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Strategic power indices: Quarrelling in coalitions[*]

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Abstract

While they use the language of game theory known measures of a priory voting power are hardly more than statistical expectations assuming the random behaviour of the players. Focusing on normalised indices we show that rational players would behave differently from the indices predictions and propose a model that captures such strategic behaviour.

Keywords and phrases: Banzhaf index, Shapley-Shubik index, a priori voting power, rational players.

1 Introduction

Since Shapley and Shubik (1954) adopted the Shapley value to measure a priori voting power game theory has contributed an enormous literature to

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this topic: established theoretical underpinnings for the existing or rediscovered indices, introduced new ones, but the plethora of power indices hints that there is no single best. What is best depends on the institutional details concerning the voting that cannot be captured by the voting game the index is applied to (Laruelle, 1999). Whether one is more interested in comparing powers of different players in the same game or the powers of the same player in different games is one crucial choice. Since we are more interested in the first we focus on normalised indices.

Game theory embraced power indices despite the fact that none of the power indices are really "game theoretical." Voting situations are games where "the acquisition of power is the payoff" (Shapley, 1962, p. 59.), but 'acquisition' is an overstatement as players have no strategies: it seems voting indices are hardly more than statistical measures of the voters' random behaviour. We like to believe that this is not a realistic model of most voting situations; we assume that voters are rational who can and want to influence (that is: maximise) their power.

Motivated by the paradox of quarrelling members (Brams, 2003) we extend voting games to *strategic voting games* where players can choose which coalitions are they willing to join. We show that all known normalised indices are affected by such strategic behaviour.

Our paper is not the first to disallow certain (winning) coalitions in values or power indices. Aumann and Drèze (1975) assume that property rights may make it impossible to form every coalition. Owen (1977, 1982) assume that coalitions are formed exactly in order to increase power. Myerson (1977, 1980) presents a model where players communicate via conferences and not all conferences may occur Faigle and Kern (1992). The application of such restrictions to power indices are more recent (Bilbao, Jiménez, and López, 1998).

The structure of the paper is as follows. We start with a brief introduction to voting games and an overview of the known indices. We briefly explain the paradox of quarrelling members, introduce a framework for strategic indices and prove some properties.

2 Power indices

A voting situation is completely determined by $(N, \mathcal{W})$, where N is the set of voters and $\mathcal{W}$ denotes the set of winning coalitions. We study *simple* voting games, that is, games, where

1. $\varnothing \notin \mathcal{W}$,

2. $N \in \mathcal{W}$, and

3. if $C \subset D \subset N$ and $C \in \mathcal{W}$ then $D \in \mathcal{W}$.

Later the last two conditions will only be applied to the *initial* set of winning coalitions.

A simple voting game is *proper* if we exclude the possibility for a motion as well as its opposite being approved simultaneously; formally:

4. If $S \in \mathcal{W}$ and $T \in \mathcal{W}$ then $S \cap T \neq \emptyset$.

Let Γ denote the collection of proper voting games.

Let $\mathcal{M}$ denote the set of *minimal winning coalitions*: the set of coalitions for which no proper subset is a winning coalition. Formally: if $S \in \mathcal{M}$ and $i \in S$, then $S \setminus \{i\} \notin \mathcal{W}$. Clearly $\mathcal{M} \subseteq \mathcal{W}$.

In this paper we motivate our examples by *weighted voting games*. Here N is a collection of n interest groups, or *parties* having $w_1, w_2, \ldots, w_n$ individual representatives $(w_i \in \mathbb{R}_+)$. Let $w = \sum_{i=1}^{n} w_i$. We assume that a quota of $w \geq q > w/2$ is *required* to pass a bill. Representatives of a party cast the same vote and therefore the game is a weighted voting game. For more on weighted voting games see Straffin (1994).

The function κ given by

$$\kappa : \quad \Gamma \quad \longrightarrow \mathbb{R}^{\mathbb{N}} \qquad (2.1)$$

$$(N, \mathcal{W}) \quad \longmapsto (\kappa_i)_{i \in N}, \qquad (2.2)$$

is a *power measure*. A power measure κ is a *power index* if $\sum_{i \in N} \kappa_i = 1$.

In the following we explain some of the well-known indices.

