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The Asset Market Game[*]

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Abstract

This paper models asset markets as a game where assets pay according to an arbitrary payoff matrix, investors decide on fractions of wealth to allocate to each asset, and prices result from market clearing. The only pure-strategy Nash equilibrium is to split wealth proportionally to the assets' expected returns, which can be interpreted as investing according to the fundamentals. Further, the equilibrium strategy is evolutionarily stable in the sense of Schaffer (1988). We also study the stability properties of the equilibrium in an evolutionary dynamics where wealth flows with higher probability into those strategies that obtain higher realized payoffs.

Keywords: asset markets, portfolio choice, market efficiency, evolutionary stability.

JEL classification: C72, G11, D83 .

1 Introduction

The classical paradigm in financial markets based on the efficient market hypothesis (Fama (1970)) has been increasingly challenged by the literature on behavioral finance initiated by the work on noise trader risk (De Long et al. (1990)) and by the recent work on evolutionary finance and adaptive belief systems (see e. g. Brock and Hommes (1998)). The efficient market hypothesis is based on the assumption that market participants are fully rational and have rational expectations, implying that equilibrium prices immediately incorporate all available information; the remaining price movements must come from purely random perturbations. This was defended on evolutionary grounds by Fama (1965), who argued that nonrational market participants would be driven out of the market by rational arbitrageurs.

In the present paper we take the evolutionary game theory approach to analyze a financial market. We view financial markets as population games where cash flows into more successful investment strategies. In our opinion evolutionary models are especially well suited for the analysis of complex market environments with frequent interactions, as it is the case in financial markets. Market performance provides a clear fitness criterion on which selective pressures are based.

We start out from the model of a financial market analyzed in Blume and Easley (1992) and Hens and Schenk-Hoppé (2001). There are finitely many states of the world and finitely

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many assets, which pay according to an arbitrary payoff matrix. In particular, we allow for incomplete markets and redundant assets. Investors decide what fraction of their wealth to allocate to each asset. Consumption decisions have already taken place and wealth can only be allocated to invest in financial markets. Asset prices result from a market clearing condition.

In the first part of the paper we consider strategic institutional investors that are aware of the effect of their investment decisions on prices. We show that this *asset market game* has a unique pure-strategy Nash equilibrium which prescribes to split wealth proportionally to the expected returns of the assets. This can be interpreted as investing according to the fundamentals.[1]

Then we move from absolute to relative performance considerations which are distinctive of evolutionary game theory. Investors are not necessarily well informed about the characteristics that define the asset market and do not necessarily behave strategically. Instead they tend to follow strategies that exhibit good performance. We find that, if all investors follow the equilibrium strategy, any one who would experiment with a new investment strategy would not only be worse off in absolute, but also in relative terms. In other words, the equilibrium strategy is evolutionarily stable.[2]

In the second part of the paper we consider an evolutionary dynamics on investment strategies and asset prices. We could think of different investment strategies as mutual funds with different compositions over the same set of assets. We assume that mutual fund flows depend on recent past performance.[3] Specifically, we postulate that wealth flows with higher probability into those strategies that obtained higher realized payoffs. Because the number of shares that can be acquired with a monetary unit depends on prices, total performance will depend both on asset returns and prices. An additional noise parameter models new, exogenous information coming into the system. Although the resulting stochastic process never gets absorbed in any population profile, we show that most of the time, a majority of traders invest according to the equilibrium strategy, and thus prices are close to fundamentals. Further, at any given period, prices are more likely to move towards their fundamental values.

From the technical point of view our dynamics is related to the one proposed by Kirman (1993), who models recruitment processes in ant communities, with the aim of explaining herding behavior in financial markets. In his model, recruitment is based only on random sampling. In contrast, here portfolio updating is based on comparisons of realized payoffs. In particular, we postulate that the probability that some investor switches to a new strategy is proportional to the observed payoff difference. This rule corresponds to the proportional imitation rule studied by Schlag (1998).

Whereas Blume and Easley (1992) and Hens and Schenk-Hoppé (2001) assume that investors hold a fixed portfolio and always reinvest their returns, focusing on long-run wealth accumulation, we let investors change strategy and focus on wealth flows instead. We do this for two reasons. One is that, traditionally, the evolutionary approach relates a strategy's reproductive success to its payoffs in the following sense. A strategy propagates faster—tends to be adopted—if it has higher payoffs. Second, in the case of financial markets,

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[1]Bell and Cover (1980) already model a stock market as a one-shot, two-player, constant-sum game where the goal of each investor is to maximize the probability of outperforming the opponent. They find that the Nash equilibrium involves exchanging the initial endowment for a random amount of wealth, and then investing so as to maximize the expected logarithm of wealth. In their model neither asset prices nor market clearing are taken into account.

[2]The concept of evolutionary stability used here, due to Schaffer (1988), refers to a finite population and differs from the usual concept in evolutionary game theory for a continuum population (cf. Section 4).

[3]Experimental and empirical evidence supporting this assumption can be found in Kliger et al. (2003) and the references therein.

investors are not necessarily interested in long-run wealth accumulation, but in terminal wealth maximization after finitely many periods. Consequently, wealth is constantly flowing into and out of the market.

If the objective is to maximize the long-run accumulated wealth, the appropriate strategy, as shown by Blume and Easley and Hens and Schenk-Hoppé, is one that allocates wealth according to the expected *relative* returns, rather than to expected returns as in our equilibrium strategy. Both strategies coincide only in the case of Arrow securities—when assets pay one unit in a particular state and zero otherwise. In the case of diagonal securities—that pay an arbitrary amount, but only in one state—investing according to expected *relative* returns would mean dividing wealth among assets proportionally to the probability of the corresponding states, independently of the assets returns in those states. Blume and Easley refer to this investment rule as *betting your beliefs*. An implication of our results is that, if all investors bet their beliefs, there are short-term incentives to deviate.

Conceptually our dynamic model is closely related to the work by De Long et al. (1990). They consider an overlapping generations economy with a risky and a riskless asset and two types of traders: noise traders, with wrong beliefs on future returns, and sophisticated traders, who try to take advantage of noise traders by exploiting the arbitrage opportunities that the existence of noise traders creates. They argue that, under certain conditions, the expected returns of noise traders might be larger than those of sophisticated traders. An imitation-based dynamics is postulated where the proportion of noise traders increases whenever their realized returns are larger. Due to the difference in expected returns, in the long run noise traders might come to dominate the market, driving the prices permanently away from fundamentals.

The motivation underlying our dynamics is very similar. Different ways of splitting wealth among assets in our model could also be interpreted as coming from different beliefs about the returns of assets. Our population dynamics can then be viewed as the evolution of the fractions of traders entertaining different beliefs. Fundamentalists split wealth according to the Nash equilibrium strategy, while "noise traders" with wrong beliefs split wealth differently. Fundamentalists in our evolutionary dynamic model, though, are not sophisticated since they simply follow a given strategy independently of the current population profile and any considerations of risk. Our conclusions could then be reinterpreted as follows: while noise traders will survive and increase price volatility, as in De Long et al., fundamentalists will always dominate and prices cannot persistently be away from their fundamental values.

The rest of the paper is organized as follows. Section 2 introduces the basic model. Section 3 identifies the Nash equilibrium of the asset market game. Section 4 examines the evolutionary stability of the equilibrium strategy. Section 5 analyzes the dynamics. Section 6 illustrates the dynamic results in terms of the evolution of asset prices. Finally, Section 7 concludes.

2 Strategic investors

2.1 The Asset Market

There are $S \geq 2$ possible states of the world, $s = 1, \ldots, S$. State s occurs with probability $q_s > 0$, with $\sum_{s=1}^{S} q_s = 1$. We consider a general asset market with $K \geq 2$ assets, $k = 1, \ldots, K$. In state s asset k yields nonnegative payoff $A_k(s) \geq 0$. We assume that

total payoff of all assets is strictly positive in each state; that is

$$\sum_{k=1}^{K} A_k(s) > 0.$$

We also assume that each asset has a strictly positive payoff for some state; that is, for each asset k, there exists at least some state s such that $A_k(s) > 0$. Note that we explicitly allow for incomplete asset markets; for instance, if $S > K$.

There are N players (also called *investors* or *traders*), $i = 1, \ldots, N$. Each of them owns initial wealth $r_0^i > 0$. Total wealth is normalized, so that $\sum_{i=1}^{N} r_0^i = 1$.

Let $\Delta^K = \{z = (z_1, \ldots, z_k) \in \mathbb{R}_+^K \mid \sum_{k=1}^{K} z_k = 1\}$ denote the $(K-1)$-dimensional simplex. Player i must choose an investment strategy, i. e. a vector $\alpha^i = (\alpha_1^i, \ldots, \alpha_K^i) \in \Delta^K$, where α_k^i denotes the fraction of i's wealth allocated to asset k. Given a strategy profile $\overline{\alpha} = (\alpha^1, \ldots, \alpha^N)$, we denote $\overline{\alpha}_k = (\alpha_k^1, \ldots, \alpha_k^N)$ for convenience. The latter gives the proportions of wealth allocated to asset k by all investors.

