, combined and viewed together with Theorem 2, give the result that (under the continuum hypothesis) the class of Banach spaces for which (taken as commodity spaces) every atomless economy fulfilling (A1) to (A5) and (C1) to (C7) has a Walrasian equilibrium and exhibits core-Walras equivalence consists of those Banach spaces that do not contain non-separable weakly compact subsets.

**Theorem 3.** *Let $(T, \mathcal{T}, \nu)$ be an arbitrary atomless complete finite measure space, and let $E$ be any Banach space containing a non-separable weakly compact subset. Assume the continuum hypothesis. Then there is an ordering $\geq$ on $E$ under which $E$ is an ordered Banach space with* $\operatorname{int} E_+ \neq \emptyset$ *and an economy $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ with commodity space $E$ such that* (A1) *to* (A5) *and* (C1) *to* (C7) *are satisfied but*

$$C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E}); \text{ moreover, } \mathcal{W}(\mathcal{E}) = \emptyset.$$

*This remains true even when* (C7) *is sharpened to*

(C7') For some $X \subset E_+$, $X(t) = X$ for all $t \in T$.

[11] Actually, in these theorems measurability of the *graph* of the preference mapping is assumed instead of condition (A5). However, by using the lemma in the appendix of our paper, the proofs of these theorems go through under (A5). It should be remarked here as well that Khan and Yannelis (1991) do not use conditions (C5) and (C6). On the other hand, it is assumed by these authors that for some norm compact subset of the commodity space, say $C$, the endowment of each trader belongs to $C$.

See Section 4.2 for the proof and see also the remarks at the end of this section. Note that when combined with Theorem 2, Theorem 3 implies, in particular, that for very large commodity spaces the notion of the core is more robust than that of a Walrasian equilibrium.

The following theorem is actually a variant of a core equivalence result of Rustichini and Yannelis (1991). The main difference is that in their model the consumption sets coincide with the positive cone of the commodity space.

**Theorem 4.** *Let $E$ be an ordered Banach space with* $\operatorname{int} E_+ \neq \emptyset$ *and let $\mathcal{E}$ be an economy with commodity space $E$ which satisfies assumptions* (A1) *to* (A3), (A5), *and* (C2) *to* (C7). *Assume in addition that the measure space $(T, \mathcal{T}, \nu)$ of agents is atomless and that*

(C8) $X(t)$ is separable for every $t \in T$.

*Then*

$$\mathcal{W}(\mathcal{E}) = C(\mathcal{E}).$$

The proof (which is an adaptation of that in Rustichini and Yannelis (1991, Theorem 4.1) to the framework dealt with here) is contained in Section 4.3. Note that (C1) and the convexity assumption (A4) are not needed in Theorem 4.[12] Also note that this theorem shows, in particular, that the result of Theorem 3 cannot be blamed on the fact that weakly compact sets may have an empty interior, but indeed depends on whether the commodity space in question contains weakly compact sets being non-separable.

We close this section with a couple of remarks concerning Theorem 3.

**Remark 7.** The result of Theorem 3 does not depend on the fact that our definition of feasible allocations allows for free disposal. In fact, in the proof of this theorem a core allocation not supportable as a Walrasian equilibrium will be produced for which the feasibility condition is satisfied with equality.

**Remark 8.** Theorem 3 continues to be valid when the positivity and continuity requirements on an equilibrium price system are dropped. See the remark following the proof of this theorem.

**Remark 9.** Let us call a quasi-equilibrium *non-trivial* if there is a set, of measure $> 0$, of consumers having a commodity bundle in the consumption set with a value smaller than that of the endowment. Thus, under a suitable irreducibility assumption a non-trivial quasi-equilibrium will turn out to be a Walrasian equilibrium. Let $\widehat{\mathcal{Q}}(\mathcal{E})$ denote the set of all allocations belonging to a non-trivial quasi-equilibrium of an economy $\mathcal{E}$. Theorem 3 remains true if in its statement "$\mathcal{W}(\mathcal{E})$" is replaced by "$\widehat{\mathcal{Q}}(\mathcal{E})$." Again, see the remark following the proof of this theorem.

[12] Would we replace "integrable" in the definition of an allocation by "strongly measurable," which would amount to a strengthening of (A5), then (C2) would be unnecessary as well.

# 4 Proofs

## 4.1 Preliminaries

The principal mathematical tool in order to prove Theorems 1 and 3 is provided by the following:

**Proposition.** *Let $X$ be a non-separable Banach space and assume the continuum hypothesis. Then, denoting by $\omega_1$ the first uncountable ordinal number, there is a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of elements of $X^*$ such that*

(a) *$f_\alpha \neq 0$ and $\|f_\alpha\| \leq 1$ for all ordinals $\alpha \in [0, \omega_1)$; but*

(b) *if $S$ is any separable subset of $X$ then there is an ordinal $\alpha_S < \omega_1$ such that for each $\alpha \in [\alpha_S, \omega_1)$, $f_\alpha(x) = 0$ for all $x \in S$.*[13]

*Proof.* Since $X$ is non-separable, $(B_{X^*}, \text{weak}^*)$ is not first-countable at 0. (See e.g. the proof of Corollary 2 in Holmes, 1975, p.72.) Therefore, and because of the continuum hypothesis, it follows from Juhász and Szentmiklóssy (1992, Corollary 2.1 and the lines before the statement of that result) that there exists a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ in $B_{X^*}$ that converges to 0 with respect to the weak$^*$ topology, but such that $f_\alpha \neq 0$ for all $\alpha \in [0, \omega_1)$. In particular, (a) of the proposition holds for such a sequence. To see that (b) holds as well, simply note that "$f_\alpha \to 0$ in $(B_{X^*}, \text{weak}^*)$ as $\alpha \uparrow \omega_1$" means that given any $x \in X$ and $n \in \mathbb{N}$ there is a $\beta < \omega_1$ such that $|f_\alpha(x)| < 1/n$ for all $\alpha < \omega_1$ with $\alpha > \beta$, and recall that if $A$ is any countable set of ordinals $\gamma < \omega_1$ then there is an ordinal $\overline{\gamma} < \omega_1$ such that for each $\gamma \in A$, $\gamma < \overline{\gamma}$. □

## 4.2 Proof of Theorem 1

Pick some $\overline{e} \in E$ with $\|\overline{e}\| = 3$ and let $C$ be the cone generated by $\{\overline{e}\} + B_E$, i.e.

$$C = \{x \in E \colon x = \lambda(\overline{e} + y), y \in B_E, \lambda \geq 0\}.$$

Then $C$ is convex, and since $\overline{e} \notin B_E$, $C$ is closed and $C \cap -C = \{0\}$. Thus $C$ generates a vector ordering on $E$ under which $E$ becomes an ordered Banach space with positive cone $E_+$ equal to $C$.[14] Of course, $\operatorname{int} E_+ \neq \emptyset$; in particular, $\overline{e} \in \operatorname{int} E_+$.

