Equilibrium plans in constrained environments

Dolores Romero Morales * and Dries Vermeulen †

January 27, 2005

Abstract

In this paper we analyze equilibria in competitive environments under constraints across players' strategies. This means that the action taken by one player limits the possible choices of the other players. In this context the classical approach, Kakutani's Fixed Point theorem, does not work. In particular, best replies against a given strategy profile may not be feasible. We extend Kakutani's Fixed Point theorem to deal with the feasibility issue.

Our main motivation to study this problem of co-dependency comes from the field of supply chain planning. A set of buyers is faced with external demand over a planning horizon, and to satisfy this demand they request inputs from a set of suppliers. Both suppliers and buyers face production capacities and the planning is made in a decentralized manner. A well-known coordination scheme for this setting is the upstream approach where the planning of the buyers is used to decide the request to the suppliers. We show the existence of equilibria for two versions of this coordination model. However, we illustrate with an example that the centralized solution is not, in general, an equilibrium, suggesting that regulation may be needed.

We also apply our Fixed Point theorem to a production economy, where both supply and demand are upper bounded.

*Saïd Business School, University of Oxford, Park End Street, Oxford OX1 1HP, United Kingdom

†Maastricht University, Dept. of Quantitative Economics, P.O. Box 616, 6200 MD Maastricht, The Netherlands

1. Introduction

In a non-cooperative game we are given a set of players and their corresponding strategy sets. Players simultaneously choose their strategies, and the payoff of each player depends on the strategies chosen. Because of this structure of the payoffs, most of the literature devoted to non-cooperative games assumes that the feasible strategies of each player are independent of the strategies chosen by the rest of the players.

However, there are many relevant situations were constraints exist across the strategies of the players. For example, Caron and Laye (2003) consider a Cournot oligopoly model where market demand puts constraints on the total quantity produced by all firms. Also, in the context of strategic voting, Saporiti and Tohmé (2003) consider a voting model where the assumption of single crossing imposes restrictions across individual preferences.

A third example, and our initial motivation to study constrained competitive environments, comes from decentralized decision making in supply chain planning, see e.g. Cachon and Netessine (2004) for non-cooperative games in supply chain analysis. We consider a two-level supply chain composed by suppliers at the first level and buyers at the second level. The buyers are faced with deterministic external demand over a planning horizon, and to satisfy this demand they request inputs from the suppliers. Both suppliers and buyers face linear production and inventory holding costs as well as production capacities. This supply chain functions in a decentralized manner where information such as unit costs and capacities are only locally known. A distribution center manages the inventory of the inputs, and the only information offered by the suppliers and the buyers are upper/lower bounds on production levels. It is obvious that the production planning of the buyers (suppliers) will be, in general, constrained by the amount of inputs offered (requested) by the suppliers (buyers).

Cachon and Netissine argue that the game theoretic analysis of such models is problematic. In this paper we show that nevertheless equilibria in competitive environments under constraints across players' strategies do exist. However, Kakutani's Fixed Point theorem, cannot be applied directly in this context. In particular, best replies against a given strategy profile may not be feasible. We extend Kakutani's Fixed Point theorem to deal with the feasibility issue. Under the usual conditions, upper semicontinuity and convex-valuedness of the best response correspondence, we show the existence of Nash equilibrium.

We apply our existence result to two different settings. The first setting is a generalization of the Cournot oligopoly model that is studied in Caron and Laye (2003). We show that also in the generalization of the model of Caron and Laye a Cournot-Nash equilibrium always exists.

The second application is in the context of decentralized decision making in a supply chain, the motivating example for this paper. We show that, for upstream planning mechanisms in the two-level supply chain, a Nash equilibrium exists, as long as best responses against the planning schedules of the competitors in the chain form a convex set. We prove that for two special cases of the upstream planning mechanism, namely *hedging against late deliveries* and *minimizing inventory levels*, this convexity requirement is indeed fulfilled and hence the existence of Nash equilibrium is guaranteed. The objective in the first variant is to satisfy the requests of the buyers as soon as possible, while in the second one the requests are produced as late as possible.

This paper is organized as follows. In Section 2 we introduce a game played in a constrained environment. Under the assumption of convex-valuedness and upper semicontinuity of the best response correspondence, we show the existence of a Nash equilibrium. In Section 3 we apply this result to a generalization of the capacity constrained Cournot model that is analyzed in Caron and Laye (2003). In Section 4, we present the decentralized supply chain setting that motivated this work. We propose the upstream coordination mechanism, and show that for two variants of this constrained environment Nash equilibrium exists. Moreover, we prove that centralized planning is not, in general, a Nash equilibrium. Section 5 concludes and addresses issues for further research.

2. Existence of equilibrium in constrained environments

In theorem 1 of this section we state and prove a generalization of the theorem of Nash (1950, 1951). The original theorem of Nash says that every n-person game in normal form has at least one equilibrium. In the normal form setting each player is supposed to make a choice from a set of mixed strategies. This choice can and has to be made independently of the other players. In our generalization we explicitly allow for constraints across the strategy spaces of the players of the game. This means that feasibility of a player's choice of strategy, in contrast with the normal form setting, may depend on the choices made by other players. As we already argued, such restrictions across strategies arise naturally in many situations.

The proof of theorem 1 is based on the fixed point theorem of Kakutani (1941). In fact theorem 1 can also be shown to be a special case of the main theorem of Debreu (1952) showing the existence of a social equilibrium in generalized games. However, since in our particular setting we developed a novel technique to derive theorem 1 relatively easily from the more generally known result of Kakutani, we decided to present the line of proof that uses Kakutani's result.

We will first formally define our model. Let $N = \{1, \ldots, n\}$ be a finite set of players. Each player i has a finite set A_i of one-dimensional decision variables at his disposal. A strategy of player i is a vector

$$x_i = (x_{ia})_{a \in A_i}$$

in $\mathbb{R}^{A_i}$. However, because we want to have feasibility restrictions explicitly available in our model, not all combinations $x = (x_i)_{i \in N}$ of strategy choices can be realized. Thus, let

$$\mathcal{D} \subset \prod_{i \in N} \mathbb{R}^{A_i}$$

be the collection of feasible strategy combinations. We assume that $\mathcal{D}$ is compact and convex. A compact and convex set $\mathcal{D}$ of feasible strategy combinations is called a *constrained environment*.

BEST RESPONSES Consider player j. Given a strategy combination $x = (x_i)_{i \in N}$ the set of feasible responses of player j is

$$\mathcal{D}_j(x) := \{y \in \mathcal{D} \mid y_i = x_i \text{ for all } i \neq j\}.$$

Suppose a *best response correspondence* $BR_j\colon \mathcal{D} \twoheadrightarrow \mathcal{D}$ is given for player j such that

(i) the graph of BR_j is a closed subset of $\mathcal{D} \times \mathcal{D}$

(ii) for each $x \in \mathcal{D}$, $BR_j(x)$ is a non-empty subset of $\mathcal{D}_j(x)$, and

(iii) for each $x \in \mathcal{D}$, $BR_j(x)$ is convex.

Notice that we slightly deviate from the usual terminology. Normally the best response set $BR_j(x)$ for player j is defined as a subset of $\mathbb{R}^{A_i}$ with additional feasibility restrictions. Then the best response set is defined as

$$BR(x) = \prod_{i \in N} BR_i(x).$$

However, in our scenario this product set is *not* necessarily a subset of $\mathcal{D}$. Thus, the usual tools like the Fixed Point theorem of Kakutani cannot be applied directly. Nevertheless, the setting of constrained environments we just introduced still enables us to prove the existence of Nash equilibria.

Definition 1. The object $G = \langle \mathcal{D}, (BR_i)_{i \in N} \rangle$ is called a *game played in a constrained environment.*

Playing the game comes down to choosing a strategy combination x in $\mathcal{D}$. Player i is said to behave optimally when x is an element of $BR_i(x)$. This justifies the following definition.

Definition 2. A strategy profile $x \in \mathcal{D}$ is called a *Nash equilibrium* of the game in a constrained environment $G = \langle \mathcal{D}, (BR_i)_{i \in N} \rangle$ if for all players i we have that x is an element of $BR_i(x)$.

Let $\mathcal{D}^n$ be the n-fold product of the set $\mathcal{D}$. An element $(x^i)_{i \in N}$ of $\mathcal{D}^n$ is called *symmetric* if $x^i = x^j$ for all players i and j. Furthermore, let σ be the cyclic permutation of the player set N defined by

$$\sigma(i) := \begin{cases} i+1 & \text{if } i \neq n \\ 1 & \text{if } i = n. \end{cases}$$

Define the map $\phi\colon \mathcal{D}^n \twoheadrightarrow \mathcal{D}^n$ by

$$\phi((x^i)_{i \in N}) := \prod_{\sigma(i)} BR_i(x^i).$$

Now we can prove

Lemma 1. *Any fixed point of ϕ is symmetric.*

Proof. Suppose that $(x^i)_{i \in N}$ is a fixed point of ϕ. Because σ is cyclic, it is sufficient to show that $x^i_j = x^{\sigma(i)}_j$ for all i and j. Since $(x^i)_{i \in N}$ is a fixed point of ϕ, we know from the definition of ϕ that

$$x^{\sigma(i)} \in BR_i(x^i) \quad \text{for all } i.$$

Therefore, since $BR_i(x^i)$ is a subset of $\mathcal{D}_i(x^i)$, we know that $x^{\sigma(i)}_j = x^i_j$ whenever $j \neq i$. Thus we can derive the desired equality for all i and j with $j \neq i$. However, for $j = i$ we get from the equalities we already deduced that

$$x^{\sigma(i)}_i = x^{\sigma^2(i)}_i = \cdots = x^{\sigma^n(i)}_i = x^i_i. \qquad \triangleleft$$

Using the lemma above we can easily prove existence of Nash equilibrium for games that are played in constrained environments.