2.2 Market Clearing and Prices

The Asset Market Game introduced in this section has a structure similar to that of a Strategic Market Game in Shapley and Shubik (1977). There is one unit of each asset k available. Given a strategy profile $\overline{\alpha}$, asset prices are determined according to the market-clearing condition

$$p_k(\overline{\alpha}_k) = \sum_{i=1}^{N} \alpha_k^i r_0^i, \tag{1}$$

provided at least some $\alpha_k^i > 0$. Let $x_k^i(\overline{\alpha}_k)$ denote the number of units of asset k purchased by investor i. If investors allocate positive wealth to any asset, i. e. if $\alpha_k^i > 0$ for any i and k, they receive a number of units of asset k which is computed as the ratio of the amount of wealth allocated to that asset to the price of the asset. If $\alpha_k^i = 0$, it means that investor i does not purchase asset k. That is,

$$x_k^i(\overline{\alpha}_k) = \begin{cases} \frac{\alpha_k^i r_0^i}{p_k(\overline{\alpha}_k)} & \text{if } \alpha_k^i > 0 \\ 0 & \text{if } \alpha_k^i = 0 \end{cases} \tag{2}$$

Provided at least some $\alpha_k^i > 0$, Equations (1) and (2) imply

$$\sum_{i=1}^{N} x_k^i(\overline{\alpha}_k) = 1. \tag{3}$$

Formally, the market-clearing price of any asset k remains undetermined when there is no trade of that asset, that is when $\alpha_k^i = 0$ for all i. In that case, though, the price is not relevant for the assignment of assets to investors, since investors who do not allocate wealth to asset k receive 0 units of that asset. For convenience, we can set $p_k(\overline{\alpha}_k) = 0$ if asset k is not traded. This will turn out to be inconsequential for the analysis.

Note that $(p_1(\overline{\alpha}_1), \ldots, p_K(\overline{\alpha}_K)) \in \Delta^K$, since $p_k(\overline{\alpha}_k) \geq 0$ and

$$\sum_{k=1}^{K} p_k(\overline{\alpha}_k) = \sum_{i=1}^{N} r_0^i \sum_{k=1}^{K} \alpha_k^i = 1. \tag{4}$$

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Note also that, if all investors allocate the same fraction of wealth $z > 0$ to a given asset, its market price will be numerically equal to that fraction:

$$\overline{\alpha}_k = (z, \ldots, z), \ z > 0 \Longrightarrow p_k(\overline{\alpha}_k) = \sum_{i=1}^{N} zr_0^i = z. \tag{5}$$

In that case, investors also acquire a number of units of the asset k numerically equal to their initial wealth. To see this we use (5):

$$\overline{\alpha}_k = (z, \ldots, z), \ z > 0 \Longrightarrow x_k^i(\overline{\alpha}_k) = \frac{zr_0^i}{p_k(\overline{\alpha}_k)} = r_0^i. \tag{6}$$

The functions $x_k^i(\overline{\alpha}_k)$ are discontinuous at the points where $\alpha_k^i = 0$ for all i. (Note that $x_k^i(z, \ldots, z) = r_0^i$ for all $z > 0$, but $x_k^i(0, \ldots, 0) = 0$.) More important is the fact that, if an asset k is currently not being traded, any single player will be able to acquire its entire supply by deviating and allocating any $\varepsilon > 0$ arbitrarily small on k, since this small investment will determine the price of the asset:

$$\alpha_k^i = \varepsilon > 0, \ \alpha_k^j = 0 \ \forall j \neq i \Longrightarrow x_k^i(\overline{\alpha}_k) = \frac{\varepsilon r_0^i}{p_k(\overline{\alpha}_k)} = \frac{\varepsilon r_0^i}{\varepsilon r_0^i} = 1. \tag{7}$$

On the other hand, at all strategy profiles where some opponent j has $\alpha_k^j > 0$—in particular, in the interior of the strategy space Δ^K—the function $x_k^i(\overline{\alpha}_k)$ is twice differentiable. We compute the following for reference.

$$\frac{\partial x_k^i(\overline{\alpha}_k)}{\partial \alpha_k^i} = \frac{r_0^i}{p_k(\overline{\alpha}_k)} \left(1 - x_k^i(\overline{\alpha}_k)\right) \tag{8}$$

$$\frac{\partial^2 x_k^i(\overline{\alpha}_k)}{\partial (\alpha_k^i)^2} = -2 \left(\frac{r_0^i}{p_k(\overline{\alpha}_k)}\right)^2 \left(1 - x_k^i(\overline{\alpha}_k)\right) \tag{9}$$

Provided $\alpha_k^j > 0$ for some $j \neq i$, the right hand side of Equation (8) is strictly positive and that of Equation (9) is strictly negative. That is, holding opponents' investment constant, the number of units of asset k purchased is a strictly increasing and strictly concave function of the fraction of wealth allocated to k. Obviously, increasing the fraction of wealth allocated to any asset would allow investors to increase their holdings if the price of the asset would not change. But the price will increase as a result of the increase in demand. The fact that x_k^i is increasing in α_k^i shows that the negative price effect cannot be so large that investors are left with less units of k after an increase in their fraction of wealth allocated to the asset. Concavity of x_k^i implies, however, that the negative price effect becomes stronger as the fraction of wealth allocated to k keeps on increasing; that is, there are decreasing returns to investment.

2.3 Payoffs

Assume that investors are risk neutral.[4] Given a strategy profile $\overline{\alpha} = (\alpha^1, \ldots, \alpha^N)$, we write $\overline{\alpha} = (\alpha^i, \alpha^{-i})$ following the standard convention in game theory. The expected payoff of player i is

$$\pi^i(\alpha^i, \alpha^{-i}) = \sum_{s=1}^{S} q_s \left(\sum_{k=1}^{K} A_k(s) x_k^i(\overline{\alpha}_k)\right). \tag{10}$$

---

[4]We concentrate here on the case of risk neutrality to make the comparison with the evolutionary results in Section 4 easier. There fitness is based on payoffs and not on utility.

Defining the expected payoff of asset k as

$$E_k = \sum_{s=1}^{S} q_s A_k(s) > 0, \tag{11}$$

Equation (10) can be rewritten as

$$\pi^i(\alpha^i, \alpha^{-i}) = \sum_{k=1}^{K} E_k x_k^i(\overline{\alpha}_k). \tag{12}$$

The tuple $(\Delta^K, r_0^i, \pi^i)_{i=1}^{N}$ defines the *Asset Market Game*.

3   Nash Equilibrium

We aim now at identifying all pure-strategy Nash Equilibria of the Asset Market Game. Due to the discontinuity of the function $x_k^i(\overline{\alpha}_k)$ when k is not traded, the analysis must avoid differential calculus at this point. Therefore, we proceed as follows. First, we show that at any Nash equilibrium all assets must be traded. This implies that all asset prices are strictly positive in equilibrium. Then, we can proceed to find the value of equilibrium prices. Finally, we will find the equilibrium allocation of wealth to assets.

The intuition why all assets must be traded in equilibrium was already mentioned. If any asset is currently not traded, an investor reallocating any fraction of wealth to invest in the non-traded asset will determine its price. Since all assets have positive expected return, this deviation is profitable for a small enough reallocation of wealth (and hence asset price).

Lemma 1. *If $\overline{\alpha}$ is a Nash equilibrium, then all assets are traded. Hence, $p_k(\overline{\alpha}_k) > 0$ for all $k = 1, \dots, K$.*

Proof. Suppose $\alpha_{k_1}^i = 0$ for all $i = 1, \dots, N$. Choose any player i and any other asset $k_2 \neq k_1$ such that $\alpha_{k_2}^i > 0$. Hence, $p_{k_2}(\overline{\alpha}_{k_2}) > 0$. Let $0 < \varepsilon < \alpha_{k_2}^i$. If i deviates by investing ε in asset k_1, then $x_{k_1}^i(0, \dots, \varepsilon r_0^i, \dots, 0) = 1$ by (7). Define $\alpha^i(\varepsilon) \in \Delta^K$ such that

$$\alpha_{k_1}^i(\varepsilon) = \varepsilon, \ \ \alpha_{k_2}^i(\varepsilon) = \alpha_{k_2}^i - \varepsilon, \ \ \alpha_k^i(\varepsilon) = \alpha_k^i \ \text{ for any } k \neq k_1, k_2.$$

The payoff of player i when deviating to strategy $\alpha^i(\varepsilon)$ is

$$\pi^i(\alpha^i(\varepsilon), \alpha^{-i}) = \sum_{k \neq k_1, k_2} E_k x_k^i(\overline{\alpha}_k) + E_{k_1} + E_{k_2} \frac{\left(\alpha_{k_2}^i - \varepsilon\right) r_0^i}{p_{k_2}(\overline{\alpha}_{k_2}) - \varepsilon r_0^i}. \tag{13}$$

