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Evaluating Approximate Equilibria of Dynamic Economic Models

Paul Pichler*

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Abstract

This paper evaluates the performances of Perturbation Methods, the Parameterized Expectations Algorithm and Projection Methods in finding approximate decision rules of the basic neoclassical stochastic growth model. In contrast to the existing literature, we focus on comparing numerical methods for a given functional form of the approximate decision rules, and we repeat the evaluation for many different parameter sets. We find that significant gains in accuracy can be achieved by moving from linear to higher-order approximations. Our results show further that among linear and quadratic approximations, Perturbation Methods yield particularly good results, whereas Projection Methods are well suited to derive higher-order approximations. Finally we show that although the structural parameters of the model economy have a large effect on the accuracy of numerical approximations, the ranking of competing methods is largely independent from the calibration.

Keywords: numerical accuracy, Perturbation, Projection Methods, Parameterized Expectations Algorithm

JEL classifictaion: C63; C68

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1 Introduction

In many applications, equilibrium decision rules of dynamic models cannot be derived analytically. The reason underlying this observation is that the solution procedure usually involves solving complex constrained intertemporal optimization problems and thus translates into finding the zeros of a high-dimensional system of non-linear functional equations. These equations are constituted, for example, by first-order conditions and market clearing conditions. Often the equilibrium conditions involve stochastic elements and expectations over non-linear functions of future variables, such that an analytical solution is generally infeasible.

Instead of solving for exact equilibria economists approximate decision rules numerically. Various different methods have been developed in recent years to conduct this task.[1] A first formal evaluation of competing numerical methods in macroeconomics can be found in Taylor and Uhlig (1990). They compare the performances of fourteen different methods in solving the neoclassical stochastic growth model without leisure choice. Taylor and Uhlig (1990) show that different numerical methods have significantly different implications for the model's equilibrium properties. They conclude that - as no clear winner can be found among the competing methods - researchers shall be aware not to use any specific method blindly.[2]

A more recent evaluation of numerical methods is provided by Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2003). They compare linear and log-linear approximations with higher-order Perturbation methods, Projection Methods and Value Function Iteration. They conclude that high-order approximation methods yield superior results than linear approximations. Taking accuracy, speed and programming burden into account, they suggest to move to at least a second-order approximation when solving dynamic equilibrium models. Novales and Pérez (2004) arrive at the opposite conclusion. They compare the performances of log-linear and linear approximations with a second-order Parameterized Expectations approach. Using three different well-known economic models they find that the results derived from log-linear approximations of the models are virtually indistinguishable from those obtained using the Parameterized Expectations Algorithm. Log-Linearization methods, they conclude, are well suited to approximate dynamic economic models in many applications. Finally, a paper by Heer and Maussner (2004) takes a position in between. They suggest using log-linearization in the first place and, if necessary, applying non-linear methods such as the Parameterized Expectations Approach and Projection Methods to improve the accuracy in highly non-linear problems.

Our impression from the existing literature is that it gives too little advice to applied macroeconomists which methods to use for approximating dynamic general equilibrium models. In particular, we think that three important questions about the performance of numerical methods have not been satisfactorily answered yet:

[1] For surveys of numerical methods in economics see Judd (1998), Marimon and Scott (1999) and Heer and Maussner (2005).

[2] A severe limitation of the evaluation exercise conducted in Taylor and Uhlig (1990) has been pointed out by Novales and Pérez (2004). Because only one solution realization was considered for each method and the probability distribution of the technology shock differs among the implemented approaches, the study by Taylor and Uhlig (1990) lacks homogeneity and robustness.

1. Does increasing the order of approximation significantly improve the accuracy of the solution?

2. For a given order of approximation of the policy function, which method selects the coefficients of the decision rules in the best way?

3. Do the performances of competing numerical methods depend strongly on the structural parameters of the model to be solved?

As outlined above, the existing literature gives contradictory answers to the first of these questions, thus we believe the topic deserves further investigation. The second question, to the best of our knowledge, has not yet been explicitly emphasized. We believe it is of special interest in applications where researchers have strong a priori beliefs about the functional form of the decision rules.[3] The problem is then to select the method which best estimates the coefficients of these decision rules. The third question has already been partly covered by all four contributions cited in the beginning of this section. However, all these papers analyze parameter sensitivity using only very few different calibration schemes. In this paper, we use 500 different parameter sets. This allows us to investigate the dependence of numerical accuracy on structural parameters in greater detail.
The remainder of the paper is organized as follows. Section two overviews the basic neoclassical growth model and defines its competitive equilibrium. Section three briefly discusses three numerical methods to solve for an approximate equilibrium of the model economy. Section four presents two formal methods to assess the quality of numerical approximations. Section five illustrates our evaluation exercise and presents its results. Section six summarizes and concludes.

2 The Model

We evaluate competing numerical methods using the basic neoclassical growth model as a testing ground. We basically choose this model because it guarantees comparability with other studies. As the model is very well-known we restrict ourselves to presenting the main formulas and defining the competitive equilibrium.
The model economy is populated by a single representative household. Time is discrete, i.e. $t \in \{0, 1, 2, \ldots\}$. The household chooses sequences of consumption $\{c_t\}_{t=0}^{\infty}$, investment $\{i_t\}_{t=0}^{\infty}$, output $\{y_t\}_{t=0}^{\infty}$ and capital $\{k_{t+1}\}_{t=0}^{\infty}$ such that it maximizes the expected discounted utility derived from its current and future consumption stream. The maximization is subject to constraints posed by a production function, a budget constraint, a law of motion for the exogenous productivity level, and by non-negativity requirements for all variables. We assume a CES utility function and a Cobb-Douglas production function. Formally, the household solves

$$\max_{c_t, i_t, y_t, k_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t (1-\gamma)^{-1} c_t^{1-\gamma} \tag{1}$$

subject to

$$y_t \quad = \quad a_t k_t^{\alpha}$$

[3] These beliefs may for example come from related models which are analytically solvable.

