On existence of rich Fubini extensions*

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Abstract

This note presents new results on existence of rich Fubini extensions. The notion of a rich Fubini extension was recently introduced by Sun (2006) and shown by him to provide the proper framework to obtain an exact law of large numbers for a continuum of random variables. In contrast to the existence results for rich Fubini extensions established by Sun (2006), the arguments in this note don't use constructions from nonstandard analysis.

Keywords: Fubini extension, exact law of large numbers.

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1 Introduction

In many contexts of economics, a large finite set is idealized by a continuum. The prototype example is Aumann's (1964) model of perfect competition, where the set of agents is specified to be an atomless measure space. In this spirit, there is also the desire to get, in models with individual risk, the conclusion that an atomless measure space of agents implies that, under some independence condition, individual risk exactly cancels out in non-negligible measurable sets of agents. This amounts to the desire to get, with a continuum of random variables, an "exact version" of the classical law of large numbers. However, as was first noted in the economic literature by Judd (1985) and Feldman and Gilles (1985), there are mathematical difficulties with this idea. In particular, there are problems concerning measurability of sample functions.

Nevertheless, there are results showing that one can have models where individual risk cancels out in the aggregate. See Al-Najjar (2004), Alós-Ferrer (2002), Anderson (1991), Green (1994), Sun (1998, 2006), and Uhlig (1996).[1]

One of the contributions in Sun (2006) is the result that an exact law of large numbers indeed holds for processes that are measurable with respect to a Fubini extension of the product measure corresponding to the index probability space and the sample space.[2] As shown by Sun (2006), this fact provides the proper mathematical foundation for models designed to have the feature that there is a cancellation of individual risk in the aggregate.

Existence of Fubini extensions that are "rich" in the sense of allowing for non-trivial measurable processes to which an exact law of large numbers applies was shown by Sun using Loeb space constructions; see Sun (1998, Theorem 6.2) and Sun (2006, Proposition 5.6).

In this note, we present new results about the existence of rich Fubini extensions. In particular, our arguments will not depend on constructions from nonstandard analysis. Thus, in view of the results in Sun (2006) on the exact law of large numbers via general Fubini extensions, the results and arguments in our note imply that non-trivial processes to which an exact law of large numbers applies can be obtained without (directly or indirectly) involving nonstandard analysis.

The rest of this paper is organized as follows. The next section contains the basic definitions concerning the notion of a rich Fubini extension. Section 3 contains some notational conventions as well as further definitions needed for the special purpose of this note. Section 4 contains the statements of our results, and Section 5 the proofs. In an appendix, some mathematical terminology used in this paper is recalled.

[1] We refer to Sun (2006) for a discussion where it is argued that there are interpretative difficulties with the framework of finitely additive measures used by Al-Najjar (2004) as well as with the model of Uhlig (1996) where the Pettis integral is used to justify an exact law of large numbers.

[2] See the next section for the precise meanings.

2 Basic definitions

We first make the following convention.

Convention. Throughout this paper, product measures are understood to be complete product measures.

The three definitions in this section are taken from Sun (2006), but slightly reformulated here concerning notation.

Definition 1. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $(X \times Y, \Lambda, \lambda)$ the corresponding product probability space. Let $\overline{\lambda}$ be a probability measure on $X \times Y$, and $\overline{\Lambda}$ its domain. Then $\overline{\lambda}$ is said to be a *Fubini extension* of λ if (a) $\overline{\Lambda} \supset \Lambda$ and (b) for each $H \in \overline{\Lambda}$—denoting by χH the characteristic function of H—the integrals $\iint \chi H(x, y)\, d\nu(y)\, d\mu(x)$ and $\iint \chi H(x, y)\, d\mu(x)\, d\nu(y)$ are well-defined and $\iint \chi H(x, y)\, d\nu(y)\, d\mu(x) = \overline{\lambda}(H) = \iint \chi H(x, y)\, d\mu(x)\, d\nu(y)$.

Note that (a) and (b) in this definition imply that $\overline{\lambda}$ must agree with λ on Λ. Also note that this definition implies that if $f\colon X \times Y \to \mathbb{R}$ is a $\overline{\Lambda}$-measurable function, then for almost all $x \in X$, the x-sections $f(x, \cdot)$ are measurable for the ν-completion of T, and similarly for the y-sections. From this it follows in turn that an analogous statement holds for functions from $X \times Y$ to any Polish space. The definition also implies that the *conclusion* of Fubini's theorem holds for $\overline{\lambda}$-integrable functions from $X \times Y$ to $\mathbb{R}$.

Definition 2. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, Z a Polish space, and $f\colon X \times Y \to Z$ a function such that for almost all $x \in X$, $f(x, \cdot)$ is measurable for the μ-completion of T and the Borel sets of Z. Then the family $\langle f(x, \cdot)\rangle_{x \in X}$ is said to be *essentially pairwise independent* if there is a null set N in X such that for each $x \in X \setminus N$ the functions $f(x, \cdot)$ and $f(x', \cdot)$ are stochastically independent for almost all $x' \in X$.

Let (X, Σ, μ), (Y, T, ν), and Z be as in this latter definition, and let λ be the product measure on $X \times Y$ given by μ and ν. As shown by Sun (2006, Theorem 2.8), if a process $f\colon X \times Y \to Z$ is measurable with respect to the domain of some Fubini extension $\overline{\lambda}$ of λ, then an exact law of large numbers holds in the sense that essentially pairwise independence of the family $\langle f(x, \cdot)\rangle_{x \in X}$ implies that, given any $E \in \Sigma$ with $\mu(E) > 0$, for almost all $y \in Y$ the distribution of $(f \restriction E \times Y)(\cdot, y)$ with respect to μ_E is equal to the distribution of $f \restriction E \times Y$ with respect to $\overline{\lambda}_{E \times Y}$, where $f \restriction E \times Y$ is the restriction of f to $E \times Y$, and μ_E and $\overline{\lambda}_{E \times Y}$ are the probability measures obtained be renormalizing the subspace measures induced by μ and $\overline{\lambda}$ on E and $E \times Y$, respectively; in particular, for almost all $y \in Y$, the distribution of $f(\cdot, y)$ with respect to μ is equal to the distribution of f with respect to $\overline{\lambda}$. We remark that Theorem 2.8 of Sun (2006) also shows that the converse of this law of large numbers is true.

Of course, the measurability and independence requirements in the law of large numbers stated above are trivially satisfied for a constant valued process.

The next definition states a criterion for a Fubini extension to yield a framework in which this law has a non-trivial meaning.

Definition 3. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure. Let $\overline{\lambda}$ be a Fubini extension of λ, and $\overline{\Lambda}$ its domain. The Fubini extension $\overline{\lambda}$ is called a *rich Fubini extension* if there is a $\overline{\Lambda}$-measurable function $f\colon X \times Y \to [0,1]$ such that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent and for almost every $x \in X$, the distribution of the function $f(x,\cdot)$ is the uniform distribution on $[0,1]$.

Let f be as in this definition, and let τ be any Borel probability measure on a Polish space Z. By a standard fact, τ is the distribution of some measurable function g defined on $([0,1], \mathcal{B}, \rho)$, where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$ and ρ is Lebesgue measure. Then the composition $f' = g \circ f$ is a $\overline{\Lambda}$-measurable function from $X \times Y$ to Z such that the family $\langle f'(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent, and for almost every $x \in X$, the distribution of $f'(x,\cdot)$ is τ. In particular, by the Fubini property of $\overline{\Lambda}$, the distribution of f' is equal to τ. Thus the word "rich" in Definition 3 is justified.

Finally, we remark that if a process f is as required in Definition 3, then f cannot be measurable already for the domain of the product measure λ (see Sun, 2006, Proposition 2.1). Thus a rich Fubini extension must always be a proper extension of the product measure in question, so the problem of existence of rich Fubini extensions is non-trivial. (See also the remark at the end of Section 5.5.)

3 Notation, conventions, and further definitions

If (X, Σ, μ) is any measure space, $\operatorname{cov}\mathcal{N}(\mu)$ denotes the least cardinal of any family of μ-null sets which covers X, provided such a family exists. We let $\operatorname{cov}\mathcal{N}(\mu)$ be undefined if no such family exists. Thus, if κ is a cardinal and it is written, e.g., "$\operatorname{cov}\mathcal{N}(\mu) \le \kappa$," then this is understood to imply that X *can* be covered by a family of μ-null sets.

For a non-empty set I, ν_I denotes the usual measure on $\{0,1\}^I$. In particular, $\nu_{\mathbb{N}}$ denotes the usual measure on $\{0,1\}^{\mathbb{N}}$; $\nu_{\mathbb{N}}^B$ denotes the restriction of $\nu_{\mathbb{N}}$ to the Borel σ-algebra of $\{0,1\}^{\mathbb{N}}$.

If (X, Σ, μ) is any measure space, "measurable" for a mapping $f\colon X \to \{0,1\}^{\mathbb{N}}$ always means measurable with respect to the Borel (= Baire) sets of $\{0,1\}^{\mathbb{N}}$.