The deviation is profitable if and only if

$$\pi^i(\alpha^i(\varepsilon), \alpha^{-i}) > \pi^i(\alpha^i, \alpha^{-i}) \iff E_{k_1} + E_{k_2} \frac{\left(\alpha_{k_2}^i - \varepsilon\right) r_0^i}{p_{k_2}(\overline{\alpha}_{k_2}) - \varepsilon r_0^i} > E_{k_2} \frac{\alpha_{k_2}^i r_0^i}{p_{k_2}(\overline{\alpha}_{k_2})} \tag{14}$$

which holds by continuity for ε small enough, since $E_{k_1} > 0$. ∎

We perform a preliminary computation in order to study the set of Nash equilibria. Consider any strategy profile $\overline{\alpha}$ where all assets are traded, implying $p_k(\overline{\alpha}_k) > 0$ for all k. Suppose that player i, currently investing a strictly positive fraction of wealth in asset k_2,

were to reallocate a fraction $0 \leq \varepsilon < \alpha_{k_2}^i$ to asset k_1. As we did in the previous proof, we define $\alpha^i(\varepsilon) \in \Delta^K$ such that

$$
\begin{aligned}
\alpha_{k_1}^i(\varepsilon) &= \alpha_{k_1}^i + \varepsilon \\
\alpha_{k_2}^i(\varepsilon) &= \alpha_{k_2}^i - \varepsilon \\
\alpha_k^i(\varepsilon) &= \alpha_k^i \text{ for any } k \neq k_1, k_2
\end{aligned} \tag{15}
$$

Given the strategies of all other players, we compute the payoffs to investor i of this deviation:

$$
\pi^i(\alpha^i(\varepsilon), \alpha^{-i}) = \sum_{k \neq k_1, k_2} E_k x_k^i(\overline{\alpha}_k) + E_{k_1} \frac{(\alpha_{k_1}^i + \varepsilon) r_0^i}{p_{k_1}(\overline{\alpha}_{k_1}) + \varepsilon r_0^i} + E_{k_2} \frac{(\alpha_{k_2}^i - \varepsilon) r_0^i}{p_{k_2}(\overline{\alpha}_{k_2}) - \varepsilon r_0^i}. \tag{16}
$$

This expression is a (differentiable) function of ε such that $\pi^i(\alpha^i(0), \alpha^{-i}) = \pi^i(\alpha^i, \alpha^{-i})$. Hence, whenever

$$
\left. \frac{d}{d\varepsilon} \pi^i(\alpha^i(\varepsilon), \alpha^{-i}) \right|_{\varepsilon=0} > 0 \tag{17}
$$

we can conclude that player i has an incentive to deviate by reallocating at least a small fraction of wealth from asset k_2 to k_1. We have that

$$
\left. \frac{d}{d\varepsilon} \pi^i(\alpha^i(\varepsilon), \alpha^{-i}) \right|_{\varepsilon=0} = r_0^i \left[\frac{E_{k_1}}{p_{k_1}(\overline{\alpha}_{k_1})} \left(1 - x_{k_1}^i(\overline{\alpha}_{k_1}) \right) - \frac{E_{k_2}}{p_{k_2}(\overline{\alpha}_{k_2})} \left(1 - x_{k_2}^i(\overline{\alpha}_{k_2}) \right) \right]. \tag{18}
$$

Let us try to interpret Equation (18). Think of an investor considering to reallocate wealth from k_2 to k_1. Given the asset prices at the profile $\overline{\alpha}$,

$$
\frac{E_{k_1}}{p_{k_1}(\overline{\alpha}_{k_1})} - \frac{E_{k_2}}{p_{k_2}(\overline{\alpha}_{k_2})} \tag{19}
$$

is the expected profit of moving one monetary unit from k_2 to k_1. If this difference is positive, this reallocation of wealth has a positive *direct* effect on expected returns. On the other hand, this has the effect of raising the price of asset k_1 and lowering the price of asset k_2. In turn, the number of affordable units of k_1 (resp. k_2) per unit of wealth invested decreases (resp. increases). At the margin, the difference in expected returns due to this change in the holdings of each asset is given by[5]

$$
E_{k_2} \frac{x_{k_2}^i(\overline{\alpha}_{k_2})}{p_{k_2}(\overline{\alpha}_{k_2})} - E_{k_1} \frac{x_{k_1}^i(\overline{\alpha}_{k_1})}{p_{k_1}(\overline{\alpha}_{k_1})}. \tag{20}
$$

For each asset this *indirect* effect is larger the larger the number of units hold, $x_{k_j}^i(\overline{\alpha}_{k_j})$. Equation (18) says that there is an incentive to reallocate wealth from k_2 to k_1 whenever the gains from the direct effect are larger than the losses from the indirect effect through prices.

Equation (18) is the key to show that in any equilibrium asset prices must not only be positive, but they must be numerically equal to the normalized (or relative) expected returns given by

$$
R_k = \frac{E_k}{E} \quad \text{where } E = \sum_{k'=1}^K E_{k'}. \tag{21}
$$

---

[5]The marginal change in the number of affordable shares per monetary unit transferred is given by

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon r_0^i} p_{k_j}(\overline{\alpha}_{k_j}) x_{k_j}^i(\overline{\alpha}_{k_j}) \left(\frac{1}{p_{k_j}(\overline{\alpha}_{k_j}) - \varepsilon r_0^i} - \frac{1}{p_{k_j}(\overline{\alpha}_{k_j})} \right) = \frac{x_{k_j}^i(\overline{\alpha}_{k_j})}{p_{k_j}(\overline{\alpha}_{k_j})}.
$$

Lemma 2. *If $\overline{\alpha}$ is a Nash equilibrium, then $p_k(\overline{\alpha}_k) = R_k$ for all $k = 1, \ldots, K$.*

Proof. Let $\overline{\alpha}$ be a Nash equilibrium. We proceed by contradiction. Note that the vector $(R_1, \ldots, R_K)$ is an element of the simplex Δ^K. Recall also that $(p_1(\overline{\alpha}_1), \ldots, p_K(\overline{\alpha}_K)) \in \Delta^K$ by (4). If these two vectors are not identical, there must exist assets k_1, k_2 such that $R_{k_1} > p_{k_1}(\overline{\alpha}_{k_1})$ and $R_{k_2} < p_{k_2}(\overline{\alpha}_{k_2})$. Given that, by Lemma 1, all prices are strictly positive, it follows that

$$\frac{R_{k_1}}{p_{k_1}(\overline{\alpha}_{k_1})} > 1 \text{ and } \frac{R_{k_2}}{p_{k_2}(\overline{\alpha}_{k_2})} < 1 \tag{22}$$

Let i be a player with $\alpha_{k_2}^i > 0$ and define $\alpha^i(\varepsilon)$ as in (15), with $0 \le \varepsilon < \alpha_{k_2}^i$. Dividing Equation (18) by the constant E and taking (22) into account, we obtain

$$\left(\frac{1}{E}\right) \frac{d}{d\varepsilon} \pi^i(\alpha^i(\varepsilon), \alpha^{-i})\bigg|_{\varepsilon=0} = r_0^i \left[\frac{R_{k_1}}{p_{k_1}(\overline{\alpha}_{k_1})} \left(1 - x_{k_1}^i(\overline{\alpha}_{k_1})\right) - \frac{R_{k_2}}{p_{k_2}(\overline{\alpha}_{k_2})} \left(1 - x_{k_2}^i(\overline{\alpha}_{k_2})\right) \right] >$$
$$r_0^i \left[\left(1 - x_{k_1}^i(\overline{\alpha}_{k_1})\right) - \left(1 - x_{k_2}^i(\overline{\alpha}_{k_2})\right) \right] = r_0^i \left[x_{k_2}^i(\overline{\alpha}_{k_2}) - x_{k_1}^i(\overline{\alpha}_{k_1}) \right] \tag{23}$$

We claim that there exists a player i with $\alpha_{k_2}^i > 0$ such that $x_{k_2}^i(\overline{\alpha}_{k_2}) - x_{k_1}^i(\overline{\alpha}_{k_1}) \ge 0$. If we can establish this, then it follows from (23) that i has an incentive to deviate, a contradiction which completes the proof.

Suppose that $x_{k_2}^i(\overline{\alpha}_{k_2}) - x_{k_1}^i(\overline{\alpha}_{k_1}) < 0$ for all i such that $\alpha_{k_2}^i > 0$. Using (3) and the fact that $x_{k_2}^j(\overline{\alpha}_{k_2}) = 0$ for all j with $\alpha_{k_2}^j = 0$ we reach the following contradiction:

$$1 = \sum_{i=1}^{N} \{x_{k_2}^i(\overline{\alpha}_{k_2}) \mid \alpha_{k_2}^i > 0\} < \sum_{i=1}^{N} \{x_{k_1}^i(\overline{\alpha}_{k_1}) \mid \alpha_{k_2}^i > 0\} \le \sum_{i=1}^{N} x_{k_1}^i(\overline{\alpha}_{k_1}) = 1. \tag{24}$$

$\blacksquare$

The intuition behind Lemma 2 is the following. By Equation (22), whenever an asset is undervalued in the sense that the expected return per monetary unit invested in that asset exceeds one, there must be another asset that is overvalued. In that case, reallocating wealth from the latter to the former yields positive direct gains as measured by Equation (19). Moreover, there must be some investor holding a too strong position on the overvalued asset. For this investor the indirect effect through prices given by Equation (20) cannot offset the direct gains of transferring wealth from the overvalued to the undervalued asset.