$$
\begin{aligned}
i_t &= y_t - c_t \\
k_{t+1} &= (1-\delta)k_t + i_t \\
\log a_{t+1} &= \rho \log a_t + \sigma_\varepsilon \varepsilon_{t+1} \\
\varepsilon_{t+1} &\sim N(0,1) \\
c_t \geqslant 0, i_t &\geqslant 0 \quad , k_t \geqslant 0 \qquad \forall t
\end{aligned}
$$

The parameters and variables have the following interpretation: a_t denotes an exogenous level of technological progress; $\beta \in (0,1)$ is the household's time discount factor; $\alpha \in (0,1)$ is a production function parameter which describes capital's share in output; $\gamma > 0$ is a utility function parameter describing the household's attitude towards risk; $\delta \in (0,1)$ is the depreciation rate of capital; ϵ_{t+1} denotes a shock to technology at time $t+1$, which is normally distributed with mean zero; the parameter σ_ε captures the standard deviation of technology shocks, ρ governs their persistence.
A necessary condition for optimality is given by the Euler equation

$$
c_t^{-\gamma} = \beta E_t c_{t+1}^{-\gamma} [\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)] \tag{2}
$$

E_t denotes the expectation operator conditional upon the information available to the decision maker at time t. When augmented with a transversality condition

$$
\lim_{t \to \infty} \beta^t c_t^{-\gamma} k_{t+1} = 0 \tag{3}
$$

the Euler equation (2) becomes sufficient for optimality.[4]
The system of equilibrium conditions is summarized by

$$
E_t R(c_{t+1}, i_{t+1}, y_{t+1}, c_t, i_t, y_t, k_{t+1}, k_t, a_{t+1}, a_t) = 0
$$

where

$$
R(\cdot) = \begin{pmatrix}
\beta {c_{t+1}}^{-\gamma} [\alpha a_{t+1} {k_{t+1}}^{\alpha-1} + (1-\delta)] - c_t^{-\gamma} \\
y_t - a_t k_t^\alpha \\
y_t - c_t - i_t \\
(1-\delta)k_t + i_t - k_{t+1} \\
\rho \log a_t - \log a_{t+1}
\end{pmatrix}
$$

Using $f_t = (c_t \quad i_t \quad y_t)'$, $s_t = k_t$ and $v_t = a_t$ the model's equilibrium conditions read

$$
E_t R(f_{t+1}, f_t, s_{t+1}, s_t, v_{t+1}, v_t) = 0 \tag{4}
$$

The vector f_t contains the model's control variables, s_t contains the endogenous state variables and v_t contains the exogenous state variables. A recursive competitive equilibrium of the model economy may be defined as follows:

Definition A recursive competitive equilibrium is a pair of policy functions $f_t = g(s_t, v_t)$ and $s_{t+1} = h(s_t, v_t)$ such that - for every initial state s_0 and exogenous process $\{v_t\}_{t=0}^\infty$ - the system of functional equations (4) and a transversality condition are satisfied.

[4]See, for example, Stokey and Lucas (1989) for a proof of the sufficiency

3 Numerical Methods

As the competitive equilibrium cannot be solved for analytically, we use numerical methods to derive an approximation. We consider three classes of methods which we think are particularly popular among researchers: Perturbation Methods, Parameterized Expectations Algorithms (PEA) and Projection Methods. The remainder of this section briefly illustrates the steps involved in the implementation of each of the methods.[5]

3.1 Perturbation

Perturbation Methods are widely used to approximate dynamic general equilibrium models. Particularly first-order Perturbation, which is often referred to as (Log-)Linearization, is a very popular method to solve for approximate decision rules of dynamic models. Following the seminal paper by Blanchard and Kahn (1980) different variants of first-order Perturbation have been developed.[6] Recent contributions include Uhlig (1999), Sims (2002) and Klein (2000). Extensions to higher orders have been introduced by Schmitt-Grohe and Uribe (2004), Swanson, Anderson, and Levin (2003) and Kim, Kim, Schaumburg, and Sims (2005). We follow Klein (2000) to derive first-order accurate solutions and apply the methods proposed by Schmitt-Grohe and Uribe (2004) to derive second-order approximations.
Our implementation of Perturbation Methods thus requires the following steps.

1. Rewrite the equilibrium equations as

$$E\tilde{R}(f^*_{t+1}, f^*_t, x^*_{t+1}, x^*_t) = 0 \tag{5}$$

   where $f^*_t = \log f_t$, $x^*_t = [\log s'_t \ \log v'_t]'$ and $\tilde{R}$ denotes the system of rewritten equilibrium conditions. Compute the non-stochastic steady state $(\bar{f^*}, \bar{x^*})$ by solving

$$\tilde{R}(\bar{f^*}, \bar{f^*}, \bar{x^*}, \bar{x^*}) = 0$$

   for $\bar{f^*}$ and $\bar{x^*}$.

2. Recall that the model's competitive equilibrium is given by a set of policy functions $f^*_t = g(x^*_t, \sigma_\varepsilon)$ and $x^*_{t+1} = h(x^*_t, \sigma_\varepsilon) + \eta\sigma_\varepsilon\varepsilon_{t+1}$ where $\eta = [0\,1]'$. Use them both to rewrite (5) as

$$E\tilde{R}(g(h(x^*_t, \sigma_\varepsilon) + \eta\sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon), g(x^*_t, \sigma_\varepsilon), h(x^*_t, \sigma_\varepsilon) + \eta\sigma_\varepsilon\varepsilon_{t+1}, x^*_t) = 0$$

   Summarize this system by $F(x^*_t, \sigma_\varepsilon) = 0$.