For convenience, we will work with the following restatement of Definition 3. (Recall for this that $[0,1]$ with Lebesgue measure and $\{0,1\}^{\mathbb{N}}$ with its usual measure are isomorphic as measure spaces.)

Definition 4. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure. Let $\overline{\lambda}$ be a Fubini extension of λ, and $\overline{\Lambda}$ its domain. The Fubini extension $\overline{\lambda}$ is called a *rich Fubini extension* if there is a $\overline{\Lambda}$-measurable function $f\colon X \times Y \to \{0,1\}^{\mathbb{N}}$ such that the family $\langle f(x,\cdot)\rangle_{x\in X}$ is essentially pairwise independent and for almost all $x \in X$, the distribution of the function $f(x,\cdot)$ is equal to $\nu_{\mathbb{N}}^B$.

Let (X, Σ, μ), (Y, T, ν), and λ be as in this definition. By Sun (2006, Theorem 4.2) (see also Theorem 3 below), there can be no rich Fubini extension of λ if one of the σ-algebras Σ and T, say Σ, has a non-negligible element A such that the trace of Σ on A is essentially countably generated. For this reason we consider probability spaces that satisfy the criterion in the following definition.

Definition 5. Let (X, Σ, μ) be a probability space and $(\mathfrak{A}, \hat{\mu})$ its measure algebra. The measure μ (or the measure space (X, Σ, μ)) is said to be *super-atomless* if each non-zero principal ideal of $\mathfrak{A}$ has uncountable Maharam type.[3] [4]

Examples of super-atomless probability spaces are $\{0, 1\}^I$ with its usual measure when I is an uncountable set, the product measure space $[0, 1]^I$ where each factor is endowed with Lebesgue measure when I is uncountable, subsets of these spaces with full outer measure when endowed with the subspace measure, atomless Loeb probability spaces. Furthermore, any atomless Borel probability measure on a Polish space can be extended to a super-atomless probability measure[5]; in particular, Lebesgue measure on $[0, 1]$ can be extended to a super-atomless probability measure.

We also need the following definition.

Definition 6. Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \hat{\mu})$. For an uncountable cardinal κ, the measure μ (or the measure space (X, Σ, μ)) is said to be *κ-super-atomless* if $\kappa = \min\{\kappa' \colon \kappa'$ is the Maharam type of some non-zero principal ideal of $\mathfrak{A}\}$.[6]

4 Results

Theorem 1. *Given any super-atomless probability space (X, Σ, μ), there is probability space (Y, T, ν) (also super-atomless) such that the product measure corresponding to μ and ν has a rich Fubini extension.*

Note that in Theorem 1, for the given probability space (X, Σ, μ) we can in particular have that $X = [0, 1]$ and that μ is any extension of Lebesgue measure

[3]We refer to Fremlin (2002) for terminology and facts concerning measure algebras. Some basic terminology is recalled in the appendix.

[4]The name "super-atomless", suggested to me by Erik Balder, is aimed to indicate that the condition in this definition is a straightforward strengthening of non-atomicity, the latter being equivalent to the property that non-zero principal ideals of the measure algebra of a probability space in question have infinite Maharam type. We remark that the notions "saturated probability space," "$\aleph_1$-atomless probability space" and "nowhere separable probability space" which appear in the literature state conditions that can be shown to be equivalent to the condition in Definition 5.

[5]As shown in Podczeck (2008), this fact is a straightforward consequence of the fact that there are countably separated probability spaces with uncountable Maharam type. For this latter fact, see Fremlin (2005, 521P).

[6]Recall that the cardinals are well-ordered, so the definition makes sense.

on $[0,1]$ to a super-atomless measure. As remarked at the end of the previous section, such extensions of Lebesgue measure do exist.

In Sun and Zhang (2008), developed simultaneously and independently from this paper, it is also shown that there are rich Fubini extensions where one of the factor spaces is $[0,1]$ with an extension of Lebesgue measure (the extension being super-atomless in our terminology). In Sun and Zhang (2008) the extension of Lebesgue measure is constructed as part of the construction of the Fubini extension. Theorem 1 of this paper shows that actually there is no need for a particular choice of such an extension, i.e., in order to get the conclusion of this theorem, any extension of Lebesgue measure on $[0,1]$ to a super-atomless measure can be taken as given. Moreover, the fact that the given space (X, Σ, μ) in Theorem 1 can be any super-atomless probability space shows, by the definition of "super-atomless," that the conclusion of this theorem actually depends only on properties of the measure algebra of (X, Σ, μ), so the result that there are rich Fubini extensions where one of the factor measures is a super-atomless extension of Lebesgue measure on $[0,1]$ is a special case of a result at a deeper level of abstraction.

We also note, writing $\mathfrak{c}$ for the cardinal of the continuum:

Remark 1. In Theorem 1, if $\#(X) \leq \mathfrak{c}$ then the probability space (Y, T, ν) can be chosen so that $\#(Y) = \mathfrak{c}$. (For an argument establishing this, see subsection 5.3.) In particular, (Y, T, ν) can be chosen with $\#(Y) = \mathfrak{c}$ if $X = [0,1]$ and μ is any extension of Lebesgue measure on $[0,1]$ to a super-atomless measure.

A concrete version of Theorem 1 is contained in the next result.

Theorem 2. *Let (X, Σ, μ) be any super-atomless probability space. Then there is a probability measure ν on $(\{0,1\}^{\mathbb{N}})^X$ such that the product probability measure on $X \times (\{0,1\}^{\mathbb{N}})^X$ corresponding to μ and ν has a rich Fubini extension, say $\overline{\lambda}$ with domain $\overline{\Lambda}$. The measure ν and the Fubini extension $\overline{\lambda}$ can be chosen in such a way that the coordinate projections function $f\colon X \times (\{0,1\}^{\mathbb{N}})^X \to \{0,1\}^{\mathbb{N}}$, given by $f(x,y) = y(x)$, has the following properties: (a) f is $\overline{\Lambda}$-measurable; (b) the family $\langle f(x,\cdot)\rangle_{x\in X}$ is i.i.d. for ν with distribution $\nu_{\mathbb{N}}^{B}$, thus, in particular, essentially pairwise independent for the marginals μ and ν of $\overline{\lambda}$.*

Theorem 2 is a generalization of Proposition 5.6 in Sun (2006) where, in our notation, X is $[0,1]$ but the measure μ is constructed in the proof of that proposition so that the resulting probability space $([0,1], \Sigma, \mu)$ is isomorphic as a measure space to an atomless Loeb probability space. We remark in this regard that if a probability space $([0,1], \Sigma, \mu)$ is isomorphic to an atomless Loeb probability space, then μ cannot be an extension of Lebesgue measure.[7]

[7]This follows from Keisler and Sun (2002) where it is shown that if (X_0, Σ_0, μ_0) is any atomless Loeb probability space, X a Polish space, $f\colon X_0 \to X$ a measurable mapping, and μ denotes the distribution of f on X, then for μ-almost every $x \in X$ the inverse image $f^{-1}(\{x\})$ has a cardinality at least as large as that of the continuum.

Can it be shown that, given *any two* super-atomless probability spaces, the corresponding product measure has a rich Fubini extension? Unfortunately, the answer is no. Consider $\{0,1\}^{\omega_1}$ with its usual measure ν_{ω_1}, where ω_1 is the least uncountable cardinal. It cannot be proved in ZFC that $\operatorname{cov}\mathcal{N}(\nu_{\omega_1}) = \omega_1$.[8] On the other hand, $\{0,1\}^{\omega_1}$ is Maharam-type-homogeneous with Maharam type ω_1. But this implies that if $\operatorname{cov}\mathcal{N}(\nu_{\omega_1}) > \omega_1$, then the product measure corresponding to two copies of $\{0,1\}^{\omega_1}$ cannot have a rich Fubini extension. In fact, the next theorem states necessary conditions for rich Fubini extensions to exist.

Theorem 3. *Let (X, Σ, μ) and (Y, T, ν) be probability spaces. If the product probability measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension, then the following hold.*

(a) *Each non-zero principal ideal of the measure algebra of ν has Maharam type $\geq \operatorname{cov}\mathcal{N}(\mu)$.*

(b) *Each non-zero principal ideal of the measure algebra of μ has Maharam type $\geq \operatorname{cov}\mathcal{N}(\nu)$.*

Theorem 3 implies, in particular, the fact already noted in Section 3 that, given probability spaces (X, Σ, μ) and (Y, T, ν), in order for the corresponding product measure to have a rich Fubini extension, it is necessary that the measure algebras of both μ and ν do not contain non-zero principal ideals with countable Maharam type, or, in other words, that both μ and ν be super-atomless. But note that Theorem 3 shows that actually more than this is needed to get a rich Fubini extension. The proof of Theorem 3 will also show, as a byproduct, that a rich Fubini extension of a product measure in question must be a proper extension (see the remark at the end of Section 5.5).

The following result provides sufficient conditions in order that the product measure corresponding to two given probability spaces have a rich Fubini extension.