Theorem 1. *The game G has at least one Nash equilibrium.*

Proof. Apply the Fixed Point theorem of Kakutani to ϕ. This yields a point $(x^i)_{i \in N}$ with $x^{\sigma(i)} \in BR_i(x^i)$ for all i. However, by the previous lemma, we know that there is a strategy combination $x \in \mathcal{D}$ such that $x^i = x$ for all i. Hence, $x \in BR_i(x)$ for all i, and x is a Nash equilibrium of the game G. $\triangleleft$

3. The constrained Cournot-Nash equilibrium

In this section we present the first application of theorem 1, which is a generalization of the capacity constrained Cournot model that is analyzed in Caron and Laye (2003). In this model a set of producers simultaneously provides a good on several markets. Restrictions on minimum consumption levels as well as saturation in these markets give rise to constraints across strategies of producers.

Consider a model in which a set N of producers provides goods to a set M of markets. Each producer i in N decides which quantity $q_{ij} \geq 0$ to produce for each market j in M. We write $q_j = (q_{ij})_{i \in N}$ and $q = (q_{ij})_{i \in N, j \in M}$. The marginal production costs for producer i are equal to $c_i \geq 0$.

Producer i has a capacity of K_i, and market j has a minimum demand R_j as well as a saturation level of S_j, for all i and j. A production plan $q = (q_{ij})_{i \in N, j \in M}$ is *feasible* if

$$\sum_{j \in M} q_{ij} \leq K_i \text{ for all } i \in N$$

and

$$R_j \leq \sum_{i \in N} q_{ij} \leq S_j \text{ for all } j \in M.$$

The set of feasible production plans is denoted by $\mathcal{D}$. Both K_i and S_j can be set to infinity, in which case the corresponding constraints are trivially fulfilled. Given a feasible production plan q, the price of one unit of good i on market j is equal to $P_{ij}(q_j) \geq 0$.

Remark 1. Notice that we allow for price differentiation within one market. Therefore, we do not assume homogeneity of the goods produced. Our framework also allows for both heterogeneity of goods and partial substitutability of goods within one market. The setting of Caron and Laye (2003) is obtained as a special case of our model by imposing $S_j = \infty$ and choosing

$$P_{ij}(q_j) = a_j - b_j \sum_{i \in N} q_{ij}.$$

In particular, their model only accounts for homogeneous goods –and therefore full substitutability– and linear pricing. $\triangleleft$

We assume that P_{ij} is continuous in q_j. Furthermore, with respect to the variable q_{ij} the function P_{ij} is assumed to be decreasing and concave. The setting of Caron and Laye (2003) satisfies these conditions. Given the feasible production plan $q = (q_{ij})_{i \in N, j \in M}$, the profit of producer i is given by

$$\Pi_i(q) = \sum_{j \in M} (P_{ij}(q_j) - c_i)\, q_{ij}.$$

The optimization problem of producer i now looks as follows. Given a production plan q the set of *feasible responses* of producer i is

$$\mathcal{D}_i(q) := \{q' \in \mathcal{D} \mid q'_{kj} = q_{kj} \text{ for all } k \neq i\}.$$

The set of *best responses* is

$$BR_i(q) := \{q' \in \mathcal{D}_i(q) \mid \Pi_i(q') \geq \Pi_i(q'') \text{ for all } q'' \in \mathcal{D}_i(q)\}.$$

We will show now that this particular generalization of the capacity constrained Cournot model presented in Caron and Laye fits the framework described in section 1. In particular we will argue that $\langle \mathcal{D}, (BR_i)_{i \in N} \rangle$ is indeed a game that is played in a constrained environment in the sense of section 1. In order to do this, first notice that the set $\mathcal{D}$ of feasible production plans is indeed compact and convex. So, we only need to verify that each BR_i satisfies the three conditions stated in section 1. This will be done by means of the next two lemmas.

Lemma 2. *The graph of the correspondence BR_i is closed.*

Proof. Suppose that for every k we have a

$$q'^k \in BR_i(q^k)$$

and that $(q'^k, q^k) \to (q', q)$ as $k \to \infty$. We will show that q' is an element of $BR_i(q)$. Take a point q'' in $\mathcal{D}_i(q)$. Notice that the set $\mathcal{D}$ of feasible production plans is polyhedral. Thus we know, see e.g. Cook et al. (1986), that there exist points q''^k in $\mathcal{D}_i(q^k)$ that converge to q'' as $k \to \infty$. However, since q'^k is an element of $BR_i(q^k)$, we know for all k that

$$\Pi_i(q'^k) \geq \Pi_i(q''^k).$$

Hence, by the continuity of Π_i, we also get that $\Pi_i(q') \geq \Pi_i(q'')$, and the desired result holds. $\triangleleft$

Lemma 3. *The set $BR_i(q)$ is not empty and convex.*

Proof. Non-emptiness easily follows from the observation that $BR_i(q)$ is obtained by maximization of the continuous function Π_i over the compact set $\mathcal{D}_i(q)$. In order to prove convexity of $BR_i(q)$, take two points q' and q'' in $BR_i(q)$ and let $\lambda \in [0,1]$. We will show that $\lambda q' + (1-\lambda)q''$ is also an element of $BR_i(q)$.

First notice that the set $\mathcal{D}$ of feasible production plans is compact and convex. Therefore the set $\mathcal{D}_i(q)$ of feasible responses is also a compact and convex set, and $\lambda q' + (1-\lambda)q''$ is an element of $\mathcal{D}_i(q)$. It is sufficient to show that Π_i is concave in the variables $(q_{ij})_{j\in M}$ that are governed by producer i.

To show that Π_i is concave in $(q_{ij})_{j\in M}$, it suffices to prove that

$$q_{ij} \mapsto (P_{ij}(q_j) - c_i)q_{ij}.$$

is concave in the variable q_{ij}. We will now more generally show that a function

$$x \mapsto xf(x) \qquad \text{for all } x \geq 0$$

is concave whenever f is non-increasing and concave. Take $0 \leq x \leq y$. So, $f(x) \geq f(y)$. Hence,

$$\begin{aligned}
& (\lambda x + (1-\lambda)y)f(\lambda x + (1-\lambda)y) \\
\geq\; & (\lambda x + (1-\lambda)y)(\lambda f(x) + (1-\lambda)f(y)) \\
=\; & \lambda x f(x) + \lambda(1-\lambda)(y-x)f(x) + (1-\lambda)(\lambda x + (1-\lambda)y)f(y) \\
\geq\; & \lambda x f(x) + \lambda(1-\lambda)(y-x)f(y) + (1-\lambda)(\lambda x + (1-\lambda)y)f(y) \\
=\; & \lambda x f(x) + (1-\lambda)y f(y).
\end{aligned}$$

The first inequality follows from the concavity of f and the assumption that x and y are both non-negative. The second inequality follows from the assumption that $y \geq x$ and the fact that $f(x) \geq f(y)$. This shows that the profit function Π_i is indeed concave. $\triangleleft$

Thus we have shown that $\langle \mathcal{D}, (BR_i)_{i \in N} \rangle$ is a game that is played in a constrained environment. From this observation we immediately get the following consequence of theorem 1.

Theorem 2. *The generalization of the capacity constrained Cournot model always has a Cournot-Nash equilibrium.*

4. The supply chain planning framework

In this section we present the application that motivated this work. We deal with a decentralized supply chain where its members keep most of their parameters, such as unit costs and capacities, locally and only exchange upper/lower bounds on production levels. We coordinate this supply chain by means of a distribution center that is in charge of making a production planning while taking into account the information provided by each of the members. The goal is to define the way the production planning is constructed, such that the corresponding game is played in a constrained environment. We propose two appealing alternatives, based on lead-times and inventory holding costs, that indeed can be modeled as games played in a constrained environment. Hence, we derive the existence of Nash equilibria for these alternatives.

Our supply chain is composed of a set S of suppliers and a set B of buyers that have to satisfy a demand pattern over a discrete planning horizon of T periods. At each period $t = 1, \ldots, T$ each buyer $b \in B$ faces a demand of $d(b,t) \geq 0$ units for the so-called end product.

In order to produce one unit of the end product each buyer $b \in B$ needs one unit of a so-called input product, which he can subsequently transform into the end product at a cost of $c(b,t)$ when production takes place at period t.

The buyers should satisfy the demand exactly in time. Therefore, when needed buyers will keep the end product in stock, where the unit inventory holding costs for buyer b at period t are equal to $h(b,t)$.