Lemma 2 shows that, in equilibrium, prices must be in accordance with the fundamental values of the assets, meaning that they must be numerically equal to the (normalized) expected returns. As a consequence all assets must yield the same expected returns per monetary unit invested. *A priori*, this is not enough to identify the equilibria, because fundamental prices will obtain whenever aggregate investment is in accordance with expected returns in the market-clearing equations (1). That is, different strategy profiles can lead to fundamental prices. We will show, though, that in fact there exists a unique equilibrium in pure strategies where all investors use the following strategy.

$$\alpha^* = (\alpha_1^*, \ldots, \alpha_K^*) \quad \text{with } \alpha_k^* = R_k \text{ for all } k = 1, \ldots, K. \tag{25}$$

In the case of Arrow securities ($S = K$ and $A_k(s) = 1$ if $s = k$, $A_k(s) = 0$ otherwise), α^* reduces to the investment rule called *betting your beliefs* by Blume and Easley (1992), given by $\alpha_s^i = q_s$. In the case of diagonal securities ($S = K$ and $A_k(s) > 0$ if $s = k$, $A_k(s) = 0$ otherwise), α^* and *betting your beliefs* differ. For example, consider two equiprobable states of the world $s = 1, 2$ and two assets $k = 1, 2$. Asset 1 pays 2 units if state 1 occurs, and 0

otherwise. Asset 2 pays 3 units if state 2 occurs, and 0 otherwise. While *betting your beliefs* would prescribe $\alpha_1 = \alpha_2 = 1/2$, α^* prescribes $\alpha_1^* = 2/5$ and $\alpha_2^* = 3/5$.

Hens et al. (2003) analyze an asset market with strategic, risk-averse investors with strictly concave utility functions of wealth. Under the additional assumption of *no aggregate risk*—when the sum of the asset returns is constant and independent of the state—they find that in a symmetric Nash equilibrium all investors split their wealth proportionally to expected *relative* returns, which coincides with relative expected returns for the case of no aggregate risk. The same result is obtained with logarithmic utility allowing for aggregate risk. For general constant relative risk aversion the equilibrium is neither expected relative returns nor relative expected returns.

Theorem 1. *The only pure-strategy Nash equilibrium of the Asset Market Game is the profile* $\overline{\alpha}^* = (\alpha^*, .\overset{N}{.}., \alpha^*)$. *Moreover, it is a strict equilibrium.*

Proof. We already know from Lemmata 1 and 2 that in any Nash equilibrium all assets are traded and $p_k(\overline{\alpha}_k) = R_k > 0$ for all k. Take any trader i. We claim that, in equilibrium, $\alpha_k^i = R_k$ for all k. Suppose otherwise. Since both α^i and $(R_1, \ldots, R_K)$ are in the simplex Δ^K, there must exist two assets k_1, k_2 such that $0 \leq \alpha_{k_1}^i < R_{k_1}$ and $\alpha_{k_2}^i > R_{k_2}$. In particular, this implies that trader i holds more units of k_2 than of k_1; since the prices are $p_k(\overline{\alpha}_k) = R_k$, we have

$$x_{k_2}^i(\overline{\alpha}_{k_2}) = \frac{\alpha_{k_2}^i r_0^i}{R_{k_2}} > r_0^i > \frac{\alpha_{k_1}^i r_0^i}{R_{k_1}} = x_{k_1}^i(\overline{\alpha}_{k_1}).$$

Define $\alpha^i(\varepsilon)$ as in (15), with $0 \leq \varepsilon < \alpha_{k_2}^i$. From Equation (18) and by Lemma 2, we have

$$\left(\frac{1}{E}\right)\frac{d}{d\varepsilon}\pi^i(\alpha^i(\varepsilon), \alpha^{-i})\bigg|_{\varepsilon=0} = r_0^i\left[\frac{R_{k_1}}{p_{k_1}(\overline{\alpha}_{k_1})}\left(1 - x_{k_1}^i(\overline{\alpha}_{k_1})\right) - \frac{R_{k_2}}{p_{k_2}(\overline{\alpha}_{k_2})}\left(1 - x_{k_2}^i(\overline{\alpha}_{k_2})\right)\right] = $$
$$r_0^i\left[x_{k_2}^i(\overline{\alpha}_{k_2}) - x_{k_1}^i(\overline{\alpha}_{k_1})\right] > 0 \quad (26)$$

Hence, player i can profitably deviate by reallocating a small fraction of wealth to asset k_1. We conclude that the only possible Nash equilibrium is given by $\alpha_k^i = R_k$ for all k and i.

It remains to show that this profile is indeed a Nash equilibrium. (So far we have only proven that no other profile can be a Nash equilibrium.) We will do this by analyzing trader i's maximization problem.[6]

Suppose, then, that $\alpha_k^j = R_k$ for all k and all $j \neq i$. We have to show that $\alpha_k^i = R_k$ for all k is a best response for player i. Player i's maximization problem is equivalent to

$$\max_{\alpha^i \in \Delta^K} \frac{1}{E} \cdot \pi^i(\alpha^i, \alpha^{-i}) \quad (27)$$

whose solution must fulfill the first order conditions

$$R_k \frac{\partial x_k^i(\overline{\alpha}_k)}{\partial \alpha_k^i} - \lambda^i = 0 \quad (28)$$

for $k = 1, \ldots, K$, where λ^i is the Lagrange multiplier associated to the constraint $\sum_k \alpha_k^i = 1$. For $\alpha_k^i = R_k$, we have that $p_k(\overline{\alpha}_k) = R_k$ and $x_k^i(\overline{\alpha}_k) = r_0^i$, and hence (recalling (8)) the first order conditions are fulfilled with

$$\lambda^i = r_0^i\left(1 - r_0^i\right) \quad (29)$$

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[6]It is not difficult to show uniqueness also from trader i's optimization problem and Lemma 2. However, we find the argument above more intuitive.

Thus, α^* fulfills the first order conditions for each trader's optimization problem with the Lagrange multiplier given by (29). Since x_k^i is strictly concave for all k whenever asset k is traded, the payoff function given by (12) is also strictly concave. Therefore, $\overline{\alpha}^*$ is a strict Nash equilibrium. ∎

The uniqueness part of Theorem 1 is rather strong. To see why, consider the following. It is straightforward to show that the Nash equilibrium is also a *competitive* equilibrium, because if traders take the prices $p_k = R_k$ as given, their profits are constant in own strategy:

$$\pi_i = \sum_k E_k \frac{\alpha_k^i r_0^i}{R_k} = r_0^i \cdot E.$$

For the same reason, *any* strategy profile inducing fundamental prices constitutes a competitive equilibrium, i.e. there are multiple competitive equilibria. This is in sharp contrast with our result: once strategic effects are taken into account, the Nash equilibrium is unique.

Intuitively, if prices equal relative expected returns, transferring wealth among assets will have no direct effects as measured by expression (19). Since this expression is independent of the traders' strategies, any profile will be a competitive equilibrium if it induces prices in accordance with fundamentals. Indirect, strategic effects through changes in prices, as measured by expression (20), are still present though. As was mentioned above, this expression does depend on the traders' strategies through the number of units of each asset hold. These effects only disappear if the number of units hold is the same for all assets. This is precisely the force at work in the previous proof. Inequality (26) above amounts to the claim that, provided prices are equal to relative expected returns, if traders holds a stronger position on some asset, they can profitably deviate by moving to a more balanced allocation.

This is the key property of the Nash equilibrium: for each trader the number of units hold from each asset is constant and numerically equal to the initial wealth.[7] Hence, when all investors follow α^* there are no incentives to deviate. Direct effects of transferring wealth among assets do not exist, because prices are equal to relative expected returns (recall equation (19)). Indirect effects do not exist either, because the number of units of each asset hold by any trader is also the same (recall equation (20)).

A priori the number of units hold is economically irrelevant, but when prices equal relative expected returns, holding the same number of units of each asset means that traders also hold an equally strong position in terms of wealth invested relative to prices: a small investment in cheap assets, a large investment in expensive assets. If this were not the case, traders could manipulate asset prices away from fundamentals to their advantage. Selling there where they have more, hence reducing those prices, and buying there where they have less, hence increasing those prices, they would gain more than they would lose. This is because there is a symmetric effect on the asset prices. See the following example for an illustration.