Theorem 4. *Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the corresponding product probability measure on $X \times Y$. Suppose that for some uncountable cardinals α and β, μ is α-super-atomless and ν is β-super-atomless. Further suppose that for some cardinal κ, with $\kappa \leq \min\{\alpha, \beta\}$, there is a non-decreasing family $\langle M_\xi \rangle_{\xi<\kappa}$ of null sets in X with $\bigcup_{\xi<\kappa} M_\xi = X$ and a non-decreasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of null sets in Y with $\bigcup_{\xi<\kappa} N_\xi = Y$. Then λ has a rich Fubini extension.*

The hypotheses in Theorem 4 can be satisfied as shown in the following example.

[8] Recall that Martin's axiom implies that $\operatorname{cov}\mathcal{N}(\nu_{\omega_1}) = \mathfrak{c}$ (see Fremlin, 2005, 523Y(f)(ii) and 517O(b) and (d)), and that it is (relatively) consistent with ZFC that Martin's axiom holds and $\omega_1 < \mathfrak{c}$. For this latter fact as well as for a statement of Martin's axiom, see e.g. Ciesielski (1997, Chapter 8.2).

Example. Let κ be any cardinal with uncountable cofinality, and consider $\{0,1\}^\kappa$ with its usual measure ν_κ. Fix any $\overline{x} \in \{0,1\}^\kappa$ and for each $\xi < \kappa$, let

$$N_\xi = \{x \in \{0,1\}^\kappa : x(\eta) = \overline{x}(\eta) \text{ for all } \eta < \kappa \text{ with } \eta \geq \xi\}.$$

Set $X = \bigcup_{\xi<\kappa} N_\xi$, let μ be the subspace measure on X induced by ν_κ, and Σ the domain of μ. As κ has uncountable cofinality, X intersects every non-empty subset of $\{0,1\}^\kappa$ that is determined by coordinates in some countably subset of κ. Thus X has full outer measure for ν_κ. This implies that μ is a probability measure and that the measure algebra of μ can be identified with that of ν_κ. According to a standard fact, ν_κ is Maharam-type-homogeneous with Maharam type κ, and it follows that μ has the same property. In our terminology, this means μ is κ-super-atomless. Note that for any $\xi < \kappa$, N_ξ is a ν_κ-null set in $\{0,1\}^\kappa$ since all of its elements agree on some infinite subset of κ. Hence for any $\xi < \kappa$, N_ξ is a μ-null set in X. Evidently the family $\langle N_\xi \rangle_{\xi<\kappa}$ is non-decreasing. Thus, a pair of two copies of the probability space (X, Σ, μ) just constructed provides an example as desired. (If $\kappa \leq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinal of the continuum, the argument can be refined to yield an X with $\#(X) = \mathfrak{c}$; c.f. the proof of Theorem 5.)

Recall that if (X, Σ, μ) is any complete atomless probability space, there is a mapping $f: X \to [0,1]$ which is inverse-measure-preserving for μ and Lebesgue measure on $[0,1]$. Hence, if $[0,1]$ can be covered by a non-deceasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of Lebesgue null sets, for some cardinal κ, then any atomless probability space (X, Σ, μ) has the property that the set X can be covered by a non-deceasing family $\langle M_\xi \rangle_{\xi<\kappa}$ of μ-null sets (with the same κ). Thus we have the following corollary of Theorem 4.

Corollary 1. *Let κ be a cardinal and suppose that $[0,1]$ can be covered by a non-decreasing family $\langle N_\xi \rangle_{\xi<\kappa}$ of Lebesgue null sets. Then given any two probability spaces (X, Σ, μ) and (Y, T, ν) such that μ is α-super-atomless with $\alpha \geq \kappa$, and ν is β-super-atomless with $\beta \geq \kappa$, the product measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension.*

If the continuum hypothesis is true then $[0,1]$ can be covered by ω_1 many Lebesgue null sets, denoting by ω_1 the least uncountable cardinal. Therefore Corollary 1 implies:

Corollary 2. *If the continuum hypothesis holds then given any two super-atomless probability spaces (X, Σ, μ) and (Y, T, ν), the product measure on $X \times Y$ corresponding to μ and ν has a rich Fubini extension.*

Recall that a weakening of the continuum hypothesis is given by Martin's axiom, but that Martin's axiom still implies that the union of fewer than $\mathfrak{c}$ many Lebesgue null sets in $[0,1]$ is a Lebesgue null set, where $\mathfrak{c}$ is the cardinal of the continuum.[9] Thus under Martin's axiom the hypothesis on $[0,1]$ in Corollary 1 holds for $\kappa = \mathfrak{c}$. Hence, by Corollary 1, the following result holds.

[9] See Ciesielski (1997, p. 145, Theorem 8.2.7).

Corollary 3. *Suppose Martin's axiom is true. Then given any two probability spaces* (X, Σ, μ) *and* (Y, T, ν) *such that* μ *is* α*-super-atomless with* $\alpha \geq \mathfrak{c}$*, and* ν *is* β*-super-atomless with* $\beta \geq \mathfrak{c}$*, the product measure on* $X \times Y$ *corresponding to* μ *and* ν *has a rich Fubini extension.*

The final result of this note will also be derived as a consequence of Theorem 4; see Section 5.7.

Theorem 5. *Let* X *and* Y *be Polish spaces,* μ *an atomless Borel probability measure on* X*, and* ν *an atomless Borel probability measure on* Y*. Then there is a super-atomless probability measure* μ' *on* X *which extends* μ*, and a super-atomless probability measure* ν' *on* Y *which extends* ν*, such that the product measure on* $X \times Y$ *corresponding to* μ' *and* ν' *has a rich Fubini extension.*

Closing this section, we notice that there is an obvious gap between the sufficient conditions for existence of a rich Fubini extension, as they are stated in Theorem 4, and the necessary conditions as stated in Theorem 3. Of course, this gap gives room for further research.

5 Proofs

Notation: If A is a subset of a product $X \times Y$ and $x \in X$, then A_x denotes the x-section of A, and if $y \in Y$ then A_y denotes the y-section of A. Thus, if $x \in X$, then $A_x = \{y \in Y : (x, y) \in A\}$; similarly, for $y \in Y$, $A_y = \{x \in X : (x, y) \in A\}$.

5.1 Lemmata

Lemma 1. *Let* (X, Σ, μ) *and* (Y, T, ν) *be probability spaces, and* $(X \times Y, \Lambda, \lambda)$ *the corresponding product probability space. Suppose there is a sequence* $\langle H^i \rangle_{i \in \mathbb{N}}$ *of subsets of* $X \times Y$ *such that:*

(a) *There is a null set* N *in* X *such that for each* $x \in X \setminus N$ *and each* $i \in \mathbb{N}$*, the section* H^i_x *is a member of* T *with* $\nu(H^i_x) = 1/2$*.*

(b) *There is a null set* N *in* Y *such that for each* $y \in Y \setminus N$ *and each* $i \in \mathbb{N}$*, the section* H^i_y *is a member of* Σ *with* $\mu(H^i_y) = 1/2$*.*

(c) *For each* $B \in \mathrm{T}$ *there is null set* N_B *in* X *such that for each* $x \in X \setminus N_B$*,* B *and the sections* H^i_x*,* $i \in \mathbb{N}$*, form a stochastically independent family in* T*.*

(d) *For each* $A \in \Sigma$ *there is null set* N_A *in* Y *such that for each* $y \in Y \setminus N_A$*,* A *and the sections* H^i_y*,* $i \in \mathbb{N}$*, form a stochastically independent family in* Σ*.*

Then λ *has a rich Fubini extension* $\overline{\lambda}$ *such that the domain of* $\overline{\lambda}$ *contains all the sets* H^i*,* $i \in \mathbb{N}$*, and such that a function* $f : X \times Y \to \{0,1\}^{\mathbb{N}}$ *which witnesses richness of* $\overline{\lambda}$ *is given by setting, for each* $(x, y) \in X \times Y$ *and* $i \in \mathbb{N}$*,*

$$f^i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in H^i \\ 0 & \text{if } (x, y) \notin H^i. \end{cases}$$