Upstream in the supply chain each supplier $s \in S$ can produce the input product.

The unit production costs faced by supplier s at period t are equal to $c(s,t)$. The capacity of supplier s at period t is equal to $m(s,t)$.

Finally, the input product can only be held in stock at the distribution center, where the unit inventory holding at period t are equal to $g(t)$. These costs are shared equally by the buyers and the suppliers.

We will assume that the unit production costs as well as the unit inventory holding costs are strictly positive.

In order to acquire the input product buyer b in B places an order $r(b,t) \geq 0$ at the distribution center for each period t. This order represents his minimum requirements at each period. Independently, supplier s in S reports quantities $q(s,t)$ to the distribution center representing the maximal amount of input product he is willing to produce at period t. All orders and reports by buyers and suppliers respectively are placed at the beginning of the planning horizon.

Feasibility of ask/bid profiles

Given the specification of each requirement $r(b,t)$ by the buyers and each capacity $q(s,t)$ by the suppliers, the distribution center first checks feasibility of the ask/bid profile as follows. Write

$$D(b,t) := d(b,1) + \cdots + d(b,t),$$

the total demand for buyer b up to and including period t. Similarly we write

$$R(b,t) := r(b,1) + \cdots + r(b,t),$$

the total requirement of buyer b up to and including period t, and

$$Q(s,t) := q(s,1) + \cdots + q(s,t),$$

the total reported capacity of supplier s up to and including period t. The feasibility restrictions we impose can now be written as follows.

For each buyer $b \in B$, for each supplier $s \in S$, and for all $t = 1, \ldots, T$

$$R(b,t) \geq D(b,t) \quad \text{and} \quad m(s,t) \geq q(s,t).$$

These conditions reflect demand satisfaction for the end product and the production restrictions on the supply side, respectively. Moreover we require that

$$\sum_{s\in S} Q(s,t) \geq \sum_{b\in B} R(b,t)$$

for all $t = 1, \ldots, T$. These conditions ensure demand satisfaction for the input product. The set of feasible ask/bid profiles (q, r) is denoted by $\mathcal{D}$.

Feasibility of coordination mechanisms

For each feasible ask/bid profile (q, r) in $\mathcal{D}$ the distribution center announces a binding coordination arrangement. For each buyer b, supplier s, and period t it specifies

$$x_s(q,r,t) \quad \text{and} \quad y_b(q,r,t).$$

Supplier s is now bound to produce $x_s(q, r, t)$ units in period t and buyer b is bound to pick up the amount $y_b(q, r, t)$ during period t and to transform this amount into end product.

Thus, a coordination arrangement, or *mechanism*, is a collection

$$(x, y) = ((x_s)_{s\in S}, (y_b)_{b\in B})$$

of continuous functions $x_s : \mathcal{D} \times T \to \mathbb{R}$, the delivery schedules for the suppliers to the distribution center, and $y_b : \mathcal{D} \times T \to \mathbb{R}$, the pick-up schedules for the buyers from the distribution center. A coordination arrangement must satisfy the following conditions. Write

$$\begin{aligned} X_s(q,r,t) &= x_s(q,r,1) + \cdots + x_s(q,r,t) \\ X(q,r,t) &= \textstyle\sum_{s\in S} X_s(q,r,t) \\ x(q,r,t) &= \textstyle\sum_{s\in S} x_s(q,r,t). \end{aligned}$$

Similarly we define $Y_b(q, r, t)$, $Y(q, r, t)$ and $y(q, r, t)$. We require the following three set of conditions. For each supplier $s \in S$,

$$x_s(q,r,t) \leq q(s,t)$$

for all $t = 1, \dots, T$. For each buyer $b \in B$,

$$y_b(q, r, t) \geq r(b, t)$$

for all $t = 1, \dots, T$. And finally, the condition that the total stock in the distribution center should always be non-negative. So, for all $t = 1, \dots, T$,

$$X(q, r, t) \geq Y(q, r, t).$$

Total production and inventory costs

As said before inventory costs at the distribution center are supposed to be shared equally by all buyers and suppliers alike. So, if we write

$$G(q, r, t) := X(q, r, t) - Y(q, r, t)$$

for the total stock available at the distribution center at period t, the costs associated with the binding coordination arrangement $(x_s(q, r, t))_{t=1}^{T}$ for supplier s are

$$V_s(q, r) := \sum_{t=1}^{T} c(s, t) x_s(q, r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q, r, t).$$

The first term is equal to the total production costs given the binding coordination arrangement imposed by the distribution center. The second term is the part of the inventory costs associated with the input product that supplier s is supposed to pay.

Similarly, given the binding coordination schedule $(y_b(q, r, t))_{t=1}^{T}$ for buyer b, his costs are equal to

$$\begin{aligned} V_b(q, r) \quad := \quad & \sum_{t=1}^{T} c(b, t) y_b(q, r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q, r, t) \\ + \quad & \sum_{t=1}^{T} h(b, t)(Y_b(q, r, t) - D(b, t)). \end{aligned}$$

So far we have only addressed the decisions of the distribution center given an ask/bid profile (q, r). However, the buyers and suppliers are thus confronted with an entire range of opportunities when they change their stated requirements or capacities. The set of *feasible responses* of supplier s given the ask/bid profile (q, r) is the set

$$\mathcal{D}_s(q, r) := \{(q', r) \in \mathcal{D} \mid q'_{s'} = q_{s'} \text{ for all } s' \neq s\}.$$

The set of *best responses* of supplier s given the ask/bid profile (q, r) is the set

$$BR_s(q, r) := \{(q', r) \in \mathcal{D}_s(q, r) \mid V_s(q', r) \leq V_s(q'', r) \text{ for all } (q'', r) \in \mathcal{D}_s(q, r)\}.$$

In the same way we can define the sets $\mathcal{D}_b(q, r)$ and $BR_b(q, r)$ for each buyer b in B.

Of course we would like to show now that $G = \langle \mathcal{D}, (BR_b)_{b \in B}, (BR_s)_{s \in S} \rangle$ is a game played in a constrained environment, because this will guarantee us the existence of Nash equilibrium. We can indeed show most of the requirements needed to make G a game played in a constrained environment, except one, namely convex-valuedness of the best response correspondences. But we will first discuss the properties we can prove. At least $\mathcal{D}$ is clearly compact and convex, because suppliers have a maximum capacity and all restrictions are linear. Furthermore, we have the following result.

Lemma 4. *The graphs of the correspondences BR_s and BR_b are closed. Moreover, both correspondences are non-empty valued.*

Proof. The proof that the graphs of both BR_b and BR_s are closed is identical to the proof of lemma 2, taking into account that we are dealing here with cost minimization instead of profit maximization. The non-emptiness of the values of both BR_b and BR_s easily follows from the observation that these values are given by the minimization of a continuous function over a compact set. $\triangleleft$

G is a game played in a constrained environment as soon as we can show that the values of both BR_b and BR_s are convex. This however is not true in general. In the remaining sections we will present a counterexample for the convexity of the values of BR_s, and show that in two special cases we can prove the desired convexity by adding a constraint to the set of best responses for the suppliers.

Upstream planning mechanisms

In an upstream planning mechanism the distribution simply implements the requirements reported by the buyers, i.e., given the feasible ask/bid profile (q, r), the distribution center sets $y_b(q, r, t) = r(b, t)$ for each b and t. The advantage of this is that, as the next lemma shows, at least on the buyer side we do not run into any additional difficulties when we want to apply theorem 1.

Lemma 5. *The correspondence BR_b only has non-empty and convex values.*

Proof. Take a buyer b in B. His best response set $BR_b(q, r)$ given a feasible ask/bid profile (q, r) is given as the set of reports $(r(b, t))_{t=1}^{T}$ that maximize the function

$$\begin{aligned} V_b(q, r) \quad &:= \quad \sum_{t=1}^{T} c(b, t) y_b(q, r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q, r, t) \\ &+ \quad \sum_{t=1}^{T} h(b, t)(Y_b(q, r, t) - D(b, t)). \end{aligned}$$

over the set

$$\mathcal{D}_b(q, r) := \{(q, r') \in \mathcal{D} \mid r'_{b'} = r_{b'} \text{ for all } b' \neq b\}.$$

However, since the distribution center uses an upstream planning mechanism, buyer b knows that $y_b(q, r, t) = r(b, t)$ for all t. Moreover $G(q, r, t)$ and $Y_b(q, r, t)$ are linear functions in the variables $y_b(q, r, t)$, which implies that the set $BR_b(q, r)$ is given as the set of reports of buyer b that maximize the linear function $V_b(q, r)$ over the compact and convex set $\mathcal{D}_b(q, r)$. Hence, $BR_b(q, r)$ is non-empty and convex. $\triangleleft$

In the next sections we will discuss two upstream planning mechanisms in more detail. First we will however show that, no matter how we define the delivery schedules x_s for the suppliers, the centralized solution –the planning that minimizes the total cost faced by both suppliers and buyers– will in general not be a Nash equilibrium. This is by the way not a feature that is restricted to upstream planning mechanisms. For virtually *all* mechanisms the centralized solution will often not be a Nash equilibrium.

