Small Sample Confidence Intervals for Multivariate Impulse Response Functions at Long Horizons

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July 2005

Abstract. Existing methods for constructing confidence bands for multivariate impulse response functions may have poor coverage at long lead times when variables are highly persistent. The goal of this paper is to propose a simple method that is not pointwise and that is robust to the presence of highly persistent processes. We use approximations based on local-to-unity asymptotic theory, and allow the horizon to be a fixed fraction of the sample size. We show that our method has better coverage properties at long horizons than existing methods, and may provide different economic conclusions in empirical applications. We also propose a modification of this method which has good coverage properties at both short and long horizons.

Keywords: Local to unity, persistence, impulse responses, VARs.

JEL Classification: C32, C12, F40.

We gratefully acknowledge comments from H. Pesaran, three referees, F. Altissimo, T. Bollerslev, G. Elliott, R. Gallant, R. Giacomini, O. Jorda, L. Kilian, D. Osborn, A. Pagan, J. Piger, A. Tarozzi, G. Tauchen, M. Watson and participants to the Universities of Houston/Rice, UC Davis, Duke, UCLA, Ohio State, Queen Mary and Westfield College, Virginia Tech, the 2003 NBER-NSF Conference on Time Series Analysis, the 2003 EABCN Conference, the 2004 SNDE Conference, and the 2004 Winter Meetings of the Econometric Society. [†]Corresponding author: Barbara Rossi, Department of Economics, Duke University, Durham NC27708, USA. E-mail: brossi@econ.duke.edu. Phone: (919) 660 1801. Fax: (919) 684 8974.

1. Introduction

Impulse response functions (IRFs) play an important role in describing the impact that shocks have on economic variables and they are generally obtained from Vector Autoregressions (VAR). It is conventional to compute the response of current and future values of economic variables to a one standard deviation increase in the current value of the structural shocks. The estimate of the IRFs and their confidence intervals are commonly based on Lütkepohl's (1990) asymptotic normal approximation or bootstrap approximations to that distribution (see Kilian (1998a, 1999)).

Existing methods for constructing IRFs and their confidence intervals, however, may provide different results depending on whether the series are assumed to be stationary, exactly integrated or exactly cointegrated. Even when standard methods of inference are justified asymptotically, confidence bands may have poor coverage properties in small samples in the presence of highly persistent variables, as shown by Kilian and Chang (2000). These authors compare the finite-sample accuracy and average length of commonly used confidence intervals for IRF coefficients for VAR models in levels based on existing empirical studies. They caution applied researchers against inference at horizons bigger than 16 quarters, as inference becomes very unreliable. Unit root pre-tests do not solve the problem, as the actual coverage of IRF bands obtained after a pre-test can be quite different from the nominal one.

This paper proposes a new method for constructing confidence bands for multivariate IRFs in the presence of highly persistent processes. We use asymptotic

approximations based on local-to-unity asymptotic theory and allow the lead time of the IRF to be a fixed fraction of the sample size. The method thus depends on the variables being highly persistent. The assumption is therefore *different* (and in many empirical examples more plausible) than the assumptions that justify existing methods, and is intended to provide better approximations in small samples in the presence of high persistence. The advantages of our method are that: *(i)* it does not require a researcher to decide whether the process has a unit root or not; *(ii)* it is easy to compute; *(iii)* the confidence bands contain the *whole* true IRF with a pre-specified confidence level (i.e. they are not pointwise); *(iv)* it is robust to the presence of deterministic components; *(v)* it allows for mildly non-stationary processes (e.g. roots about 1.001). Due to the nature of our approximation, our confidence bands are appropriate at long lead times and as long as the process is highly persistent (including a unit root). How long the lead time in practice has to be for our approximation to be accurate is investigated in a Monte Carlo experiment.

The empirical literature commonly estimates VARs either in levels or in first differences, sometimes after unit root pre-test procedures, whose results are usually sensitive to the order of integration of the economic variables. This problem in practice is either ignored or ad-hoc robustness checks are performed. That is, researchers check if the same results hold whether one uses a specification in levels or in first differences. However, even when the results remain largely unchanged, this approach will not give any indication of the overall coverage of the procedure and, as we show,

pre-tests create considerable coverage distortions. This paper not only quantifies the size distortions that researchers face when conducting inference about IRFs after pre-tests, but it also shows when and by how much standard methods may be improved upon by the method proposed in this paper. Our method generally works well for horizons bigger than or equal to ten percent of the sample size. Depending on the degree of persistence, our method performs better than pre-test-based IRFs also for smaller horizons. We also discuss a simple way for constructing confidence bands that allow reliable (although conservative) inference at both longer and shorter horizons.

Alternative methods are available in the literature. Andrews (1993) and Andrews and Chen (1994) provide bias-corrected parameter estimates for IRFs in univariate time series. Their method is an important improvement over normal sampling methods, but the coverage is poor at long lead times and it is computationally demanding. Another available method is Hansen (1999) grid-bootstrap method. However, to date there is no extension of the aforementioned methods to deal with multivariate processes. Kilian (1998a) provides a useful, improved bias-corrected bootstrap method that explicitly accounts for the bias and skewness of the small-sample distribution of the IRF estimator. However, these methods may not be robust to the presence of roots equal to one or mildly explosive, or deterministic terms, and it is important to investigate whether there are alternative methods that can provide better coverage or smaller length of the confidence bands.[1] Phillips (1998) studies IRFs

[1] For a comparison of these methods in a univariate setup, see Pesavento and Rossi (2005). Sims and Zha (1999) propose an alternative Bayesian method.

in the presence of local-to-unity, formally providing their limiting distributions and establishing their non-normality in small samples; however, he does not provide an empirical methodology to construct the confidence bands in practice. Recent solutions have been proposed by Wright (2000) and Gospodinov (2002), although only for univariate processes. Wright (2000) relies on a local-to-unity approximation, which inspired our proposal for a method that is robust at short horizons (discussed in Section 6), but he proposes a Bonferroni method, which renders the confidence bands conservative. Gospodinov (2002) relies on the inversion of a LR test in which the constrained estimate exploits a null hypothesis on the value of the IRF at some horizon of interest. His method has the correct size pointwise and confidence bands have small length. Compared to his, our method is much less computationally intensive, has uniform coverage at medium to long horizons, and, unlike Gospodinov (2002), it does not require a specific null hypothesis which may not necessarily be available in most economic applications.[2] On the other hand, our method explicitly relies on long horizon asymptotics, so it might have worse size properties at short horizons.

2. Motivation and Preview of the Results

Consider a researcher interested in analyzing whether monetary shocks have an effect on the real exchange rate. An answer to this question would provide an important empirical contribution to the long-standing debate on flexible versus sticky price models of exchange rate determination. Researchers working on this topic typically

[2] We thank a referee for pointing this out.

run VARs to estimate IRFs, and have to face the choice of using variables in levels or in first differences. For example, Eichenbaum and Evans (1995) estimate a VAR in levels, whereas Rogers (1999) relies on unit root pre-tests. Neither approach may be satisfactory.

To document this problem, Figure 1 shows Monte Carlo simulations for a bivariate VAR where one variable has a root that is close to unity. The experiment, explained in more detail in Section 4, is representative of the practical situation outlined above, where the researcher needs to include a "key" variable (here, the real exchange rate), but neither theory nor unit root pre-tests provide conclusive evidence on whether this variable has a unit root or not.[3] For expository purposes, below we focus on one IRF only. The methods of constructing confidence intervals are described in details in Section 4.

Figure 1(a) shows that IRFs based on both VARs in levels and unit root pre-test-based VARs provide unreliable inference. The figure shows one minus the actual coverage rate of the various methods used to construct IRFs confidence bands. The nominal (desired) coverage rate is 0.90, so that a method performs well when its line is around 0.10. These lines are reported as a function of δ, a parameter that denotes the horizon of the IRF as a fraction of the total sample size, and plays an important role in this paper. For example, in a sample of 100 monthly observations, a horizon of

[3] In some cases there may be more than one large root. The results of Section 4 show that our method is robust to misspecification of the largest root of the other variables in the VAR.