We will now construct an economy for the given measure space $(T, \mathcal{T}, \nu)$ of agents. Concerning consumption sets and endowments, for each individual $t$ in $T$ we let the endowment $e(t)$ be equal to $\overline{e}$ and the consumption set $X(t)$ equal

[13] If $\alpha, \beta$ are ordinal numbers then $[\alpha, \beta)$ denotes the ordinal interval $\{\gamma \colon \alpha \leq \gamma < \beta\}$.

[14] It is easily seen that the cone $E_+$ so constructed is normal, i.e. has not an excessive "width;" in particular, order intervals are norm bounded, and every element of $E^*$ is the difference of two elements of $E^*_+$. Cf. Kelley and Namioka (1976, pp. 227 and 228).

to $E_+$. Clearly this assignment of endowments and consumption sets satisfies assumptions (B1) and (B2) (hence also (A1)). Further, since the measure $\nu$ is finite, the endowment mapping $t \mapsto \bar{e}$ is integrable, as required in our definition of an economy.

With regard to the construction of preferences, first recall that, by assumption, the continuum hypothesis is in force and the measure space $(T, \mathcal{T}, \nu)$ is atomless. Therefore, by using Proposition 3.3 in Tourky and Yannelis (1998), we may write $T = \bigcup_{\alpha<\omega_1} N_\alpha$ where $\omega_1$ stands for the first uncountable ordinal number and $(N_\alpha)_{\alpha<\omega_1}$ is a family of pairwise disjoint null sets in $T$. Denote by $\phi\colon T \to [0, \omega_1)$ the mapping that takes a $t \in T$ to that ordinal number $\alpha$ for which $t \in N_\alpha$.

Now let $(f_\alpha)_{\alpha<\omega_1}$ be a family of elements of $E^*$ chosen according to the proposition in Section 4.1, and for each $t \in T$ let $q_t = f_{\phi(t)}$. Then according to (a) of the proposition

$$q_t \neq 0 \text{ and } \|q_t\| \leq 1 \text{ for all } t \in T, \tag{4.1}$$

and from (b) of the proposition:

(4.2) For any given separable subset $S$ of $E$ the set

$$\{t \in T\colon q_t(s) \neq 0 \text{ for some } s \in S\} \text{ is a null set in } T,$$

because for each ordinal number $\alpha < \omega_1$ we have $\phi^{-1}([0, \alpha)) = \bigcup_{\alpha'<\alpha} N_{\alpha'}$, each $N_{\alpha'}$ is a null set, and for each $\alpha < \omega_1$ the set $[0, \alpha)$ is countable.

Finally, use the Hahn Banach theorem to find a $\hat{q} \in E^*$ with $\|\hat{q}\| = 1$ and $\hat{q}(\bar{e}) = 3$—as is possible since $\|\bar{e}\| = 3$—and then for each $t \in T$, define a utility function $u_t\colon E_+ \to \mathbb{R}$ by

$$u_t(x) = \hat{q}(x) + q_t(x),\ x \in E_+.$$

The family of preferences so defined satisfies assumptions (A2) to (A5) as well as (B3). Indeed, this is clear for (A2) to (A4). Concerning (B3), first note that in view of (4.2), we may as well assume that $q_t(\bar{e}) = 0$ holds for *all* $t \in T$ (by modifying the mapping $t \mapsto q_t$ on a null set if necessary). Then for all $t \in T$ and each $y \in B_E$:

$$(\hat{q} + q_t)(\bar{e} + y) = \hat{q}(\bar{e}) + \hat{q}(y) + q_t(y) \geq 3 - 1 - 1 > 0$$

whence $(\hat{q} + q_t)(x) > 0$ for each $x \in E_+ \setminus \{0\}$. That is, if $x, x' \in E_+$ satisfy $x \geq x'$ and $x \neq x'$ (i.e. $x - x' \in E_+ \setminus \{0\}$) then $(\hat{q} + q_t)(x - x') > 0$ i.e. $u_t(x) > u_t(x')$. Thus we have (B3). For (A5), recall that by definition, an allocation is a function that is Bochner integrable and hence almost separably valued. Thus from (4.2):

(4.3) If $h\colon T \to E_+$ is an allocation then

$$\text{for almost all } t \in T,\ u_t(h(t)) = \langle \hat{q}, h(t) \rangle.$$

Consequently, since a Bochner integrable function is weakly measurable, and since the measure space $(T, \mathcal{T}, \nu)$ is complete, given any allocation $h: T \to E_+$ the mapping $t \mapsto u_t(h(t))$ is measurable. It is plain that this implies (A5).

Record for future reference that the element $\hat{q}$ of $E^*$ is positive. Indeed, for any $y \in B_E$, $\hat{q}(\overline{e} + y) > 3 - 1 > 0$ whence $\hat{q}(x) \geq 0$ for each $x \in E_+$.

To sum up, an economy for the given measure space of traders has been found such that the assumptions listed in the statement of Theorem 1 all hold. We are now going to show that this economy has no Walrasian equilibrium. Arguing by contradiction, suppose there is a Walrasian equilibrium, say $(p, f)$. Then $\int f(t)\, d\nu(t) \leq \nu(T)\overline{e}$ and hence

$$\left\langle \hat{q}, \int f(t)\, d\nu(t) \right\rangle \leq \langle \hat{q}, \nu(T)\overline{e} \rangle \tag{4.4}$$

since $\hat{q} \in E^*_+$. Also, since for each $t \in T$ the endowment point $\overline{e}$ belongs to the budget set, we must have $u_t(f(t)) \geq u_t(\overline{e})$ for almost all $t \in T$. Now from (4.3), for almost every $t \in T$ both $u_t(f(t)) = \hat{q}(f(t))$ and $u_t(\overline{e}) = \hat{q}(\overline{e})$. Consequently, $\hat{q}(f(t)) \geq \hat{q}(\overline{e})$ for almost all $t \in T$, whence, from (4.4), $\hat{q}(f(t)) = \hat{q}(\overline{e})$ for almost all $t \in T$. It follows that for almost every $t \in T$, $u_t(\overline{e}) = u_t(f(t))$, whence the pairing of $p$ with the initial allocation $t \mapsto \overline{e}$ is a Walrasian equilibrium also. But $\overline{e}$ is in the interior of the consumption set $E_+$, so the equilibrium conditions for the pair $(p, t \mapsto \overline{e})$ imply that for almost all $t \in T$, $\ker p \subset \ker(\hat{q} + q_t)$; that is, by a standard fact from linear algebra, for almost every $t \in T$ there exists a real number $\lambda_t$ such that $\hat{q} + q_t = \lambda_t p$. However, this means there must be a $z \in E$ for which $q_t(z) \neq 0$ for almost all $t \in T$. Indeed, in case $\hat{q} \neq \gamma p$ for every real number $\gamma$, take any $z \in E$ with $\hat{q}(z) \neq 0$ but $p(z) = 0$. Consider the other case, i.e. assume that $\hat{q} = \gamma p$ for some $\gamma$. Then for almost every $t \in T$, $q_t = \gamma_t p$ for some $\gamma_t$, and since according to (4.1), $q_t \neq 0$ for all $t \in T$, it follows again that for some $z \in E$, $q_t(z) \neq 0$ for almost all $t \in T$. But "$q_t(z) \neq 0$ for almost all $t \in T$" is impossible by virtue of (4.2), and we have thus arrived at a contradiction, proving that $\mathcal{W}(\mathcal{E}) = \emptyset$ where $\mathcal{E}$ stands for the economy constructed.