The *Shapley-Shubik index* (Shapley and Shubik, 1954) is an application of the Shapley value (Shapley, 1953) to measure voting power, motivated by the story that parties throw their support at a motion in some order until a winning coalition is reached. The last, *pivotal* party gets all the credit; the Shapley-Shubik index is then the proportion of orderings where it is pivotal

$$\phi_i = \frac{\# \text{ times } i \text{ is pivotal}}{n!}.$$

The *Banzhaf measure* (Penrose, 1946; Banzhaf, 1965) is the probability that a party is *critical* for a coalition, that is, the probability that it can turn winning coalitions into losing ones.

$$\psi_i = \frac{\# \text{ times } i \text{ is critical}}{2^{n-1}}.$$

The *Banzhaf index* β (Coleman, 1971) is the Banzhaf measure normalised to 1 – already in the spirit of the Shapley-Shubik index.

There are a few variants of the (normalised) Banzhaf index. In the *Johnston index* γ (Johnston, 1978) the credit a critical player gets is inversely proportional to the number of critical players in the coalition. In effect, coalitions of different sizes have the same contribution to the distribution of power. The *Deegan-Packel index* ρ (Deegan and Packel, 1978) is a further modification that only considers minimal winning coalitions, motivated by the idea that only minimal winning coalitions should form so that the benefits from winning should be least divided (Riker, 1962). Finally the Holler-Packel or *Public Good Index h* (Holler and Packel, 1983) modifies the Deegan-Packel index: here the benefit of forming a winning coalition is given to each and every player in the coalition. With the normalisation in simple games the index is nothing but a normalised Banzhaf index, where only minimal coalitions are taken into account.

3   Strategic voting

All existing indices assume an exogenously given set of winning coalitions and that players join winning coalitions at all times. This seems indeed

natural – why would players give up part of their power? If for instance two players start to "quarrel" and refuse to cooperate making any coalition they both belong to losing their power should decrease. Not necessarily. The "Paradox of Quarrelling Members" (Kilgour, 1974; Brams, 2003) arises when two players benefit from refusing to cooperate with each other.

Whether this is truly paradoxical is subject to a debate, what matters for us is that players can *acquire* power by approving/rejecting certain coalitions. In this paper we extend voting games to allow for such strategic considerations and define strategic power indices.

Consider the following game.

Example 1. The game G_1 consists of four players represented by their weights[1]: $3_1, 3_2, 2_1, 2_2$ and voting has a quota of 6. The winning coalitions are the following: $\mathcal{W} = \{\underline{3_13_2}, \underline{3_13_2}2_1, \underline{3_13_2}2_2, \underline{3_12_12_2}, \underline{3_22_12_2}, 3_13_22_12_2\}$. (Critical players are underlined.) Given this set we can calculate the Banzhaf index $\beta = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\}$. Notice that in coalition $3_13_22_1$ player 2_1 is not critical; suppose 2_1 rejects participation in this coalition. The recalculated Banzhaf index for $\mathcal{W}' = \{\underline{3_13_2}, \underline{3_13_2}2_2, \underline{3_12_12_2}, \underline{3_22_12_2}, 3_13_22_12_2\}$ is $\beta' = \{\frac{3}{10}, \frac{3}{10}, \frac{1}{5}, \frac{1}{5}\}$. Player 2_1's rejection increased its relative power. It is therefore *not* in player 2_1's interest to join every winning coalition it is invited to. This finding is not really surprising. In coalition $3_13_22_1$ player 2_1 assisted players 3_1 and 3_2 in forming a winning coalition, but without getting any credit for it.

Minimal winning coalitions may also be subject to blocks:

Example 2. G_2 is a 9-player game with players $5_1, 5_2, 5_3, 1_1, 1_2, 1_3, 1_4, 1_5, 1_6$ and a quota of 11. Here $\mathcal{M} = \{5_15_25_3, 5_i5_j1_k, 5_i1_11_21_31_41_51_6\}$, where $k \in \{1, 2, 3, 4, 5, 6\}$ and $i, j \in \{1, 2, 3\}$. Let $\mathcal{W} = \mathcal{M}$. Then the Banzhaf index is given by $\beta = \{\frac{7}{39}, \frac{7}{39}, \frac{7}{39}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}\}$.
Now consider $\mathcal{W}' = \{5_15_25_3, 5_i5_j1_k, 5_l1_11_21_31_41_51_6\}$, where $k \in \{1, 2, 3, 4, 5, 6\}$, $i, j \in \{1, 2, 3\}$ and $l \in \{2, 3\}$. Then $\beta' = \{\frac{13}{71}, \frac{14}{71}, \frac{14}{71}, \frac{5}{71}, \frac{5}{71}, \frac{5}{71}, \frac{5}{71}, \frac{5}{71}, \frac{5}{71}\}$. The set $\mathcal{W}'$ does not contain the minimal winning coalition $5_11_11_21_31_41_51_6$, yet the critical player 5_1 is better off as $\frac{13}{71} > \frac{7}{39}$.