Proof. Let $\mathcal{F}$ denote the set of all subsets $F \subset X \times Y$ such that the integrals $\int_X \nu(F_x)\,d\mu(x)$ and $\int_Y \mu(F_y)\,d\nu(y)$ are well-defined and equal. Then $\mathcal{F}$ is a Dynkin class (i.e. $\varnothing \in \mathcal{F}$ and $\mathcal{F}$ is closed against forming complements and unions of disjoint sequences) as may easily be checked. Also, (a) to (d) imply that whenever $A_1 \times B_1, \dots, A_n \times B_n$ are finitely many measurable rectangles in $X \times Y$ and $H^{i_1}, \dots, H^{i_m}$ is a finite subfamily of $\langle H^i \rangle_{i \in \mathbb{N}}$, then the intersection

$$(A_1 \times B_1) \cap \cdots \cap (A_n \times B_n) \cap H^{i_1} \cap \cdots \cap H^{i_m}$$

belongs to $\mathcal{F}$. Therefore, by the monotone class theorem, there is a σ-algebra $\Lambda' \subset \mathcal{F}$ which contains all measurable rectangles in $X \times Y$ and all the sets H^i, $i \in \mathbb{N}$. Define $\lambda' \colon \Lambda' \to \mathbb{R}$ by setting $\lambda'(F) = \int_X \nu(F_x)\,d\mu(x)$ for $F \in \Lambda'$. Using the monotone convergence theorem, it follows that λ' is a probability measure on $X \times Y$. Let $\overline{\lambda}$ be its completion, and $\overline{\Lambda}$ the domain of $\overline{\lambda}$. Then since $\mathcal{F}$ contains all measurable rectangles in $X \times Y$, we have $\overline{\Lambda} \supset \Lambda$. By construction, the Fubini property holds for the characteristic functions of the elements of Λ', which in particular implies that if N is a λ'-null set in $X \times Y$, then for μ-almost every $x \in X$, the x-section of N is a ν-null set in Y, and for ν-almost every $y \in Y$, the y-section of N is a μ-null set in X. Consequently, the Fubini property holds for the characteristic functions of the elements of $\overline{\Lambda}$. In particular, $\overline{\lambda}$ coincides with λ on Λ. Thus $\overline{\lambda}$ is a Fubini extension of λ such that the domain $\overline{\Lambda}$ of $\overline{\lambda}$ contains all the sets H^i, $i \in \mathbb{N}$. Note that we have $\overline{\lambda}(H^i) = 1/2$ for all $i \in \mathbb{N}$.

Now consider the function $f \colon X \times Y \to \{0,1\}^{\mathbb{N}}$ defined in the statement of the lemma. Since $H^i \in \overline{\Lambda}$ for each $i \in \mathbb{N}$, f is measurable for $\overline{\Lambda}$ and the Borel sets of $\{0,1\}^{\mathbb{N}}$.

It remains to show that the family $\langle f(x,\cdot) \rangle_{x \in X}$ is essentially pairwise independent, and that for almost every $x \in X$, $f(x,\cdot)$ is inverse-measure-preserving for ν and $\nu^B_{\mathbb{N}}$. To this end, for each $x \in X$ let T_x denote the σ-algebra on Y generated by the set $\{H^i_x \colon i \in \mathbb{N}\}$, and let $\overline{N}$ be a null set in X chosen according to condition (a). In particular, then, for each $x \in X \setminus \overline{N}$, T_x is a sub-σ-algebra of T. Also, in view of (c), we may assume that for each $x \in X \setminus \overline{N}$, the family $\langle H^i_x \rangle_{i \in \mathbb{N}}$ is stochastically independent (applying (c) e.g. to $B = Y$ and replacing $\overline{N}$ by a larger null set, if necessary).

Fix any $\overline{x} \in X \setminus \overline{N}$. Applying (c) to each finite intersection of elements of the family $\langle H^i_{\overline{x}} \rangle_{i \in \mathbb{N}}$, we can see that there is a null set $N_{\overline{x}}$ in X such that for each $x \in X \setminus N_{\overline{x}}$, the family of all the sets H^i_x, $i \in \mathbb{N}$, and $H^i_{\overline{x}}$, $i \in \mathbb{N}$, is a stochastically independent family in T. But this implies that for each $x \in X \setminus N_{\overline{x}}$, the σ-algebras $\mathrm{T}_{\overline{x}}$ and T_x are stochastically independent. Now the definition of f implies that for each $x \in X$, $f(x,\cdot)$ is measurable for T_x and the Borel sets of $\{0,1\}^{\mathbb{N}}$, and it follows that for each $x \in X \setminus N_{\overline{x}}$, $f(x,\cdot)$ and $f(\overline{x},\cdot)$ are stochastically independent.

Since this argument applies to each fixed $\overline{x} \in X \setminus \overline{N}$, it follows that the family $\langle f(x,\cdot) \rangle_{x \in X}$ is essentially pairwise independent. Finally, note that if $x \in X \setminus \overline{N}$, then since $\langle H^i_x \rangle_{i \in \mathbb{N}}$ is stochastically independent for such an x,

$f(x, \cdot)$ is inverse-measure-preserving for ν and $\nu_{\mathbb{N}}^{B}$, by the definition of f and since $\nu(H_x^i) = 1/2$ for all $i \in \mathbb{N}$ and all $x \in X \setminus \overline{N}$. This completes the proof. $\square$

Lemma 2. *Let (X, Σ, μ) be a κ-super-atomless probability space. Then there is a family $\langle E_\xi \rangle_{\xi<\kappa}$ in Σ, with $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$, such that for each $A \in \Sigma$ there is a countable set $D_A \subset \kappa$ such that A and the sets E_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in Σ.*

Proof. Suppose first that μ is Maharam-type-homogeneous, and let $(\mathfrak{A}, \hat{\mu})$ denote the measure algebra of μ. Then by Maharam's theorem, there is a measure algebra isomorphism between $(\mathfrak{A}, \hat{\mu})$ and the measure algebra of the usual measure ν_κ on $\{0, 1\}^\kappa$. Denote this latter measure algebra by $(\mathfrak{C}_\kappa, \hat{\mu}_\kappa)$. For each $\xi < \kappa$ let $F_\xi = \{x \in \{0, 1\}^\kappa : x(\xi) = 1\}$. Then $\langle F_\xi \rangle_{\xi<\kappa}$ is a stochastically independent family in the domain of ν_κ, with $\nu_\kappa(F_\xi) = 1/2$ for each $\xi < \kappa$. Thus the family $\langle F_\xi^\bullet \rangle_{\xi<\kappa}$, where $F_\xi^\bullet$ is the element in $\mathfrak{C}_\kappa$ determined by F_ξ, is a stochastically independent family in $\mathfrak{C}_\kappa$, with $\hat{\nu}_\kappa(F_\xi^\bullet) = 1/2$ for each $\xi < \kappa$. By a standard fact, the set $\{F_\xi^\bullet : \xi < \kappa\}$ completely generates $\mathfrak{C}_\kappa$. Consequently, since $(\mathfrak{A}, \hat{\mu})$ and $\mathfrak{C}_\kappa$ are isomorphic as measure algebras, there is a stochastically independent family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}$, with $\hat{\mu}(a_\xi) = 1/2$ for each $\xi < \kappa$, such that the set $\{a_\xi : \xi < \kappa\}$ completely generates $\mathfrak{A}$. For each $\xi < \kappa$ select an element E_ξ in Σ which determines a_ξ. In particular, then, $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$. Now pick any $A \in \Sigma$. Let $A^\bullet$ be the element in $\mathfrak{A}$ determined by A. Since the set $\{a_\xi : \xi < \kappa\}$ completely generates $\mathfrak{A}$, there is a countable set $D_A \subset \kappa$ such that $A^\bullet$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by the set $\{a_\xi : \xi \in D_A\}$.[10] But this subalgebra of $\mathfrak{A}$ and the closed subalgebra of $\mathfrak{A}$ generated by the set $\{a_\xi : \xi \in \kappa \setminus D_A\}$ are stochastically independent, because the family $\langle a_\xi \rangle_{\xi<\kappa}$ is stochastically independent.[11] It follows that $A^\bullet$ and the elements a_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in $\mathfrak{A}$, whence A and the sets E_ξ, $\xi \in \kappa \setminus D_A$, form a stochastically independent family in Σ.

Now suppose μ is not Maharam-type-homogeneous. Since μ is a probability measure, Maharam's theorem implies that there is a countable partition $\langle S_i \rangle_{i \in I}$ of X, with $S_i \in \Sigma$ and $\mu(S_i) > 0$ for each $i \in I$, such that, denoting by μ_i the subspace measure on S_i induced by μ, μ_i is Maharam-type-homogeneous for each $i \in I$. Let κ_i be the Maharam type of μ_i and note that $\kappa = \min\{\kappa_i : i \in I\}$ (by the definition of "κ-super-atomless"). For each $i \in I$, let Σ_i denote the domain of μ_i (i.e. Σ_i is the trace of Σ on S_i) and let $\overline{\mu}_i$ denote the normalization of μ_i so that $\overline{\mu}_i$ is a probability measure. (Thus $\overline{\mu}_i$ is the measure on S_i given as $\overline{\mu}_i = \frac{1}{\mu_i(S_i)} \mu_i$.) Note that for each $i \in I$, $\overline{\mu}_i$ is again Maharam-type-homogeneous with Maharam type κ_i.

Now for each $i \in I$, considering the probability space $(S_i, \Sigma_i, \overline{\mu}_i)$, let $\langle E_\xi^i \rangle_{\xi<\kappa_i}$ be a family in Σ_i, constructed according to the first paragraph of this proof. Recalling that $\kappa = \min\{\kappa_i : i \in I\}$, for each $i \in I$ let $\langle E_\xi^i \rangle_{\xi<\kappa}$ be a subfamily of

[10] See Fremlin (2002, 331G(d) and 331G(e)).
[11] See Fremlin (2002, 325X(e) and 325X(f)).

the family $\langle E^i_\xi\rangle_{\xi<\kappa_i}$, and then let $\langle E_\xi\rangle_{\xi<\kappa}$ be the family in Σ defined by setting $E_\xi = \bigcup_{i\in I} E^i_\xi$ for each $\xi < \kappa$. Note that we must have $\mu(E_\xi) = 1/2$ for each $\xi < \kappa$.