To complete the proof, we must show that $C(\mathcal{E})$ is non-empty. In fact, we show that $C(\mathcal{E})$ contains the initial allocation $t \mapsto \overline{e}$. To this end, fix any coalition $C \in \mathcal{T}$ with $\nu(C) > 0$ and let $f: T \to E_+$ be an allocation which is feasible for $C$. Thus, $\int_C f(t)\, d\nu(t) \leq \nu(C)\overline{e}$. In particular, $\langle \hat{q}, \int_C f(t)\, d\nu(t) \rangle \leq \langle \hat{q}, \nu(C)\overline{e} \rangle$ (recall: $\hat{q}$ is positive), whence $\int_C \langle \hat{q}, f(t) \rangle\, d\nu(t) \leq \nu(C)\langle \hat{q}, \overline{e} \rangle$. But therefore, in view of (4.3), $u_t(f(t)) > u_t(\overline{e})$ cannot hold for almost all $t \in C$, and it follows that the initial allocation $t \mapsto \overline{e}$ indeed belongs to $C(\mathcal{E})$ as predicted. The proof of Theorem 1 is complete.

### 4.3 Proof of Theorem 2

Assume first that $E$ is separable. In this case, the assumptions of the theorem guarantee that one can use the arguments of Khan and Yannelis (1991) or Podczeck (1997, Theorem 5.1), together with part (i) of the lemma in the Appendix, to get an allocation $f\colon T \to E$ and some $p \in E^*_+ \setminus \{0\}$ such that

(i) $\int f(t)\, d\nu(t) \leq \int e(t)\, d\nu(t)$; and

(ii) for almost all $t \in T$: if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x\rangle > \langle p, e(t)\rangle$.

(With respect to Khan and Yannelis (1991) note that their assumption of a norm compact subset of $E$ that contains the endowments of all agents can be dropped by using a suitable approximation argument. With respect to Podczeck (1997, Theorem 5.1) note that his assumptions (B-4) and (B-5) are not needed if the price/allocation pair $(p, f)$ is required to satisfy only (i) and (ii) above but not the budget conditions of a Walrasian equilibrium; also, his conditions (A-1) and (A-2) can be dropped if, as in the present theorem, preferences are assumed to be convex.)

Our proof now proceeds by application of the just made observation to the restrictions of the economy $\mathcal{E}$ to the elements of some directed set of separable subspaces of $E$, followed by a limit argument to obtain a core allocation.

To begin with, consider the endowment mapping $t \mapsto e(t)$. By definition of an economy, it is integrable, hence almost separably valued; we may as well modify the economy $\mathcal{E}$ on a null set of $T$ if necessary and assume that for some separable subset $H$ of $E$, $e(t)$ is in $H$ for *each* $t \in T$. Now let $J$ be a subset of $E_+$ as guaranteed by Assumption (C4). In particular, $J$ is separable and hence the same is true of $H \cup J$. Let $\mathcal{F}$ denote the collection of all separable closed linear subspaces $F$ of $E$ with $F \supset H \cup J$. Note that by choice of $H$ and $J$:

$$\text{For each } t \in T \text{ there is an } x^\circ_t \in X(t) \text{ such that} \qquad \text{for all } F \in \mathcal{F},\ x^\circ_t \in F \text{ and } e(t) - x^\circ_t \in F \cap \operatorname{int} E_+. \tag{4.5}$$

We provide each $F \in \mathcal{F}$ with the relative ordering of $E$. Then each $F \in \mathcal{F}$ is also an ordered Banach space, and we have $F_+ = F \cap E_+$. Now for each $F \in \mathcal{F}$ and each $t \in T$, let

$X^F(t) = X(t) \cap F$, and let

$\succcurlyeq^F_t$ be the restriction of $\succcurlyeq_t$ to $X^F(t)$.

In particular, then, $e(t) \in X^F(t) \subset F_+$, and thus $\mathcal{E}^F$ given by

$$\mathcal{E}^F = [(T, \mathcal{T}, \nu), (X^F(t), \succcurlyeq^F_t, e(t))_{t \in T}]$$

is an economy with commodity space $F$. (Evidently, the endowment mapping $t \mapsto e(t)$, being integrable as a mapping into $E$, is integrable when considered as mapping into $F$, too.) Given any $F \in \mathcal{F}$, the economy $\mathcal{E}^F$ satisfies the same assumptions as the original economy $\mathcal{E}$. Indeed, this is obvious for (A1) to (A4) and (C2). As for (A5), simply note that an integrable mapping from $T$ into $F$ is also integrable when viewed as mapping taking values in $E$. That (C1) holds is immediate from the fact that the weak topology of $F$ coincides with the relative weak topology of $F$ as a subspace of $E$. That (C3) holds follows from the fact that the inclusion of $T \times F$ into $T \times E$ is $\mathcal{T} \otimes \mathcal{B}(F) - \mathcal{T} \otimes \mathcal{B}(E)$ measurable (because $\mathcal{B}(F) = \{B \cap F : B \in \mathcal{B}(E)\}$). Concerning (C4), finally, note that $F \cap \operatorname{int} E_+ \subset \operatorname{int} F_+$ ($\operatorname{int} F_+$ meaning the interior of $F_+$ in $F$, of course). Hence (4.5) can be reformulated to state:

(4.6) For each $t \in T$ there is an $x_t^\circ \in E_+$ such that for all $F \in \mathcal{F}$,
$$x_t^\circ \in X(t) \cap F \text{ and } e(t) - x_t^\circ \in \operatorname{int} F_+ \quad (\operatorname{int} F_+ \text{ the interior of } F_+ \text{ in } F).$$

Thus, (C4) is satisfied for the economies $\mathcal{E}^F$, too. We may therefore use the observation made at the beginning of this proof to choose, for each $F \in \mathcal{F}$, an integrable mapping $f_F : T \to E$ and a point $p_F \in F_+^* \setminus \{0\}$ such that

$$\int f_F(t)\, d\nu(t) \le \int e(t)\, d\nu(t), \tag{4.7a}$$

$$f_F(t) \in X(t) \quad \text{for almost all } t \in T, \text{ and} \tag{4.7b}$$

(4.7c) for almost all $t \in T$,
$$\text{if } x \in X(t) \cap F \text{ satisfies } x \succ_t f_F(t) \text{ then } \langle p_F, x\rangle > \langle p_F, e(t)\rangle.$$

For future reference we remark:

(4.8) There is a $u \in E$ such that for each $F \in \mathcal{F}$, $u \in \operatorname{int} F_+$ and $p_F(u) > 0$ ($\operatorname{int} F_+$ meaning the interior of $F_+$ in $F$).