3. Approximate the system of equilibrium conditions $F(x^*_t, \sigma_\varepsilon)$ with a first-order Taylor series around the non-stochastic steady state $(\bar{x^*}, 0)$. Denote the system of approximate conditions by $\tilde{F}^1(x^*_t, \sigma_\varepsilon)$. Use the algorithm

[5] Detailed outlines of these methods can be found in Judd (1998), Marimon and Scott (1999) and Heer and Maussner (2005).

[6] A detailed survey and comparison of these methods can be found in Anderson (2000).

by Klein (2000) to derive the stable solution of $\tilde{F^1}(x_t^*, \sigma_\varepsilon) = 0$. This translates into solving a system of equations

$$\begin{aligned} F(\bar{x_t^*}, 0) &= 0 \\ F_x(\bar{x_t^*}, 0) &= 0 \\ F_{\sigma_\varepsilon}(\bar{x_t^*}, 0) &= 0 \end{aligned} \tag{6}$$

for first-order accurate decision rules $f_t = \tilde{g}^1(s_t, v_t)$ and $s_{t+1}^1 = \tilde{h}(s_t, v_t)$, which are consistent with the transversality condition.

4. To derive second-order accurate decision rules, approximate the system of equilibrium conditions with a second-order Taylor series, $\tilde{F^2}(x_t^*, \sigma_\varepsilon)$. Setting $\tilde{F^2}(x_t^*, \sigma_\varepsilon) = 0$ requires that (6) and

$$\begin{aligned} F_{xx}(\bar{x_t^*}, 0) &= 0 \\ F_{x\sigma_\varepsilon}(\bar{x_t^*}, 0) &= 0 \\ F_{\sigma_\varepsilon x}(\bar{x_t^*}, 0) &= 0 \\ F_{\sigma_\varepsilon\sigma_\varepsilon}(\bar{x_t^*}, 0) &= 0 \end{aligned} \tag{7}$$

hold. Given $f_t = \tilde{g}^1(s_t, v_t)$ and $s_{t+1}^1 = \tilde{h}(s_t, v_t)$, the system of equations (7) is linear. Solve this system for second-order accurate decision rules $f_t = \tilde{g}^2(s_t, v_t)$ and $s_{t+1}^2 = \tilde{h}(s_t, v_t)$.

In principle, we can repeat the last step to derive approximations of any desired order: given the approximation of order n, the approximation of order $n+1$ can be derived by solving a linear system of equations. However, as the matrix algebra involved becomes very complex for orders of three and more, we confine ourselves to Perturbation Methods of order one and two.

3.2 The Parameterized Expectations Algorithm

The Parameterized Expectations Algorithm (PEA) has been introduced into economics by Marcet (1988) and den Haan and Marcet (1990). Modifications can be found in Christiano and Fisher (2000) and Maliar and Maliar (2003), among others. The underlying idea is to approximate the conditional expectation arising in the stochastic Euler equation (2) by a parametric function in the model's state variables, $\psi(s_t, v_t; \kappa)$. The functional form ψ and the parameter vector κ then imply decision rules which solve the system of model equations. Using these decision rules and a sequence of shocks for the exogenous process allows for the generation of artificial time series of the model's variables. Furthermore we can build a series of 'forecast errors' that the household commits by using $\psi(s_t, v_t; \kappa)$ to form expectations. An updated κ can be found by minimizing the sum of squared forecast errors. This procedure is repeated until κ converges. The final κ then constitutes the best parameter vector a household can use to form expectations.

To facilitate comparison with the previous methods we again rewrite the model in natural logarithms. Our implementation of the PEA then requires the following steps.

1. Rewrite the model as

$$\tilde{R}(E_t\phi(f_{t+1}^*, f_t^*, s_{t+1}^*, s_t^*, v_{t+1}^*, v_t^*), f_t^*, s_{t+1}^*, s_t^*, v_{t+1}^*, v_t^*) = 0 \tag{8}$$

where the asterisk again indicates the natural logarithm of a variable and where the function ϕ explicitly captures the part of the system where expectations have to be taken. $\tilde{R}$ is essentially the same function as R but accounts for the new arguments.

2. Choose a function $\psi(s_t^*, v_t^*; \kappa)$ to approximate $E_t\phi(f_{t+1}^*, f_t^*, s_{t+1}^*, s_t^*, v_{t+1}^*, v_t^*)$. Make an initial guess for the vector κ, denote it by κ_0.

3. Replace $E_t\phi(f_{t+1}^*, f_t^*, s_{t+1}^*, s_t^*, v_{t+1}^*, v_t^*)$ in (8) by $\psi(s_t^*, v_t^*; \kappa_0)$ and solve for the implied decision rules $\tilde{g}(s_t^*, v_t^*; \psi, \kappa_0)$ and $\tilde{h}(s_t^*, v_t^*; \psi, \kappa_0)$.[7]

4. Pick an initial state s_0 and generate a sequence of length T for the exogenous state, $\{v_t^*\}_{t=0}^T$. Generate artificial series $\{f_t^*\}_{t=0}^T$ and $\{s_{t+1}^*\}_{t=0}^T$ using $\tilde{g}(s_t^*, v_t^*; \psi, \kappa_0)$ and $\tilde{h}(s_t^*, v_t^*; \psi, \kappa_0)$.