Consider any $A \in \Sigma$. Set $A_i = A \cap S_i$ for each $i \in I$. By choice of the families $\langle E^i_\xi\rangle_{\xi<\kappa}$, for each $i \in I$ there is a countable set $D^i_A \subset \kappa$ such that A_i and the sets E^i_ξ, $\xi \in \kappa\setminus D^i_A$, form a stochastically independent family in Σ_i for $\overline{\mu}_i$. Set $D_A = \bigcup_{i\in I} D^i_A$ and consider any finite subfamily $E_{\xi_1},\ldots,E_{\xi_n}$ of $\langle E_\xi\rangle_{\xi<\kappa}$ with $\xi_j \notin D_A$ for $j = 1,\ldots,n$. Using the fact that $\langle S_i\rangle_{i\in I}$ is a partition of X, it follows that

$$\begin{aligned}\mu(A\cap E_{\xi_1}\cap\cdots\cap E_{\xi_n}) &= \sum_{i\in I}\mu_i(A_i\cap E^i_{\xi_1}\cap\cdots\cap E^i_{\xi_n})\\ &= \sum_{i\in I}\mu_i(S_i)\overline{\mu}_i(A_i\cap E^i_{\xi_1}\cap\cdots\cap E^i_{\xi_n})\\ &= \sum_{i\in I}\mu_i(S_i)\overline{\mu}(A_i)2^{-n}\\ &=\Big(\sum_{i\in I}\mu_i(A_i)\Big)2^{-n}\\ &= \mu(A)\prod_{j=1}^{n}\mu(E_{\xi_j}).\end{aligned}$$

Thus, A and the sets E_ξ, $\xi \in \kappa\setminus D_A$, form a stochastically independent family in Σ. □

Lemma 3. *Let X be an uncountable set, and let $\overline{\nu}$ be the product measure on $(\{0,1\}^{\mathbb{N}})^X$ obtained by giving each factor $\{0,1\}^{\mathbb{N}}$ its usual measure $\nu_{\mathbb{N}}$. For each $i \in \mathbb{N}$ and each $x \in X$, let*

$$K^i_x = \left\{y \in (\{0,1\}^{\mathbb{N}})^X : y^i(x) = 1\right\}.$$

Finally, let Y be a subset of $(\{0,1\}^{\mathbb{N}})^X$ with full outer measure for $\overline{\nu}$, and let ν be the subspace measure on Y induced by $\overline{\nu}$. Then:

(i) *Let* T *be the domain of ν and set $H^i_x = K^i_x \cap Y$ for $i \in \mathbb{N}$ and $x \in X$. Then:*

  (1) *For each $i \in \mathbb{N}$ and each $x \in X$, $H^i_x \in \mathrm{T}$ and $\nu(H^i_x) = 1/2$.*

  (2) *Given any $B \in \mathrm{T}$, there is countable set $J_B \subset X$ such that B and the sets H^i_x, $i \in \mathbb{N}$, $x \in X\setminus J_B$, form a stochastically independent family in* T.

(ii) *Let ν' be the image measure of ν under the inclusion of Y into $(\{0,1\}^{\mathbb{N}})^X$, and* T' *its domain. Then:*

  (1) *For each $i \in \mathbb{N}$ and each $x \in X$, $K^i_x \in \mathrm{T}'$ and $\nu'(K^i_x) = 1/2$.*

  (2) *Given any $B \in \mathrm{T}'$, there is countable set $J_B \subset X$ such that B and the sets K^i_x, $i \in \mathbb{N}$, $x \in X\setminus J_B$, form a stochastically independent family in* T'.

Proof. Write $\overline{\mathrm{T}}$ for the domain of $\overline{\nu}$. Note first that the family $\langle K_x^i \rangle_{x \in X, i \in \mathbb{N}}$ is a stochastically independent family in $\overline{\mathrm{T}}$ with $\overline{\nu}(K_x^i) = 1/2$ for each $x \in X$ and $i \in \mathbb{N}$ (which follows directly from the definition of product measure). Next note that if E and F are elements of $\overline{\mathrm{T}}$ such that E is determined by coordinates in some subset $J \subset X$, and F by coordinates in the complement $X \setminus J$, then E and F are stochastically independent.[12] Also note that if C is any element of $\overline{\mathrm{T}}$, there is a $C' \in \overline{\mathrm{T}}$ which differs from C by a null set and is determined by coordinates in some countable $J \subset X$. Combining these three facts, we can see that given any $C \in \overline{\mathrm{T}}$, there is a countable $J \subset X$ such that C and the sets K_x^i, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in $\overline{\mathrm{T}}$.

It is now straightforward to see that (i) and (ii) of the lemma hold. Indeed, fix any $x \in X$ and $i \in \mathbb{N}$. Since $K_x^i \in \overline{\mathrm{T}}$, we have $H_x^i \in \mathrm{T}$ and therefore also $K_x^i \in \mathrm{T}'$. Since $\overline{\nu}(K_x^i) = 1/2$ and Y has full outer measure for $\overline{\nu}$, it follows that $\nu(H_x^i) = 1/2$ and from this that $\nu'(K_x^i) = 1/2$. Thus (i)(1) and (ii)(1) hold.

As for (i)(2), pick any $B \in \mathrm{T}$. For some $C \in \overline{\mathrm{T}}$, $B = C \cap Y$ and $\overline{\nu}(C) = \nu(B)$ (by the definition of T and since Y has full outer measure for $\overline{\nu}$). From above, there is a countable set $J \subset X$ such that C and the sets K_x^i, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in $\overline{\mathrm{T}}$. Let L be any non-empty finite subset of $\mathbb{N} \times (X \setminus J)$. Then, using the fact that Y has full outer measure for $\overline{\nu}$,

$$\begin{aligned}
\nu\Big(B \cap \bigcap_{(i,x) \in L} H_x^i\Big) &= \nu\Big(\Big(C \cap \bigcap_{(i,x) \in L} K_x^i\Big) \cap Y\Big) \\
&= \overline{\nu}\Big(C \cap \bigcap_{(i,x) \in L,} K_x^i\Big) \\
&= \overline{\nu}(C) \prod_{(i,x) \in L,} \overline{\nu}(K_x^i) \qquad \text{because } L \subset \mathbb{N} \times (X \setminus J) \\
&= \nu(B) \prod_{(i,x) \in L} \nu(H_x^i).
\end{aligned}$$

It follows that B and the sets H_x^i, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T. Thus (i)(2) holds.

Finally, consider any $B \in \mathrm{T}'$. By the definition of T', $B \cap Y \in \mathrm{T}$. Hence, from the previous paragraph, there is a countable set $J \subset X$ such that $B \cap Y$ and the sets H_x^i, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T. Let L be any non-empty finite subset of $\mathbb{N} \times (X \setminus J)$. Then, by definition of ν',

$$\begin{aligned}
\nu'\Big(B \cap \bigcap_{(i,x) \in L} K_x^i\Big) &= \nu\Big(\Big(B \cap \bigcap_{(i,x) \in L} K_x^i\Big) \cap Y\Big) \\
&= \nu\Big((B \cap Y) \cap \bigcap_{(i,x) \in L,} H_x^i\Big) \\
&= \nu(B \cap Y) \prod_{(i,x) \in L,} \nu(H_x^i) = \nu'(B) \prod_{(i,x) \in L} \nu'(K_x^i).
\end{aligned}$$

[12] This follows e.g. from the general fact that if $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of probability spaces and J is any subset of I then the product measure on $\prod_{i \in I} X_i$ can be identified with the product of the product measures on $\prod_{i \in J} X_i$ and $\prod_{i \in I \setminus J} X_i$ via the bijection $x \mapsto (x \restriction J, x \restriction X \setminus J)$ where $x \in \prod_{i \in I} X_i$; for this fact, see Fremlin (2001, Theorem 254N).

It follows that B and the sets K_x^i, $i \in \mathbb{N}$, $x \in X \setminus J$, form a stochastically independent family in T'. Thus (ii)(2) holds. $\square$

5.2 Proof of Theorem 1

Since (X, Σ, μ) is super-atomless, and since for any infinite cardinal κ there is a bijection between κ and $\kappa \times \mathbb{N}$, Lemma 2 implies that we may select an uncountable cardinal κ and a family $\langle E_\xi^i \rangle_{\xi<\kappa, i\in\mathbb{N}}$ in Σ, with $\mu(E_\xi^i) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $A \in \Sigma$ there is a countable set $J_A \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_A$, A and the sets E_ξ^i, $i \in \mathbb{N}$, form a stochastically independent family in Σ.