(Indeed, $p_F \in F_+^* \setminus \{0\}$ means $p_F(x) > 0$ for all $x \in \operatorname{int} F_+$ and according to (4.6) there is a $u \in E$ such that $u \in \operatorname{int} F_+$ for each $F \in \mathcal{F}$.)

Preparing for a limit argument, consider the Banach space $L^1(\nu, E)$ of all ($\nu$-equivalence classes of) Bochner integrable functions $h$ from $(T, \mathcal{T}, \nu)$ into $E$, the norm being given by $\|h\|_1 = \int \|h(t)\|\, d\nu(t)$. Let

$$B = \{h \in L^1(\nu, E) : h(t) \in X(t) \text{ for almost all } t \in T\}.$$

We claim $B$ is a weakly compact set in $L^1(\nu, E)$. Indeed, by Assumption (C2), $B$ is both bounded in $L^1(\nu, E)$ and uniformly integrable. But these facts together with the fact that each $X(t)$ is weakly compact in $E$ (Assumption (C1)) imply that $B$ is weakly relatively compact in $L^1(\nu, E)$ (see Diestel, Ruess, and Schachermayer, 1993, Corollary 2.6). Now $B$ is a (norm) closed subset of $L^1(\nu, E)$ (since each $X(t)$ is closed, and since a (norm) convergent sequence in $L^1(\nu, E)$, say with limit $g$, has a subsequence that converges to $g$ pointwise almost everywhere in the norm of $E$). But since each $X(t)$ is convex (Assumption (A1)), $B$ is too, and hence $B$ is in fact weakly closed. We may conclude that $B$ is indeed weakly compact in $L^1(\nu, E)$, as advertised.

According to (4.7b), $B$ contains $f_F$ for each $F \in \mathcal{F}$ (identifying each $f_F$ with its $\nu$-equivalence class). Evidently $\mathcal{F}$ is a directed set under inclusion (since the closed linear span of two elements of $\mathcal{F}$ is again separable, hence in $\mathcal{F}$). Thus the family $(f_F)_{F\in\mathcal{F}}$ is a net in $B$, and since this latter set was shown to be weakly compact in $L^1(\nu, E)$, we may assume there is an $f \in B$ such that $f_F \to f$ weakly in $L^1(\nu, E)$ (by passing to a subnet if necessary).

We assert that $f$ is a core allocation for the economy $\mathcal{E}$. To see this, first observe that $f$ being in $B$ we have $f(t) \in X(t)$ for almost all $t \in T$. That is, $f$ is an allocation. Next, note that the operator $h \mapsto \int h\, d\nu$, $h \in L^1(\nu, E)$, is (norm) continuous, thus (by a standard fact—see Dunford and Schwartz, 1958, Theorem 15, p. 422) continuous for the weak topologies of $E$ and $L^1(\nu, E)$, too. Therefore from (4.7a) and the fact that $E_+$ is closed and convex, hence weakly closed,

$$\int f(t)\, d\nu(t) \le \int e(t)\, d\nu(t),$$

i.e. the allocation $f$ is feasible. Now to see that $f$ is in fact in the core, we shall establish the following

*Claim*: Given any $G \in \mathcal{F}$, there is a $p \in G^*_+$ such that for almost every $t \in T$, if $x \in X(t) \cap G$ satisfies $x \succ_t f(t)$ then $p(x) > p(e(t))$.

Assuming for the time being that the claim has been verified, we can continue as follows. Suppose $f$ is not in the core. Then, since $f$ is a feasible allocation, there is a coalition $S$ with $\nu(S) > 0$ and an integrable function $g\colon T \to E$ such that $g(t) \succ_t f(t)$ for almost all $t \in S$ but $\int_S g(t)\, d\nu(t) \le \int_S e(t)\, d\nu(t)$. The mapping $g$ being integrable, hence almost separably valued, an element $G$ can be chosen from $\mathcal{F}$ so that for almost all $t \in T$, $g(t) \in G$. (Indeed, whenever $A$ is a separable subset of $E$, the closed linear span of $A \cup H \cup J$ is an element of $\mathcal{F}$, $H$ and $J$ the subsets of $E$ from the definition of $\mathcal{F}$ above.) In particular, then, both $\int_S g(t)\, d\nu(t)$ and $\int_S e(t)\, d\nu(t)$ are in $G$. Now choosing a $p \in G^*$ in accordance with the claim above, we obtain on the one hand

$$\left\langle p, \int_S g(t)\, d\nu(t) \right\rangle = \int_S \langle p, g(t)\rangle\, d\nu(t) > \int_S \langle p, e(t)\rangle\, d\nu(t) = \left\langle p, \int_S e(t)\, d\nu(t) \right\rangle$$

and on the other one, since $p$ is positive,

$$\left\langle p, \int_S g(t)\, d\nu(t) \right\rangle \le \left\langle p, \int_S e(t)\, d\nu(t) \right\rangle,$$

and we thus arrive at a contradiction, proving that $f$ is a core allocation.

It remains to show that the above claim is correct. Thus let $G \in \mathcal{F}$. Returning to the net $(f_F)_{F\in\mathcal{F}}$, consider the set $Z \subset L^1(\nu, E)$ given by $Z = \{f_F\colon F \supset G\}$. Evidently $Z$ is weakly relatively compact in $L^1(\nu, E)$, the $f_F$'s belonging to the weakly compact subset $B$ of $L^1(\nu, E)$. In addition, $f$ is a weak cluster point of $Z$ because the net $(f_F)_{F\in\mathcal{F}}$ converges to $f$ weakly in $L^1(\nu, E)$. Recall now that every Banach space is angelic[15] in its weak topology. Applying this fact to the present situation, find a sequence $(f_n)_{n=1}^{\infty}$ in $Z$ with $f_n \to f$, weakly in $L^1(\nu, E)$.

From above, for each $n$ we have a $p_n \in G_+^*$ such that:

(4.9) For almost every $t \in T$, for all $n$,
if $x \in X(t) \cap G$ satisfies $x \succ_t f_n(t)$ then $\langle p_n, x\rangle > \langle p_n, e(t)\rangle$.

(4.10) For some $u \in \operatorname{int} G_+$, $p_n(u) = 1$ for all $n$
($\operatorname{int} G_+$ the interior of $G_+$ in $G$).