12 months would correspond to $\delta = 0.12$.[4] The upper panel in Figure 1(a) shows that when the root is exactly one, estimation in levels (the line with diamonds) produces IRFs bands that undercover. In this case, a pre-test (the line with circles) would be a much better choice, as it is approximately around 0.10 at any δ. However, when the root is close to unity, say 0.97, the bottom panel in Figure 1(a) shows that the situation is completely different. Unit root pre-testing lacks power and induce the researcher to estimate a mis-specified model – the model in first differences – which causes the coverage to be extremely poor. On the other hand, estimation in levels is now better for relatively large horizons, $\delta \geq 0.05$, although the coverage is still not equal to the nominal level. Thus, the applied researcher faces the problem of choosing between levels and pre-test-based methods, knowing that neither method is superior to the other, and that their relative performance crucially depends on the magnitude of the unknown root. To our knowledge, there is no method that successfully solves this problem. To overcome this difficulty, Rogers (1999) estimates a VAR with the real exchange rate both in levels and in first differences. While this seems a pragmatic solution, it is not satisfactory, as nothing guarantees that the overall coverage will be correct.

In this paper we propose methods that have asymptotically correct coverage at

[4] We chose to report δ rather than the horizon because our analysis focuses on small samples, so that the horizon per se is less important than its ratio to the available sample size. This happens because, as explained later, the sample size determines the degree of imprecision of the estimate of the unit root, and the horizon determines how much this imprecision is blown up. See Rossi (2001) for more details.

medium to long horizons no matter whether the root is unity or highly persistent, whereas both the level and the pre-test based IRF bands may not. The dotted line in Figure 1(a) shows one such method. It is clear that the method has approximately correct coverage at medium to long horizons ($\delta \geq 0.10$), and that its coverage properties are robust to whether there is an exact unit root or not, which is exactly in the region where the usual pre-test methods lead to unreliable inference.

In the next section we present these methods, and discuss how to implement them. More details and Monte Carlo results are available in Section 4. Section 5 extends this method to more than one root local to unity, and Section 6 provides a simple modification that ensures good coverage at both short and long horizons. Section 7 discusses an empirical application to real versus nominal sources of fluctuations in exchange rates, and shows that shocks are more persistent than commonly found in VARs estimated in levels; thus, according to Eichenbaum and Evans (1995), shocks would seem to disappear more quickly than they really do. Section 8 concludes.

3. The Model

Let the data generating process (hereafter DGP) be: $(I - \Phi L)\, w_t = \widetilde{u}_t$, where w_t is a $(m \times 1)$ vector of variables.[5] Without loss of generality, we can express Φ in terms of its eigenvalues and eigenvectors: $\Phi = V^{-1}\Lambda V$, so that we can rewrite (1) in terms

[5] w_t may contain deterministic components (constants and time trends) but they are irrelevant, as the IRFs are defined as deviations from the deterministic components (see Phillips (1998)).

of rotated variables $y_t \equiv V w_t$ and $u_t \equiv V\widetilde{u}_t$:

$$(I - \Lambda L)\, y_t = u_t \tag{1}$$

$$u_t = \Theta(L)\epsilon_t \tag{2}$$

The structural shocks in this VAR, denoted by η_t, are such that $\epsilon_t = A_0\eta_t$ where A_0 is the matrix that identifies the structural shocks. $\Theta(L) \equiv I_m + \sum_{i=1}^{\infty}\Theta_i L^i$, is a matrix polynomial in the lag operator L, and the following assumptions hold:

Assumption 1: ϵ_t is a martingale difference sequence with covariance Σ and finite fourth moments.

Assumption 2: $|\Theta(z)| = 0$ has roots outside the unit circle, and $\sum_{i=0}^{\infty} i\,\|\Theta_i\| < \infty$, where $\|\Theta_i\| = \max_{r,s}|\Theta_i^{(r,s)}|$, where r, s denote the row and the column position of an element of Θ_i.

Assumption 3: $\Lambda = I + \frac{1}{T}\mathbf{C}$, where $\mathbf{C}$ is a diagonal matrix with fixed elements $c_1, c_2, ...c_m$ along the main diagonal.

Assumption 4: The matrix V is known**.**

Assumption 5: $\frac{h}{T} \underset{T\to\infty}{\to} \delta$.

Assumption 1 and 2 are fairly standard. Assumption 1 implies that $\{\epsilon_t\}$ satisfies a functional central limit theorem, while Assumption 2 ensures that u_t is a stationary process satisfying a Beveridge and Nelson decomposition, and with spectral density

at frequency zero $\Omega \equiv \Theta(1)\Sigma\Theta(1)'$ such that $\Omega^{1/2} \equiv \Theta(1)\Sigma^{1/2}$ is invertible. Assumption 3 assumes that Λ is diagonal with the largest roots of the process on the main diagonal. These roots are distinct and persistent (close to $I(1)$) such that their real parts follow a local-to-unity process. This allows us to use multivariate local-to-unity asymptotic theory (see Stock (1996) and Phillips (1998)) to improve the asymptotic approximation in small samples. By allowing $c_1, c_2, .., c_m$ to be either negative or positive, our approximation encompasses both stationary but persistent, and mildly explosive processes. The diagonality of Λ rules out processes that behave like I(2) in small samples, whereas the eigenvectors V describe possible cointegrating vectors. Assumption 4 emphasizes that throughout this paper we assume that the cointegrating vectors are known (as in Elliott, Jansson and Pesavento (2002))[6], which is not a limitation as long as the researcher is interested in the structural shocks to y_t, like we assume. To obtain better asymptotic approximations to the distribution of IRF coefficients in small samples, Assumption 5 allows the lead time of the IRF to be a fixed fraction of the sample size.

From (1), the DGP can be rewritten as (see Rossi (2005) for a proof of this result):

$$y_{t+h} = \sum_{j=0}^{h} \Lambda^j \left(I + \Lambda^{-1}\Theta_1 + \Lambda^{-2}\Theta_2 + ...\right) \epsilon_{t+h-j} + \Lambda^{h+1} y_{t-1} + a_{t,h} \qquad (3)$$

[6] The general model (1) nests some popular models as particular cases as described in an Appendix available upon request.

where $a_{t,h} \equiv \left(\Lambda^{h+1} - L^{-(h+1)}\right) \sum_{j=0}^{\infty} \Lambda^j \sum_{s=j+1}^{\infty} \Lambda^{-s} \Theta_s \epsilon_{t-1-j}$.[7] For the purposes of approximating the IRF at horizon h, under Assumptions 1-5, $\left(I + \Lambda^{-1}\Theta_1 + \Lambda^{-2}\Theta_2 + ...\right)$ can be approximated by $\Theta(I)$, and $a_{t,h}$ can be ignored. In fact, $\Lambda^{h+1} \sum_{j=0}^{\infty} \Phi^j \sum_{s=j+1}^{\infty} \Lambda^{-s} \Theta_s \epsilon_{t-1-j}$ does not depend upon ϵ_t, whereas $L^{-(h+1)} \sum_{j=0}^{\infty} \Lambda^j \sum_{s=j+1}^{\infty} \Lambda^{-s} \Theta_s \epsilon_{t-1-j}$ contains information on ϵ_t only through $\sum_{s=h+1}^{\infty} \Lambda^{h-s} \Theta_s$, which, from the summability property in Assumption 2, goes to zero when $h \to \infty$.[8] Hence, the reduced form and structural IRFs can be approximated respectively as $\frac{\partial y_{t+h}}{\partial \epsilon_t} \simeq \Lambda^h \Theta(I)$ and $\frac{\partial y_{t+h}}{\partial \eta_t} = \frac{\partial y_{t+h}}{\partial \epsilon_t} \frac{\partial \epsilon_t}{\partial \eta_t} \simeq \Lambda^h \Theta(I) A_0$.

Considering assumptions 3 and 5 together, we have that $\Lambda^h \underset{T \to \infty}{\to} e^{\mathbf{C}\delta}$, where $e^{\mathbf{C}\delta}$ denotes a diagonal matrix with $(e^{c_1\delta}, e^{c_2\delta}, ...e^{c_m\delta})$ on the main diagonal. Note that, due to the nature of our assumptions, our confidence bands will be appropriate at long lead times and as long as the process is highly persistent (including a unit root). How long the lead time in practice has to be in order for our approximation to be accurate will be investigated in a Monte Carlo experiment.

The IRF of the effect of a unitary $j-th$ structural shock on the $k-th$ variable

[7] Note that (3) is a device for obtaining better asymptotic approximations in small samples. Assumptions 3 and 5 govern its validity. Also, (3) follows directly from eq. (15) in Rossi (2005) by letting $q \to \infty$.