Example 1. There are two equally likely states of nature $s = 1, 2$, and two assets $k = 1, 2$. Asset 1 pays 2 units in state 1 and zero otherwise, while asset 2 pays 3 units in state 2 and zero otherwise. Hence $E_1 = 1$ and $E_2 = \frac{3}{2}$. These are diagonal securities, but not Arrow securities. Consider two investors, Adam and Eve, endowed with the same initial wealth $r_0^A = r_0^E = \frac{1}{2}$. In a Nash equilibrium both should invest a fraction $(1/(\frac{5}{2})) = 0.4$ of their wealth in asset 1 and the remaining 0.6 in asset 2, which yields prices $p_1 = 0.4$ and $p_2 = 0.6$.

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[7]The fact that share holdings for a given trader are constant across assets comes from the symmetry across traders' investments (recall (6)). It is obvious that the reverse implication also holds, and thus the Nash equilibrium can be characterized as *the only symmetric profile inducing fundamental prices.*

Suppose instead that Adam invests 0.7 of his wealth in asset 1, while Eve only invests 0.1 of her wealth in the same asset. The price is still given by $p_1 = \frac{1}{2} \cdot (0.7 + 0.1) = 0.4$; thus $p_2 = 0.6$. So the described profile is a competitive equilibrium, since prices are equal to relative expected returns. Eve receives 0.125 units of asset 1 and 0.75 units of asset 2. Each monetary unit invested yields an expected return of 2.5 and total expected payoffs for both, Adam and Eve, equal initial wealth times the expected return, 1.25.

Suppose now that Eve reallocates 0.1 of her wealth from asset 2 to asset 1, so that she invests 0.2 of her wealth in asset 1 and 0.8 in asset 2. The price of asset 1 increases to $p_1(\varepsilon) = \frac{1}{2}(0.7 + 0.2) = 0.45$ and the price of asset 2 decreases to 0.55 (the change in prices is of course symmetric, since price vectors always add up to one). Consequently, Eve now receives $\left(\frac{1}{2} \cdot 0.2\right)/0.45 \approx 0.222$ units of asset 1 and $\left(\frac{1}{2} \cdot 0.8\right)/0.55 \approx 0.727$ units of asset 2. Her new expected payoff is

$$\pi^E \approx 0.222 \cdot 1 + 0.727 \cdot 1.5 \approx 1.313 > 1.25$$

larger than with her previous allocation of wealth. With her new investment plan, Eve has acquired additional units of asset 1, although she has increased its price from 0.4 to 0.45. The return per monetary unit invested in that asset, though, is now $1/0.45 \approx 2.222$ rather than 2.5. By doing this, Eve is losing $2.5 - 2.222 \approx 0.277$ with every monetary unit invested in asset 1, i. e. with 0.2 of her wealth. Still the price of asset 2 has dropped from 0.6 to 0.55, and although she is now receiving less units of asset 2, the return per monetary unit invested in that asset is now $(\frac{3}{2})/0.55 \approx 2.727$ rather than 2.5. She is now earning 0.227 more than before per unit invested in asset 2, i. e. with 0.8 of her initial wealth. Even though her additional per-monetary-unit earnings in asset 2 are smaller than her per-monetary-unit losses in asset 1, she has a much larger investment in asset 2. The total payoff change is $0.2 \cdot (-0.272) + 0.8 \cdot (+0.227) \approx 0.127 > 0$.

By transferring wealth between the two assets, Eve has successfully manipulated prices to her advantage, strategically increasing her earnings in an asset where her share holdings are large at the expense of an asset where her share holdings are small.

4   Evolutionary Stability

So far we have adopted the classical approach in game theory, assuming that all investors know the assets distributions of payoffs and maximize their expected utility given their opponents' investment strategy. In that setting we have shown that the profile where all investors follow α^* is a symmetric Nash equilibrium. In the present section we adopt an evolutionary approach with boundedly rational investors that have limited information. In particular, they need not be informed about the assets distributions of payoffs and they do not act strategically. An evolutionarily stable strategy (ESS) is then defined as an investment strategy such that, once adopted by all investors, it cannot be outperformed by any different investment strategy. The idea is that no investor who would experiment with a new strategy could obtain a larger return per monetary unit invested than the investors still using the status quo strategy. If this were possible, other investors would follow the successful experimenter and the original strategy would not be stable.

The definition of evolutionary stability we use here is due to Schaffer (1988) and applies to any finite number of players. Although based on the same principle of non-invadability, it differs from the standard definition of Maynard Smith (1982). The latter, which is a refinement of Nash equilibrium, applies to a continuum population of players who are randomly drawn in pairs to play a two-person game. Schaffer's (1988) definition is better suited to model market settings, where the payoffs to any player depend on the actions of all market

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participants, and not only on the action chosen by a randomly picked opponent. It is important to note, however, that this concept of evolutionary stability is not related to Nash equilibrium in general.[8] The reason for this important difference is due to finite population effects (see Vega-Redondo (1996) for a discussion). In particular, as we will see in more detail below, a Schaffer ESS aims at maximizing the difference between own and opponents' payoffs, a feature that is known as *spiteful* behavior. A deviation from a Nash equilibrium strategy to an ESS may be worth undertaking if, even when it reduces own payoffs, it weakens the opponent in relative terms. This responds to an idea of selection of strategies based on relative performance.

Taking returns per monetary unit as the relevant payoffs, our asset market game is symmetric. We now proceed to adapt Schaffer's (1988) concept to this framework.

Definition 1. We say that $\alpha \in \Delta^K$ is an *evolutionarily stable strategy* in the asset market game (abbreviated ESS) if, for any i and for any strategy profile $\overline{\alpha}$ such that $\alpha^i \neq \alpha$ and $\alpha^j = \alpha$ for all $j \neq i$,

$$\frac{1}{r_0^i} \pi^i(\alpha^i, \alpha^{-i}) \leq \frac{1}{r_0^j} \pi^j(\alpha^j, \alpha^{-j}) \tag{30}$$

for all $j \neq i$. The ESS is strict if inequality (30) holds strictly.

Note that the returns per monetary unit for any two investors choosing α is the same. Hence only one comparison between the experimenter and any status quo investor is necessary in (30). That is, for any $j, h \neq i$, with $\alpha^j = \alpha^h = \alpha$, we have that $\alpha^{-j} = \alpha^{-h}$ up to a permutation. Thus

$$\frac{1}{r_0^j} \pi^j(\alpha^j, \alpha^{-j}) = \frac{1}{r_0^h} \pi^h(\alpha^h, \alpha^{-h}).$$

It follows directly from Equation (30) that an ESS α solves the following problem

$$\max_{\alpha^i \in \Delta^K} \left(\frac{1}{r_0^i} \pi^i(\alpha^i, \alpha^{-i}) - \frac{1}{r_0^j} \pi^j(\alpha^j, \alpha^{-j}) \right) \tag{31}$$

for any $j \neq i$. It is in this sense that an ESS maximizes relative performance.

We remark that our concept of ESS is *exactly* that in Schaffer (1988) if we redefine the payoffs to be per-unit returns. Such normalization of payoffs would not affect the set of Nash equilibria. Thus the strategy profile where all traders invest according to α^* is still the only equilibrium of the game. Proposition 1 below shows that, in our asset market game, the Nash equilibrium strategy α^* is also evolutionarily stable.

In general, the payoff of an experimental strategy could increase with the number of players that adopt it. Thus, even if a strategy is an ESS and hence resistant to the appearance of a single experimenter, it need not be resistant to the appearance of a larger fraction of experimenters.

Definition 2. We say that an ESS $\alpha \in \Delta^K$ is *globally stable* if inequality (30) holds for any strategy profile $\overline{\alpha} = (\alpha', \overset{m}{\ldots}, \alpha', \alpha, \overset{N-m}{\ldots}, \alpha)$, or permutation thereof, where $\alpha' \neq \alpha$, $\alpha^i = \alpha'$, and $\alpha^j = \alpha$. The strategy α is strictly globally stable if the inequality holds strictly.

A globally stable ESS is resistant against any fraction of experimenters in the population.[9] Proposition 1 shows that α^* also fulfills this stronger stability condition. The intuition for this result is simply that all investors deviating to the same $\alpha' \neq \alpha^*$ will change prices to their disadvantage and the advantage of their opponents.

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[8] This paper will provide an example of a Schaffer ESS that is also a Nash equilibrium.

[9] Definition 2 differs slightly from the one by Schaffer (1988), who calls an ESS globally stable if it fulfills the inequality strictly for $m \geq 2$ (see Crawford (1991) and Tanaka (2000) for closely related concepts).