(Note: each $f_n$ equals $f_F$ for some $F \in \mathcal{F}$ with $F \supset G$. The $p_n$'s are obtained by restricting the appropriate $p_F$'s to $G$; by virtue of (4.8) we can renormalize so as to get (4.10).) Each $p_n$ being in $G_+^*$, (4.10) implies that the sequence $(p_n)$ is (norm) bounded. Therefore, since $G$ is separable, it has a weak* convergent subsequence, say $(p_{n_k})$ with limit $p$; let $(f_{n_k})$ be the corresponding subsequence of $(f_n)$. Note that $p \in G_+^* \setminus \{0\}$ and that, for each $k$, $f_{n_k}(t) \in X(t)$ for almost all $t \in T$.

Since $(f_{n_k})$, being a subsequence of $(f_n)$, converges to $f$ weakly in $L^1(\nu, E)$, there exists a sequence $(g_k)$ with $g_k \in \operatorname{co}\{f_{n_m} : m \geq k\}$ such that $(g_k(t))$ converges to $f(t)$ in the norm of $E$ for almost every $t \in T$. (See Diestel, Ruess, and Schachermayer, 1993, Theorem 2.1.) By convexity of the sets $X(t)$ (Assumption (A1)), for almost every $t \in T$ we must have $g_k(t) \in X(t)$ for all $k$. Thus by completeness and upper semicontinuity of preferences (Assumptions (A2) and (A3)), for almost every $t \in T$, if $x \in X(t)$ satisfies $x \succ_t f(t)$ then there is a $k_0$ (possibly depending on $x$ and $t$) such that $x \succ_t g_k(t)$ for all $k \geq k_0$. Since preferences are convex (Assumption (A4)) it follows that, for almost every $t \in T$, if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $x \succ_t f_{n_k}(t)$ for infinitely many $k$. Hence from (4.9), for almost every $t \in T$, if $x \in X(t) \cap G$ satisfies $x \succ_t f(t)$ then $\langle p_{n_k}, x\rangle > \langle p_{n_k}, e(t)\rangle$ for infinitely many $k$. But therefore, for almost every $t \in T$, if $x \in X(t) \cap G$ satisfies $x \succ_t f(t)$ then $\langle p, x\rangle \geq \langle p, e(t)\rangle$ because $p_{n_k} \to p$ weak* in $G^*$. Now $p$ belongs to $G_+^* \setminus \{0\}$ so for all $v \in \operatorname{int} G_+$, $p(v) > 0$ (as before, $\operatorname{int} G_+$ meaning the interior of $G_+$ in $G$). Hence, using (4.6) (together with the hypotheses that preferences are lower semicontinuous and consumption sets are convex), we can conclude that, for almost all $t \in T$, if $x \in X(t) \cap G$ satisfies $x \succ_t f(t)$ then, in fact, $\langle p, x\rangle > \langle p, e(t)\rangle$. This completes the verification of the claim above, thus finishing the proof of Theorem 2.

**Remark 10.** Observe that the allocation $f$ obtained in the proof just given need not be Walrasian. This would be so even if the above claim were to be validated

[15] Recall that a topological space $Y$ is called *angelic* if for every relatively countably compact set $A \subset Y$ the following are true: (a) $A$ relatively compact, and (b) for each $x$ in the closure of $A$ there is a sequence in $A$ that converges to $x$.

for (the restrictions to the $G \in \mathcal{F}$ of) a single $p \in E^*$ and, in addition, the budget condition were to hold for almost all $t \in T$. The reason is that the exceptional set of measure zero of agents not optimizing relative to $G$ at $p$ may vary with $G \in \mathcal{F}$.

### 4.4 Proof of Theorem 3

This proof closely follows that of Theorem 1, so many of the details will be left out. As in the proof of Theorem 1, we start by selecting some $\overline{e}$ from $E$ with $\|\overline{e}\| = 3$ and endow $E$ with the vector ordering—denoted by $\geq$—induced by the cone generated by $\{\overline{e}\} + B_E$. Thus $E$ becomes an ordered Banach space with $\operatorname{int} E_+ \neq \emptyset$; note that $\overline{e} \in \operatorname{int} E_+$.

Choose some non-separable weakly compact subset of $E$; this is possible by hypothesis. Denoting this set by $C'$, let $C$ be the closed absolutely convex hull of $C' \cup \{\overline{e}\}$; of course, $C$ is non-separable and weakly compact, too.

Denote the closed linear span of $C$ by $Y$. Endowing $Y$ with the relative norm and the relative ordering of $E$, $Y$ also becomes an ordered Banach space and we have $Y_+ = E_+ \cap Y$. Observe that if an economy $\mathcal{E}$ with commodity space $Y$ is given, $\mathcal{E}$ can be viewed as an economy with commodity space $E$ as well. Moreover, as may readily be verified, if $\mathcal{E}$ as economy with commodity space $Y$ fulfills the assumptions listed in Theorem 3, then so does $\mathcal{E}$ viewed as an economy with commodity space $E$. (For Assumption (C4), note that $\operatorname{int} Y_+ = Y \cap \operatorname{int} E_+$, $\operatorname{int} Y_+$ meaning the interior with respect to $Y$, of course.) It is therefore enough to prove Theorem 3 with $Y$ as commodity space; so we may as well assume that $Y = E$, i.e that the linear span of $C$ is dense in $E$.

Now since $C$, being weakly compact, is in particular (norm) bounded, and since $\overline{e} \in \operatorname{int} E_+$, for some number $\beta$ with $0 < \beta < 1$ we have $\{\overline{e}\} + \beta C \subset \operatorname{int} E_+$. Choose and fix such a $\beta$ and let $K' = \beta C$. Clearly $K'$ is weakly compact and convex. Now let

$$K = \{(1-\lambda)(\overline{e} + z) + \lambda(2\overline{e}) \colon z \in K', 0 \leq \lambda \leq 1\}.$$

The following properties of the set $K$ will be needed. First, $K \subset \operatorname{int} E_+$ because $2\overline{e} \in \operatorname{int} E_+$ and $\{\overline{e}\} + K' \subset \operatorname{int} E_+$. Second, $K$ is convex and weakly compact since $K'$ is. Further, since $K'$, being absolutely convex, contains both the point 0 and the point $-\beta\overline{e}$, we have $\overline{e} \in K$ and for $\gamma = (1-\beta)$, $\gamma\overline{e} \in K$ as well; observe that $\overline{e} - \gamma\overline{e} \in \operatorname{int} E_+$. Finally,

$$(4.11) \qquad 2\overline{e} \in K, \text{ and for all } x \in K,\ 2\overline{e} \geq x.$$

Indeed, $2\overline{e} \in K$ holds by the definition of $K$. To check the second assertion, pick any $x \in K$. Then $2\overline{e} - x = (1-\lambda)(\overline{e} - z)$ for some $0 \leq \lambda \leq 1$ and some $z \in K'$. By construction, $\{\overline{e}\} + K' \subset E_+$, and since $K'$ is absolutely convex, $K' = -K'$. Hence $\overline{e} - z \geq 0$ whence $2\overline{e} \geq x$.