[8] This clearly follows as $\left\| \sum_{s=h+1}^{\infty} \Lambda^{h-s} \Theta_s \right\| \leq \sum_{s=h+1}^{\infty} \left\| \Lambda^{h-s} \right\| \|\Theta_s\| \leq \left(\sum_{s=h+1}^{\infty} \left\| \Lambda^{h-s} \right\| \right) \left(\sum_{s=h+1}^{\infty} \|\Theta_s\| \right)$. The first factor in the last expression is bounded given Assumption 3, while $\left(\sum_{s=h+1}^{\infty} \|\Theta_s\| \right) \underset{h \to \infty}{\to} 0$ as it is absolute summable by Assumption (2).

in y_{t+h} can then be approximated by:

$$\frac{\partial y_{t+h}^{(k)}}{\partial \eta_t^{(j)}} \simeq \mathbf{i}_k' \Lambda^h \Theta(I) A_0 \mathbf{i}_j \underset{T\to\infty}{\to} \mathbf{i}_k' e^{\mathbf{C}\delta} \Theta(I) A_0 \mathbf{i}_j \tag{4}$$

where $\mathbf{i}_s$ denotes the $s-th$ column of the $m \times m$ identity matrix. Note that the limiting IRF depends on the largest roots, described by $\mathbf{C}$, and on the cumulated short run dynamics, described by $\Theta(I)$. At long horizons, the uncertainty associated with the short run parameters is of smaller order than the uncertainty associated with the largest roots, and we can ignore the uncertainty in the estimation of $\Theta(I)$ by simply replacing it with a consistent estimator.

Equation (4) is an approximation to the IRF that is valid under our assumptions 1-5, and that can be used to construct confidence intervals for the IRFs of y_t. Although C cannot be consistently estimated, methods for constructing valid confidence intervals for the $c_k's$ are available (e.g. Stock (1991) or Elliott and Jansson (2003)). Let the confidence interval for c_k obtained by one of such methods be denoted by $(c_{L,k}, c_{U,k})$, for $k = 1, ..m$. Since the elements in (4) are monotone functions of c_k, we propose to construct confidence intervals for the IRF coefficients from the confidence intervals for the c_k's as $(e^{\delta c_{L,k}} \mathbf{i}_k' \widehat{\Theta}(I) \widehat{A}_0 \mathbf{i}_j, e^{\delta c_{U,k}} \mathbf{i}_k' \widehat{\Theta}(I) \widehat{A}_0 \mathbf{i}_j)$ where $\widehat{\Theta}(I)$ and $\widehat{A}_0$ are any consistent estimates of $\Theta(I)$ and A_0. More examples on how to implement this method in practice are provided in Sections 4 and 5.

As usual, different types of identification result in different IRFs. The long-

run identification (Blanchard and Quah (1989)) imposes a triangular structure to $\Theta(I) A_0$. The Wold ordering identification (Sims (1980)) imposes constraints on A_0 such that A_0 is the Choleski factor of Σ**.** We will be agnostic about the identification procedure and will take it as given, as our goal is to provide a method for constructing IRF bands that have correct coverage, not to propose new methods to identify shocks. It is also important to note that, while this paper focuses on structural IRFs, we expect that our method could also be applied to the analysis of the Generalized IRFs proposed by Koop, Pesaran and Potter (1996), and Pesaran et al. (1998, 2004).

4. Multivariate impulse response functions with a possible unit root

This section compares various methods for the construction of confidence bands for IRFs in multivariate models in the presence of a root local to unity, both in terms of coverage and length. For ease of exposition, we focus on a bivariate VAR, where one variable has an exact unit root while the other has a large root that is close to one. This corresponds to the common situation in which the researcher "knows" that one variable is I(1) but is unsure about whether the other variable is stationary or not (e.g. Rogers (1999)). While this provides a useful approach when the researcher is unsure about the persistence in only one of the variables (and, as we will show, anyway is appropriate in case the researcher is interested in the IRFs for just that variable), nevertheless it clearly does not represent the most general situation. At the end of this section, we show that our method is robust to mis-specification in the largest root of the second variable (provided that it is persistent). The next section

will instead focus on the more general situation in which there can be more than one local-to-unity root. The DGP is:

$$\begin{aligned} w_{1t} &= \mu + \rho w_{1t-1} + u_{1t} \\ w_{2t} &= w_{2t-1} + u_{2t} \end{aligned} \tag{5}$$

where $\rho = 1 + \frac{c}{T}$. The dynamics of the model is determined by $\Psi(L) u_t = \varepsilon_t$, where $\Psi(L) = I - \Psi L$. Equation (5) is a simplified version of our model (1) where $\Phi = \Lambda = \begin{bmatrix} 1+\frac{c}{T} & 0 \\ 0 & 1 \end{bmatrix}$ and $\Theta(L) = \Psi(L)^{-1}$. The elements of the spectral density at frequency zero of $u_{t,}$, $\Omega = \Psi(1)^{-1} \Sigma \Psi(1)^{-1'}$, are denoted as $\omega_{ij}, i, j = 1, 2$. The structural IRFs are computed by using a Wold causal ordering identification where shocks to w_{2t} do not contemporaneously affect w_{1t} (i.e. $\Sigma^{1/2}$ is lower triangular). The structural IRFs at long horizons are (up to an irrelevant $o(T^{-1/2})$):

$$\frac{\partial w_{t+h}}{\partial \eta_t} = \Lambda^h \Omega^{1/2} \underset{T \to \infty}{\to} \begin{bmatrix} e^{c\delta} & 0 \\ 0 & 1 \end{bmatrix} \Omega^{1/2} \tag{6}$$

Let $R^2 \equiv \omega_{11}^{-1} \omega_{12} \omega_{22}^{-1} \omega_{21}$ be the square of the frequency zero correlation between the innovation in Δw_{2t} and the innovation in the quasi difference of w_{1t}. As discussed in Elliott and Jansson (2003), R^2 is an important nuisance parameter that affects the power of the EJ test, one of the tests that we use in our simulations. We are

interested in the structural response of w_{1t} to a unitary shock to η_{1t}:

$$\frac{\partial w_{1t+h}}{\partial \eta_{1t}} \underset{T\to\infty}{\to} \omega_{[11]}^{1/2} e^{c\delta} \tag{7}$$

where $\omega_{[11]}^{1/2}$ is the element $[1,1]$ of the matrix $\Omega^{1/2}$.

To compare the coverage rates of different methods, we simulated model (5) with $\Psi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}$, $R^2 = 0.5$,[9] and $T = 100$ over 5000 Monte Carlo replications. The IRF bands for the VAR in levels and in differences are computed by using Lütkepohl's (1990) method with simulated standard errors based on 500 replications.

Here is how we implement our method in detail. First, we construct a confidence interval for c. We then obtain the confidence bands for the IRF at long horizons by using the confidence interval for c and a consistent estimator of $\omega_{[11]}^{1/2}$ in the closed-form formula (7). To construct confidence intervals for c, we invert the acceptance region of the following tests for a unit root in w_{1t}: the ADF, the Elliott et al. (1996, ERS thereafter), and the Elliott and Jansson (2003, EJ thereafter) tests. These tests differ in their power properties, and the most powerful tests will usually lead to smaller confidence bands (often at the cost of additional computations). Thus, each of these methods will provide different IRF bands, all of which will have the correct coverage, but may have different lengths. To give an example, suppose that the researcher is interested in $\frac{\partial w_{1t+h}}{\partial \eta_{1t}}$. In this case, the method is implemented in

[9] We choose $R^2 = 0.5$, a case in which EJ test has significantly higher power than ERS. The reader is referred to Elliott and Jansson (2003) for a comparison of the different tests.

practice as follows. (i) Construct a confidence interval for c (denoted by $(c_L; c_U)$) by inverting the acceptance region of a unit root test. For the sake of simplicity, suppose that the researcher is interested in inverting the ADF test and that the estimated statistic is -2.068. Thus, directly from Stock's (1991) Table A1, p. 455-6, the confidence interval for c is $(-13.73, 2.411)$. (ii) Run a VAR in differences to estimate $\Theta(L)$, A_0, and use these to estimate $\Omega^{1/2}$. (iii) Finally, the confidence band is obtained by plugging the estimate from (ii) into eq. (7). If $T = 100$ and $h = 20$, then $\delta = 0.2$, $e^{c_L \delta} = 0.064$ and $e^{c_U \delta} = 1.619$. Thus, the confidence interval for the IRF is $(\widehat{\omega}_{[11]}^{1/2} e^{c_U \delta}, \widehat{\omega}_{[11]}^{1/2} e^{c_L \delta}) = (0.064\widehat{\omega}_{[11]}^{1/2}, 1.619\widehat{\omega}_{[11]}^{1/2})$, where $\widehat{\omega}_{[11]}^{1/2}$ is the estimate from the identified structural VAR.