Example 1. Consider a planning horizon consisting of six time periods, i.e. $T = 6$, one single supplier and one single buyer. The data are as follows. For the buyer we have

$t =$	1	2	3	4	5	6
$d(b,t) =$	2	4	8	10	8	4
$c(b,t) =$	5	20	5	20	5	5
$h(b,t) =$	3	3	3	3	3	3

and for the supplier we have

$t =$	1	2	3	4	5	6
$c(s,t) =$	10	10	20	20	20	13

The unit inventory holding costs for the input product faced by the distribution center are equal to 1, equally paid by the supplier and the buyer ($\frac{1}{2}$ per period each).

The centralized solution $(x^*(s,t), y^*(b,t))$ (the feasible plan that minimizes total cost, given that the demand is satisfied) is given by

$t =$	1	2	3	4	5	6
$x^*(s,t) =$	6	26				4
$y^*(b,t) =$	6		18		8	4
$G(t) =$		26	8	8		
$O(t) =$	4		10			

where $G(t)$ is the amount of the input product stored at the distribution center at period t, $O(t)$ is the amount of end product stored at period t, $x^*(s,t)$ is the amount of input product the distribution center requires the supplier to deliver at period t and $y^*(b,t)$ is the amount of input product the distribution center requires the buyer to collect (and subsequently transform into end product). The total cost of this production schedule equals 636, of which

$$6 \times 10 + 26 \times 10 + 4 \times 13 + \frac{1}{2}(26 + 8 + 8) = 393$$

is paid by the supplier and

$$6 \times 5 + 18 \times 5 + 8 \times 5 + 4 \times 5 + \frac{1}{2}(26 + 8 + 8) + 3(4 + 10) = 243$$

is paid by the buyer.

This production schedule $(x^*(s,t), y^*(b,t))$ can never be an equilibrium outcome of the upstream planning mechanism for the following reason. Let (q,r) be a feasible ask/bid profile that has the centralized solution as an outcome of the upstream planning mechanism, i.e. $x_s(q,r,t) = x^*(s,t)$ and $y_b(q,r,t) = y^*(b,t)$.

Since the distribution center simply copies the requirements of the buyer, we know that $r(b,t) = y_b(q,r,t) = y^*(b,t)$. However, given these requirements reported by the buyer, the supplier has a feasible bid for his capacities that results in lower costs than in the centralized solution. If the supplier reports

$t =$	1	2	3	4	5	6
$q'(s,t) =$	6	30	0	0	0	0

the resulting production schedule from the upstream planning mechanism is

$t =$	1	2	3	4	5	6
$x_s(q',r,t) =$	6	30				
$y_b(q',r,t) =$	6		18		8	4
$G(t) =$		30	12	12	4	
$O(t) =$	4		10			

with a total cost of 640, of which

$$6 \times 10 + 30 \times 10 + \frac{1}{2}(30 + 12 + 12 + 4) = 389$$

is paid by the supplier and

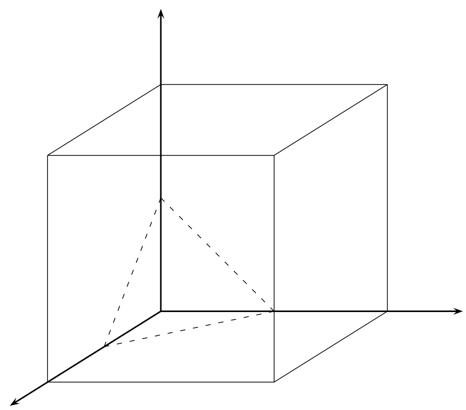
$$6 \times 5 + 18 \times 5 + 8 \times 5 + 4 \times 5 + \frac{1}{2}(30 + 12 + 12 + 4) + 3(4 + 10) = 251$$

is paid by the buyer. Thus, the centralized production schedule can never be the outcome of a decentralized optimal choice for the supplier under any upstream planning mechanism. ◁

Minimal best responses for suppliers

Our aim is to prove the existence of Nash equilibria in upstream planning mechanisms by means of theorem 1. In the following we will illustrate with an example why we cannot directly use the best reply correspondences BR_s.

Example 2. Consider a single supplier and a single buyer. The supplier has three days to produce one unit of input product for the buyer. Each day his capacity is equal to 2. A reported capacity (q_1, q_2, q_3) for the supplier is feasible if each q_i is between 0 and 2, and moreover $q_1 + q_2 + q_3 \geq 1$. Graphically this can be represented by the cube $[0, 2]^3$ with one corner chopped off.



Now, given a feasible reported capacity (q_1, q_2, q_3), consider the upstream planning mechanism returning the schedule

$$(x_1, x_2, x_3) = \begin{cases} (1, 0, 0) & \text{if } q_1 \geq 1 \\ (q_1, 1 - q_1, 0) & \text{if } q_1 < 1 \text{ and } q_1 + q_2 \geq 1 \\ (q_1, q_2, 1 - q_1 - q_2) & \text{else,} \end{cases}$$

where the demand is produced as soon as possible.

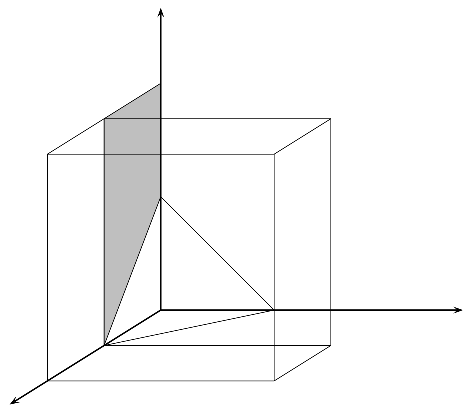
Moreover, suppose that the production and inventory costs are such that producing in periods 1 and 3 is equally expensive, while production in period 2 is very costly. Then the set of best replies is $\mathcal{S} \cup \mathcal{T}$ where

$$\mathcal{S} = \{(q_1, 0, q_3) \mid q_1 + q_3 \geq 1, 0 \leq q_1 \leq 1, 0 \leq q_3 \leq 2\}$$

and

$$\mathcal{T} = \{(q_1, q_2, q_3) \mid 1 \leq q_1 \leq 2, 0 \leq q_2 \leq 2, 0 \leq q_3 \leq 2\}.$$

Graphically this is the shaded area $\mathcal{S}$ together with the box $\mathcal{T}$ depicted in the front of the picture below.



However, this is clearly not a convex set and hence $\langle \mathcal{D}, (BR_b)_{b \in B}, (BR_s)_{s \in S} \rangle$ is not a game played in a constrained environment. $\triangleleft$

In the following we construct a subcorrespondence of $BR_s(q, r)$. First, observe that given a feasible ask/bid profile in the above example, say $(q, r) \in \mathcal{D}$, supplier s gets the same planning back from the distribution center when reporting q' or $x_s(q', r, \cdot)$. Therefore, if $(q', r) \in BR_s(q, r)$, then $(x_s(q', r, \cdot), r) \in BR_s(q, r)$. In the above example it can easily be seen that convexity is not a problem when we only consider best responses of the latter type. This observation is the motivation for the following definitions.

Definition 3. Let (q, r) be an element of $\mathcal{D}$ and let s be a supplier in S. An element (q', r) of $BR_s(q, r)$ is called a *minimal best response* of supplier s if supplier s has reported such that $x_s(q', r, t) = q'(s, t)$, for all t. The set of minimal best responses of supplier s given (q, r) is denoted by $MBR_s(q, r)$. $\triangleleft$

Definition 4. An upstream planning mechanism is called *monotonic* if for all $(q', r) \in BR_s(q, r)$ and all $(q'', r) \in \mathcal{D}_s(q, r)$ with $x_s(q', r, t) \leq q''(s, t) \leq q'(s, t)$

it holds that

$$x_\ell(q', r, t) = x_\ell(q'', r, t),$$

for all suppliers $\ell \in S$. ◁

Not all upstream mechanisms are monotonic. For example proportional allocation mechanisms are not. For monotonic mechanisms we can show the following result.

Theorem 3. *Suppose that the upstream planning mechanism under consideration is monotonic. If the values of the correspondence MBR_s are convex, then $G = \langle \mathcal{D}, (BR_b)_{b\in B}, (MBR_s)_{s\in S}\rangle$ is a game played in a constrained environment.*

Proof. It remains to be shown that the graph of MBR_s is closed and that its values are nonempty.

First we will show that the graph of MBR_s is closed. To see this, notice that, by the definition of a minimal best response, the graph of MBR_s is equal to the intersection of the graphs of BR_s and the correspondence φ_s defined by

$$\varphi_s(q, r) = \{(q', r) \in \mathcal{D}_s(q, r) \mid x_s(q', r, t) = q'(s, t), \text{ for all } t\}.$$

The graph of BR_s is closed according to lemma 4, while the closedness of the graph of φ_s follows immediately from the fact that $\mathcal{D}_s$ is a continuous correspondence, the continuity of x_s and the maximum theorem (see e.g. Berge (1966)). Hence, the graph of MBR_s is also closed.