Now toward the construction of an economy, for each individual $t$ in the given set $T$ of traders we let the endowment be equal to $\overline{e}$ and the consumption set equal to $K$. Evidently, then, as required in Theorem 3, assumptions (C7) and, in particular, (C7') are satisfied, and from the previous paragraph, the same is true of (A1) and (C1) to (C4). Furthermore, as required in our definition of an economy, the endowment of each agent belongs to the consumption set, and the endowment mapping $t \mapsto \overline{e}$ is integrable (since the measure space $(T, \mathcal{T}, \nu)$ is finite).

Concerning preferences, choose elements $q_t$, $t \in T$, and $\hat{q}$ from $E^*$ precisely as in the proof of Theorem 1, and then for each $t \in T$ let $u_t \colon K \to \mathbb{R}$ be the utility function defined by

$$u_t(x) = \hat{q}(x) + q_t(x),\ x \in K.$$

Clearly, assumptions (A2), (A3), and, since $K$ is convex (and each $u_t$ is the restriction to $K$ of a function that is linear), (A4) and (C6) hold for this specification of preferences. In addition—see the proof of Theorem 1—(A5) is satisfied, and in particular we have:

(4.12) If $h \colon T \to K$ is any allocation then for almost all $t \in T$, $u_t(h(t)) = \langle \hat{q}, h(t) \rangle$.

Finally, regarding (C5), recall from the proof of Theorem 1 that one may assume that for all $t \in T$, $(\hat{q} + q_t)(x) > 0$ for each $x \in E_+ \setminus \{0\}$. But then, from (4.11), we have $u_t(2\overline{e}) > u_t(x)$ for each $x \in K \setminus \{2\overline{e}\}$; that is, for all $t \in T$, the only satiation point of $u_t$ is $2\overline{e}$. Thus, since $\overline{e} \in E_+$ and consequently $2\overline{e} \geq \overline{e}$, (C5) is satisfied, too.

We have thus found an economy $\mathcal{E}$ for the given measure space of traders such that all the assumptions listed in the statement of Theorem 3 hold. In particular, by Theorem 2, $C(\mathcal{E}) \neq \emptyset$. (In fact, using (4.12) it may be seen that $C(\mathcal{E})$ contains the initial allocation $t \mapsto \overline{e}$—cf. the last paragraph in the proof of Theorem 1.) It remains to show that $\mathcal{W}(\mathcal{E}) = \emptyset$. Proceeding by contradiction, suppose there is a Walrasian equilibrium, say $(p, f)$. But then, in view of (4.12), the pair $(p, t \mapsto \overline{e})$ must be a Walrasian equilibrium as well—see the argument in the paragraph containing (4.4) for details. Recall now the set $C$ from above, and in particular, recall that (a) $C$ is absolutely convex; (b) span $C$ is dense in $E$; and (c) for some real number $\beta > 0$, $\{\overline{e}\} + \beta C$ belongs to the consumption set $K$. From (a), if $z \in \operatorname{span} C$ then for some $\gamma > 0$, $\gamma z \in C$ and hence for some $\gamma' > 0$, $\gamma' z \in \beta C$.[16] Therefore, because of (c), the equilibrium conditions with respect

[16]Indeed, if $z \in \operatorname{span} C$ then $z$ can be written in the form

$$z = \left(\sum_{i=1}^{n} |\alpha_i|\right) \cdot \sum_{i=1}^{n} \frac{\alpha_i}{\sum_{i=1}^{n} |\alpha_i|} v_i$$

for some non-zero numbers $\alpha_1, \ldots, \alpha_n$ and some $v_1, \ldots, v_n \in V$, and since $V$ is absolutely convex, $\sum_{i=1}^{n} (\alpha_i / \sum_{i=1}^{n} |\alpha_i|) v_i$ must belong to $V$.

to $(p, t \mapsto \bar{e})$ imply that for almost all $t \in T$,

$$\ker p \cap \operatorname{span} C \subset \ker(\hat{q} + q_t).$$

By virtue of (b), and since $p$ and $\hat{q} + q_t$ are continuous, it follows that in fact, for almost all $t \in T$, $\ker p \subset \ker(\hat{q} + q_t)$.[17] By the arguments in the proof of Theorem 1, this yields a contradiction. Proof complete.

**Remark 11.** Note that the argument in the above proof leading to the conclusion $\mathcal{W}(\mathcal{E}) = \emptyset$ is independent of whether or not the price system $p$ is positive. Thus, as announced in Remark 8 in Section 3.3, Theorem 3 is valid without the requirement that an equilibrium price system be positive.

Concerning the requirement that price systems be continuous, suppose the economy constructed in the proof of Theorem 3 would have a Walrasian equilibrium with a discontinuous price system, again say $p$. As above, it would follow that the pair $(p, t \mapsto \bar{e})$ is a Walrasian equilibrium, too, and from this that for almost every $t \in T$, $\ker p \cap \operatorname{span} C \subset \ker(\hat{q} + q_t)$. (For both these conclusions, continuity of the price system is not needed.) Now by construction, for all $t \in T$, $\hat{q} + q_t$ is continuous so $\ker(\hat{q} + q_t)$ is closed. Also by construction, $\hat{q} + q_t \neq 0$; in particular, $\hat{q} + q_t$ is non-zero on $\operatorname{span} C$, this latter set being dense in $E$. Letting $p'$ be the restriction of $p$ to $\operatorname{span} C$, our preceding observations show that $\ker p'$ cannot be dense in $\operatorname{span} C$. In other words, $p'$ is continuous; let $p''$ be the (uniquely determined) extension of $p'$ to an element of $E^*$. Evidently, then, $\ker p'' \cap \operatorname{span} C \subset \ker(\hat{q} + q_t)$ for almost all $t \in T$, and we thus end up as in the original proof of Theorem 3. Consequently, as claimed in Remark 8 in Section 3.3, Theorem 3 continues to hold when equilibrium price systems are allowed to be discontinuous.

Finally, for the economy $\mathcal{E}$ constructed in the proof of Theorem 3, a non-trivial quasi-equilibrium as defined in Remark 9 must in fact be a Walrasian equilibrium because endowments are the same for all agents. Thus, since $\mathcal{W}(\mathcal{E}) = \emptyset$, the set of non-trivial quasi-equilibria is empty, too. (From the preceding remarks, this is so even when price systems may be non-positive or discontinuous).[18] Thus the claim in Remark 9 is true.

[17] To see this, recall the standard fact that if $Z$ is a real linear topological space, $\phi$ a continuous linear functional on $Z$, and $A$ a subset of $Z$ which is convex and dense in $Z$, then $A \cap \ker \phi$ is dense in $\ker \phi$.