5. Compute a parameter vector κ_1 as

$$\kappa_1 = \arg\min_{\kappa} \sum_{t=0}^{T-1} \left(\psi(s_t^*, v_t^*; \kappa) - \phi(f_{t+1}^*, f_t^*, s_{t+1}^*, s_t^*, v_{t+1}^*, v_t^*)\right)^2$$

6. Select an updating rule μ and derive a new κ_0 as $\kappa_0^{new} = \mu(\kappa_0, \kappa_1)$. Replace κ_0 by κ_0^{new}. Go back to Step 3 until κ has converged.

Our implementation of the PEA uses artificial series of length $T = 5.000$ and a complete updating scheme, i.e. $\mu(\kappa_0, \kappa_1) = \kappa_1$. The maximum number of iterations is set to 500. To find the initial guess we use an estimation-based log-linear homotopy approach as described in Pérez (2004).
A potential problem with the PEA is that convergence may fail if the initial guess κ_0 is too far away from the true solution. Another problem is that the nonlinear regression involved in the PEA may suffer from multicollinearity. This is because the regressors become highly correlated when the order of approximation increases.[8] We can largely circumvent the first problem by making an educated guess using the results derived from a log-linear approximation of the model. However, the second problem turns out to be severe. We find that the accuracy of the PEA does not significantly improve when we move from first to second-order approximation. We believe this result comes from the multicollinearity introduced by adding second-order polynomial terms.

3.3 Galerkin Projection

Projection Methods have been introduced into economics by Judd (1992). Another important contribution is Christiano and Fisher (2000) who combine the Parameterized Expectations Algorithm and Projection Methods to solve models with occasionally binding constraints.
Projection Methods differ from the PEA in three important aspects. First, instead of approximating the expectations function ϕ, Projection Methods approximate the decision rules g and h directly. Secondly, Projection Methods use families of orthogonal polynomials for approximation, such as Chebyshev

[7] Implicitly we assume that for a given ϕ the system of functional equations is invertible with respect to the decision variables. This condition is usually met in economic models.

[8] It is easy to check that $Corr(x^i, x^{i+1}) \to 1$ for $i \to \infty$

Polynomials or Legendre polynomials. Finally, Projection Methods differ from the PEA in the choice of minimization criteria. Whereas the PEA minimizes the sum of squared forecast errors, Projection Methods generally solve a problem

$$\int E_t \tilde{R}(s_t, \varepsilon_{t+1}; \tilde{g}; \kappa)\omega_i(x_t)dx_t = 0 \quad i = 1, \ldots, n \tag{9}$$

where $x_t = (s_t' \quad v_t')'$ denotes the model's state variables and $\omega_i(x_t)$ is a weighting function that determines the minimization criterion used. Depending on the specific choice for $\omega_i(x_t)$ we distinguish between Least Squares Projection, Collocation and Galerkin Projection.[9] In this paper we use a Galerkin approach with Chebyshev Polynomials as basis functions. The latter are defined recursively by

$$T_0(x) = 1,\ T_1(x) = x,\ T_i(x) = 2xT_{i-1}(x) - T_{i-2}(x) \text{ for } i = 2, 3, \ldots$$

Chebyshev Polynomials constitute a family of orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $\omega(x) = 1/\sqrt{1-x^2}$. Formally, for all $i \neq j$ it holds that $\int_{-1}^{1} T_i(x)T_j(x)\omega(x)dx = 0$. This orthogonality property allows us to increase the order of approximation without introducing multicollinearity.

The steps involved in our implementation of Galerkin Projection are as follows.

1. Rewrite the model as

$$E_t \tilde{R}\left(g(h(x_t^*, \varepsilon_{t+1})), g(x_t^*), h(x_t^*, \varepsilon_{t+1}), x_t^*\right) = 0 \tag{10}$$

   where $x_t^* = (\log s_t' \quad \log v_t')'$ summarizes the model's state variables transformed to natural logarithms. The function $f_t^* = g(x_t^*)$ denotes the true policy function that solves (10). Recall that once the true g has been found, the policy function $h(x_t^*, \varepsilon_{t+1})$ is uniquely determined by the system of equilibrium conditions.

2. Select an order of approximation, n, and use a complete basis of Chebyshev polynomials to form $\tilde{g}(x_t^*; \kappa)$. As the vector x_t^* contains only two variables, the approximate decision rule is given by

$$\tilde{g}(x_t^*; \kappa) = \sum_{\substack{i=0,\ldots,n \\ j=0,\ldots,n \\ i+j \leqslant n}} \kappa_{ij} T_i(\xi(x_t^{*1}))T_j(\xi(x_t^{*2}))$$

   where x_t^{*1} gives the first element of x_t^* and x_t^{*2} gives the second element, and where ξ maps the state space into the interval $[-1, 1]$.

3. Use $\tilde{g}(x_t; \kappa)$ to derive $\tilde{h}(x_t, \varepsilon_{t+1}; \tilde{g}, \kappa)$ from the equilibrium system. Replace g and h in (10) by their approximate counterparts to derive a residual function

$$E_t \tilde{R}(x_t^*, \varepsilon_{t+1}; \tilde{g}, \kappa) \tag{11}$$

   For the true policy functions the residual function is zero everywhere. However, this does not hold for approximate policy functions.

[9] Because of the specific problem Projection Methods solve, they are equivalently referred to as Weighted Residual Methods. Further information can be found in Judd (1998).