Next, take an ask/bid profile (q, r) in $\mathcal{D}$. By lemma 4 we know that $BR_s(q, r)$ is not empty, so we can take a $(q', r) \in BR_s(q, r)$. Define q'' by, for $t \in T$, $q''(\ell, t) = q(\ell, t)$ for all $\ell \neq s$, and $q''(s, t) = x_s(q', r, t)$. We will show that (q'', r) is an element of $MBR_s(q, r)$.

It is straightforward to check that (q'', r) is an element of $\mathcal{D}_s(q, r)$. Furthermore, since $x_s(q', r, t) \leq q''(s, t) \leq q'(s, t)$ clearly holds, monotonicity of the mechanism implies that $x_\ell(q', r, t) = x_\ell(q'', r, t)$ for all $\ell \in S$. From this we get that $V_s(q', r) = V_s(q'', r)$ and (q'', r) is an element of $BR_s(q, r)$. ◁

In the following we will discuss two variants of the upstream planning mechanism for which we can use theorem 3 to show the existence of a Nash equilibrium. This is in some sense bad news since the system may be trapped in a suboptimal planning schedule in which no one has an inclination to change his behavior. To define the coordination mechanism completely, it remains to describe the way the distribution center constructs a planning for the suppliers. In the first one, the objective is to satisfy the requests of the buyers as soon as possible, and thus hedging against delayed deliveries when exceptions happen in the supply chain, while in the second variant the requests are produced as late as possible, and thus keeping the inventory levels as low as possible.

In the remaining sections we will assume that the suppliers are put in a queue. This queue is modeled by the sequence $s(1), s(2), \ldots, s(n)$, where $s(1)$ is the supplier in the front of the queue, $s(2)$ is the second supplier in the queue, et cetera. The specific order is not crucial in what follows. The queue is mainly used as a tie breaking tool so that we can make unambiguous assignments of quantities to players within a period. Thus we will without loss of generality assume that $s(i) = i$ for all i.

Hedging against late deliveries

Recall that, due to the market clearing condition, by the end of period T the suppliers need to have produced the amount

$$Y(q, r) = Y(q, r, 1) + \cdots + Y(q, r, T).$$

Definition 5. The *critical report* is the unique pair (i^*, t^*) such that

$$\sum_{t<t^*} \sum_{s \in S} q(s, t) + \sum_{i<i^*} q(i, t^*) < Y(q, r) \leq \sum_{t<t^*} \sum_{s \in S} q(s, t) + \sum_{i \leq i^*} q(i, t^*).$$

In fact (i^*, t^*) is chosen such that the total demand $Y(q, r)$ is fulfilled as soon as possible, i.e., letting all suppliers produce to full capacity until period t^* where $Y(q, r)$ is fulfilled. Now write

$$L(q, r) := Y(q, r) - \sum_{t<t^*} \sum_{s \in S} q(s, t) + \sum_{i<i^*} q(i, t^*)$$

and choose

$$x_i(q,r,t) = \begin{cases} q(i,t) & \text{if } t < t^* \text{ or if } t = t^* \text{ and } i < i^* \\ L(q,r) & \text{if } t = t^* \text{ and } i = i^* \\ 0 & \text{else.} \end{cases}$$

The mechanism thus defined is called *hedging against late deliveries*. Notice that this is in fact the mechanism used in example 2. In the remainder of this section we will show that $\langle \mathcal{D}, (BR_b)_{b\in B}, (MBR_s)_{s\in S}\rangle$ is a game played in a constrained environment by using theorem 3.

In order to use theorem 3, first observe that hedging against late deliveries is indeed a monotonic mechanism, so we can use theorem 3. Next, in order to prove convexity of the values of MBR_s, we need to establish two technical facts. Consider a feasible ask/bid profile, say $(q,r) \in \mathcal{D}$. With this profile we can associate the non-negative real number $M_s(q,r)$ defined as

$$M_s(q,r) := \min \left\{ \sum_{t=1}^{T} q'(s,t) \mid (q',r) \in \mathcal{D}_s(q,r) \right\}.$$

We have the following two observations.

Lemma 6. *Given an element (q',r) in $MBR_s(q,r)$, it holds*

$$\sum_{t=1}^{T} q'(s,t) = M_s(q,r).$$

Proof. Let us define

$$\Delta_s(q,r,t) \quad = \quad \min \left\{ 0, \sum_{\tau=1}^{t} r(b,\tau) - \sum_{\tau=1}^{t} \sum_{\ell \neq s} q(\ell,\tau) - \sum_{\tau=1}^{t-1} \Delta_s(q,r,\tau) \right\}.$$

For each t, the value $\sum_{\tau=1}^{t} \Delta_s(q,r,\tau)$ is the minimum production required from supplier s up to and including period t to ensure feasibility given the ask/bid profile (q,r). Therefore, we have that $\sum_{\tau=1}^{t} x_s(q',r,\tau) \geq \sum_{\tau=1}^{t} \Delta_s(q,r,\tau)$ for all t. Notice that it is sufficient to prove that

$$\sum_{\tau=1}^{T} x_s(q',r,\tau) = \sum_{\tau=1}^{T} \Delta_s(q,r,\tau)$$

since $x_s(q',r,t) = q'(s,t)$ for all t and Δ_s is independent of q'. Suppose that

$$\sum_{\tau=1}^{T} x_s(q',r,\tau) > \sum_{\tau=1}^{T} \Delta_s(q,r,\tau). \tag{1}$$

Going backwards in the planning horizon, we take the first period, say t^1, where $q'(s,t) > 0$ (and hence also $x_s(q',r,t) > 0$). We will show that we can decrease $q'(s,t^1)$, and therefore also $x_s(q',r,t^1)$, by

$$\varepsilon = \min \left\{ \sum_{\tau=1}^{T} q'(s,\tau) - \sum_{\tau=1}^{T} \Delta_s(q,r,\tau), q'(s,t^1) \right\}.$$

To see this, recall that (q',r) is an element of $MBR_s(q,r)$. From (1) and the definition of Δ_s, we know that ε can be produced by the rest of the suppliers, during the same period t^1 and if necessary in future periods. Therefore, we can reduce the capacity that supplier s reports during period t^1 to $q'(s,t^1) - \varepsilon$. Let us denote this reduced capacity bid by q''. It is easy to show that again $x_s(q'',r,t) = q''(s,t)$, for all t, since this was already true for q' and we have only reduced the capacity of supplier s in period t^1.

For supplier s, the new bid of capacities is better than the previous one, because his production during period t^1 decreases, being produced by others in the same period or maybe in later periods. Thus, this contradicts the fact that (q',r) is an element of $BR_s(q,r)$. $\triangleleft$

Lemma 7. *Given (q',r) and (q'',r) in $MBR_s(q,r)$, it holds*

$$x_\ell(q',r,t) = x_\ell(q'',r,t), \quad \ell \neq s,\ t = 1,\ldots,T.$$

Proof. Let t^1 be the last period where

$$\sum_{\tau=1}^{t} r(b,\tau) > \sum_{\tau=1}^{t} \sum_{\ell \neq s} q(\ell,\tau).$$

From the proof in lemma 6, we know that $x_\ell(q',r,t) = x_\ell(q'',r,t) = q(\ell,t)$ for $\ell \neq s$ and $t \leq t^1$ and $x_s(q',r,t) = x_s(q'',r,t) = 0$ for $t > t^1$. These two together with lemma 6 imply the desired result. $\triangleleft$

Now we are able to prove that $\langle \mathcal{D}, (BR_b)_{b\in B}, (MBR_s)_{s\in S}\rangle$ is indeed a game played in a constrained environment, and therefore has a Nash equilibrium. From theorem 3, we know that it is enough to show the following result.

Theorem 4. *The values of the correspondence MBR_s are convex.*

Proof. Take two elements (q', r) and (q'', r) in $MBR(q, r)$. Write

$$q(\lambda) := \lambda q' + (1-\lambda)q''$$

for $0 \leq \lambda \leq 1$. We have to prove that $(q(\lambda), r)$ is an element of $MBR(q, r)$. We will check (i) feasibility, (ii) best response and (iii) $x_s(q(\lambda), r, t) = q(\lambda)(s, t)$.

(i) To see that $(q(\lambda), r)$ is an element of $\mathcal{D}_s(q, r)$, simply observe that this is a convex set.