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[1]Subscripts are used to distinguish players with identical weights from each other.

While the aforementioned indices claim to measure power, it seems, players have actually little power to influence their power: hence they are no more than probabilistic values. The paradox of quarrelling members as well as the above examples illustrate that players *can* increase their power by refusing to participate in certain coalitions. If a player credibly refuses to participate in a coalition neither him nor his colleagues should get credit for being critical to a coalition that never forms.

3.1 General model

The idea of quarrelling is generalised to coalitions: a coalition C is blocked if the subset $Q \subseteq C$ quarrels.

Definition 1. Player i's *strategy* s_i describes subsets of players i wants to quarrel with. Formally $s_i \subseteq 2^{N \setminus \{i\}}$ such that if $C, D \in s_i$ then $C \nsubseteq D$ (otherwise quarrelling with C implies quarrelling with D).
Player i's strategy space S_i collects its voting strategies, $s = \{s_1, s_2, \ldots, s_N\}$ is a particular strategy profile and $S = \prod_{i \in N} S_i$ the strategy space.

Definition 2 (Strategic voting game)**.** A quadruple $(N, \mathcal{W}, S, \kappa)$ consisting of a set of players N, a collection of initial winning coalitions $\mathcal{W}$, a strategy space S and a power index κ is called a *strategic voting game*.

Players noncooperatively accept/reject coalitions. Given strategies s the set of winning coalitions $\mathcal{W}(s)$ is the quarrel-free subset:

$$\mathcal{W}(s) = \{w \in \mathcal{W} \mid c \nsubseteq w \; \forall c \in s_i, \; \forall i \in N\}. \tag{3.1}$$

A quarrel is for good: we must think of an offence that "burns" the possibility to reconvene. Subsets of a blocked coalition may form without restrictions.

Players maximise their payoff, given by their power $\kappa(s) = \kappa(N, \mathcal{W}(s))$.

A *strategic power index* is then given by $\kappa(N, \mathcal{W}(s^*))$, where s^* is a Nash equilibrium: for all $i \in N$ and all $s_i \subseteq s_i^*$, $s_i \in S_i$ we have $\kappa_i(s^*) \geq \kappa_i(s_i, s_{-i}^*)$. Such a strategic power index always exists ($\mathcal{W}(s^*) = \varnothing$ is an equilibrium) but is generally not unique. In the sequel we provide a unique refinement for certain indices.

3.2 Properties

The first group of results applies to all indices discussed in Section 2. Such indices give no credit to non-critical or *surplus* players and give the same credit to all critical (or swing) players within a coalition. The differences are mostly in giving different winning coalitions C different weights a^C, where $\sum_{C \in \mathcal{W}} a^C = 1$ and $a^C = 0$ for coalitions containing surplus players only.

A power index $\kappa(N, \mathcal{W})$ can be written as

$$\kappa_i = \sum_{C \in \mathcal{W}} a^C \mu_i^C, \quad \text{where} \tag{3.2}$$

$$\mu_i^C = \begin{cases} \frac{1}{k^C} & \text{if } i \text{ is critical} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

is the credit player i gets for being in the coalition if the number of critical players is k^C (therefore $\sum_{i \in C} \mu_i^C = 1$). For instance for the normalised Banzhaf index $a^C = \frac{k^C}{K}$, where $K = \sum_{C \in \mathcal{W}} k^C$.

Blocking a coalition B affects a player in two ways. On the one hand for all $C \supseteq B$ the coalition's weight becomes $(a^C)' = 0$ and hence the player loses $\sum_{C \supseteq B} a^C \mu_i^C$, on the other hand, due to the normalisation the weight of other coalitions increases, and hence the credit it gets from other coalitions is scaled up by

$$\frac{\sum_{C \in \mathcal{W}} a^C}{(\sum_{C \in \mathcal{W}} a^C) - (\sum_{C \supseteq B} a^C)}. \tag{3.4}$$

Null players do not benefit from this, and remain null in strategic indices. For this reason they will be ignored in our analysis.