4. For a given functional form $\tilde{g}$ find the $(n+1)(n+2)/2 \times 1$ parameter vector κ that minimizes the error that results from using $\tilde{g}(x_t^*;\kappa)$ instead of the true $g(x_t^*)$ in (11). For a Galerkin Method this translates into solving

$$\int\int E_t \tilde{R}(x_t^*, \varepsilon_{t+1}; \tilde{g}; \kappa)\omega_i(x_t^{*1})\omega_j(x_t^{*2})dx_t^{*1}dx_t^{*2} = 0 \quad i = 1, \ldots, n \quad (12)$$

where the weights are given by

$$\omega_i(x_t^{*1}) = \frac{T_i(\xi(x_t^{*1}))}{\sqrt{1-\xi(x_t^{*1})^2}} \qquad \omega_j(x_t^{*2}) = \frac{T_j(\xi(x_t^{*2}))}{\sqrt{1-\xi(x_t^{*2})^2}}$$

5. Approximate the integral in (12) numerically using Gauss-Chebyshev quadrature. To this end, compute $m > n$ nodes for each state variable as the zeros of the Chebyshev polynomial of order m. Then approximate (12) by

$$\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} E_t \tilde{R}(x_{l_1 l_2}^*, \varepsilon_{t+1}; \tilde{g}; \kappa)T_i(\xi(x_{l_1}^{*1}))T_j(\xi(x_{l_2}^{*2})) = 0 \quad (13)$$
$$i = 0, \ldots, n \quad j = 0, \ldots, n \quad i + j \leqslant n$$

where $x_{l_1 l_2}^* = (x_{l_1}^{*1}\, x_{l_2}^{*2})'$.

6. Evaluate the conditional expectation in (13) using Gauss-Hermite quadrature and solve this system of $(n+1)(n+2)/2$ equations for the coefficient vector κ.

Our implementation of the Galerkin Projection Method uses $m = 7$ nodes for each of the two state variables. We use a linear mapping $\xi(x) = 2(x - \underline{x})/(\bar{x} - \underline{x}) - 1$ to transform the state variables into the interval $[-1, 1]$. The upper and lower regions of the state space, $\bar{x}$ and $\underline{x}$, are taken from the simulated series derived from first-order Perturbation. These series are also used to initialize the non-linear equations solver.[10]

The Galerkin Projection method seems to be particularly well suited to derive high-order approximations. We see two reasons for that. First, because Galerkin Projection does not suffer from multicollinearity due to the orthogonality properties of the Chebyshev regressors. Secondly, because the complexity of the problem increases only moderately with the order of approximation. For example, for our model economy a fourth-order approximation requires solving a system of fifteen nonlinear equations, whereas a sixth-order approximation translates into solving a system of twenty-eight equations. Standard computer software should give solutions to such problems within seconds.

4 Accuracy of Numerical Approximations

In choosing a certain numerical method researchers trade off computational time, programming burden and the accuracy of the approximate equilibrium. Due to the steady improvement of computer hardware and the public availability of computer code, computational time and programming burden are often

[10] Further technical details can be found in the MATLAB codes, which are available on the author's website http://homepage.univie.ac.at/paul.pichler

claimed to become negligible.[11] Thus, the accuracy of the solution becomes an even more important criterion.
Economists usually apply informal and formal methods to assess the quality of approximate equilibria. Particularly popular informal methods are the comparison of decision rules, sample moments and impulse response functions. Among the formal methods economists typically consider Euler Equation Errors introduced by Judd (1992) and further developed by Santos (2000), and a χ^2-Accuracy-Test developed by den Haan and Marcet (1994). The remainder of this section illustrates these two approaches.

4.1 Euler Equation Residuals

Judd (1992) proposes to evaluate numerical methods by the size of error the representative household would make if it used the approximate decision rule instead of the true one. This error is given by

$$u_t = R(E_t[\tilde{g}[\tilde{h}(s_t, v_t), v_{t+1}], \tilde{g}(s_t, v_t), \tilde{h}(s_t, v_t), s_t, v_{t+1}, v_t])$$

where $\tilde{g}$ and $\tilde{h}$ are approximate decision rules. For the optimal growth model this corresponds to

$$u_t = c_t - [E_t \beta c_{t+1}^{-\gamma}(\alpha a_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)]^{1-\gamma}$$

A scale free measure of the error is provided by dividing u_t by the level of consumption. The quality of a numerical method can then be assessed by simulating artificial time series of a pre-specified length T for all variables, and by judging the size of

$$EE^1 = \log_{10}\left(\frac{1}{T}\sum_{t=1}^{T}|\frac{u_t}{c_t}|\right) \tag{14}$$

$$EE^2 = \log_{10}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{u_t}{c_t}\right)^2\right) \tag{15}$$

$$EE^3 = \log_{10}\left(\max|\frac{u_t}{c_t}|\right) \tag{16}$$

Measure (14) calculates the average error an individual makes by using approximate instead of true decision rules, (15) gives a measure of the volatility of the error and (16) gives its maximum. Obviously, the smaller these measures are the better is the approximation.

4.2 The χ^2-Accuracy Test

den Haan and Marcet (1994) propose to check the accuracy of numerical approximations by testing whether the expectation errors satisfy the martingale difference property. This property is implied by the assumption that the household forms expectations rationally. Consider the general model rewritten in a way such that

$$\tilde{R}(E_t\phi(f_{t+1}, f_t, s_{t+1}, s_t, v_{t+1}, v_t), f_t, s_{t+1}, s_t, v_{t+1}, v_t) = 0 \tag{17}$$

[11] We believe this statement is only partly true. Because economic models grow in complexity very fast, computational time may still remain a topic in applied work. Furthermore, computer code may not be available in many applications.