In Figure 1 we compare our method with the use of the conventional asymptotic normal approximations based on VARs estimated either in first differences or in levels, where the decision between a VAR in levels or in first differences is based on a unit root pre-test on w_{1t}. If the pre-test fails to reject a unit root at a 5% level, then the VAR is estimated with w_{1t} in first differences, otherwise w_{1t} is used in levels. We report both cases in which the ADF and the ERS test statistics are used in the pre-test, respectively labeled "PRET_ADF" and "PRET_ERS". Since pre-testing is known not to work well (Cavanagh et al. (1995)), we also report results for a VAR in levels without pre-testing, labeled "LEV".[10] Figure 1 displays one minus the

[10] Given that we are pre-testing y_t for a unit root, if the two stages were independent, the probability that the confidence interval contain the true IRF when the null of $c = 0$ is true would be $(1-0.05)^2$. Using 95% confidence intervals for the IRFs allows us to do a fairer comparison of the empirical coverage rates of the different methods. However, the two stages are not independent, because of

coverage rate of the various IRFs bands for $\rho = 1$, 0.97, 0.90, 0.80. For $\rho = 1$, the results are close to the nominal level (0.10) for all methods, except for the VAR in level, for which it is around 40%. This result reflects the bias in the estimation of ρ from a regression in levels. Our method performs very well for various values of ρ: all three variants (ADF, ERS, EJ) have coverage rates that are close to the nominal level at long horizons across different values of ρ. On the other hand, as ρ moves away from unity, confidence intervals computed from the VAR in first differences start to behave poorly, with coverage that approaches zero as the horizon increases. In fact, for large enough (though less than one) values of ρ, pre-tests have low power to reject the hypothesis of a unit root, and select a VAR in first difference most of the times. As expected, different pre-tests have significantly different coverage properties. The better coverage rate of ERS relative to ADF reflects the higher power of ERS test against alternatives that are close to one. As ρ moves further away from unity, the pre-tests are able to reject the hypothesis of a unit root more often, and their coverage improves. As ρ becomes very small (say $\rho = .80$), Assumption 3 is no longer a good approximation and our method starts to worsen: ADF and ERS have a coverage rate around 60% while EJ has a coverage rate around 70%.[11]

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the correlation between the residuals (as in Cavanagh et al. (1995)), so the pre-test coverage is not 0.90, even asymptotically. The $\overline{c}$ used here is -7, as ERS and EJ recommend.

[11] In unreported results, we investigate the case in which $\rho = 1.01$ (a mildly explosive root) within the same Monte Carlo design as above. We find that, as expected, our method is robust in this case as well. The VAR in levels performs worse in general than in the other cases depicted in Figure 1 (because the process becomes more non-stationary). The pre-test procedures also perform worse relative to the case in which $\rho = 1$, because we depart from the exact unit root case. In unreported Monte Carlo simulations, we also verified the robustness of our method to the presence of a possible deterministic trend. The results do not change if the true DGP has a deterministic trend, provided

Table 1 reports median confidence interval lengths for the various methods. The higher power of the EJ test is reflected in the shorter median confidence interval length. For values of ρ that are close to one, the inversion of the EJ test produces confidence intervals that can be a little more than half the length of confidence intervals obtained by inverting ADF. At the same time, confidence intervals constructed inverting ERS and EJ are not symmetric (results are not reported). The smaller length of the interval comes at the cost of not having a median unbiased confidence interval. The interval length for VAR in levels is also very small but, as we just noted, having short intervals is irrelevant if the probability of including the true IRF is zero.

To check the robustness of our method to some of the assumptions in the model, we compute the coverage rates for all methods when model (5) is estimated even though the second root is not exactly equal to one. Figure 2 shows that, when the second root is close to one (0.99), our method still produces confidence intervals with coverage close to their nominal value. Confidence intervals from VAR either in levels or in differences still have bad coverage at long horizons. As the second root moves away from one, ERS and ADF still perform well, while EJ shows significant size distortions. The robust behavior of ERS and ADF is due to the fact that they use information only on w_{1t}. Since we are assuming that Φ is diagonal, the misspecification in the estimation of the second row of $\Theta(I)$ does not affect the IRF for w_{1t}. On the other hand, the EJ test uses information contained in Δw_{2t}. Since in this

that the researcher takes this into account in estimating the parameters.

case w_{2t} is not really I(1), we are using incorrect information in calculating the test statistic. Overall, our method performs well even if the other variables in the model do not have exact unit roots. This suggests that it is possible to extend this method to a situation in which there are many unknown unit roots, as discussed next.

FIGURES 1, 2 AND TABLE 1 HERE

5. Multivariate impulse response functions with many unit roots

It is possible to generalize our method and analyze the coverage of multivariate confidence bands for IRFs when there is more than one root local to unity. We explore this possibility in a bivariate VAR with two roots local to unity.[12] The DGP is now:

$$\begin{aligned} w_{1t} &= \mu + \rho_1 w_{1t-1} + u_{1t} \\ w_{2t} &= \rho_2 w_{2t-1} + u_{2t} \end{aligned} \tag{8}$$

where $\rho_i = 1 + \frac{c_i}{T}$, $i = 1, 2$, $\Psi(L) u_t = \varepsilon_t$, and the values for $\Psi(L)$, Σ and the identification strategy are the same as in the previous Section. Now the structural IRFs at long horizons are (again, up to an irrelevant $o_p(T^{-1/2})$ term):

$$\frac{\partial w_{t+h}}{\partial \eta_t} = \Lambda^h \Omega^{1/2} \underset{T \to \infty}{\to} \begin{bmatrix} e^{c_1 \delta} & 0 \\ 0 & e^{c_2 \delta} \end{bmatrix} \Omega^{1/2} \tag{9}$$

[12] Note, however, that the method proposed in this paper can theoretically be applied to m local-to-unity VARs, $m > 2$.

Thus:

$$IRF_{11} \equiv \frac{\partial w_{1t+h}}{\partial \eta_{1t}} \rightarrow \omega_{[11]}^{1/2} e^{c_1 \delta}, \quad IRF_{12} \equiv \frac{\partial w_{1t+h}}{\partial \eta_{2t}} \rightarrow \omega_{[12]}^{1/2} e^{c_1 \delta} \tag{10}$$

$$IRF_{21} \equiv \frac{\partial w_{2t+h}}{\partial \eta_{1t}} \Rightarrow \omega_{[21]}^{1/2} e^{c_2 \delta}, \quad IRF_{22} \equiv \frac{\partial w_{2t+h}}{\partial \eta_{2t}} \Rightarrow \omega_{[22]}^{1/2} e^{c_2 \delta} \tag{11}$$

The confidence intervals for the IRF are then estimated as explained in Section 4. Here we focus on the following methods: (i) the method proposed in this paper, where the first stage confidence interval for c is obtained by inverting the ADF test (labeled "ADF"); (ii) confidence bands obtained after an ERS unit root pre-test on both variables in the VAR (labeled "PRET_ERS"); (iii) confidence bands obtained from a VAR in levels by using Lütkepohl's (1990) method (labeled "LEV").

Figure 3 shows the results. It displays one minus the coverage rate of various confidence bands for IRFs for the above DGP with $\rho_1 = 0.98$ and $\rho_2 = 0.95$. Our method delivers confidence bands with the correct coverage for $\delta \geq 0.06$, as before. Its coverage is better than that of a VAR in levels even for very short horizons ($\delta \geq 0.04$), while the coverage of the IRFs in the pre-test case is worse for IRF_{11} than for IRF_{22}, as the second root is closer to unity than the first root.

Overall, the results confirm the intuition provided in Figure 2 above: since $\Phi = \Lambda$, which is diagonal as it contains the roots of the process, it is possible to accurately approximate the IRF bands by repeating the procedure discussed in Section 4 separately for all the variables in the VAR. The diagonality assumption is important, and it has to be verified for the particular empirical application at hand, although it is

reasonable in many macroeconomic applications (e.g. Pesavento and Rossi (2003)).

