Bundling in exchange markets with indivisible goods[*]

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Abstract

We study efficient and individually rational exchange rules for markets with heterogeneous indivisible goods that exclude the possibility that an agent benefits by bundling goods in her endowment. Even if agents' preferences are additive, no such rule exists.

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1   Introduction

We consider exchange markets with heterogeneous indivisible objects where each agent is endowed with a set of objects and monetary transfers are not possible. Examples are the exchange of equipment and tasks among workers or departments of a firm (Papái, 2005; Atlamaz and Klaus, 2005). Our model of exchange includes housing markets (Shapley and Scarf, 1974: each agent owns exactly one object, which can be traded) and multiple-type housing markets (Moulin, 1995; Konishi *et al.*, 2001: each agent's endowment consists of one object of each type and agents are allowed to trade different types of indivisible objects). An exchange rule assigns to each trader a set of objects. We are interested in exchange rules satisfying appealing properties such as efficiency, individual rationality (no agent is worse off after trading) and robustness to manipulations. A strong non-manipulation requirement, strategy-proofness (no agent benefits from misrepresenting her preferences), is known to be incompatible with efficiency and individual rationality (Sönmez, 1999; Konishi *et al.*, 2001).

In exchange markets agents may also have an incentive to manipulate the outcome using their endowments, for example, by hiding, destroying, or transferring part of them. The study of hiding-proofness and destruction-proofness in the context of classical exchange economies goes back to Postlewaite (1979), while transfer-proof rules were studied by Sertel and Özkal-Sanver (2002) for two-sided matching with endowments. Atlamaz and Klaus (2005) analyzed these three manipulation possibilities for exchange markets with indivisible goods. In particular, they show that efficient and individually rational

rules are generally not immune to manipulations via endowments, with some exceptions for two-agent exchange markets.

We are also interested in exchange rules that prevent the possibility of manipulation via endowments, but focus on *bundling-proofness*: no agent benefits from bundling goods in her endowment. To fix ideas, we assume that bundling and decomposition of bundles are costless and the only effect of bundling is that bundled objects have to be traded together. To keep the model simple we assume that agents' endowments consist of disjoint sets of objects and interpret the objects in each set as being bundled. Now, an exchange rule assigns to each agent a set of objects such that trade restrictions given by the agents' endowment distribution are respected. Since bundling-proofness can be interpreted as a much weaker requirement than strategy-proofness (Remark 1), there is hope to restore compatibility of "non-manipulation" with efficiency and individual rationality. Unfortunately, even for additive preferences, no bundling-proof, efficient, and individually rational exchange rule exists (Theorem 1).

2 Exchange markets with indivisible objects

Let K be a finite *set of heterogeneous objects* with $|K| \geq 2$ and $N = \{1, \ldots, n\}$ a finite *set of agents* $(n \geq 2)$. We denote by 2^K the set of all subsets of K. Each agent $i \in N$ has an *endowment* B_i that consists of disjoint sets of objects B_i^k, *i.e.*, $B_i = (B_i^1, \ldots, B_i^{l(i)})$ is such that (a) for all $k \in \{1, \ldots, l(i)\}$, $B_i^k \in 2^K$, and (b) for all $k, k' \in \{1, \ldots, l(i)\}$ $(k \neq k')$, $B_i^k \cap B_i^{k'} = \emptyset$. We say that *objects in B_i^k are bundled*. By $E_i = \bigcup_{k=1}^{l(i)} B_i^k \in 2^K$ we denote the *set of objects agent i owns*. The set of all endowments for agent i is denoted by $\mathcal{B}_i$. For example, $B_i = (B_i^1, B_i^2, B_i^3) = (\{a, b\}, \{c\}, \{d, e, f\})$ means that agent i owns objects a, b, c, d, e, f and $E_i = \{a, b, c, d, e, f\}$. Furthermore, objects a and b are bundled, and objects d, e, and f are bundled as well. An *endowment distribution* $B = (B_1, \ldots, B_n)$ satisfies the following two conditions: (i) for all $i \in N$, $B_i \in \mathcal{B}_i$ and (ii) for all $i, j \in N$ $(i \neq j)$, $E_i \cap E_j = \emptyset$. We denote the *set of all endowment distributions* by $\mathcal{B}$.

Each agent $i \in N$ has complete and transitive preferences R_i over 2^K with P_i being the associated strict preference relation. We let $\mathcal{R}_i$ denote the *set of all preferences for agent i*, $R = (R_i)_{i \in N}$ a *preference profile*, and $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_n$ the *set of all preference profiles*.