For each $\xi < \kappa$, define a function y_ξ from X to $\{0, 1\}^{\mathbb{N}}$ by setting

$$y_\xi^i(x) = \begin{cases} 1 & \text{if } x \in E_\xi^i \\ 0 & \text{if } x \notin E_\xi^i \end{cases}$$

for $i \in \mathbb{N}$ and $x \in X$. Attach a countably infinite subset $D_\xi \subset X$ to each $\xi < \kappa$ in such a way that for each countably subset $D \subset X$ there is a $\xi < \kappa$ such that $D \cap D_\xi = \emptyset$. (Since both X and κ are uncountable, this is possible. Indeed, X being uncountable implies that we may select a disjoint family $\langle D_i \rangle_{i\in I}$ of countably infinite subsets of X such that $\#(I) = \omega_1$. Now since κ is uncountable, there is a surjection from κ onto I, say ϕ. Let $D_\xi = D_{\phi(\xi)}$.)

Now for each $\xi < \kappa$ let

$$N_\xi = \left\{ y \in \left(\{0,1\}^{\mathbb{N}} \right)^X : \begin{array}{l} \text{there is a null set } N \subset X \text{ such that} \\ \qquad y \restriction X \setminus N = y_\xi \restriction X \setminus N \text{ and } N \cap D_\xi = \emptyset \end{array} \right\}$$

and then let $Y = \bigcup_{\xi<\kappa} N_\xi$. Let $\overline{\nu}$ be the product measure on $(\{0,1\}^{\mathbb{N}})^X$, giving each copy of $\{0, 1\}^{\mathbb{N}}$ its usual measure $\nu_{\mathbb{N}}$. Note that for each $\xi < \kappa$, N_ξ is a $\overline{\nu}$-null set, since all of its elements agree on the infinite set D_ξ. On the other hand, Y has full outer measure for $\overline{\nu}$. Indeed, let W be any non-negligible $\overline{\nu}$-measurable subset of $(\{0,1\}^{\mathbb{N}})^X$. Then $W \supset W'$ for some non-empty subset W' of $(\{0,1\}^{\mathbb{N}})^X$ which is determined by coordinates in some countable subset of X, say J. By construction, there is a $\xi < \kappa$ such that $J \cap D_\xi = \emptyset$. Since the countable set J is a null set in X, it follows that, for such a ξ, the set

$$\left\{ y \in \left(\{0,1\}^{\mathbb{N}} \right)^X : y \restriction X \setminus J = y_\xi \restriction X \setminus J \right\}$$

is included in N_ξ and intersects the set W'. Thus Y intersects every non-negligible $\overline{\nu}$-measurable subset of $(\{0,1\}^{\mathbb{N}})^X$, i.e., Y has full outer measure for $\overline{\nu}$.

Let ν be the subspace measure on Y induced by $\overline{\nu}$, and T its domain. Then since Y has full outer measure for $\overline{\nu}$, (Y, T, ν) is a probability space. Note also that for each $\xi < \kappa$, N_ξ is a ν-null set in Y. Hence for any $A \in \Sigma$, $\bigcup_{\xi\in J_A} N_\xi$ is a ν-null set in Y since J_A is countable.

For each $i \in \mathbb{N}$ let

$$H^i = \{(x, y) \in X \times Y : y^i(x) = 1\}.$$

We may assume that the σ-algebra Σ is complete.[13] Then Lemma 1 applies to the sequence $\langle H^i \rangle_{i \in \mathbb{N}}$. Indeed, note that by construction, for any $y \in Y$ there is a $\xi < \kappa$ such that for each $i \in \mathbb{N}$ the section H^i_y differs from E^i_ξ by a null set. By the choice of the family $\langle E^i_\xi \rangle_{\xi < \kappa, i \in \mathbb{N}}$, it follows that for each $y \in Y$ and $i \in \mathbb{N}$, H^i_y belongs to Σ, with $\mu(H^i_y) = 1/2$, and that given any $A \in \Sigma$ and any $y \in Y$, if y does not belong to the null set $\bigcup_{\xi \in J_A} N_\xi$ then A and the sections H^i_y, $i \in \mathbb{N}$, form a stochastically independent family in Σ. Thus (b) and (d) of Lemma 1 hold for the family $\langle H^i \rangle_{i \in \mathbb{N}}$. By Lemma 3(i), (a) and (c) of Lemma 1 hold, too. Thus, by Lemma 1, the product measure corresponding to μ and ν has a rich Fubini extension. This completes the proof. □

5.3 Proof of Remark 1

In the proof of Theorem 1, define the sets N_ξ, $\xi < \kappa$, alternatively as

$$N_\xi = \Big\{ y \in \left(\{0,1\}^{\mathbb{N}}\right)^X : \text{there is a countable } D \subset X \text{ such that } y \restriction X \setminus D = y_\xi \restriction X \setminus D \text{ and } D \cap D_\xi = \emptyset \Big\}.$$

Observe that the arguments of the proof of Theorem 1 continue to hold with this new definition of the sets N_ξ. Now if $\#(X) \leq \mathfrak{c}$, then the set of all countable subsets of $X \setminus D_\xi$ has cardinal $\mathfrak{c}$ (note that $X \setminus D_\xi$ is an infinite set in any case), and it follows that $\#(N_\xi) = \mathfrak{c}$ for each $\xi < \kappa$, under the new definition of N_ξ. Clearly, we may choose κ in the proof of Theorem 1 so as to have $\kappa \leq \mathfrak{c}$. But if $\kappa \leq \mathfrak{c}$ and $\#(N_\xi) = \mathfrak{c}$ for each $\xi < \kappa$, then we have $\#(Y) = \mathfrak{c}$, by the definition of Y as $Y = \bigcup_{\xi < \kappa} N_\xi$. □

5.4 Proof of Theorem 2

As in the proof of Theorem 1, let $\overline{\nu}$ denote the product measure on $(\{0,1\}^{\mathbb{N}})^X$ when each factor $\{0,1\}^{\mathbb{N}}$ is given its usual measure. Construct a subset Y of $(\{0,1\}^{\mathbb{N}})^X$ in the same way as in the proof of Theorem 1, and then define the probability measure ν on Y as in the proof of Theorem 1. Let ν' denote the image measure of ν under the inclusion of Y into $(\{0,1\}^{\mathbb{N}})^X$, and let T' denote the domain of ν'. Observe that ν' extends the product measure $\overline{\nu}$. For each $i \in \mathbb{N}$ let

$$K^i = \left\{ (x, y) \in X \times \left(\{0,1\}^{\mathbb{N}}\right)^X : y^i(x) = 1 \right\}.$$

As in the proof of Theorem 1, we may assume the σ-algebra Σ on X to be complete. Then Lemma 1—with $\left((\{0,1\}^{\mathbb{N}})^X, \mathrm{T}', \nu'\right)$ in place of (Y, T, ν)—applies to

[13] Note that in Definition 1, only the completions of the factor spaces matter.

the family $\langle K^i\rangle_{i\in\mathbb{N}}$. To see this, observe that the complement of Y in $(\{0,1\}^{\mathbb{N}})^X$ and the sets N_ξ, $\xi < \kappa$, appearing in the construction of Y are ν'-null sets and conclude from this that (b) and (d) of Lemma 1 hold for the family $\langle K^i\rangle_{i\in\mathbb{N}}$ (cf. the last paragraph of the proof of Theorem 1). From Lemma 3(ii) it may be seen that (a) and (c) of Lemma 1 hold for the family $\langle K^i\rangle_{i\in\mathbb{N}}$. Thus, by Lemma 1, the product measure corresponding to μ and ν' has a rich Fubini extension $\overline{\lambda}$ whose domain $\overline{\Lambda}$ contains the sets K^i, $i \in \mathbb{N}$. In particular, the function f defined in the statement of the theorem is $\overline{\Lambda}$-measurable. Finally, since ν' is an extension of $\overline{\nu}$, it is plain that (b) in the statement of the theorem holds. This completes the proof. □

5.5 Proof of Theorem 3

Suppose the product measure corresponding to μ and ν has a rich Fubini extension, with domain $\overline{\Lambda}$ say. We may assume that the σ-algebras Σ and T are complete. Then, by Definitions 1, 2, and 4, there are an element $H \in \overline{\Lambda}$ and null sets $N^X \subset X$ and $N^Y \subset Y$ such that (a) for each $x \in X \setminus N^X$ the section H_x is a member of T with $\nu(H_x) = 1/2$, (b) given any $x \in X \setminus N^X$ we have $\nu(H_x \cap H_{x'}) = 1/4$ for almost all $x' \in X \setminus N^X$, and (c) for each $y \in Y \setminus N^Y$ the section H_y is a member of Σ.

Then by Sun (2006, Theorem 2.8) it follows that given any $A \in \Sigma$, there is a null set $N_A \subset Y$ such that $\mu(H_y \cap A) = (1/2)\mu(A)$ for all $y \in Y \setminus N_A$. In particular, then, given any $A \in \Sigma$ and any $y \in Y \setminus N_A$, there is a null set $N_{y,A} \subset Y$ such that $\mu(H_{y'} \cap (H_y \cap A)) = (1/2)\mu(H_y \cap A)$ for all $y' \in Y \setminus N_{y,A}$. Thus, given $A \in \Sigma$, if $y \in Y \setminus N_A$ and $y' \in Y \setminus N_{y,A}$, then $\mu(H_{y'} \cap H_y \cap A) = (1/4)\mu(A)$.