Again, the function ϕ collects all equilibrium conditions where expectations have to be taken. Let q denote the dimension of ϕ. Given simulated time series of length T for all variables, we can construct a series of expectation errors $\{u_t\}_{t=1}^{T}$ defined by

$$u_{t+1} = E_t\phi(f_{t+1}, f_t, s_{t+1}, s_t, v_{t+1}, v_t) - \phi(f_{t+1}, f_t, s_{t+1}, s_t, v_{t+1}, v_t) \qquad t = 0, \ldots, T-1 \tag{18}$$

where u_{t+1} is a vector of size $q \times 1$. Because expectations are rational, the errors $\{u_t\}_{t=0}^{T-1}$ should be innovations. Hence they should have a zero mean, be serially uncorrelated and should be uncorrelated with any instrumental variable in the household's information set.[12] Let $z(s_t, v_t)$ be a multidimensional function that generates sequences of n_z instrumental variables obtained from the simulated state variables $\{s_t\}_{t=0}^{T}$ and $\{v_t\}_{t=0}^{T}$. Then the rational expectations hypothesis requires

$$E_t[u_{t+1} \otimes z(s_t, v_t)] = 0 \quad \forall t = 0, \ldots, T-1$$

The sample counterpart to this expression is given by

$$B_T = \frac{1}{T}\sum_{t=0}^{T-1}[u_{t+1} \otimes z(s_t, v_t)]$$

If the approximation is good, then B_T should be very close to zero. In order to assess the quality of the approximation we thus need a criterion to decide whether B_T is significantly different from zero. den Haan and Marcet (1994) provide such a criterion. They demonstrate that under the null hypothesis $H0 : B_T = 0$ the statistic

$$J_T = TB_T' A_T^{-1} B_T$$

is distributed as a χ^2 variable with $q \cdot n_z$ degrees of freedom. A_T is a consistent estimate of the matrix

$$S_W = \sum_{i=-\infty}^{\infty} E[u_{t+1} \otimes z(s_t, v_t)] \cdot [u_{t+1-i} \otimes z(s_{t-i}, v_{t-i})]'$$

Discussions of how to derive A_T can be found in the GMM literature, e.g. in Hansen (1982) and Newey and West (1987).
An approximation can be considered as good if the statistic J_T is within the uncritical region of a $\chi^2_{qn_z}$ distribution. den Haan and Marcet (1994) propose to repeat the test for different realizations of the stochastic processes and report the percentage of statistics in the upper and lower regions of a $\chi^2_{qn_z}$ distribution. This way, they conclude, one can further reduce the probability that an accurate solution is rejected or that an inaccurate solution is accepted.

5 The Evaluation Exercise

Throughout our evaluation exercise we use seven different numerical methods to solve for approximate decision rules of the basic neoclassical growth model. First

[12] Instrumental variables are usually state variables or functions of state variables.

we apply the three methods illustrated in Section 3 to derive (log-)linear decision rules. We then repeat the procedure using quadratic decision rules. Finally, we use Galerkin Projection of order four to derive a high-order approximation.
Because the model's structural parameters determine the non-linearity of the problem and the deviations of model variables around the steady state, we guess that the quality of approximation depends crucially on these parameters. Thus we consider 500 different calibration sets for our evaluation. We derive the parameter vectors by drawing randomly from a postulated uniform distribution. This distribution is chosen such that it covers the range typically considered in the literature for parameters of the neoclassical growth model. It is given in Table 5. Using a large number of different parameter sets guarantees robustness

Table 1: Postulated Distributions for Structural Parameters

Parameter	Distribution
α	$U[0.24, 0.36]$
β	$U[0.95, 0.99]$
γ	$U[1.5, 3]$
δ	$U[0.025, 0.1]$
ρ	$U[0.85, 0.95]$
σ_ε	$U[0.01, 0.1]$

of our results with respect to variations in the structural parameters.

We start our evaluation exercise by generating artificial series of the model's exogenous state variable, the technology level, for every parameter constellation. We then apply all seven numerical methods to solve for approximate equilibria of the model and generate simulated time series for all endogenous variables. These series are used to compute Euler Equation Errors, first and second moments of simulated model variables and forecast errors together with their correlation properties. The following sections discuss each of these three steps in turn and present our results.

5.1 Euler Equation Errors

Given simulated time paths for the model's variables we calculate Judd's (1992) Euler Equation Errors and derive the measures (14)-(16). Along the way we evaluate the conditional expectation in the Euler equation applying Gauss-Hermite quadrature. Our results, together with the mean computer time[13] associated with each numerical method, are summarized in Table 2. We observe that for the case of a linear approximation the PEA and Perturbation Method deliver almost equally accurate solutions as measured by the Euler Equation Errors. Galerkin Projection performs poorly. The associated Mean Absolute Euler Equation Error is approximately seven times larger than those associated with the competing methods.

Our results for the second-order approximation differ substantially. The Perturbation Method performs best, followed by the Galerkin Projection. The

[13]Our calculations were conducted using an Acer Aspire 1622LM notebook with 3.0 GHz Pentium 4 processor and 512MB DDR333 SDRAM

Table 2: Mean Computer Time and Euler Equation Errors

	CPU	EE^1	EE^2	EE^3
Perturbation 1	0.3587	-3.2248	-6.2665	-2.2989
PEA 1	53.9945	-3.2041	-6.1297	-2.2321
Galerkin 1	1.4121	-2.3803	-4.7373	-1.9831
Perturbation 2	0.4362	-4.8259	-9.2923	-3.6626
PEA 2	196.0976	-3.2212	-6.1372	-2.1871
Galerkin 2	3.2370	-4.3595	-8.5526	-3.7939
Galerkin 4	8.1613	-6.1182	-12.0908	-5.7091