Proposition 3. *Surplus coalitions containing critical players are blocked.*

Proof. Consider a coalition B containing a surplus player i. If i is not critical in B, it is also not critical in $C \supset B$ (as, by monotonicity if $B \setminus \{i\}$ is winning, so is $C \setminus \{i\} \supset B \setminus \{i\}$) and therefore $\mu_i^C = 0$ for all $C \supseteq B$. On the other hand for all C with $a^C > 0$ there are other players that are critical, and so when blocking B the power of player i is scaled up according to Expression 3.4 making the block profitable. $\qquad \square$

Corollary 4. *For power* indices *we have* $\mathcal{M} \supseteq \mathcal{W}^*$.

Note, however that not all minimal coalitions survive as our Example 2 has already illustrated.

3.3 Equilibrium selection

In the following we study indices that are based on the set of minimal winning coalitions rather than on all winning coalitions. Holler and Packel (1983) argue that "since a non-critical member is not decisive for the winning of his preferred coalition, i.e. his preferred policy, he has no incentive to vote. ... [This] does not imply that only these coalitions will form. It merely suggest that only these coalitions should be considered for measuring a priori voting power" (Holler and Packel, 1983, p. 24.). Two points are to be noted: Firstly, only those coalitions should count where all players are positively interested in joining. Secondly, this should not exclude the possibility of other coalitions forming, however, no player should count on the formation of other coalitions, as that will not be due to his or her power. A similar prediction is made by *aspiration* solution concepts (Bennett, 1983, p. 15.).

By the definition of blocking it is clear that $\mathcal{W} = \varnothing$ is a Nash-equilibrium, while it is clear that this is neither the only equilibrium nor the one we want (not the least because power indices are undefined here). In general there are many Nash equilibria and in the following we make a selection of these. We take a conservative approach in two ways: (i) While we do not insist on players to select the most profitable block, we do on one thing: if the block of a superset of the blocked coalition would yield a better improvement then we assume that the player does not block quite so aggressively and goes for the smaller block (quarrelling with a superset). (ii) We assume that players are friendly, open to form a coalition unless there are good reasons to do otherwise. The *friendly set* $\mathcal{F}$ is defined as follows:

$$
\mathcal{W}(s) \in \mathcal{F} \text{ if } \begin{cases} \forall i \in N, s_i = \varnothing \\ \exists i \in N, \exists \mathcal{W}(s'_i, s_{-i}) \in \mathcal{F}, \text{ such that } \kappa_i(s) > \kappa_i(s'_i, s_{-i}). \end{cases}
$$

We select *friendly equilibria* s^*, such that $\mathcal{W}(s) \in \mathcal{F}$ that are minimal for inclusion. The *equilibrium set of winning coalitions* is $\mathcal{W}^* = \mathcal{W}(s^*)$ and the *strategic κ power index* is defined as

$$\kappa^* = \kappa(N, \mathcal{W}^*).$$

Now observe that for minimal winning coalitions $C \neq D$ we have neither $C \subset D$ nor $D \subset C$, therefore by blocking C a player will not block D and vice versa. Therefore a player has the possibility to block each minimal winning coalition separately and not in a larger block. As in general these minimal winning coalitions may have different sizes and consequently different μ_i^C values for i, some will be more attractive than others. By our assumption (i) players will first only block the worst coalition(s). (As the block increases the player's power, coalitions that have been unattractive before, remain unattractive. Therefore in case there are several coalitions of the same size, we may also assume that they are blocked one-by one.

In sum, our model can be reduced to players picking which coalitions they do not want to form. This result makes it particularly easy to work with coalitions rather than strategies. Then an equilibrium is simply $\mathcal{W}^*$ instead of $\mathcal{W}(s^*)$.

Player i profitably blocks coalition B iff

$$\kappa_i(N, \mathcal{W} \setminus \{B\}) \;\; > \;\; \kappa_i(N, \mathcal{W}) \qquad (3.5)$$

$$\frac{\sum_{C \in \mathcal{W}} a^C}{\sum_{C \in \mathcal{W}} a^C - a^B} \left(\sum_{C \in \mathcal{W}} a^C \mu_i^C - a^B \mu_i^B \right) \;\; > \;\; \sum_{C \in \mathcal{W}} a^C \mu_i^C \qquad (3.6)$$

After some rearrangements we get

$$\frac{\sum_{C \in \mathcal{W}} a^C \mu_i^C}{\sum_{C \in \mathcal{W}} a^C} = \kappa_i(N, \mathcal{W}) > \mu_i^B, \qquad (3.7)$$

which gives the following result.

Lemma 5. *A block by player i is profitable if and only if the blocked coalition gives less credit to player i than the average credit it gets, that is, than its power index.*

Proposition 3 can also be seen as a corollary of this lemma.