Taking $A = X$, the previous paragraph shows in particular that each $y \in Y$ is contained in some null set of Y, i.e. Y can be covered by some family of ν-null sets. Set $\alpha = \operatorname{cov}\mathcal{N}(\nu)$.

Fix any $A \in \Sigma$ with $\mu(A) > 0$. By transfinite induction, choose a family $\langle y_\xi\rangle_{\xi<\alpha}$ in Y as follows. Let y_0 be an arbitrarily element of $Y \setminus N_A$. Given that $\langle y_\eta\rangle_{\eta<\xi}$ has been chosen, where $\xi < \alpha$, let y_ξ be chosen in $Y \setminus (N_A \cup \bigcup_{\eta<\xi} N_{y_\eta,A})$. Such a choice is possible for each $\xi < \alpha$ because $\xi < \alpha = \operatorname{cov}\mathcal{N}(\nu)$ implies $Y \setminus (N_A \cup \bigcup_{\eta<\xi} N_{y_\eta,A}) \neq \emptyset$.

Then for any two ordinals $\xi, \xi' < \alpha$ with $\xi \neq \xi'$, we have

$$\begin{aligned}\mu\big((H_{y_\xi} \cap A) \cap (H_{y_{\xi'}} \cap A)\big) &= \mu(H_{y_\xi} \cap H_{y_{\xi'}} \cap A)\\ &= \frac{1}{4}\mu(A)\\ &= \frac{1}{2}\mu(H_{y_\xi} \cap A) = \frac{1}{2}\mu(H_{y_{\xi'}} \cap A)\end{aligned}$$

whence $\mu\big((H_{y_\xi} \cap A) \triangle (H_{y_{\xi'}} \cap A)\big) = (1/2)\mu(A)$. Thus since $\mu(A) > 0$, writing $(\mathfrak{A}, \hat{\mu})$ for the measure algebra of μ, and $\mathfrak{A}_A$ for the principal ideal of $\mathfrak{A}$ determined by A, $\mathfrak{A}_A$ has a subset that is discrete for the measure metric of $(\mathfrak{A}, \hat{\mu})$

and has cardinal α.[14] In particular, the Maharam type of μ cannot be finite, and hence by Fremlin (2002, 323A(d), and 2005, 524D) it follows, considering $(\mathfrak{A}_A, \hat{\mu}\restriction\mathfrak{A}_A)$ as a measure algebra in its own right, that the Maharam type of $\mathfrak{A}_A$ is, in fact, at least α. Thus (b) of the theorem holds.

As for (a), note that for each $A \in \Sigma$ and $B \in \mathrm{T}$ we have $(A \times B) \cap H \in \overline{\Lambda}$ and hence, by the Fubini property, $\int_A \nu(H_x \cap B)\, d\mu(x) = \int_B \mu(H_y \cap A)\, d\nu(y)$. From the second paragraph of this proof, $\int_B \mu(H_y \cap A)\, d\nu(y) = (1/2)\mu(A)\nu(B)$ for each $A \in \Sigma$ and $B \in \mathrm{T}$. Consequently, for each fixed $B \in \mathrm{T}$,

$$\int_A \nu(H_x \cap B)\, d\mu(x) = \frac{1}{2}\nu(B)\mu(A) \text{ for all } A \in \Sigma.$$

Hence, for each $B \in \mathrm{T}$ there is a null set $N_B \subset X$ such that $\nu(H_x \cap B) = (1/2)\nu(B)$ for all $x \in X \setminus N_B$. From this it follows that (a) of the theorem holds, using an argument analogous to that which had led to (b) of the theorem. □

Remark 2. The above proof shows in particular that a rich Fubini extension must be a proper extension of the product measure in question. Indeed, in the notation of that proof, for any null set $N \subset Y$ let

$$K_{Y\setminus N} = \{a \in \mathfrak{A}\text{: there is a } y \in Y \setminus N \text{ such that } a \text{ is determined by } H_y\}.$$

Further, let Λ denote the domain of the product measure on $X \times Y$ that is given in the context of the above proof. By a standard fact, were H an element of Λ, then there would be a null set $N \subset Y$ such that $K_{Y\setminus N}$ were a separable subset of $\mathfrak{A}$ for the measure metric on $\mathfrak{A}$ (see Fremlin, 2003, 418S, and 2002, 367R). Now observe that in the construction in the fourth paragraph of the above proof, N_A may be replaced by any null set $N \subset Y$ with $N \supset N_A$. But this implies that, given any null set $N \subset Y$, the set $\mathfrak{A}_A$ in the fifth paragraph of that proof has an uncountable subset that is discrete for the measure metric and such that each of its elements is determined by a section H_y with $y \in Y \setminus N$. Thus, taking $A = X$, we can see that for any null set $N \subset Y$, $K_{Y\setminus N}$ is non-separable for the measure metric on $\mathfrak{A}$. We may conclude that H cannot be an element of Λ.

5.6 Proof of Theorem 4

As $\alpha \geq \kappa$, and since there is a bijection between κ and $\kappa \times \mathbb{N}$, using Lemma 2 we may select a family $\langle E^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$ in Σ, with $\mu(E^i_\xi) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $A \in \Sigma$ there is a countable set $J_A \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_A$, A and the sets E^i_ξ, $i \in \mathbb{N}$, form a stochastically independent family in Σ. Similarly, as $\beta \geq \kappa$, we may select a family $\langle F^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$ in T, with $\nu(F^i_\xi) = 1/2$ for each $\xi < \kappa$ and $i \in \mathbb{N}$, such that given any $B \in \mathrm{T}$ there is a

[14] Recall that if (Z, Υ, ρ) is a finite measure space and $(\mathfrak{C}, \hat{\rho})$ its measure algebra, the measure metric on $\mathfrak{C}$ is just the metric that assigns, to every pair $E^\bullet$, $F^\bullet$ of elements of $\mathfrak{C}$, the number $\rho(E \triangle F)$ where E and F are any elements of Υ determining $E^\bullet$ and $F^\bullet$, respectively.

countable set $J_B \subset \kappa$ such that for each $\xi < \kappa$ with $\xi \notin J_B$, B and the sets F^i_ξ, $i \in \mathbb{N}$, form a stochastically independent family in T.

For each $\xi < \kappa$ set $M'_\xi = M_\xi \setminus \bigcup_{\eta<\xi} M_\eta$ and $N'_\xi = N_\xi \setminus \bigcup_{\eta<\xi} N_\eta$. Then $\langle M'_\xi \rangle_{\xi<\kappa}$ is a disjoint family of null sets in X which covers X, and $\langle N'_\xi \rangle_{\xi<\kappa}$ a disjoint family of null sets in Y which covers Y. For each $i \in \mathbb{N}$ set

$$H^i = \Big(\bigcup_{\xi<\kappa} M'_\xi \times (F^i_\xi \setminus N_\xi) \Big) \cup \Big(\bigcup_{\xi<\kappa} (E^i_\xi \setminus M_\xi) \times N'_\xi \Big).$$

We want to see that Lemma 1 applies to the family $\langle H^i \rangle_{i\in\mathbb{N}}$. To this end, for each $x \in X$ let ξ_x be the least ordinal $\xi < \kappa$ such that $x \in M_\xi$. Thus ξ_x is also the uniquely determined ordinal $\xi < \kappa$ such that $x \in M'_\xi$. Observe that for each $x \in X$ and each $i \in \mathbb{N}$ the section H^i_x satisfies

$$F^i_{\xi_x} \setminus N_{\xi_x} \subset H^i_x \subset F^i_{\xi_x} \cup N_{\xi_x} .$$

Thus for each $x \in X$ and each $i \in \mathbb{N}$, H^i_x differs from $F^i_{\xi_x}$ by a null set. We may assume that T is complete. Then by the choice of the family $\langle F^i_\xi \rangle_{\xi<\kappa, i\in\mathbb{N}}$, it follows that for each $x \in X$ and $i \in \mathbb{N}$, H^i_x belongs to T, with $\nu(H^i_x) = 1/2$, and that given any $B \in$ T and any $x \in X$, if $\xi_x \notin J_B$—where J_B is the countable subset of κ that was associated with B at the beginning of this proof—then B and the sections H^i_x, $i \in \mathbb{N}$, form a stochastically independent family in T; that is, B and the sections H^i_x, $i \in \mathbb{N}$, form a stochastically independent family in T whenever x does not belong to the null set $\bigcup_{\xi\in J_B} M'_\xi$. Thus (a) and (c) of Lemma 1 hold. Similarly it follows that (b) and (d) of Lemma 1 hold. Thus, by Lemma 1, the product measure corresponding to μ and ν has a rich Fubini extension. This completes the proof. □

5.7 Proof of Theorem 5

Let $\mathfrak{c}$ denote the cardinal of the continuum. By Theorem 4, it suffices to show that if Z is any Polish space and ν an atomless Borel probability measure on Z, then there is an extension of ν to a measure ν' on Z such that ν' is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$ and such that there is a non-decreasing family $\langle N_\xi \rangle_{\xi<\mathfrak{c}}$ of ν'-null sets which covers Z, i.e. such that $\bigcup_{\xi<\mathfrak{c}} N_\xi = Z$.