FIGURE 3 HERE

6. A method robust at short horizons

While the main method proposed in this paper applies to long horizons, namely horizons longer than ten percent of the sample size, it is also possible to refine it in order to make it robust to short horizons too. We discuss one such method in this section. The method is inspired by the work of Wright (2000). To overcome the difficulties associated with the construction of confidence bands for IRFs in a univariate context, Wright (2000) proposes a Bonferroni method that ensures that coverage is at least equal to the nominal level at every horizon. A main advantage in our framework is that we can apply his insight to the largest roots of a multivariate process, thus making the method feasible for implementation in a multivariate context.[13] Another advantage of our method is that it controls coverage exactly at both short and long horizons, while the coverage at intermediate horizons is conservative. The method is slightly more complicated than those proposed in the previous sections, but still easy and fast to use. However, it builds on Bonferroni methods, so it will be conservative at medium horizons, and not median unbiased.[14]

The method is as follows: (a) compute a $(1-\alpha_1)\,\%$ confidence interval for c, (c_L, c_U), by inverting an ADF test, $\alpha_1 = 0.10$. Use this confidence interval to compute

[13] Computationally, there is a small difference between Wright (2000) and our method, which is implemented by not re-estimating the Θ_i associated to different values of c in step (a) below.

[14] In general, it is more median unbiased the closer the root is to unity.

$e^{c_L\delta}$ and $e^{c_U\delta}$; (b) estimate a VAR in quasi differences and construct a $(1-\alpha_2)\,\%$ confidence interval for Θ_i by using Lütkepohl's (1990) method, where $\alpha_2 = 0.10$; (c) for each limit of the confidence interval for Θ_i at each horizon, compute $e^{c_L\delta}\ \Theta_i$ and $e^{c_U\delta}\ \Theta_i$; (d) the overall confidence interval is: $(\min_i\ e^{C_L\delta}\ \Theta_i;\ \max_i\ e^{C_U\delta}\ \Theta_i)$. By the Bonferroni inequality, its coverage should be at least $(1-\alpha_2-\alpha_1)\,\%$ at each horizon h. By construction, this method is now pointwise at any horizon.[15]

To offer more insights on the method, consider a simple univariate ARMA(1,q) process: $(1-\rho L)\,y_t = u_t$, $u_t = \sum_{j=0}^{q}\theta_j\epsilon_{t-j}$, $\theta_0 = 1$. Note that $u_{t+h} = \sum_{j=0}^{q}\theta_j\epsilon_{t+h-j}$. The IRF of the level of y_{t+h} to the shock ϵ_t, $h \geq 0$, is $\rho^h\left(1+\sum_{j=1}^{\min(h,q)}\rho^{-j}\theta_j\right)$ $\simeq \rho^h\left(1+\sum_{j=1}^{\min(h,q)}\theta_j\right)$. Note that $\left(1+\sum_{j=1}^{\min(h,q)}\theta_j\right)$ is also the cumulative IRF of u_{t+h} onto ϵ_t. In the long horizon approximation, we only considered uncertainty on ρ^h by estimating the confidence interval for $e^{c\delta}$ and using a consistent estimate for $\left(1+\sum_{j=1}^{\min(h,q)}\theta_j\right)$. At short horizons, however, the uncertainty on the $\theta's$ is important, so we cannot simply plug-in an estimate of $\left(1+\sum_{j=1}^{\min(h,q)}\theta_j\right)$, but need to evaluate its variability. To take the latter into account, we construct confidence intervals for $\left(1+\sum_{j=1}^{\min(h,q)}\theta_j\right)$ in step (b).[16] The confidence intervals are obtained by

[15] We choose $\alpha_1 = 0.10$ because this guarantees that the coverage at long horizons is 0.90, so that the results are comparable with our long horizon method at a nominal coverage of 0.90.

[16] These can be approximated by confidence intervals for the cumulative IRFs from an AR for the quasi-differences, $(1-\rho L)\,y_t$. An approximation to the latter can be obtained directly as the cumulative IRF from an AR for the first differences of y_t, $\Delta y_t \simeq u_t$, with the advantage of being easier to implement in practice. In fact, the latter will turn out to be extremely useful in the general VAR(p) case, where these cumulative IRFs can be obtained by standard packages as the IRF to the *level* of the variables when the model is estimated in *first differences*. When the persistence is not very high, it might be safer to implement this method by using quasi-differences (i.e. the residuals from the estimation of a VAR(1) in levels) rather than first differences, as discussed in Pesavento and Rossi (2003).

Monte Carlo simulation. Then we combine the two confidence intervals as described in steps (c) and (d) by using the Bonferroni inequality.

If $(1-\alpha_1)\,\%$ is the nominal coverage level for the confidence interval in step (a) – the confidence interval for c – and $(1-\alpha_2)\,\%$ is the nominal coverage level for step (b) – the confidence interval for Θ_i –, then the short-run method we propose will have a coverage level equal to $(1-\alpha_2)\,\%$ at very short horizons, and equal to $(1-\alpha_1)\,\%$ at very long horizons. For intermediate values of the horizon, the coverage level will be determined by the Bonferroni inequality, and will be at least $[1-(\alpha_1+\alpha_2)]\%$.

The reason why we can exactly control coverage at both very short and very long horizons is because the uncertainty on the estimate of ρ is of order $O_p\left(T^{-1}\right)$, whereas the uncertainty on Θ_i is of order $O_p\left(T^{-1/2}\right)$ and the uncertainty on the estimate of ρ^h (when h is large) is of order $O_p\left(1\right)$ (cfr. the discussion in Rossi (2001)). IRFs at the one period ahead horizon depend on ρ and Θ_i, where the uncertainty over ρ is asymptotically irrelevant relative to the uncertainty over Θ_i. Thus, this implies that the coverage at very short horizons is equal to $(1-\alpha_2)\,\%$. On the other hand, IRFs at very long horizons depend on ρ^h and Θ_i. The uncertainty over ρ^h is of order $O_p\left(1\right)$ whereas the uncertainty on Θ_i (and on its powers) is of order $O_p\left(T^{-1/2}\right)$. Thus, at very long horizons, the uncertainty over Θ_i is asymptotically irrelevant, and inference is driven by the uncertainty over ρ^h. This implies that the coverage level at very long horizons is equal to $(1-\alpha_1)\,\%$. Therefore, one of the advantages of the method we propose here relative to the approach taken in Wright (2000) is that our method

asymptotically has a size equal to its nominal value at both very short and very long horizons. In contrast, Wright's (2000) method is conservative at every horizon, and the researcher would not know the actual nominal coverage at any horizon (although the researcher would know a bound on it, given by the Bonferroni inequality).

We perform a simple Monte Carlo experiment to evaluate the performance of this method relative to the one proposed in the main part of the paper. The DGP is the same as described in (5), where $\rho = 0.97$, $T = 100$. Table 2 shows the results. The new method is labeled "Small h". Subscripts "L" and "R" are used to denote the empirical rejection probabilities of the true IRF laying respectively on the left and on the right of the proposed confidence interval, which should ideally be 0.05. For comparison, we also report the same probabilities based on our "Large h" method. Both methods are based on confidence intervals obtained by inverting a simple ADF test. Note that the "Small h" method improves the empirical rejection probabilities at short horizons relative to the "Large h" method. However, it is not median unbiased at short horizons, and it is slightly conservative. Nevertheless, its overall coverage properties are quite good at every horizon.

TABLE 2 HERE

7. An empirical application to exchange rate dynamics

As an empirical application, we analyze the nominal versus real sources of fluctuations in real and nominal exchange rates. There is a large literature on this topic. Eichenbaum and Evans (1995) influential paper use an identification *à* la Sims; Clar-

ida and Gali (1994), Lastrapes (1992) and Rogers (1999), among others, impose a long-run identification. Different types of identification methods will result in different restrictions on $\Theta(I) A_0$ in (4) and therefore will result in different IRF. Although our method works regardless of the type of identification method used, here we will focus on a short run identification based on Wold ordering.