Proposition 1. *The investment strategy α^* is a strictly globally stable ESS in the asset market game.*

Proof. Assume α^* has been adopted by all investors, and $1 \leq m < N$ of them experiment with $\alpha' \neq \alpha^*$. Without loss of generality, suppose the experimenters are the investors $i = 1, \ldots, m$. The resulting strategy profile is $\overline{\alpha} = (\alpha', \overset{m}{\ldots}, \alpha', \alpha, \overset{N-m}{\ldots}, \alpha)$. The price of any asset k after experimentation is given by

$$p_k(\overline{\alpha}_k) = \alpha'_k \sum_{i=1}^{m} r_0^i + \alpha_k^* \sum_{i=m+1}^{N} r_0^i = R_k + (\alpha'_k - \alpha_k^*) \sum_{i=1}^{m} r_0^i$$

Again without loss of generality, we can reorder the assets in such a way that the price ratio R_k/p_k is increasing in k. Let $\tilde{k}$ be the largest k such that $(R_k/p_k) \leq 1$. Note that $\alpha'_k - \alpha_k^* \geq 0$ for all $k = 1, \ldots, \tilde{k}$ and $\alpha'_k - \alpha_k^* < 0$ for all $k = \tilde{k}+1, \ldots, K$. For any $i = 1, \ldots, m$ and $j = m+1, \ldots, N$, we have

$$\left(\frac{1}{E}\right)\left[\frac{1}{r_0^i}\pi^i(\alpha^i, \alpha^{-i}) - \frac{1}{r_0^j}\pi^j(\alpha^j, \alpha^{-j})\right] = \sum_{k=1}^{\tilde{k}} \frac{R_k}{p_k}(\alpha'_k - \alpha_k^*) + \sum_{k=\tilde{k}+1}^{K} \frac{R_k}{p_k}(\alpha'_k - \alpha_k^*) <$$

$$\frac{R_{\tilde{k}}}{p_{\tilde{k}}} \sum_{k=1}^{\tilde{k}} (\alpha'_k - \alpha_k^*) + \frac{R_{\tilde{k}+1}}{p_{\tilde{k}+1}} \sum_{k=\tilde{k}+1}^{K} (\alpha'_k - \alpha_k^*) < 0 \quad (32)$$

The last inequality follows from the fact that $\sum_{k=1}^{\tilde{k}} (\alpha'_k - \alpha_k^*) = -\sum_{k=\tilde{k}+1}^{K} (\alpha'_k - \alpha_k^*) > 0$ because $\alpha', \alpha^* \in \Delta^K$, and $(R_{\tilde{k}}/p_{\tilde{k}}) \leq 1 < (R_{\tilde{k}+1}/p_{\tilde{k}+1})$.

With $m = 1$, this shows that α^* is a strict ESS. With arbitrary m, this shows that α^* is strictly globally stable. ∎

Recall that x_k^i is increasing in α_k^i. Therefore, intuitively, when m investors deviate from α^*, they increase the prices of those assets where they hold a stronger position after deviation and decrease the price of those where they hold a weaker position. Hence, relative to the traders still investing according to α^*, experimenters are more affected by the negative effects of overvalued assets, and less affected by the positive effects of undervalued ones, which leaves them in a worse relative position.[10]

A word of caution is necessary here. The concept of ESS (both Maynard Smith's and Schaffer's) is a static one, based on one-shot comparisons. The evolutionary literature usually proceeds from such static definitions to later establish their dynamic properties. Alós-Ferrer and Ania (2002) establish dynamic stability properties for globally stable Schaffer ESS. Hens and Schenk-Hoppé (2001) define a different concept of evolutionary stability, directly based on a stochastic wealth-accumulation dynamics and unrelated to the classic, static definitions. No confusion should arise between these two approaches. In the next section, we illustrate the dynamic properties of α^* in the framework of an evolutionary dynamics different to those in the papers just mentioned.

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[10]It can be shown that strict equilibria of zero-sum games are evolutionarily stable in the sense used here. This gives an alternative proof of the fact that α^* is an ESS, because the Asset Market Game is constant sum in the interior of the strategy space. We thank Fernando Vega-Redondo and Alex Possajennikov for this observation.

5 Dynamics

In this section, we postulate a dynamic evolutionary model where agents repeatedly play the Asset Market Game. The purpose is to check the dynamic stability of α^* against any other given portfolio α. That is, we want to show that α^* is "pairwise dynamically attracting" in a well-defined sense to be clarified below.

Every period, there are exactly N traders active in the market, all of them with the same wealth. Each trader stands simply for a fraction $r_0^i = \frac{1}{N}$ of the total wealth invested. Our focus will therefore not be on wealth-accumulation dynamics as in Blume and Easley (1992) and Hens and Schenk-Hoppé (2001), but rather on the proportion of total wealth which is invested according to a given strategy. Instead of choosing a particular behavioral model at the individual level, we postulate a population dynamics which specifies directly how wealth flows between investment strategies, depending on realized payoffs, and the effects of these flows on prices. A possible interpretation will be that the realized returns are not necessarily reinvested each period. Also, some traders may take their wealth and leave the market, while new investors enter.

Each trader can invest wealth according to either α^* or α. In a given period t, the number of traders following α^* is denoted by n_t, and the number of those following α is $N - n_t$. We drop the t-subscript when no confusion can arise. Given n, the price of asset k is

$$p_k(n) = \frac{n}{N}\alpha_k^* + \frac{N-n}{N}\alpha_k.$$

Let $\Delta(s,n)$ be the difference between the realized payoffs (per monetary unit) of α^*-investors and α-investors, conditional on the occurrence of state s. This is a random variable given by

$$\Delta(s,n) = \frac{1}{(1/N)}\left(\sum_{k=1}^{K}\frac{\alpha_k^*(1/N)A_k(s)}{p_k(n)} - \sum_{k=1}^{K}\frac{\alpha_k(1/N)A_k(s)}{p_k(n)}\right) = \sum_{k=1}^{K}\frac{(\alpha_k^* - \alpha_k)A_k(s)}{p_k(n)}$$

with expected value

$$E\Delta(n) = \sum_{s=1}^{S}q_s\sum_{k=1}^{K}\frac{(\alpha_k^* - \alpha_k)A_k(s)}{p_k(n)}.$$

By Proposition 1, for any $\alpha \neq \alpha^*$ we have that $E\Delta(n) > 0$.

We postulate a "Darwinian" dynamic process as follows. Every period, the state of the world is realized and assets pay according to $A_k(s)$. The realized payoffs of the portfolios α^* and α are observed, and some trader updates his portfolio. The probability that this revision results in a trader switching from α to α^*, rather than the opposite, is directly proportional to the difference in realized payoffs between α^* and α, i. e. $\Delta(s,n)$. A period is therefore interpreted as the time interval necessary for $1/N$ of total market wealth to be reallocated.

The probability that there will be $n + 1$ α^*-traders next period, given that there are $n \in \{1, \ldots, N\}$ in the current period, is

$$Q_{n,n+1}^s = \frac{1}{2} + \sigma\Delta(s,n)$$

where σ is a normalization parameter.[11] Symmetrically,

$$Q_{n,n-1}^s = \frac{1}{2} - \sigma\Delta(s,n)$$

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[11]The parameter σ is analogous to ζ in De Long et al. (1990) and to β in Brock and Hommes (1998).

Hence, if in any given period α^* has higher realized payoffs than α, $\Delta(s,n)$ is strictly positive and thus $Q_{n,n+1}^s > \frac{1}{2}$, i. e. it is more probable that some α-investor switches to α^* and less probable that the opposite occurs. The transition probabilities are more favorable to the first event the higher the realized payoff advantage of α^*. Of course, the opposite is true if α is the portfolio with higher realized payoffs. Observe that, if and only if for a certain state s the two portfolios were to yield *exactly* the same realized payoff, the probabilities would be $1/2$ in each direction. Note that

$$Q_{n,n+1}^s - Q_{n,n-1}^s = 2\sigma\Delta(s,n)$$

i. e. the difference in transition probabilities is directly proportional to the difference in realized payoffs.[12] If we were to take these probabilities as the description of an individual learning rule, this would correspond to the *Proportional Imitation Rule* of Schlag (1998). The parameter σ is taken equal to $1/(2\Delta)$, where Δ is the largest observable payoff difference.[13]

We complete the model by postulating a noise parameter $\varepsilon < \frac{1}{2}$. At any period, with probability $(1-\varepsilon)$, a transition takes place according to the rule above. With probability ε, a mistake occurs and exactly the opposite transition results. This might be due e. g. to new, exogenous information coming into the system. That is, the transition probabilities (conditional on s) are:

$$P_{n,n+1}^s = (1-\varepsilon)\left(\frac{1}{2} + \sigma\Delta(s,n)\right) + \varepsilon\left(\frac{1}{2} - \sigma\Delta(s,n)\right) \tag{33}$$

$$P_{n,n-1}^s = (1-\varepsilon)\left(\frac{1}{2} - \sigma\Delta(s,n)\right) + \varepsilon\left(\frac{1}{2} + \sigma\Delta(s,n)\right) \tag{34}$$

When $n = 0$ or $n = N$, no payoff comparison is possible since only one portfolio is observed and, thus, there are no evolutionary pressures. Yet noisy transitions are possible, with probabilities given by

$$P_{0,0}^s = 1-\varepsilon \qquad P_{0,1}^s = \varepsilon \qquad P_{N,N-1}^s = \varepsilon \qquad P_{N,N}^s = 1-\varepsilon \tag{35}$$

independently of the realized state. Hence, for low values of ε, the process tends to stay for a significant amount of time at situations where all traders invest according to the same strategy.