Lemma 5 also suggests a relation to the theory of *aspirations* (Bennett, 1983), although this relation turns out to be superficial. In the theory of aspirations it is not some coalition's payoff that is bargained over: it is the players that make their claims and unless their claims are satisfied certain coalitions will or will not form. Here this claim is expressed by their power index, the "credit they receive in general" and players demand the same credit in coalitions. Unfortunately the link between the two concepts does not go much beyond that. While a power index satisfies $\sum_{i \in N} \kappa_i^* = 1$ a vector of aspirations will almost always be larger. Bennett (1983, p. 15.) provides the following example:

Example 3. A game with 5 players with weights 2, 2, 1, 1, and 1, and a quota of 5. Here the unique partnered, balanced, equal gains aspiration is $(0.4, 0.4, 0.2, 0.2, 0.2)$, while the public good index is $h = (\frac{4}{17}, \frac{4}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17})$.

Now we move on to our main result.

Theorem 6. *The friendly equilibrium is uniquely defined and is given by*

$$\mathcal{W}^* = \bigcap_{w \in \mathcal{F}} w. \tag{3.8}$$

In order to prove this theorem we need some additional results.

Proposition 7. *Let $C_i, C_j \in \mathcal{W}$ be coalitions that both contain both i and j and such that i and j want to block C_i and C_j respectively. Then either i wants to block C_j or j wants to block C_i.*

Proof. Assume that the proposition is false. This means the following. Player j blocks C_j, hence $\mu_j^{C_j} < \kappa_j(\mathcal{W})$. By our assumption i does not block, hence $\mu_i^{C_j} \geq \kappa_i(\mathcal{W})$. Therefore $\mu_j^{C_j} < \mu_i^{C_j}$. Similarly i blocks C_i, hence $\mu_i^{C_i} < \kappa_i(\mathcal{W})$. By our assumption j does not block, hence $\mu_j^{C_i} \geq \kappa_j(\mathcal{W})$. In sum $\mu_i^{C_i} < \mu_i^{C_j}$ and $\mu_j^{C_i} < \mu_j^{C_j}$. Since C_i and C_j are minimal coalitions $\mu_i^{C_i} = \mu_j^{C_i} = \frac{1}{|C_i|}$ and $\mu_j^{C_j} = \mu_i^{C_j} = \frac{1}{|C_j|}$. Contradiction $\qquad \square$

Proposition 8. *For all $\mathcal{W}_i, \mathcal{W}_j \in \mathcal{F}$ we have $\mathcal{W}_i \cap \mathcal{W}_j \in \mathcal{F}$.*

Proof. The proof is by induction on the differences between $\mathcal{W}_i$ and $\mathcal{W}_j$.

First we deal with the elementary step. Assume $\mathcal{W}_i = \{A, C_1, C_2, \ldots C_m\}$, $\mathcal{W}_j = \{B, C_1, C_2, \ldots C_m\}$, that is, the two sets only differ in 1 element each. This ensures that their intersection is non-trivial. $\mathcal{W}_i$ and $\mathcal{W}_j$ are descendants of a common ancestor $\mathcal{W}_0 = \{A, B, C_1, C_2, \ldots C_m\}$, but after blocking B and A, respectively by some players i and j. The proposition merely states that either blocking A is profitable from $\mathcal{W}_i$ or blocking B is profitable from $\mathcal{W}_j$.

$\mathcal{W}_i$ is the result of blocking B by i. If $j \notin B$ then $\kappa_j(\mathcal{W}_0) \leq \kappa_j(\mathcal{W}_i)$. We know that j blocks A at $\mathcal{W}_0$ and hence $\kappa_j(\mathcal{W}_0) > \mu_j^A$. Hence $\kappa_j(\mathcal{W}_i) > \mu_j^A$, which implies that j also blocks A at $\mathcal{W}_i$. Thus $\mathcal{W}_{ij} = \{C_1, C_2, \ldots C_m\} \in \mathcal{F}$.

The symmetric case gives the corresponding result for i and B at $\mathcal{W}_j$.

Finally we must consider the case where none of the previous two cases applied, that is where $j \in B$ and $i \in A$. As only a member can block a coalition, we also have $j \in A$ and $i \in B$. Therefore we can apply Proposition 7 to show that i blocks at $\mathcal{W}_j$ or j at $\mathcal{W}_i$, which, as before, gives the result.