We focus on the Eichenbaum and Evans' (1995) influential paper, which has been used for illustrative purposes before (e.g. Kilian (1998b)). We use the ratio of the log of non-borrowed reserves to the log of total reserves as a measure of the policy instrument. The five-variables VAR also includes U.S. industrial production, U.S. CPI, the difference between U.S. and foreign short-term rates and the real exchange rate. The sample starts in 1973:1, with different ending dates for each country (2001:12 for Germany, 2002:9 for Japan and the United Kingdom and 2001:5 for Italy). Following Rogers (1999), in the VAR we use first differences of all the variables other than the real exchange rate. Here, an increase in the real exchange rates represents a depreciation of the U.S. real exchange rate. The structural IRFs are calculated using the Wold ordering: output, prices, reserves, interest rate differential and real exchange rate. An exogenous contractionary monetary shock is identified as the component of a negative innovation in $NBRX$ that is orthogonal to prices and output.[17]

[17] Data for the U.S. (industrial production, 3 months T-bill rates, Total Reserves and Non Borrowed Reserves with extended credit) and bilateral monthly nominal exchange rates are from the Federal Reserve Database. Data for Industrial Production and short term money markets rates for each foreign country and CPI for all countries including the U.S. are from the IFS database. All variables are in logarithms except for the interest rates. Since VARs are known to be sensitive to the selection of the lag length (see Kilian (2001) and Ivanov and Kilian (2005)), in order to make our results comparable with Eichenbaum and Evans (1995), we choose 6 lags, which is the same VAR lag length

The descriptive statistics and unit root tests reported in Table 3 suggest that the real exchange rate is a highly persistent variable. For most countries, we cannot reject a unit root. However, we cannot reject many stationary, although persistent, alternatives either. This prompted us to apply the method proposed in this paper. Since we are interested in the medium to long horizon response of the real exchange rate to a contractionary monetary policy shock, we focus on the method discussed in Section 4. If instead the researcher were interested in the immediate response of the real exchange rate then the method described in Section 6 would be more appropriate. Both methods are feasible here, and they are expected to give similar results at long horizons, but they will significantly differ at short horizons. As we showed in Section 6, the method robust at short horizons is pointwise and conservative, so if the researcher is interested in the response at medium to long horizons, as we are, the method proposed in Section 4 is more appropriate.

Figure 4 compares various confidence intervals for the response of the real exchange rate to a contractionary monetary policy shock for different countries. The two outer solid lines are the confidence bands computed with our method by inverting ERS test for a unit root on the real exchange rate, and the middle solid line is the median unbiased estimate of the IRF.[18] The figure also reports confidence bands from a VAR with all variables in levels (solid line with diamonds) and with all variables

that they have.

[18] Although inverting the ERS test produces confidence bands that are less precise than the ones obtained by inverting EJ test, the simulations in the previous section show that EJ is also less robust to the true DGP of the other variables.

in first differences (dotted line with stars). Some interesting results emerge. The confidence bands for VARs in first differences show more persistent effects than those based on VARs in levels, as intuition may suggest. In fact, the former may remain bounded away from zero (as for Japan and the U.K.), showing that the effects of the shock are much more persistent and may never disappear, even in the long run. On the other hand, the confidence bands for the VAR in levels include zero after a few quarters. Our method is somewhere in the middle, but in general it shows that shocks are more persistent than a VAR in levels would predict, with a dollar not depreciating even after a long period of time for most currencies (e.g. more than 4 years for Germany, Japan and U.K.). As in the previous example, our method suggest a slightly less persistent response of the real exchange than that estimated with a VAR in first difference. As in Eichenbaum and Evans (1995), a contractionary shock to U.S. monetary policy leads to a persistent appreciation of the real exchange rates for Germany, Japan, and UK. For Italy, both the VAR in first differences and in levels suggest an initial appreciation that dies out in the long run, while the inversion of ERS suggests a small, but persistent, depreciation of the real exchange rate.

TABLE 3 AND FIGURE 4 HERE

8. Conclusions

Whether shocks have long run effects on economic variables, and how persistent these effects are, is a highly debated issue. Results are sensitive to the order of integration of the variables. This problem is either ignored, or ad-hoc robustness checks that lack

theoretical justifications are routinely performed. We propose a simple method to estimate the long run effects of the shocks and the uncertainty around these estimates. The method has the advantage of controlling coverage over the *whole* IRF trajectory at long horizons (i.e. it is not pointwise). With this method, researchers do not have to take a stand on whether the process is I(1) or I(0) before doing inference. Thus, the method provides a feasible alternative to unit root pre-tests, which we show imply considerable coverage distortions.

The method that we propose for long horizons will perform well only at medium to long horizons, so we also provide a modification based on Bonferroni methods that allows pointwise inference at both short and long horizons. We would recommend the former to an applied researcher who is interested only at medium to long horizons (say horizons that are more or equal to ten percent of the sample size), but the latter if the researcher is interested in inference at both short and long horizons. The method proposed in this paper is a first step in the analysis of confidence bands for multivariate processes in the presence of high persistence. More simulation evidence will be needed to assess its finite-sample accuracy in multivariate systems and compare it to existing methods such as conventional and bias-corrected boostrap methods. We leave these issues for future research.

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10. Tables and Figures

Table 1: Median confidence interval length

δ	ADF	ERS	EJ	Pret_ADF	Pret_ERS	Level
			$c = 0$			
0.05	0.487	0.362	0.302	0.515	0.519	0.524
0.10	0.855	0.698	0.583	0.520	0.526	0.683
0.15	1.172	1.038	0.854	0.516	0.524	0.808
0.20	1.499	1.422	1.143	0.513	0.520	0.898
0.30	2.300	2.351	1.777	0.512	0.518	1.022
			$c = -3$			
0.05	0.544	0.443	0.333	0.511	0.535	0.531
0.10	0.904	0.725	0.562	0.512	0.543	0.699
0.15	1.198	0.966	0.699	0.503	0.527	0.789
0.20	1.484	1.187	0.796	0.499	0.516	0.814
0.30	2.133	1.558	0.924	0.498	0.502	0.798
			$c = -10$			
0.05	0.570	0.472	0.442	0.487	0.577	0.502
0.10	0.856	0.608	0.562	0.464	0.607	0.579
0.15	1.048	0.640	0.570	0.433	0.509	0.505
0.20	1.190	0.615	0.544	0.418	0.449	0.414
0.30	1.417	0.527	0.452	0.413	0.386	0.277
			$c = -20$			
0.05	0.464	0.453	0.457	0.501	0.540	0.444
0.10	0.431	0.415	0.542	0.375	0.424	0.337
0.15	0.341	0.321	0.524	0.267	0.270	0.191
0.20	0.255	0.237	0.478	0.179	0.181	0.111
0.30	0.139	0.126	0.372	0.082	0.083	0.040

Note: Results are for various horizons δ and are obtained by inverting the following tests: ADF (labeled "ADF"), ERS (labeled "ERS"), EJ (labeled "EJ"), pre-test based on ADF (labeled "PRET_ADF"), pre-test based on ERS (labeled "PRET_ERS") and VAR in level (labeled "LEV").

Table 2: Short and long horizon comparison

δ	Small h_L	Small h_R	Small h	Large h_L	Large h_R	Large h
0.01	0.016	0.080	0.096	0.244	0.388	0.633
0.02	0.003	0.057	0.060	0.101	0.342	0.443
0.03	0.001	0.041	0.042	0.068	0.238	0.306
0.04	0.001	0.037	0.038	0.057	0.164	0.221
0.05	0.002	0.035	0.036	0.049	0.114	0.163
0.06	0.002	0.036	0.039	0.048	0.086	0.135
0.07	0.004	0.038	0.042	0.047	0.072	0.119
0.08	0.005	0.041	0.046	0.046	0.067	0.113
0.09	0.007	0.042	0.049	0.046	0.064	0.110
0.10	0.010	0.043	0.053	0.047	0.063	0.110
0.15	0.020	0.049	0.069	0.049	0.062	0.111
0.2	0.028	0.051	0.079	0.050	0.061	0.111
0.25	0.033	0.053	0.086	0.051	0.061	0.111
0.3	0.037	0.054	0.092	0.052	0.060	0.112

Note: Comparison of the long horizon method of Section 5 (labeled "Large h") and the short-run robust method of Section 6 (labeled "Small h"). The table reports the percentage of samples in which the true value of the whole IRF lays above (subscript "L"), below (subscript "R") or either above or below the confidence interval. The nominal values should be, respectively,