Equations (33)-(35) give all relevant transition probabilities; all other transitions have zero probability. The unconditional transition probabilities are given by

$$P_{n,n+1} = \sum_{s=1}^S q_s P_{n,n+1}^s = \frac{1}{2} + (1-2\varepsilon)\sigma E\Delta(n) \tag{36}$$

$$P_{n,n-1} = \sum_{s=1}^S q_s P_{n,n-1}^s = \frac{1}{2} - (1-2\varepsilon)\sigma E\Delta(n) \tag{37}$$

for $n \neq 0, N$, and $P_{0,n} = P_{0,n}^s$, $P_{N,n} = P_{N,n}^s$ for all n.

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[12]In a recent paper Kliger et al. (2003) find experimental evidence indicating that investors' tendency to delegate money to a fund manager increases with the managers past performance and it decreases with the performance of the managers competitors.

[13]The *Proportional Imitation Rule* was introduced by Schlag (1998) in the context of a large population of agents independently choosing among actions with uncertain payoffs. He shows that proportional imitation has certain optimality properties, which do not apply to the present strategic context (see Ania (2000)). Our motivation to use that rule here is rather its relation to evolutionary dynamics (on this see also Björnerstedt and Weibull (1996)).

The stochastic process just described constitutes a discrete-time birth-death process and will be referred to as the *portfolio dynamics*. In this simple Markov chain the invariant distribution, μ, is unique, strictly positive, and can be explicitly calculated using the so-called detailed balance condition.[14] The value $\mu(n)$ will give the probability that exactly n traders are investing according to α^* in the long run; $\mu(n)$ also gives the fraction of time that this event occurs along any sample path of the process.

In the next theorem we show that, along any given sample path, most of the time we will observe that a majority of traders follow α^* rather than any other α. This results from the following two properties of the invariant distribution of the *portfolio dynamics*. First, the occasions where all investors follow α^* are the most frequent; in particular, more frequent than those where they all follow α. Second, whenever both investment strategies coexist, any population profile with more α^*-traders is observed more frequently than any other with less α^*-traders.

Theorem 2. *For any $\alpha \neq \alpha^*$, the invariant distribution μ of the portfolio dynamics verifies*

   (i) for all $n = 1, \ldots, N - 1$, $\mu(n) < \mu(n + 1)$, and

   (ii) $\mu(0) < \mu(N)$.

Hence, $\mu(N) > \mu(n)$ for all $n \neq N$.

Proof. The invariant distribution μ fulfills the detailed balance condition

$$\mu(n)P_{n,n+1} = \mu(n+1)P_{n+1,n} \tag{38}$$

Substituting (36) and (37) yields that, for all $1 \leq n \leq N - 2$,

$$\mu(n+1) = \mu(n) \cdot \frac{\frac{1}{2} + (1 - 2\varepsilon)\sigma E\Delta(n)}{\frac{1}{2} - (1 - 2\varepsilon)\sigma E\Delta(n+1)}$$

Since $E\Delta(n) > 0$ by Proposition 1, we obtain that $\mu(n + 1) > \mu(n)$ and (i) is proved for $1 \leq n \leq N - 2$. For $n = N - 1$,

$$\mu(N) = \mu(N-1)\frac{\frac{1}{2} + (1 - 2\varepsilon)\sigma E\Delta(N - 1)}{\varepsilon} > \mu(N - 1)$$

since $\varepsilon < 1/2$. Iterating (38) we obtain that

$$\mu(N) = \mu(0) \prod_{n=1}^{N-1} \frac{\frac{1}{2} + (1 - 2\varepsilon)\sigma E\Delta(n)}{\frac{1}{2} - (1 - 2\varepsilon)\sigma E\Delta(n)}$$

and (ii) follows. ∎

Figure 1 illustrates the last result. In both diagrams there we consider a market with two equiprobable states and two diagonal assets that pay 2 and 1 in the respective states. For this market $\alpha^* = (2/3, 1/3)$. The number of investors is fixed to $N = 100$ and the noise parameter is $\varepsilon = 0.1$. In diagram (a) we let α^* compete against the strategy $\lambda^* = (1/2, 1/2)$,

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[14]Discrete-time birth-death processes are sometimes also called general one-dimensional random walks. They are defined as Markov chains such that (i) the state space is either the nonnegative integers or a finite set $\{0, \ldots, N\}$, and (ii) for every state n, the only positive-probability transitions are to states $n, n - 1$, and $n + 1$, with the last two being strictly positive. The detailed balance condition follows directly from the definition of invariant distribution applied to this particular case. See e. g. Feller (1968, p.396).
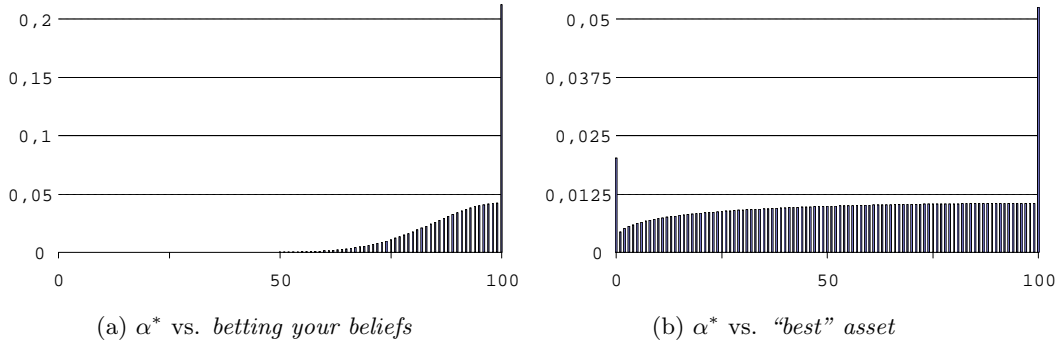
(a) α^* vs. *betting your beliefs*   (b) α^* vs. *"best" asset*

Figure 1: Invariant distribution of the *portfolio dynamics*. The number of traders using α^* appears on the x-axis.

which corresponds to *betting your beliefs* in Blume and Easley (1992). The number n of investors that follow α^* is plotted in the x-axis. The fraction of periods where n is observed, $\mu(n)$, is plotted in the y-axis. Observe that the fraction of periods where all investors follow α^* is significantly higher and that the invariant distribution is increasing. In this example, $\mu(100) \approx 0,21$, while e. g. $\mu(99) \approx 0,04$. Although it cannot be perceived in the diagram, $\mu(n)$ is strictly positive for all n, but it is extremely small for $n \leq 50$, with $\mu(0) \approx 3 \times 10^{-10}$.

In diagram (b) we let α^* compete against the strategy $\alpha = (0.999, 0.001)$. We could think of this as the one-shot optimal strategy that concentrates wealth on the asset with maximum expected payoff.[15] Investing in the "best" asset gives α a certain advantage. Still the figure shows that α^* is used more frequently. Intuitively, when many investors follow α, asset 1 (resp. asset 2) is overvalued (resp. undervalued). Any investor who would change to α^* could make a profit on asset 2. In contrast, when many investors follow α^*, prices are close to fundamentals, and arbitrage opportunities are very limited. In summary, the problem of α is that it ignores prices.

Diagram (b) displays an interesting feature that will be discussed later on (see Remark 1), and could not be perceived in diagram (a). Namely, in general, $\mu(0) > \mu(n)$ for small n, even though $\mu(n)$ is increasing for $n \geq 1$. In this particular case, it is even true that $\mu(0) > \mu(99)$.

6 Asset prices

In what follows we present sample paths of the dynamics analyzed in Section 5 from the perspective of asset prices. In this way, our results will be viewed from the usual financial markets standpoint. Figure 2 shows the evolution of asset prices in the market considered for the construction of Figure 1. In this example one period corresponds to the time interval necessary for 1% of total market wealth to be reallocated. Here, it is useful to think of one period as being approximately equal to one trading day, which leads to an estimate of 4

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[15]We allow for a small investment on the inferior asset for two reasons. First, in order to avoid the indeterminacy of prices if all investors follow a strategy with zero weight on some asset. Second, the maximal observable payoff difference Δ, which is used to compute the normalization parameter σ, is bounded independently of N if α is completely mixed, but increases with N otherwise.