To this end, note first that if I is any infinite set with $\#(I) \le \mathfrak{c}$ then there is a subset $A \subset \{0,1\}^I$, with $\#(A) = \mathfrak{c}$, such that A has full outer measure for the usual measure ν_I on $\{0,1\}^I$(see Fremlin, 2005, 523B together with 523D(d)).

Now consider $\{0,1\}^{\mathfrak{c}}$ with its usual measure $\nu_{\mathfrak{c}}$. Fix any $\overline{x} \in \{0,1\}^{\mathfrak{c}}$. For each $\xi < \mathfrak{c}$, let $J_\xi = \{\eta < \mathfrak{c} \colon \eta \le \xi\}$. By the fact stated in the previous paragraph, for each $\xi < \mathfrak{c}$ we may choose a set $N'_\xi \subset \{0,1\}^{\mathfrak{c}}$ so that (a) $x \restriction \mathfrak{c} \setminus J_\xi = \overline{x} \restriction \mathfrak{c} \setminus J_\xi$ for each $x \in N'_\xi$, (b) N'_ξ intersects every non-negligible measurable subset of $\{0,1\}^{\mathfrak{c}}$ which is determined by coordinates in J_ξ, and (c) $\#(N'_\xi) = \mathfrak{c}$ if ξ is infinite. For each $\xi < \mathfrak{c}$, let $N_\xi = \bigcup_{\eta\le\xi} N'_\eta$. Then $\langle N_\xi \rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of subsets of $\{0,1\}^{\mathfrak{c}}$ such that N_ξ is finite if ξ is finite, and $\#(N_\xi) = \mathfrak{c}$ for each infinite $\xi < \mathfrak{c}$.

Let $Y = \bigcup_{\xi<\mathfrak{c}} N_\xi$. Then $\#(Y) = \mathfrak{c}$. Since $\mathfrak{c}$ has uncountable cofinality, (b) implies that Y has full outer measure for $\nu_\mathfrak{c}$ (because every non-negligible measurable subset of $\{0,1\}^\mathfrak{c}$ includes a non-negligible measurable subset of $\{0,1\}^\mathfrak{c}$ which is determined by coordinates in some countable set $J \subset \mathfrak{c}$). Finally, because of (a), N_ξ is a $\nu_\mathfrak{c}$-null set in $\{0,1\}^\mathfrak{c}$ for each $\xi < \mathfrak{c}$.

Let μ denote Lebesgue measure on $[0,1]$ and let λ be the product measure on $\{0,1\}^\mathfrak{c} \times [0,1]$ corresponding to $\nu_\mathfrak{c}$ and μ. By Fremlin (2005, 334X(g)), λ is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$. Now since $\#(Y) = \mathfrak{c}$ and Y has full outer measure for $\nu_\mathfrak{c}$, the arguments in the proof of Proposition 521P(b) in Fremlin (2005) show that there is a subset $C \subset Y \times [0,1] \subset \{0,1\}^\mathfrak{c} \times [0,1]$ such that

(1) C has full outer measure for λ;

(2) the subspace measure λ_C on C induced by λ is countably separated.

(1) implies that λ_C is a probability measure on C and that the measure algebra of λ_C can be identified with that of λ. Thus, as λ is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$, so is λ_C. In particular, λ_C is atomless.

Observe that $\langle N_\xi \times [0,1]\rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of λ-null sets in $\{0,1\}^\mathfrak{c} \times [0,1]$ whose union is $Y \times [0,1]$. Thus setting $M_\xi = C \cap (N_\xi \times [0,1])$ for each $\xi < \mathfrak{c}$, we obtain a non-decreasing family $\langle M_\xi\rangle_{\xi<\mathfrak{c}}$ of λ_C-null sets which covers C.

Now let Z be any Polish space, and ν an atomless Borel probability measure on Z. Then, since λ_C is atomless, (2) implies that there is an injection $\phi\colon C \to Z$ which is inverse-measure-preserving for λ_C and ν. To see this, note first that (2) means there is an injection $\phi_1\colon C \to \mathbb{R}$ which is measurable for the domain of λ_C and the Borel sets of $\mathbb{R}$. Let ν_1 be the Borel measure on $\mathbb{R}$ given by setting $\nu_1(B) = \lambda_C(\phi_1^{-1}(B))$ for each Borel set B in $\mathbb{R}$. Since λ_C is atomless and ϕ_1 is an injection, ν_1 is zero on singletons and therefore atomless because Z is a separable and metrizable topological space. Now by a standard fact, since both $\mathbb{R}$ and Z are Polish spaces, and both ν_1 and ν are atomless Borel measures, there is a bijection $\phi_2\colon \mathbb{R} \to Z$ which is inverse-measure-preserving for ν_1 and ν in both directions (see Fremlin, 2003, 433X(f)). Set $\phi = \phi_2 \circ \phi_1$.

Let ν' be the image measure of λ_C under ϕ. Then, because ϕ is an injection, ϕ induces an isomorphism between the measure algebras of λ_C and ν'. Hence, as λ_C is Maharam-type-homogeneous with Maharam type $\mathfrak{c}$, so is ν'. Of course, ν' is an extension of ν, since ϕ is inverse-measure-preserving for λ_C and ν. Finally, if we set $M'_\xi = \phi(M_\xi) \cup (Z \setminus \phi(C))$ then, again by the fact that ϕ is an injection, $\langle M'_\xi\rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family of ν'-null sets which covers Z. This completes the proof. □

Appendix

In this appendix, we recall some basic terminology concerning measure algebras. Let (X, Σ, μ) be a measure space, and let $\mathcal{N}(\mu)$ denote the ideal of null sets in X.

(a) The *measure algebra* of (X, Σ, μ) (or, for short, of μ) is the pair $(\mathfrak{A}, \hat{\mu})$ given as follows:

- $\mathfrak{A}$ is the quotient Boolean algebra $\Sigma/(\mathcal{N}(\mu) \cap \Sigma)$. That is, denoting by $\sim$ the equivalence relation on Σ given by $E \sim F$ if and only if $E \triangle F \in \mathcal{N}(\mu)$, $\mathfrak{A}$ is the set of equivalence classes in Σ for $\sim$, endowed with binary operations $\cap^\bullet$, $\cup^\bullet$, $\setminus^\bullet$, $\triangle^\bullet$, and a partial ordering $\subset^\bullet$, inherited from Σ as follows: If $E^\bullet, F^\bullet \in \mathfrak{A}$ and E, F are any elements of Σ determining $E^\bullet$ and $F^\bullet$, respectively, then $E^\bullet \subset^\bullet F^\bullet$ if and only if $E \setminus F \in \mathcal{N}(\mu)$, $E^\bullet \cap^\bullet F^\bullet = (E \cap F)^\bullet$, and analogously for $\cup^\bullet$, $\setminus^\bullet$, and $\triangle^\bullet$.
- $\hat{\mu}\colon \mathfrak{A} \to [0, \infty]$ is the functional given by $\hat{\mu}(E^\bullet) = \mu(E)$ where E is any element of Σ determining $E^\bullet$.

(b) A *principal ideal* of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ of the form $\{b \in \mathfrak{A}\colon b \subset^\bullet a\}$ where $a \in \mathfrak{A}$; it is called a non-zero principal ideal of $\mathfrak{A}$ if $\hat{\mu}(a) > 0$. Observe that any principal ideal of $\mathfrak{A}$, with the binary operations and the partial ordering inherited from $\mathfrak{A}$, is a Boolean algebra in its own right.

(c) A *subalgebra* of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ that contains $X^\bullet$ (the element of $\mathfrak{A}$ determined by X) and that is closed under $\cup^\bullet$ and $\setminus^\bullet$ (thus also under $\cap^\bullet$ and $\triangle^\bullet$). A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is called *order-closed* if, with respect to $\subset^\bullet$, any non-empty upwards directed subset of $\mathfrak{B}$ has its supremum in $\mathfrak{B}$ in case the supremum is defined in $\mathfrak{A}$.

(d) The *Maharam type* of $\mathfrak{A}$ is the least cardinal of any subset $B \subset \mathfrak{A}$ which completely generates $\mathfrak{A}$, i.e. of any $B \subset \mathfrak{A}$ such that the smallest order-closed subalgebra of $\mathfrak{A}$ containing B is $\mathfrak{A}$ itself. Similarly, the Maharam type of a principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ is the least cardinal of any subset $B \subset \mathfrak{A}_a$ which completely generates $\mathfrak{A}_a$ (considering $\mathfrak{A}_a$ as a Boolean algebra in its own right).

(e) The Maharam type of the measure space (X, Σ, μ), or of the measure μ, is defined to be the Maharam type of $\mathfrak{A}$.

(f) (X, Σ, μ), or the measure μ, is said to be *Maharam-type-homogeneous* if each non-zero principal ideal of $\mathfrak{A}$ has a Maharam type equal to that of μ.

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