In order to prove (ii) and (iii) we first need to show the following claim

$$x_\ell(q', r, t) = x_\ell(q'', r, t) = x_\ell(q(\lambda), r, t) \tag{2}$$

for all ℓ and t. Notice that

$$x_\ell(q', r, t) = x_\ell(q'', r, t) = x_\ell(q(\lambda), r, t)$$

immediately follows for all $\ell \neq s$ and t from lemma 7. So, we only need to prove (2) for $\ell = s$ and all t. For this, notice that

$$\begin{aligned}
\sum_{t=1}^{T} x_s(q(\lambda), r, t) &\leq \sum_{t=1}^{T} q(\lambda)(s, t) \\
&= \sum_{t=1}^{T} [\lambda q'(s, t) + (1-\lambda) q''(s, t)] \\
&= \lambda \sum_{t=1}^{T} x_s(q', r, t) + (1-\lambda) \sum_{t=1}^{T} x_s(q'', r, t) \\
&= M(q, r).
\end{aligned}$$

On the other hand, from the definition of $M(q, r)$ we know that

$$\sum_{t=1}^{T} x_s(q(\lambda), r, t) \geq M(q, r).$$

Hence, $x_s(q(\lambda), r, t) = q(\lambda)(s,t)$ for all t, and therefore

$$x_s(q(\lambda), r, t) = \lambda x_s(q', r, t) + (1-\lambda) x_s(q'', r, t).$$

(ii) We have that

$$\begin{aligned}
V_s(q(\lambda), r) \quad &:= \quad \sum_{t=1}^{T} c(s,t) x_s(q(\lambda), r, t) + \frac{1}{|B|+|S|} \sum_{t=1}^{T} g(t) G(q(\lambda), r, t) \\
&= \quad \left[\lambda \sum_{t=1}^{T} c(s,t) x_s(q', r, t) + \frac{1}{|B|+|S|} \sum_{t=1}^{T} g(t) G(q', r, t)\right] \\
&\quad + (1-\lambda) \quad \left[\sum_{t=1}^{T} c(s,t) x_s(q'', r, t) + \frac{1}{|B|+|S|} \sum_{t=1}^{T} g(t) G(q'', r, t)\right] \\
&= \quad \lambda V_s(q', r) + (1-\lambda) V_s(q'', r).
\end{aligned}$$

Hence, since (q', r) and (q'', r) are elements of $BR_s(q, r)$, and $(q(\lambda), r)$ has the same cost for supplier s, we see that $(q(\lambda), r)$ is indeed an element of $BR_s(q, r)$.

(iii) In order to prove that $x_s(q(\lambda), r, t) = q(\lambda)(s,t)$ for all t, notice that from (2), only using $\ell = s$, immediately implies that

$$\begin{aligned}
x_s(q(\lambda), r, t) \quad &= \quad \lambda x_s(q', r, t) + (1-\lambda) x_s(q'', r, t) \\
&= \quad \lambda q'(s,t) + (1-\lambda) q''(s,t) \\
&= \quad q(\lambda)(s,t)
\end{aligned}$$

which finishes our proof. ◁

Minimizing inventory levels

In the following, we propose another variation of the upstream mechanism where the requirements are produced by the suppliers as late as possible. Thus, given a feasible ask/bid profile (q, r) in this variation, the distribution center sets

$$x_s(q, r, T) = \begin{cases} q(s, T) & \text{if } s < s^T \\ \sum_{b \in B} r(b, T) - \sum_{\ell=1}^{s-1} q(\ell, T) & \text{if } s = s^T \\ 0 & \text{if } s > s^T \end{cases}$$

where s^T is defined as the first index s such that

$$\sum_{b\in B} r(b,T) < \sum_{\ell=1}^{s} q(\ell,T),$$

or ∞ when such an index does not exists. Further, for any $t = T-1,\ldots,1$ the distribution center sets

$$x_s(q,r,t) = \begin{cases} q(s,t) & \text{if } s < s^t \\ \sum_{b\in B}\sum_{\tau=t}^{T} r(b,\tau) - \sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q,r,\tau) - \sum_{\ell=1}^{s-1} q(\ell,t) & \text{if } s = s^t \\ 0 & \text{if } s > s^t, \end{cases}$$

where s^t is defined as the first index s such that

$$\sum_{b\in B}\sum_{\tau=t}^{T} r(b,\tau) < \sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q,r,\tau) + \sum_{\ell=1}^{s} q(\ell,t),$$

or ∞ when such an index does not exists. If $s^t < \infty$, we say that t is a critical period and that s^t is its critical supplier.

Similarly to the ‘hedging against late deliveries’ variant, we cannot use directly the best reply correspondences to show the existence of Nash equilibria by means of theorem 1.

Example 3. Consider the problem instance given in example 2. Given a feasible reported capacity (q_1,q_2,q_3), the mechanism returns the schedule

$$(x_1,x_2,x_3) = \begin{cases} (0,0,1) & \text{if } q_3 \geq 1 \\ (0,1-q_3,q_3) & \text{if } q_3 < 1 \text{ and } q_2+q_3 \geq 1 \\ (1-q_2-q_3,q_2,q_3) & \text{else.} \end{cases}$$

Then the set of best replies is $\mathcal{S}\cup\mathcal{T}$ where

$$\mathcal{S} = \{(q_1,0,q_3) \mid q_1+q_3 \geq 1, 0 \leq q_1 \leq 1, 0 \leq q_3 \leq 1\}$$

and

$$\mathcal{T} = \{(q_1,q_2,q_3) \mid 0 \leq q_1 \leq 2, 0 \leq q_2 \leq 2, 1 \leq q_3 \leq 2\},$$

which is clearly not a convex set. ◁

Nevertheless, notice that also 'minimizing inventory levels' is a monotonic mechanism, so we can again try to use theorem 3. However, in contrast to the 'hedging against late deliveries' variant, $\sum_{t=1}^{T} x_s(\cdot, r, t)$ is not constant in the set of minimal best responses $MBR_s(q, r)$. Thus, lemma 6 no longer holds in this setting.

Example 4. Consider a planning horizon consisting of four time periods, i.e. $T = 4$, two suppliers and one single buyer. The requirements of the buyer are equal to $(3, 6, 6, 8)$. For the suppliers we have

$$\begin{array}{rcccc} t = & 1 & 2 & 3 & 4 \\ \\ c(1,t) = & 10 & \frac{1}{3} & \frac{2}{3} & 1 \\ m(1,t) = & 5 & 5 & 5 & 5 \\ \\ c(2,t) = & 15 & 15 & 15 & 15 \\ m(2,t) = & 5 & 5 & 5 & 5. \end{array}$$

The unit inventory holding costs for the input product faced by the distribution center are equal to 1, equally paid by the suppliers and the buyer ($\frac{1}{3}$ per period each).

Suppose that the second supplier bids his actual capacities, i.e., $(5, 5, 5, 5)$, then the following ask/bids are best responses

$$(0, 2, 0, 3), (0, 0, 4, 0) \text{ and } (0, 0, 0, 3).$$

Therefore, the total production is not constant. This example also illustrates that, given the bid $(5, 5, 5, 5)$ of supplier 2, supplier 1 may produce more than enough to satisfy the requirements, i.e., more than three units. ◁

Given an ask/bid vector (q, r), the planning horizon decomposes into blocks by using the critical periods. A block is defined as a collection of consecutive periods such that the first period is the only period where the (backward) cumulative requirements are at most the (backward) cumulative capacities. To be precise,

$[t_1, t_2]$ is a block if and only if

$$\sum_{b\in B}\sum_{\tau=t}^{t_2} r(b,\tau) > \sum_{\ell\in S}\sum_{\tau=t}^{t_2} q(\ell,\tau)$$

for each $t = t_1+1, \ldots, t_2$, and

$$\sum_{b\in B}\sum_{\tau=t_1}^{t_2} r(b,\tau) \leq \sum_{\ell\in S}\sum_{\tau=t_1}^{t_2} q(\ell,\tau).$$

It is easy to see that the critical periods decompose the planning horizon into blocks. In example 3, the best responses $(0,2,0,3), (0,0,4,0), (0,0,0,3)$ yield three different decompositions of the planning horizon into blocks, namely

$$\{[1,1],[2,3],[4]\}, \{[1,2],[3,4]\}, \{[1,3],[4]\}.$$

We will now show that in $MBR_s(q,r)$, the set of production plans the distribution center returns to supplier s is convex, and thus the desired result that $\langle \mathcal{D}, (BR_b)_{b\in B}, (MBR_s)_{s\in S}\rangle$ has a Nash equilibrium follows. We first need a couple of technical results.

Remark 2. Let (q',r) be an element of $MBR_s(q,r)$ and let $[t_1,t_2]$ be one of the blocks of the decomposition of (q,r). If supplier s bids some capacity in the first period of the block $q'(s,t_1) > 0$, then $x_\ell(q',r,t_1) = q'(\ell,t_1)$ for all $\ell \neq s$. Otherwise, part of this production could have been done by some of the other suppliers involving less production costs for supplier s and the same inventory holding costs.

Lemma 8. *Let (q',r) and (q'',r) be elements of $MBR_s(q,r)$, and $(q(\lambda),r)$ where $q(\lambda) := \lambda q' + (1-\lambda)q''$ for $0 \leq \lambda \leq 1$. Then, any common critical period for both (q',r) and (q'',r) is also critical for $(q(\lambda),r)$.*

Proof. It is enough to show that

$$\sum_{\ell\in S}\sum_{\tau=t}^{T} x_\ell(q(\lambda),r,\tau) \geq \lambda \sum_{\ell\in S}\sum_{\tau=t}^{T} x_\ell(q',r,\tau) + (1-\lambda)\sum_{\ell\in S}\sum_{\tau=t}^{T} x_\ell(q'',r,\tau) \quad (3)$$

for all t. Suppose that these inequalities are true. Take a common critical point for both (q',r) and (q'',r), say t, then we know that there exists a supplier s^t

such that

$$\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q',r,\tau)+\sum_{\ell=1}^{s^t} q'(\ell,t) > \sum_{b\in B}\sum_{\tau=t}^{T} r(b,\tau)$$

and

$$\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q'',r,\tau)+\sum_{\ell=1}^{s^t} q''(\ell,t) > \sum_{b\in B}\sum_{\tau=t}^{T} r(b,\tau).$$

Now by taking the corresponding convex combination,

$$\begin{aligned}\lambda\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q',r,\tau) \quad &+ \quad (1-\lambda)\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q'',r,\tau)+\sum_{\ell=1}^{s^t} q(\lambda)(\ell,t)\\ &> \quad \sum_{b\in B}\sum_{\tau=t}^{T} r(b,\tau),\end{aligned}$$

and the result follows from (3).