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(a) α^* vs. *betting your beliefs*

(b) α^* vs. *"best" asset*

(c) $t = 1000\ldots2000$ in (a), 4 years
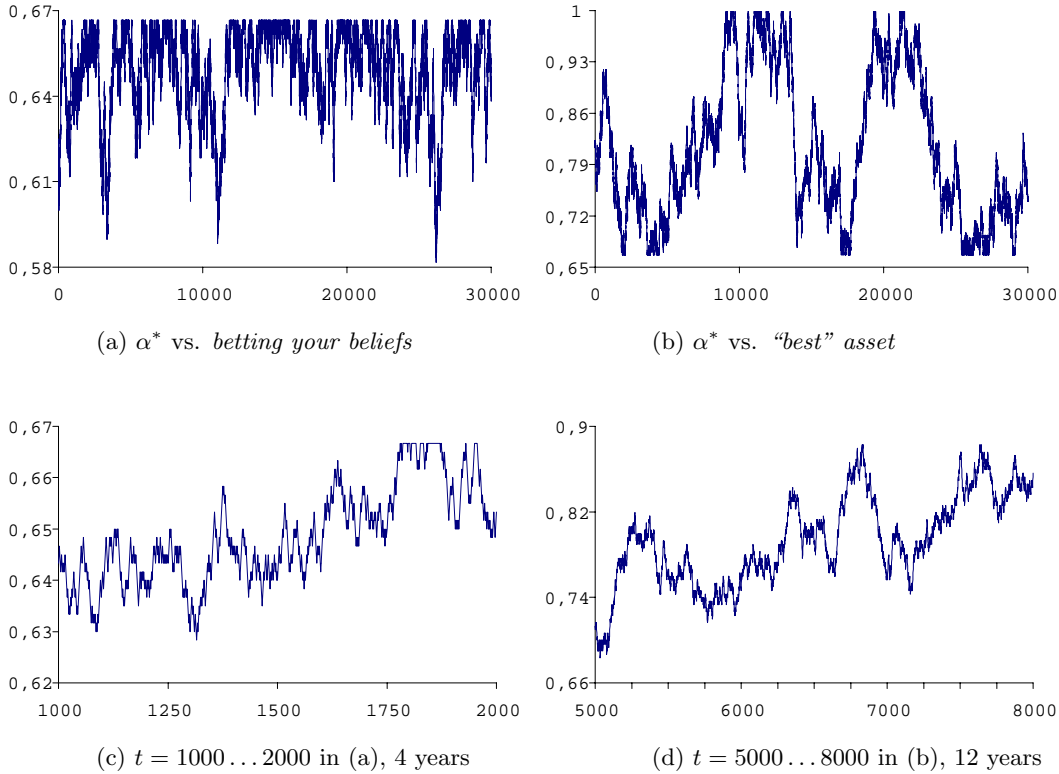
(d) $t = 5000\ldots8000$ in (b), 12 years

Figure 2: Evolution of the price of asset 1 with $N = 100$ and $\varepsilon = 0.1$. The x-axis approximately correspond to trading days. Price appears in the y-axis.

years every 1,000 periods if years have 250 trading days.[16]

Diagrams (a) and (c) plot the price of asset 1 in the *portfolio dynamics* where $\alpha^* = (2/3, 1/3)$ competes against *betting your beliefs*; i. e. $\alpha = (1/2, 1/2)$. Diagrams (b) and (d) correspond to the dynamics where α^* competes against the *"best"-asset strategy* $\alpha = (0.999, 0.001)$. In the former case the price of asset 1 is always between $1/2$ and $2/3$, while in the latter it must lie between $2/3$ and 0.999 depending on the fraction of α^*-traders in the population. If all traders follow α^*, the price of the asset must be $2/3$, the fundamental price.[17]

Diagram (a) is the analogue of Figure 1(a). Note that the price never goes below 0.58. This is because there are always more traders using α^*. Observe how the numerical value $2/3$ is actually attained a significant fraction periods–recall that $\mu(100) \approx 0.21$. Diagram (b) is the corresponding plot for Figure 1(b). In this case both $2/3$ and 0.999 are attained, which reflects the higher value of $\mu(0)$ in this example. The sharp difference in the value

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[16]In 2001 the NASDAQ average daily trading volume was $\$44 \times 10^9$, approximately 1.5% of the total listings' market value of $\$2.9 \times 10^{12}$. This corresponds to an estimate of 0.66 days per period in our examples. Similar data for the NYSE (1.441×10^9 average daily stock volume traded over 349.9×10^9 shares listed in 2002) yield an estimate of 2.4 days per period.

[17]Since we only allow for two different strategies to compete, the price cannot overshoot the fundamental price in both directions.
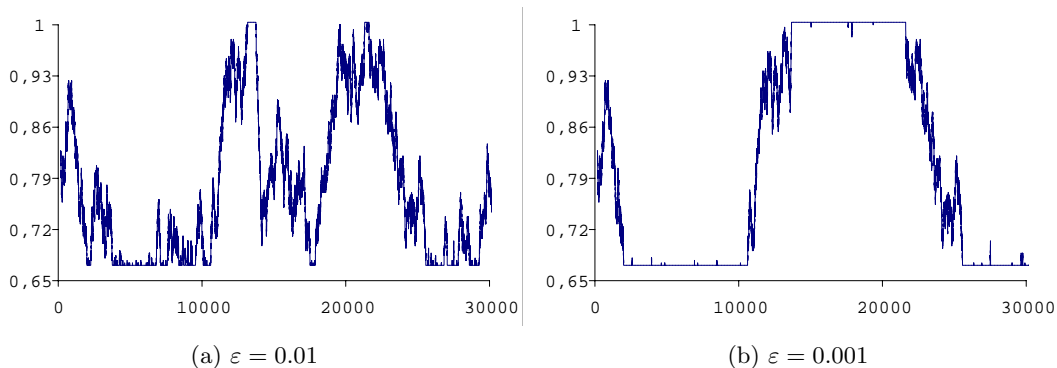
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(a) $\varepsilon = 0.01$ (b) $\varepsilon = 0.001$

Figure 3: Effect of the noise parameter ε.

that each strategy assigns to asset 1 result in much higher volatility.

Diagrams (c) and (d) show much shorter time scales, which are probably more relevant for the analysis of financial markets. Diagram (c) corresponds to periods $t = 1000, \ldots, 2000$ in diagram (a), showing a typical run where the price moves towards the fundamental price. Essentially what Theorem 2 says is that such runs are more frequent than those where the price moves away from the fundamental price. Diagram (d), on the other hand, shows that it is still possible that prices move away from fundamentals for quite long time intervals. There we take periods $t = 5000, \ldots, 8000$ in (b) to illustrate a bubble that goes on for more than a decade, preceding a crash that can be observed in diagram (b).

Remark 1. In the market example of Figures 1 and 2 we took $\varepsilon = 0.1$, significantly above zero. In the proof of Theorem 2, one can see that the invariant distribution depends on the noise parameter; i. e. $\mu = \mu^{\varepsilon}$. It is easy to show that, as $\varepsilon \to 0$, μ^{ε} concentrates on the profiles where all traders invest according to the same strategy. Formally, the support of the *limit invariant distribution* $\lim_{\varepsilon \to 0} \mu^{\varepsilon}$ contains only 0 and N. In the terminology of the literature on learning in games these are the only *stochastically stable states*,[18] which implies that, in the long run and for negligible ε, these are the only two profiles which could be observed with positive probability. It does not say anything about how much weight is placed on each of them. In Figure 1 this would mean that all the weights $\mu(n)$, $1 \leq n \leq N - 1$ converge to zero when $\varepsilon \to 0$. The approach in this paper allows to explicitly compute the invariant distribution and show that, in some cases, the probability that we observe all investors using $\alpha \neq \alpha^*$ is actually negligible. This can also be seen in Figure 1(a).

Figure 3 illustrates how the process changes when the noise parameter becomes smaller for the sample path plotted in Figure 2(b) above. Observe how the frequency of the profiles where all investors use the same strategy increases to the expense of all other mixed profiles.

7   Conclusions

In the present paper, asset markets are modelled as a game, where assets pay according to an arbitrary payoff matrix, investors decide on fractions of wealth to allocate to each asset, and prices result from market clearing.

In the first part of the paper, it is found that the only Nash equilibrium is to split wealth proportionally to the expected returns of the assets, which can be interpreted as investing

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[18]See e. g. Kandori et al. (1993) and Young (1993).

according to the fundamentals. In equilibrium, asset prices are numerically equal to relative expected returns. Uniqueness implies that, whenever prices are away from these fundamental values, there must be arbitrage opportunities. Furthermore, it is shown that the equilibrium strategy is evolutionarily stable. At first sight, this could be read as a restatement of the efficient market hypothesis.

The second part of the paper analyzes an evolutionary dynamics where wealth flows with higher probability into those strategies that obtain higher realized payoffs. An additional noise parameter models new, exogenous information coming into the system. It is shown that, most of the time, a majority of traders invest according to the equilibrium strategy, and thus prices are close to fundamentals. Further, at any given period, prices are more likely to move towards their fundamental values.

Still, the stochastic nature of the system allows prices to be away from fundamental values during long periods of time, and hence there might be persistent arbitrage opportunities. This is not only due to noise, but also because strategies which are wrong in their perceptions about the values of assets may have lucky runs which can drive investors away from the equilibrium strategy. Therefore, one could observe persistent violations of the efficient market hypothesis as suggested by De Long et al. (1990). The main difference between our view and theirs is the fact that, since α^* is a strictly globally stable ESS, noise traders cannot have larger expected payoffs than fundamentalists, that is α^*-traders. As a consequence, in the long run the frequency of periods where all traders invest according to the fundamentals are much higher and noise traders cannot dominate.

In a sense we try to reconcile the two apparently opposed views that markets are efficient, on the one hand, with prices containing all relevant information and no arbitrage opportunities, and on the other hand that noise traders may be successful and survive in the market.

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