PEA delivers only poor results. It is particularly interesting that the performance of the PEA improves only slightly when we move from a linear to a quadratic approximation of the expectations function. A possible explanation is that the second-order PEA fails to converge within 500 iterations in approximately twenty percent of the cases considered. We then use the last estimate for further analysis. As these last estimates are found to be very close to the final parameter vectors, however, we think that failed convergence causes only a minor deterioration in accuracy. The relatively poor performance is rather a consequence of the multicollinearity introduced by adding quadratic terms, and of the relatively small number of artificial data points we use.[14] On the contrary, the quality of the Galerkin Projection increases substantially when higher-order terms are added in the decision rules. Moving from first to second-order approximation reduces the Mean Absolute Euler Equation Error by a factor of almost hundred. Another significant gain in accuracy is achieved by moving from second to fourth order. We believe this substantial gain is a consequence of the orthogonality property of the regressors, and of the particularly good performance of Chebyshev polynomials in fitting non-linear functions. Finally, we observe that the accuracy of the approximation increases by a factor of almost forty when moving from first-order to second-order Perturbation. Our results demonstrate that increasing the order of approximation yields significant gains in accuracy when using Perturbation Methods and the Galerkin Projection Method, however, not when using the Parameterized Expectations Algorithm. This, we believe, explains why Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2003) and Novales and Pérez (2004) arrive at opposite conclusions. The first paper compares (log-)linear approximations with high-order Projection and Perturbation Methods. As the latter perform particularly well the authors conclude that non-linear methods are to be preferred over linear approximations. Novales and Pérez (2004) compare linear approximations with a second-order Parameterized Expectations Algorithm. Because the second-order PEA performs only poorly, the authors conclude that log-linearization methods yield satisfactory results. Our exercise explains both results in a unified framework.

[14] Increasing the length of artificial series would improve the performance of the PEA, however, at the cost of dramatically increased computational burden. For manageable T we are convinced that Perturbation and Projection Methods still perform better than the PEA, thus we keep our length of artificial series at $T = 5.000$.

Table 2 shows further that Perturbation Methods most efficiently select the coefficients of the decision rules for a fixed order of approximation. Obviously, first and second-order Perturbation yield the best results among linear and quadratic approximations, respectively. High-order Perturbation Methods thus seem particularly promising for approximation purposes. Incorporating them into the evaluation exercise would be an interesting extension of our work. Finally, we find that whereas the accuracy of numerical methods depends strongly on the structural parameters, the relative performances do not. Figure 1 plots Logged Mean Absolute Euler Equation Errors for all parameter constellations under consideration. Obviously the approximation accuracy depends strongly on the structural parameters, as the Mean Euler Equation Errors vary by a factor of far over hundred, even over 10.000 for the fourth-order Galerkin method. Figure 2 plots pairwise differences in logged Mean Absolute Euler Equation Errors for each parameter set. We see that whether a particular method performs better than another is largely independent from the structural parameters. For example, the Galerkin Projection Method of order four yields better results than all other approximations for all parameter vectors.

5.2 Comparison of Simulated Moments

We continue our evaluation by visually inspecting first and second moments of simulated model variables. Table 3 reports unconditional means and standard deviations for consumption and capital series implied by the seven methods under consideration. One method, the first-order Galerkin Projection, implies

Table 3: First and Second Moments of Simulated Consumption and Capital Series

	Consumption		Capital	
	Mean	Std	Mean	Std
Perturbation 1	1.4003	0.2055	6.6450	1.6703
PEA 1	1.4147	0.2053	6.9027	1.6557
Galerkin 1	1.3697	0.2102	5.6038	1.4012
Perturbation 2	1.4177	0.2057	6.9861	1.7725
PEA 2	1.4144	0.2051	6.8954	1.6570
Galerkin 2	1.4151	0.2039	6.9427	1.7460
Galerkin 4	1.4155	0.2040	6.9510	1.7337

apparently different means and standard deviations for the simulated time series. We think that this is the consequence of the particularly poor performance of the Galerkin Projection when linear decision rules are postulated. Furthermore, we observe that unconditional means implied by first-order Perturbation tend to be smaller than their counterparts implied by other approaches. Among the remaining methods we can hardly visually identify any differences. We thus conclude that the inspection of simulated moments may serve well as a tool to detect very bad approximations, but is not of great help in selecting among good approximations.