[18] Of course, in the economy constructed in the proof of Theorem 3, if the linear span of the consumption set $K$ is not equal to the commodity space $E$ then there are *trivial* quasi-equilibria. Indeed, in this case, to obtain such a quasi-equilibrium take any feasible allocation and as price system a linear functional on $E$ which is non-zero but zero on the linear span of $K$. However, if the linear span of $K$ is dense in $E$ then such a price system cannot be continuous.

### 4.5 Proof of Theorem 4

Since we have required a Walrasian equilibrium price system to be positive, it is clear that $\mathcal{W}(\mathcal{E}) \subset C(\mathcal{E})$. To prove the reverse inclusion, first note that in view of assumptions (C7) and (C8), we may as well assume $E$ to be separable. (Indeed, let $Y$ denote the closed linear span of the union $\bigcup_{t\in T} X(t)$; by (C7) and (C8), $Y$ is separable. Endowing $Y$ with the ordering induced from $E$, we can view $\mathcal{E}$ as an economy with commodity space $Y$. All assumptions of the theorem continue to hold—cf. the arguments in the fourth paragraph of the proof of Theorem 2—and in particular, $Y \cap \operatorname{int} E_+ \neq \emptyset$ whence, with regard to price systems, every element of $Y_+^*$ has an extension to an element of $E_+^*$ by the Krein-Rutman theorem.) Now suppose $f \in C(\mathcal{E})$. In particular, $f(t) \in X(t)$ for almost all $t \in T$ and $\int_T f(t)\, d\nu(t) \leq \int_T e(t)\, d\nu(t)$; by modifying $f$ if necessary on a null set we may assume that in fact $f(t) \in X(t)$ for *each* $t \in T$.

Define a correspondence $\varphi\colon T \to 2^E$ by

$$\varphi(t) = \{x \in X(t)\colon x \succ_t f(t)\} \cup \{e(t)\},\ \ t \in T.$$

Then the integral $\int_T \varphi(t)\, d\nu(t)$ is non-empty. (E.g. $\int_T e(t)\, d\nu(t)$ belongs to this set.) Moreover, since the measure space $(T, \mathcal{T}, \nu)$ is atomless (by hypothesis), $\mathrm{c}\ell \int_T \varphi(t)\, d\nu(t)$ is a convex set in $E$. (See e.g. Yannelis, 1991, Theorem 6.2, p. 22). Finally, we must have

$$\left(\mathrm{c}\ell \int_T \varphi(t)\, d\nu(t) - \left\{\int_T e(t)\, d\nu(t)\right\}\right) \cap -\operatorname{int} E_+ = \emptyset.$$

Indeed, suppose the contrary. Then, since $\operatorname{int} E_+$ is an open set, there are a $v \in \operatorname{int} E_+$—in particular $v \neq 0$ because $E_+ \cap -E_+ = \{0\}$)—and an integrable function $g\colon T \to E$ such that $g(t) \in \varphi(t)$ for almost all $t \in T$ and

$$\int_T g(t)\, d\nu(t) = \int_T e(t)\, d\nu(t) - v.$$

Let $S = \{t \in T\colon g(t) \succ_t f(t)\}$. By Assumption (A5), $S \in \mathcal{T}$ and by definition of $\varphi$ we have $g(t) = e(t)$ for almost all $t \in T \setminus S$. Thus we may write

$$\begin{aligned}
\int_S g(t)\, d\nu(t) &= \int_T g(t)\, d\nu(t) - \int_{T\setminus S} e(t)\, d\nu(t) \\
&= \int_T e(t)\, d\nu(t) - v - \int_{T\setminus S} e(t)\, d\nu(t) \\
&= \int_S e(t)\, d\nu(t) - v.
\end{aligned}$$

Since $v \neq 0$, it follows that $\nu(S) > 0$ and hence, since $v \geq 0$, that $f \notin C(\mathcal{E})$.[19] But this is a contradiction, establishing the claim.

[19]Recall that according to our core definition, all coalitions have the possibility of free disposal.

Since the set $-\operatorname{int} E_+$ is open and convex, it follows from the separation theorem that there is a $p \in E_+^* \setminus \{0\}$ such that

$$\inf \left\langle p, \mathrm{c}\ell \int_T \varphi(t)\, d\nu(t) - \left\{ \int_T e(t)\, d\nu(t) \right\} \right\rangle \geq 0$$

i.e. such that

$$\inf \left\langle p, \mathrm{c}\ell \int_T \varphi(t)\, d\nu(t) \right\rangle \geq \int_T \langle p, e(t) \rangle\, d\nu(t)\,. \tag{4.13}$$

Observe now that the function $f$ being integrable is $\mathcal{T} - \mathcal{B}(E)$ measurable since the measure space $(T, \mathcal{T}, \nu)$ is complete. Thus, since $f(t) \in X(t)$ for all $t \in T$, and since we have assumed $E$ to be separable, according to part (ii) of the lemma in the appendix we have

$$\{(t, x) \in T \times E \colon x \succ_t f(t)\} \in \mathcal{T} \otimes \mathcal{B}(E).$$

For the same reasons as for $f$, the endowment mapping $t \mapsto e(t)$ is $\mathcal{T} - \mathcal{B}(E)$ measurable, and it follows that

$$\{(t, x) \in T \times E \colon x \in \varphi(t)\} \in \mathcal{T} \otimes \mathcal{B}(E). \tag{4.14}$$

Recall from above that $\int_T \varphi(t)\, d\nu(t)$ is non-empty. Hence, by a well known fact, since $E$ is separable, (4.14) implies that[20]

$$\inf \left\langle p, \int_T \varphi(t)\, d\nu(t) \right\rangle = \int_T \inf \langle p, \varphi(t) \rangle\, d\nu(t)\,.$$

Consequently, from (4.13),

$$\int_T \inf \langle p, \varphi(t) \rangle\, d\nu(t) \geq \int_T \langle p, e(t) \rangle\, d\nu(t)\,.$$

But by the definition of $\varphi$, $\inf\langle p, \varphi(t)\rangle \leq \langle p, e(t) \rangle$ for all $t \in T$, and combining this with the previous equation we find that $\inf\langle p, \varphi(t)\rangle = \langle p, e(t)\rangle$ for almost all $t \in T$. In other words, again by the definition of $\varphi$:

(4.15) For almost every $t \in T$,
if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x \rangle \geq \langle p, e(t) \rangle$.

Now note that, for any $t \in T$, if $f(t)$ is not a satiation point for $\succcurlyeq_t$, then by Assumption (C6) ("local non-satiation at non-satiation points") the statement "$\langle p, x \rangle \geq \langle p, e(t) \rangle$ whenever $x \succ_t f(t)$" implies that $\langle p, f(t) \rangle \geq \langle p, e(t) \rangle$. If, on the other hand, $f(t)$ happens to be a satiation point, then according to (C5),

[20] See e.g. Hildenbrand (1974, Proposition 6, p. 63). Actually, this latter result is stated in terms of $\mathbb{R}^n$-valued correspondences. However, as can be seen from its proof, it generalizes directly to the context of a separable Banach space.