We can show (3) using backwards induction on t. First, we prove that the result is true for $t = T$. We have that

$$\begin{aligned}&\sum_{\ell\in S} x_\ell(q(\lambda),r,T)\\ &\quad = \quad \min\{\sum_{\ell\in S} q(\lambda)(\ell,T), \sum_{b\in B} r(b,T)\}\\ &\quad = \quad \min\{\lambda\sum_{\ell\in S} q'(\ell,T)+(1-\lambda)\sum_{\ell\in S} q''(\ell,T), \sum_{b\in B} r(b,T)\}\\ &\quad \geq \quad \lambda\min\{\sum_{\ell\in S} q'(\ell,T), \sum_{b\in B} r(b,T)\}+(1-\lambda)\min\{\sum_{\ell\in S} q''(\ell,T), \sum_{b\in B} r(b,T)\}\\ &\quad = \quad \lambda\sum_{\ell\in S} x_\ell(q',r,T)+(1-\lambda)\sum_{\ell\in S} x_\ell(q'',r,T).\end{aligned}$$

Now suppose that (3) is true for period $t+1$ and we want to prove it for t. In that case

$$\begin{aligned}&\sum_{\ell\in S}\sum_{\tau=t}^{T} x_\ell(q(\lambda),r,\tau)\\ &\quad = \quad \min\{\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q(\lambda),r,\tau)+\sum_{\ell\in S} q(\lambda)(\ell,t), \sum_{b\in B}\sum_{\tau=t}^{T} r(b,t)\}\\ &\quad \geq \quad \min\{\lambda(\sum_{\ell\in S}\sum_{\tau=t+1}^{T} x_\ell(q',r,\tau)+\sum_{\ell\in S} q'(\ell,t))\end{aligned}$$

$$+(1-\lambda)(\sum_{\ell\in S}\sum_{\tau=t+1}^{T}x_\ell(q'',r,\tau)+\sum_{\ell\in S}q''(\ell,t)),\sum_{b\in B}\sum_{\tau=t}^{T}r(b,t)\}$$

and the proof continues in a similar fashion as above. ◁

Lemma 9. *Let (q',r) and (q'',r) be elements of $MBR_s(q,r)$, and $(q(\lambda),r)$ where $q(\lambda) := \lambda q' + (1-\lambda)q''$ for $0 \le \lambda \le 1$. Then, we have*

1. *$x_\ell(q',r,t) = q'(\ell,t)$, $x_\ell(q'',r,t) = q''(\ell,t)$ and $x_\ell(q(\lambda),r,t) = q(\lambda)(\ell,t)$, for each $\ell \in S$ and $t = 2,\dots,T$,*
2. *$x_s(q(\lambda),r,1) = q(\lambda)(s,1)$, and*
3. *$\sum_{\ell\neq s} x_\ell(q(\lambda),r,1) = \lambda\sum_{\ell\neq s} x_\ell(q',r,1) + (1-\lambda)\sum_{\ell\neq s} x_\ell(q'',r,1)$.*

Proof. Let $\{t^k\}_{k=1}^K$ be the joint set of the critical periods of (q',r) and (q'',r), given in decreasing order.

Because of feasibility we know that $t^K = 1$. Without loss of generality, we will assume that there is only one period being critical for both (q',r) and (q'',r), namely $t^K = 1$. Otherwise, and by using lemma 8, $(q(\lambda),r)$ will share this critical point as well and the planning horizon can therefore be split using this common critical period. The same proof can be applied to each of the two new planning horizons.

From the definition of a block, Claim 1 follows trivially for (q',r) for any period between 2 and the first critical one, and similarly for (q'',r). For the rest of periods, it is enough to show that for $k = 1,\dots,K-1$

$$\sum_{b\in B}\sum_{\tau=t^k}^{T} r(b,\tau) = \sum_{\ell\in S}\sum_{\tau=t^k}^{T} q'(\ell,\tau) \tag{4}$$

when t^k is a critical period for (q',r), and therefore $x_\ell(q',r,t) = q'(\ell,t)$ for all $\ell \in S$ and $t \ge t^k$, or

$$\sum_{b\in B}\sum_{\tau=t^k}^{T} r(b,\tau) = \sum_{\ell\in S}\sum_{\tau=t^k}^{T} q''(\ell,\tau) \tag{5}$$

when t^k is a critical period for (q'', r), and therefore $x_\ell(q'', r, t) = q''(\ell, t)$ for all $\ell \in S$ and $t \geq t^k$.

To show (4) and (5) we will use backwards induction. We will first show the result for $k = 1$. Suppose that t^1 is a critical period of (q', r) and that (4) does not hold. Then we have

$$\sum_{b \in B} \sum_{\tau = t^1}^{T} r(b, \tau) < \sum_{\ell \in S} \sum_{\tau = t^1}^{T} q'(\ell, \tau).$$

Thus, $q'(s, t^1) = 0$ by remark 2. Moreover, because t^1 is not a critical point of (q'', r)

$$\sum_{\ell \in S} \sum_{\tau = t^1}^{T} q''(\ell, \tau) < \sum_{b \in B} \sum_{\tau = t^1}^{T} r(b, \tau),$$

from which follows that

$$\sum_{\tau = t^1}^{T} q''(s, \tau) < \sum_{\tau = t^1}^{T} q'(s, \tau). \tag{6}$$

We will show that supplier s can reduce the costs associated with (q'', r), contradicting the fact that (q'', r) is a best response for supplier s. First observe that the inventory levels in periods $t^1 - 1, \ldots, T$ are strictly positive because these periods belong to the same block. From (6), we know that there exists a period τ where $m(s, \tau) \geq q'(s, \tau) > q''(s, \tau)$. (We may observe that $\tau > t^1$ because $q'(s, t^1) = 0$.) In the following we will show that we can decrease the costs associated with (q'', r) by decreasing the inventories levels in periods $t^1 - 1, \ldots, \tau - 1$ and producing the corresponding amount in period τ. We notice that the unit production costs in period τ for supplier s are at most as expensive as the unit inventory costs incurred from period t^1 up to period τ. This is because in (q', r) the other suppliers have capacity left during period t^1. We can construct (q''', r) such that $q'''(\ell, t) = q''(\ell, t)$ for all $(\ell, t) \neq (s, \tau)$ and $q'''(s, \tau) = q''(s, \tau) + \varepsilon$. It is easy to show that $(q''', r) \in \mathcal{D}_s(q, r)$ and the costs supplier s are strictly lower in (q''', r) than in (q'', r). This yields the desired reduction on costs.

Now suppose that the result is true for $k - 1$ and we want to prove it for k. Again and without loss of generality, suppose that t^k is a critical point of (q', r).

If (4) does not hold for t^k, we have

$$\sum_{b\in B}\sum_{\tau=t^k}^{T} r(b,\tau) < \sum_{\ell\in S}\sum_{\tau=t^k}^{T} q'(\ell,\tau),$$

where $q'(s,t^k)=0$ by using remark 2, i.e., the rest of the suppliers have capacity left during period t^k.

Let t^{low} be the lowest critical period of (q'',r) in $t^1,\ldots,t^{k-1}$, or $t^{\text{low}}=T+1$ if such a period does not exists. Using (5) for t^{low} we have that

$$\sum_{\tau=t^{\text{low}}}^{T} q''(s,\tau) \geq \sum_{\tau=t^{\text{low}}}^{T} q'(s,\tau). \tag{7}$$

Moreover, because t^k is not a critical period of (q'',r),

$$\sum_{\ell\in S}\sum_{\tau=t^k}^{T} q''(\ell,\tau) < \sum_{b\in B}\sum_{\tau=t^k}^{T} r(b,\tau)$$

and thus

$$\sum_{\tau=t^k}^{T} q''(s,\tau) < \sum_{\tau=t^k}^{T} q'(s,\tau). \tag{8}$$

Again, we will show that we can reduce the costs associated with (q'',r). We have that (i) the inventory levels in periods $t^k-1,\ldots,t^{\text{low}}-1$ are strictly positive because these periods belong to the same block, (ii) there exists a period τ, $\tau=t^k+1,\ldots,t^{\text{low}}-1$ where $q'(s,\tau)>q''(s,\tau)$ since (7) and (8) hold and $q'(s,t^k)=0$, (iii) the unit production costs in period τ are at most as expensive as the unit inventory costs incurred from period t^k up to period τ because in (q',r) the other suppliers have capacity left during period t^k. In a similar fashion as before, we can show that we can reduce the costs associated with (q'',r), being in contradiction with the fact that (q'',r) is a best response.