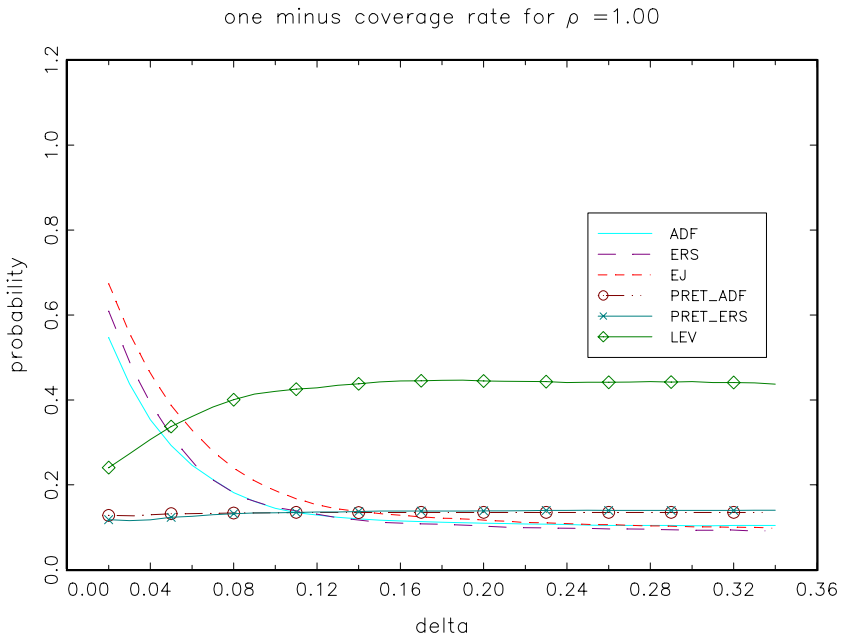
$$0.05,\ 0.05,\ 0.10.$$

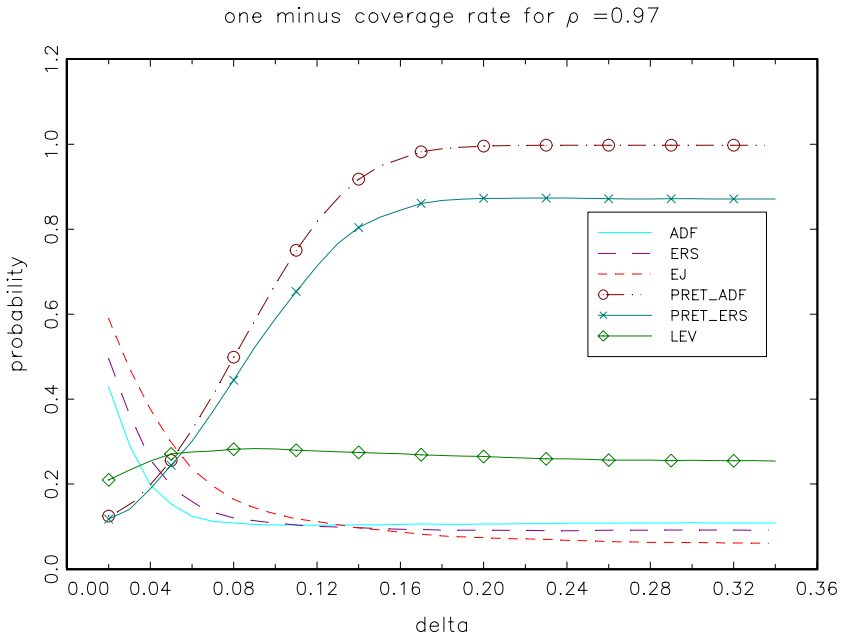
Table 3: Unit root tests for real exchange rates (short-run identification example)

	GER	JAP	UK	ITA
ADF	-1.542	-1.850	-2.470	-2.310
ERS	5.611	27.102	4.755	5.820
EJ	39.389	34.303	37.034	81.453
$\hat{R}^2$	0.134	0.202	0.192	0.126
EJ 5% c.v.	3.454	3.545	3.530	3.444
N. lags	6	6	6	6

Note: The 5% critical values for ADF and ERS are respectively -2.890 and 3.11. The 5% critical values for EJ are reported and computed for each $\hat{R}^2$. All the tests reject for values smaller than the critical values. The covariates used in the EJ test are all variables in the VAR other than the real exchange rate.

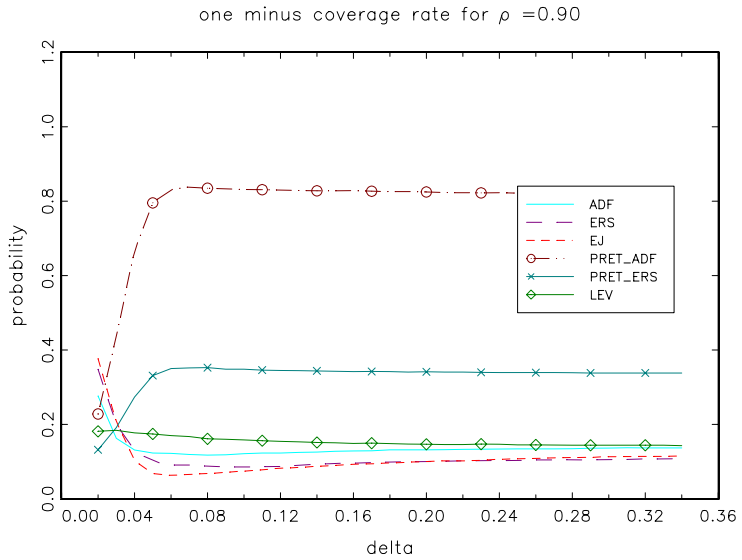
Figure 1(a): One minus coverage rate for various values of ρ, $R^2 = 0.5$

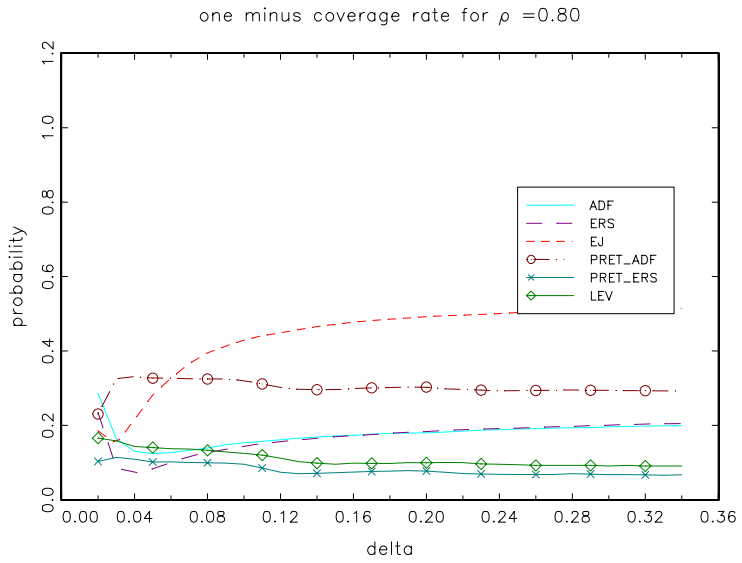




Note: The DGP is as in (5), simulated with a constant for 100 observations.

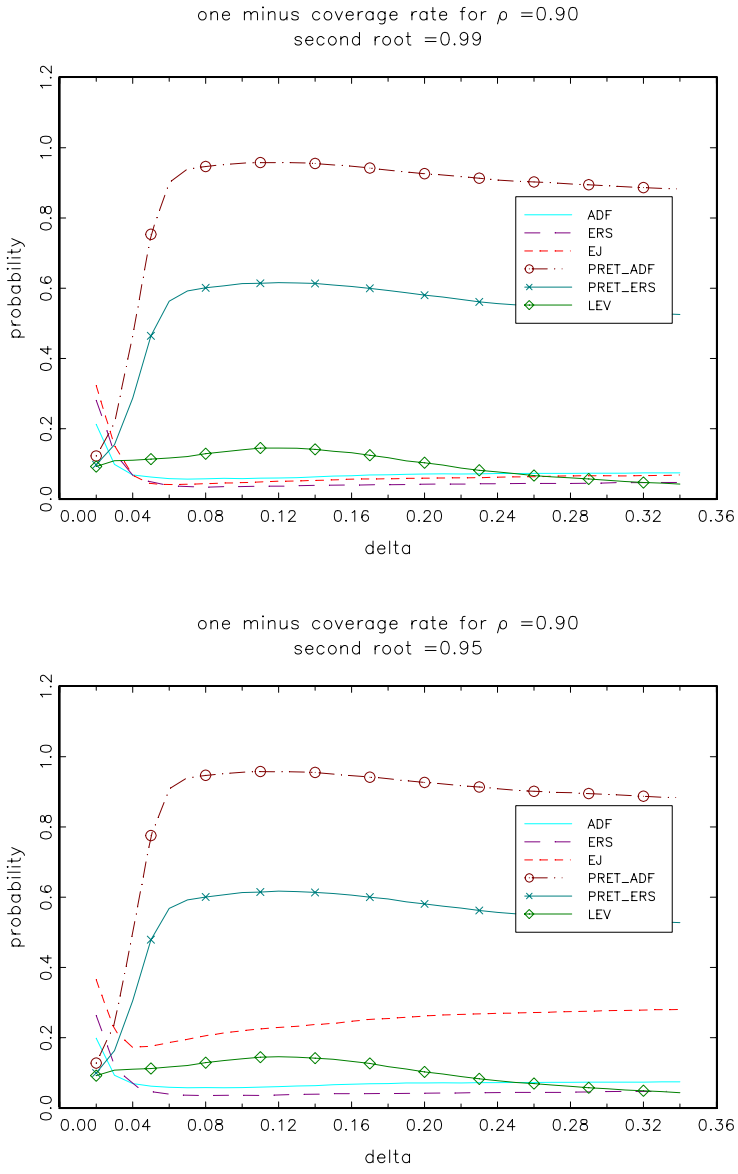
Figure 1(b): One minus coverage rate for various values of ρ, $R^2 = 0.5$