$f(t) \geq e(t)$ and thus, since $p \geq 0$, $\langle p, f(t)\rangle \geq \langle p, e(t)\rangle$ in this case as well. Hence, from (4.15),

$$\langle p, f(t)\rangle \geq \langle p, e(t)\rangle \text{ for almost all } t \in T$$

whence, since $\int_T f(t)\,d\nu(t) \leq \int_T e(t)\,d\nu(t)$ and $p \geq 0$,

$$\langle p, f(t)\rangle = \langle p, e(t)\rangle \text{ for almost all } t \in T. \tag{4.16}$$

Finally, since $p$ is positive and non-zero, assumptions (C4) and (A3) combine to imply that (4.15) may be rephrased to state:

(4.17) For almost every $t \in T$,
if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x\rangle > \langle p, e(t)\rangle$.

In view of (4.17) and (4.16), we may conclude that the allocation $f$ belongs to $\mathcal{W}(\mathcal{E})$. This completes the proof of Theorem 4.

## Appendix

**Lemma.** *Let $Y$ be an ordered Banach space and let $\mathcal{E}$ be an economy with commodity space $Y$ which satisfies assumptions* (A1) *to* (A3), (C2) *and* (C3). *Suppose $Y$ is* separable. *Then given any $\mathcal{T}-\mathcal{B}(Y)$ measurable mapping $f$ from $T$ into $Y$ with $f(t) \in X(t)$ for all $t \in T$, the measurability assumption* (A5) *implies*:

(i) *The set $\{(t,x) \in T \times Y : x \succcurlyeq_t f(t)\}$ belongs to $\mathcal{T} \otimes \mathcal{B}(Y)$.*

(ii) *The set $\{(t,x) \in T \times Y : x \succ_t f(t)\}$ belongs to $\mathcal{T} \otimes \mathcal{B}(Y)$.*

*Proof.* First note that for each $t \in T$, the consumption set of $t$, $X(t)$, being convex according to assumption (A1), is connected. Therefore, from assumptions (A2) and (A3), for each $t \in T$:

(4.18) If $D$ is any dense subset of $X(t)$ then given $x, y \in X(t)$ with $x \succ_t y$, there is a $d \in D$ such that $x \succ_t d \succ_t y$.

(See e.g. Debreu, 1959, p. 57.)

Next consider the consumption sets correspondence $t \mapsto X(t)$. Denote its graph by $G_{X(\cdot)}$. Thus

$$G_{X(\cdot)} = \{(t,x) \in T \times Y : x \in X(t)\}.$$

According to Assumption (C3), $G_{X(\cdot)} \in \mathcal{T} \otimes \mathcal{B}(Y)$ and by definition of an economy each $X(t)$ is non-empty. Therefore, since the Banach space $Y$ is separable, and since $(T, \mathcal{T}, \nu)$ is complete, the correspondence $t \mapsto X(t)$ has a Castaing representation. That is, there is a countable family $(h_i)_{i=1}^{\infty}$ of $\mathcal{T}-\mathcal{B}(Y)$ measurable mappings $h_i$ from $T$ into $Y$ such that for each $t \in T$

$$\text{c}\ell\{h_i(t)\colon i = 1, 2, \ldots\} = X(t). \tag{4.19}$$

(See e.g. Castaing and Valadier, 1977, Theorem III.22, p. 74, and recall that each $X(t)$ is closed by assumption.)

Now suppose a $\mathcal{T}-\mathcal{B}(Y)$ measurable mapping $f\colon T \to Y$ has been given so that for all $t \in T$, $f(t) \in X(t)$. Consider any $t \in T$ and let $x \in X(t)$. If $x \succ_t f(t)$ then (a), from (4.19) and (4.18), there is an $i \in \mathbb{N}$ such that $x \succ_t h_i(t) \succ_t f(t)$; and (b), by continuity of $\succcurlyeq_t$ and again by (4.19), for each $n \in \mathbb{N} \setminus \{0\}$ there is a $j \in \mathbb{N}$ such that $\|x - h_j(t)\| \le 1/n$ and $h_j(t) \succ_t h_i(t)$. Conversely, if for some $i \in \mathbb{N}$: (a) $h_i(t) \succ_t f(t)$ and (b) for each $n \in \mathbb{N} \setminus \{0\}$ there is a $j \in \mathbb{N}$ such that $\|x - h_j(t)\| \le 1/n$ and $h_j(t) \succ_t h_i(t)$, then $x \succ_t f(t)$ because (b) implies $x \succcurlyeq_t h_i(t)$ and $\succcurlyeq_t$ is transitive. Thus

$$\begin{aligned}
&\{(t, x) \in T \times Y\colon x \succ_t f(t)\} \\
&= G_{X(\cdot)} \cap \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \Big( \Big[ \big( \{t \in T\colon h_i(t) \succ_t f(t)\} \cap \{t \in T\colon h_j(t) \succ_t h_i(t)\} \big) \times Y \Big] \\
&\qquad \cap \{(t, x) \in T \times Y\colon \|x - h_j(t)\| \le 1/n\} \Big).
\end{aligned}$$

As already noted, $G_{X(\cdot)} \in \mathcal{T} \otimes \mathcal{B}(Y)$. Further, according to Assumption (C2), the correspondence $t \mapsto X(t)$ is integrably bounded, and hence the functions $f$ and $h_i, i = 1, 2, \ldots$, being $\mathcal{T}-\mathcal{B}(Y)$ measurable selections of this correspondence, are integrable since $Y$ is separable. Consequently, (A5) implies that the sets $\{t \in T\colon h_i(t) \succ_t f(t)\}$ and $\{t \in T\colon h_j(t) \succ_t h_i(t)\}$, $i, j = 1, 2, \ldots$ belong to $\mathcal{T}$. Finally, by Castaing and Valadier (1977, Theorem III.41, p. 88), the sets $\{(t, x) \in T \times Y\colon \|x - h_j(t)\| \le 1/n\}$, $j, n = 1, 2, \ldots$ are in $\mathcal{T} \otimes \mathcal{B}(Y)$, again because the $h_j$'s are $\mathcal{T}-\mathcal{B}(Y)$ measurable. Thus (ii) of the lemma follows. Analogously, the set $\{(t, x) \in T \times Y\colon f(t) \succ_t x\}$ is in $\mathcal{T} \otimes \mathcal{B}(Y)$, and taking the complement of this set relative to $G_{X(\cdot)}$, we see that (i) holds as well, using the facts that $G_{X(\cdot)} \in \mathcal{T} \otimes \mathcal{B}(Y)$ and $\succcurlyeq_t$ is complete for each $t \in T$. □

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