In the following we will show Claim 2. We will distinguish three cases.

- Case 1: $q'(s,1)$ and $q''(s,1)$ are strictly positive. From remark 2, we have that $x_\ell(q',r,1)=q'(\ell,1)$, $x_\ell(q'',r,1)=q''(\ell,1)$, for all $\ell\in S$ and together with Claim 1

$$\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) = \sum_{\ell\in S}\sum_{\tau=1}^{T} q'(\ell,\tau) = \sum_{\ell\in S}\sum_{\tau=1}^{T} q''(\ell,\tau).$$

Therefore,

$$\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) = \sum_{\ell\in S}\sum_{\tau=1}^{T} q(\lambda)(\ell,\tau),$$

and thus, again using Claim 1, $x_\ell(q(\lambda), r, 1) = q(\lambda)(\ell, 1)$, for all $\ell \in S$.

- Case 2: $q'(s,1)\cdot q''(s,1) = 0$ and $q'(s,1) + q''(s,1) > 0$. Without loss of generality we analyze the case $q'(s,1) = 0$ and $q''(s,1) > 0$. From remark 2, we know that the capacity during period 1 in (q'', r) is fully used. This together with Claim 1 means that

$$\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) = \sum_{\ell\in S}\sum_{\tau=1}^{T} q''(\ell,\tau).$$

We will show that

$$\sum_{\ell\in S}\sum_{\tau=1}^{T} q'(\ell,\tau) = \sum_{\ell\in S}\sum_{\tau=1}^{T} q''(\ell,\tau), \tag{9}$$

and therefore the capacity in (q', r) is also fully used. The desired result follows similarly as in Case 1.

Suppose that (9) is not true, i.e., $\sum_{\ell\in S}\sum_{\tau=1}^{T} q''(\ell,\tau) < \sum_{\ell\in S}\sum_{\tau=1}^{T} q'(\ell,\tau)$, and thus

$$\sum_{\tau=1}^{T} q''(s,\tau) < \sum_{\tau=1}^{T} q'(s,\tau). \tag{10}$$

In a similar fashion as above, let t^{low} be the lowest critical period of (q'', r) in $t^1, \ldots, t^{K-1}$, or $t^{\text{low}} = T + 1$ if such a period does not exists. We have that

$$\sum_{\tau=t^{\text{low}}}^{T} q''(s,\tau) \geq \sum_{\tau=t^{\text{low}}}^{T} q'(s,\tau). \tag{11}$$

We will show that we can reduce the costs for supplier s associated with (q'', r), which is a contradiction with the fact that this is a best response. We have that (i) the inventory levels during periods $1, \ldots, t^{\text{low}} - 1$ are strictly positive because these periods belong to the same block, (ii) part of this inventory is produced by supplier s during period 1 because $q''(s,1) > 0$ and the capacity of (q'', r) is fully used, (iii) there exists a period τ,

$\tau = 2, \ldots, t^{\text{low}} - 1$ where $q'(s,\tau) > q''(s,\tau)$ since (10) and (11) hold and $q'(s,1) = 0$, (iv) the unit production costs in period τ are at most as expensive as the unit inventory costs incurred from period 1 up to period τ because in (q', r) the other suppliers have capacity left during period 1 and $x_s(q', r, \tau) = q'(s,\tau) > 0$. We can construct (q''', r) such that $q'''(\ell, t) = q''(\ell, t)$ for all $(\ell, t) \neq (s, \tau)$ and $q'''(s,\tau) = q''(s,\tau) + \varepsilon$. It is easy to show that $(q''', r) \in \mathcal{D}_s(q, r)$ and the costs supplier s are strictly lower in (q''', r) than in (q'', r). This yields the desired reduction on costs.

- Case 3: $q'(s,1) = q''(s,1) = 0$. We have then $q(\lambda)(s,1) = 0$ and the desired result follows trivially.

It remains to show Claim 3. This easily follows from Claims 1 and 2:

$$\begin{aligned}
\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) &= \sum_{\ell\in S}\sum_{\tau=1}^{T} x_\ell(q(\lambda), r, \tau) \\
&= \sum_{\ell\neq s} x_\ell(q(\lambda), r, 1) + x_s(q(\lambda), r, 1) + \sum_{\ell\in S}\sum_{\tau=2}^{T} x_\ell(q(\lambda), r, \tau) \\
&= \sum_{\ell\neq s} x_\ell(q(\lambda), r, 1) + \lambda(x_s(q', r, 1) + \sum_{\ell\in S}\sum_{\tau=2}^{T} x_\ell(q', r, \tau)) \\
&\quad +(1-\lambda)(x_s(q'', r, 1) + \sum_{\ell\in S}\sum_{\tau=2}^{T} x_\ell(q'', r, \tau)) \\
&= \sum_{\ell\neq s} x_\ell(q(\lambda), r, 1) + \lambda(\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) - \sum_{\ell\neq s} x_\ell(q', r, 1)) \\
&\quad +(1-\lambda)(\sum_{b\in B}\sum_{\tau=1}^{T} r(b,\tau) - \sum_{\ell\neq s} x_\ell(q'', r, 1))
\end{aligned}$$

and the desired equality follows. ◁

Theorem 5. *The values of the correspondence MBR_s are convex.*

Proof. Take two elements (q', r) and (q'', r) in $MBR(q, r)$. Write

$$q(\lambda) := \lambda q' + (1-\lambda) q''$$

for $0 \leq \lambda \leq 1$. We have to prove that $(q(\lambda), r)$ is an element of $MBR(q, r)$. We will check (i) feasibility, (ii) best response and (iii) $x_s(q(\lambda), r, t) = q(\lambda)(s, t)$.

Feasibility follows in the same way as in the proof of theorem 4, and (iii) has been proved in lemma 9. It remains to show that that $(q(\lambda), r)$ is an element of $BR_s(q, r)$.

Lemma 9 ensures that the inventory levels in $(q(\lambda), r)$ are a convex combination of the ones in (q', r) and (q'', r), i.e.,

$$G(q(\lambda), r, t) = \lambda G(q', r, t) + (1-\lambda) G(q'', r, t).$$

Therefore,

$$\begin{aligned} V_s(q(\lambda), r) \quad &:= \quad \sum_{t=1}^{T} c(s,t) x_s(q(\lambda), r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q(\lambda), r, t) \\ &= \quad \left[\lambda \sum_{t=1}^{T} c(s,t) x_s(q', r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q', r, t) \right] \\ &+ \quad (1-\lambda) \left[\sum_{t=1}^{T} c(s,t) x_s(q'', r, t) + \frac{1}{|B| + |S|} \sum_{t=1}^{T} g(t) G(q'', r, t) \right] \\ &= \quad \lambda V_s(q', r) + (1-\lambda) V_s(q'', r), \end{aligned}$$

and the desired result follows. $\triangleleft$

5. Conclusion and further research

In this paper we showed the existence of Nash equilibrium in game theoretic models with constraints across players' strategies. We used this result in three different applications. The first application is a generalization of the model of Caron and Laye.

The second and third application are in the context of a two-level supply chain. We showed that for both the 'hedging against late deliveries' mechanism and the 'minimizing inventory levels' mechanism we can apply the existence result to a subcorrespondence of the best response correspondence. We also showed that in the two-level supply chain setting the centralized solution is in general not a Nash equilibrium.

The observation that Nash equilibrium is in general not optimal seems to suggest that, when decision making is decentralized in the supply chain, that the chain may get caught in a non-optimal solution. One way to resolve this is by monetary transfers. Is it possible to improve upon the Nash equilibrium outcome by means of payment schemes or subsidies? Under what circumstances is it possible to guarantee full optimality with such payment schemes? One other direction for further research would be to see whether we can apply the existence result also in a more general setting, for example allowing for setup costs in production.

References

Berge, C. (1966) *Espaces Topologiques*, Dunod, Paris (2^e édition).

Cachon, G.P. and S. Netessine (2004) *Game theory in supply chain analysis*, In D. Simchi-Levi, S.D. Wu, and Z-J. Shen, editors, *Handbook of Quantitative Supply Chain Analysis*, 13 – 65, Kluwer Academic Publishers, Boston.

Caron, C. and J. Laye (2003) *A fast algorithm for capacity constrained Cournot-Nash equilibrium*, Working paper.

Cook, W., A.M.H. Gerards, A. Schrijver and É. Tardos (1986) *Sensitivity theorems in integer linear programming*, Mathematical Programming **34**, 48 – 61.

Debreu, G. (1952) *A social equilibrium existence theorem*, Proceedings of the National Academy of Sciences **38**, 886 – 893.

Kakutani, S. (1941) *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal **8**, 457 – 459.

Nash, J.F. (1950) *Equilibrium points in n-person games*, Proceedings from the National Academy of Science, U.S.A. **36**, 48 – 49.

Nash, J.F. (1951) *Noncooperative games*, Annals of Mathematics **54**, 286 – 295.

Saporiti, A. and F. Tohmé (2003) *Single-crossing, strategic voting and the median choice rule*, CEMA Working Paper no. 237, Universidad del CEMA.