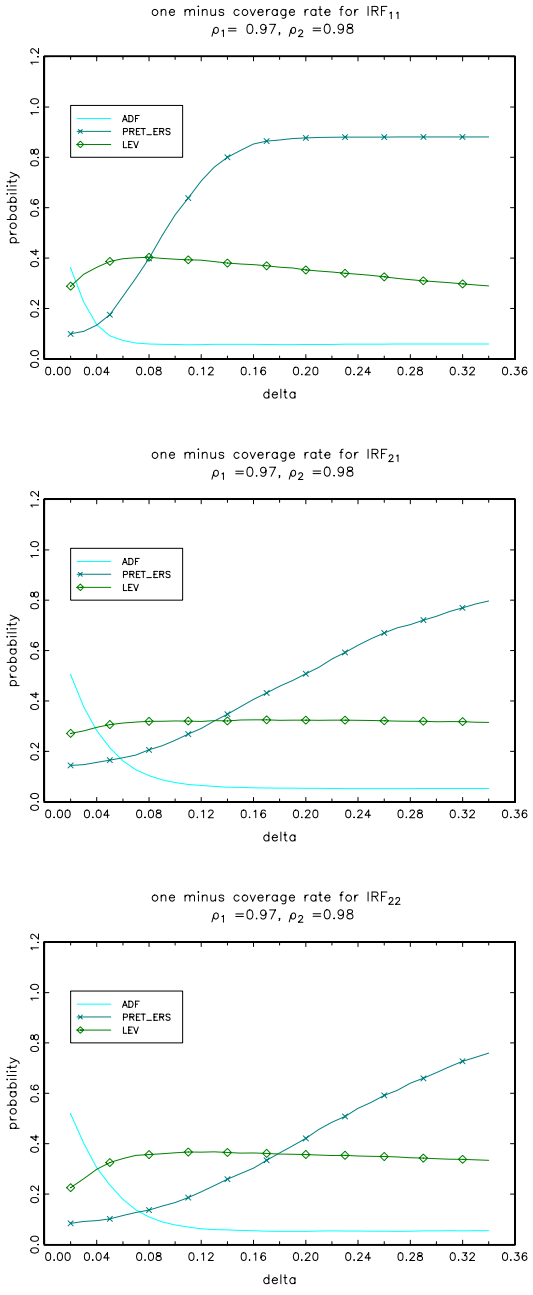
Note: The DGP is as in (5), simulated with a constant for 100 observations

Figure 2: One minus coverage rate for various values of $\rho = 0.90$, $R^2 = 0.5$ for different values of the second root



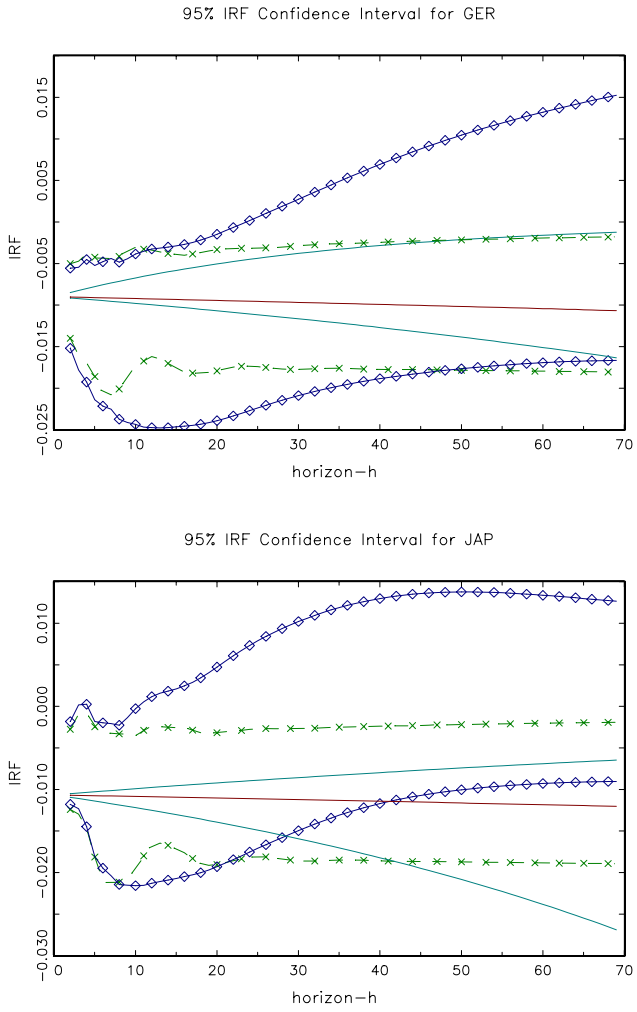
Note:The DGP is as in Section 4 simulate with a constant with 100 observations.

Figure 3: One minus coverage rate for two roots local to unity



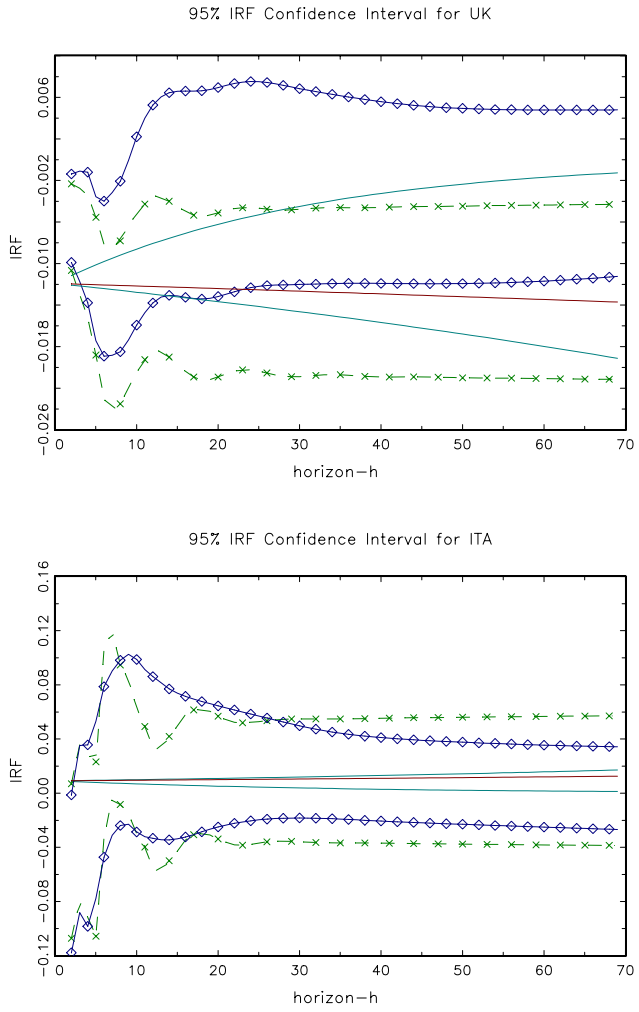
Note:The DGP is as in Section 4 simulate with a constant with 100 observations.

Figure 4(a): Confidence Intervals for a response of $q_t$to a monetary shock[†]



Note:The confidence bands are the following: VAR in first differences (dotted line with stars), ERS/Elliott and Stock (2001) (solid line – the central, thickest line is the median unbiased estimate of the IRF) and a VAR in levels (solid line with diamonds). Identification is obtained using the Wold ordering $\{y_t, p_t, NBRX_t, i_t^* - i_t, q_t\}$.

Figure 4(b): Confidence Intervals for a response of $q_t$to a monetary shock[†]



Note:The confidence bands are the following: VAR in first differences (dotted line with stars), ERS/Elliott and Stock (2001) (solid line – the central, thickest line is the median unbiased estimate of the IRF) and a VAR in levels (solid line with diamonds). Identification is obtained using the Wold ordering $\{y_t, p_t, NBRX_t, i_t^* - i_t, q_t\}$.