Figure 1: *Logged Mean Absolute Euler Equation Errors*

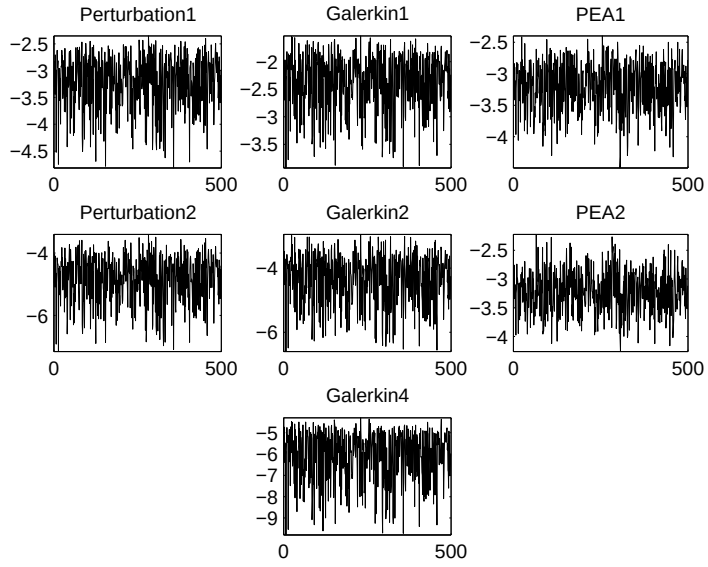
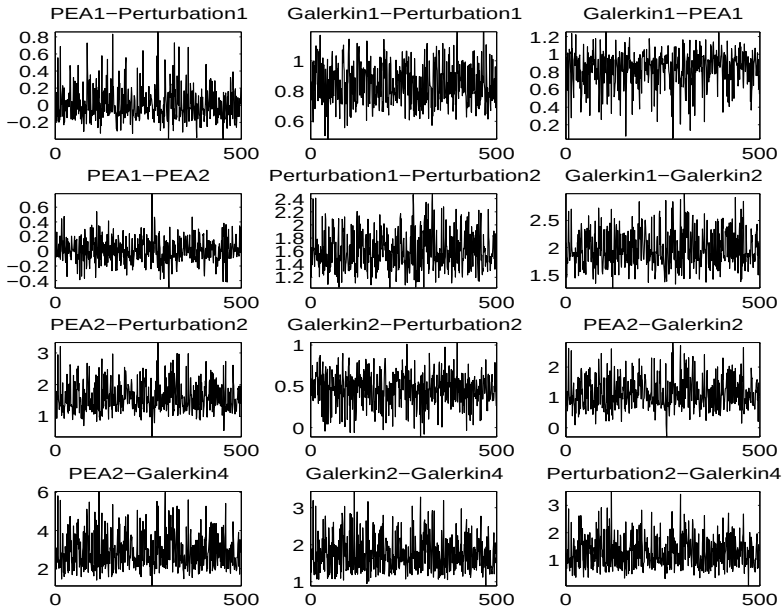


Figure 2: *Pairwise Differences in Logged Mean Absolute Euler Equation Errors (A value below zero indicates that the first method yields better results than the second)*



5.3 Forecast Error Properties

To check whether our solutions satisfy the rational expectations hypothesis we compute sequences of the household's forecast errors. Table 4 reports their unconditional means, mean autocorrelation coefficients of lag one and mean correlation coefficients with three instrumental variables: the capital stock, the technology level and the product of the both. Table 5 reports the respective maximum in absolute value. As a matter of time we do not apply the den Haan and Marcet (1994) accuracy test to check for the martingale difference property. The test requires solving and simulating the model many hundred times for each parameter set and thus increases the computational burden dramatically.[15]

Table 4: Mean Correlation Properties of Forecast Errors

	Mean	AR(1)	k	a	k*a
Perturbation 1	-0.0005	-0.0050	-0.0003	-0.0047	0.0006
PEA 1	-0.0000	-0.0005	-0.0011	-0.0002	-0.0008
Galerkin 1	0.0000	-0.0011	-0.0011	-0.0003	-0.0009
Perturbation 2	-0.0000	-0.0016	0.0029	-0.0006	0.0019
PEA 2	-0.0000	0.0022	-0.0007	0.0003	-0.0004
Galerkin 2	-0.0000	-0.0007	-0.0010	-0.0002	-0.0008
Galerkin 4	-0.0000	-0.0006	-0.0010	-0.0002	-0.0008

Table 5: Maximum Absolute Correlation Properties of Forecast Errors

	AR(1)	k	a	k*a
Perturbation 1	0.0632	0.0399	0.0411	0.0348
PEA 1	0.0524	0.0447	0.0325	0.0397
Galerkin 1	0.0734	0.0453	0.0340	0.0409
Perturbation 2	0.0586	0.0511	0.0434	0.0449
PEA 2	0.6748	0.1508	0.0341	0.1069
Galerkin 2	0.0600	0.0476	0.0332	0.0429
Galerkin 4	0.0593	0.0478	0.0330	0.0429

We find that all methods under consideration seem to fulfill the rational expectations hypothesis reasonably well. The mean correlation coefficients are close to zero for every method considered. The maximum absolute correlation coefficients are well below ten percent except for two cases of the second-order PEA. Similarly to the last section we conclude that the properties of forecast errors seem to be a good device to detect bad approximations, whereas they are hardly helpful in selecting among the better approximations.

[15]For example, if we used only 100 replications the computational time required by our evaluation exercise would increase from approximately two days to more than half a year.

6 Summary and Conclusion

This paper evaluates seven different implementations of numerical methods to solve for approximate decision rules of the neoclassical growth model. Our first result is that major gains in accuracy can be achieved by increasing the order of approximation. Moving from a first-order to a second-order approximation reduces the Mean Absolute Euler Equation Error substantially for Perturbation and Projection Methods. These methods are still computationally fast and relatively easy to implement, as computer codes are publicly available. In addition we find that using a fourth-order Galerkin Projection method may further increase accuracy significantly at low computational costs. We find that the Parameterized Expectations approach is not well suited for higher-order approximations, as it is particularly slow and performs rather poorly as long as the initial guess is not very close to the true solution.

Our second result is that - given the order of approximation - Perturbation Methods select the coefficients of the decision rules in the most efficient way. In cases where a priori information restricts the decision rule to be linear or quadratic, first and second-order Perturbation Methods are best suited to derive the unknown coefficients, whereas Projection Methods may yield very inaccurate results. Thus we suggest to use Projection Methods only when the order of approximation is sufficiently large.

Finally, we find that whether a certain numerical method outperforms another is largely independent of the structural parameters. We find that local approximation methods, such as Perturbation, may still outperform global methods, such as Galerkin Projection, even when the volatility of the model around the steady state is relatively high.

There are some obvious limitations and possible extensions of our analysis. We use only a very simple model throughout our evaluation exercise and cannot guarantee that our results carry over to more complex economies. To assure robustness, the evaluation exercise should be repeated using more complex models. In addition we could extend the analysis to comprise further numerical approaches, such as higher-order Perturbation Methods. We leave both points for further research.

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