We have discussed all possible cases which completes the first part of the proof. Now we move on to the general case. Assume that we have shown the result for all pairs of sets with differences up to $k - 1$.

Now consider $\mathcal{W}_i = \{A_1, A_2, \ldots, A_k, C_1, C_2, \ldots C_m\}$ as well as $\mathcal{W}_j = \{B_1, B_2, \ldots, B_l, C_1, C_2, \ldots C_m\}$, where $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_l$ represent the blocks that did *not* take place and $l \leq k$. (Possibly $A_p = B_q$ for some p and q.) The question is whether this difference can be eliminated.

By definition if $\mathcal{W}_i \in \mathcal{F}$ there exists a sequence of blocks starting from $\mathcal{W}_0$ that lead to $\mathcal{W}_i$ and a similar sequence exists to $\mathcal{W}_j$. Let $\mathcal{W}_i^0$ and $\mathcal{W}_j^0$ be the first elements that are not common, without loss of generality, as results of blocking B_1 and A_1 respectively. By the elementary step $\mathcal{W}_j^1 = \mathcal{W}_i^0 \cap \mathcal{W}_j^0$ belongs to $\mathcal{F}$.[2] Now take the next set $\mathcal{W}_2$ along the path to $\mathcal{W}_i$, $\mathcal{W}_i^1$. By the same argument $\mathcal{W}_i^1 \cap \mathcal{W}_j^1$ also belongs to $\mathcal{F}$. Repeating this argument we travel parallel to the path and in the penultimate step we get $\mathcal{W}_j^p \in \mathcal{F}$. For the last time by the same argument $\mathcal{W}_i \cap \mathcal{W}_j^p = \{A_2, \ldots, A_k, C_1, C_2, \ldots C_m\}$

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[2]Our notation is slightly misleading as $\mathcal{W}_j^1$ is not necessarily on the path to $\mathcal{W}_j$, but this should not lead to confusion.

also belongs to $\mathcal{F}$. If $l < k$, our inductive assumption can be used to complete the proof.

In case $l = k$ it is necessary to apply the same argument once more, but on the other side: to show that $\{B_2, \ldots, B_l, C_1, C_2, \ldots C_m\} \in \mathcal{F}$. $\qquad \square$

Proof of Theorem. By Proposition 8 pairwise intersections of elements of $\mathcal{F}$ also belong to $\mathcal{F}$. As the number of winning coalitions is finite the result on pairwise intersections implies that $\mathcal{W}^*$ as defined in Equation 3.8 belongs to $\mathcal{F}$. Clearly $\mathcal{W}^* \subseteq w$ for all $w \in \mathcal{F}$. Therefore $\mathcal{W}^*$ is the smallest friendly set and it is trivially an equilibrium. $\qquad \square$

Corollary 9. *The strategic power index κ^* is well-defined.*

4 Conclusion

We have developed a model that measures power taking the rational, utility maximising behaviour of players into account. We have also shown that none of the well-known power indices account for this behaviour. It appears that these supposedly game theoretic concepts are not more than statistical measures of random behaviour.

There are at least two possibilities to resolve this conflict. The one we chose is to modify existing power indices so that no credit is given for coalitions that do not form. The advantage of this solution is that it is directly motivated by the problem and gives a perfect answer to it without affecting the concepts a great deal.

While this is the option we choose here there is an interesting alternative. Observe that blocking a winning coalition may be advantageous to some players, but it will hurt others in the coalition. The only players whose power will surely increase are those *outside* the coalition. This indicates that overall members of the coalition loose by not forming the coalition. Hence forming the coalition increases the power of the members and therefore there exists distributions of this power that benefit all members. Giving room for renegotiation would lead us to cooperative, probably set-like solutions and would make us lose the advantages of a single-point solution concept.

Two other choices we have made are to assume that blocking coalition C also blocks $D \supset C$ and to work with power indices defined over minimal winning coalitions only. Blocking single coalitions would not preserve null players who could gain power for "mediation" (turning a blocked coalition into a winning one by their entry – of course this coalition would be blocked soon, too) and would allow non-minimal winning coalitions that are not surplus coalitions as they would only consist of critical players. While our original model considered a variant of this alternative, in order to avoid such odd phenomena one has to separate the notions of winning a feasible coalition.

Finally, the uniqueness of the friendly equilibrium for power indices also looking at surplus coalitions remains an open problem. With the aforementioned model counterexamples can be presented here a systematic search for them was in vain, now we believe the result to hold, but the present proof does not directly extend to those indices.

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