Given a preference profile $R \in \mathcal{R}$ and an endowment distribution $B \in \mathcal{B}$, we denote an *(exchange) market* by (R, B). Since in the remainder of the article we assume that the preference profile remains fixed while endowment distributions may vary, we simply denote a market by its endowment distribution $B \in \mathcal{B}$. An *allocation* for a market $B \in \mathcal{B}$ is a list $(S_1, \ldots, S_n)$ such that

1. each agent $i \in N$ receives some subset $S_i \subseteq \bigcup_{i=1}^n E_i$,

2. no two agents receive the same object, *i.e.*, for all $i, j \in N$ $(i \neq j)$, $S_i \cap S_j = \emptyset$, and

3. objects that are bundled in an agent's endowment are assigned jointly, *i.e.*, for all $i \in N$, $B_i = (B_i^1, \ldots, B_i^{l(i)})$, and all $k \in \{1, \ldots, l(i)\}$, either $B_i^k \subseteq S_j$ for some $j \in N$ or $B_i^k \cap \bigcup_{i \in N} S_i = \emptyset$.

The first two conditions are standard, while the third condition is specific to our setting and simply requires that an allocation respects all trade restrictions that bundling in the endowments creates. Note that we allow for *free disposal*, that is, $\bigcup_{i=1}^{n} S_i \subsetneq \bigcup_{i=1}^{n} E_i$ is possible.

Finally, notice that on the one hand agents are endowed with bundles of objects while on the other hand preferences are not defined over bundles but over subsets. The reason for this way of modeling is that we assume that bundles impose trade restrictions but the fact that objects are bundled does not change these objects physically.

3 Efficiency, individual rationality, and bundling-proofness

An *(exchange) rule* is a function φ that associates with each market $B \in \mathcal{B}$ an allocation $\varphi(B) = (S_i)_{i \in N}$. Given $i \in N$, we call $\varphi_i(B)$ the *allotment* of agent i at $\varphi(B)$.

We assume that a rule only chooses (Pareto) *efficient* allocations: for all $B \in \mathcal{B}$ there is no allocation $(S_i)_{i \in N}$ such that for all $i \in N$, $S_i \, R_i \, \varphi_i(B)$, with strict preference holding for some $j \in N$.

To express voluntary participation or *individual rationality*, we assume that agents find their allotments at least as good as their endowments: for all $B \in \mathcal{B}$ and all $i \in N$, $\varphi_i(B) \, R_i \, E_i$.

The main property we focus on in this paper is *bundling-proofness*. We assume that any agent can costlessly increase the degree of bundling in her endowment. By increasing the degree of bundling we mean that an agent can merge some of the subsets in her endowment to larger bundles, *e.g.*, agent i with $B_i = (B_i^1, B_i^2, B_i^3) = (\{a,b\}, \{c\}, \{d,e,f\})$ could bundle $\{c\}$ with $\{d,e,f\}$ and announce $\hat{B}_i = (\{a,b\}, \{c,d,e,f\})$. This bundling implies that now objects c, d, e, and f have to be assigned to the same agent after trade.

More generally, for any agent i with endowment B_i a *bundled endowment* $\hat{B}_i = (\hat{B}_i^1, \ldots, \hat{B}_i^{\hat{l}(i)})$ $[\hat{l}(i) \leq l(i)]$ is a partition of E_i such that for all $k \in \{1, \ldots, l(i)\}$ there exists $\hat{k} \in \{1, \ldots, \hat{l}(i)\}$ such that $B_i^k \subseteq \hat{B}_i^{\hat{k}}$, which means that $\hat{B}_i$ is a coarser partition than B_i. Given $B \in \mathcal{B}$, $(\hat{B}_i, B_{-i})$ denotes the endowment distribution obtained from B by replacing B_i by $\hat{B}_i$.

We require that if agent i bundles her endowment, then she finds her allotment without bundling at least as good as her allotment with bundling, given all other agents do not manipulate their endowments. A rule φ is *bundling-proof* if for all $B \in \mathcal{B}$, all $i \in N$, and all bundled endowments $\hat{B}_i$, $\varphi_i(B) \, R_i \, \varphi_i(\hat{B}_i, B_{-i})$.

Remark 1. As mentioned in the Introduction, it is well-known that efficiency, individual rationality, and strategy-proofness are not compatible in our model (Sönmez, 1999).

Assume that a rule is efficient and individually rational. Then, the power an agent has by misrepresenting her preferences is to reduce the set of efficient and individually rational allocations, possibly to her benefit. By bundling objects in her endowment, an agent also can reduce the set of efficient and individually rational allocations. Note that through preferences complementarities between any set of objects can be expressed, which in combination with efficiency and individual rationality may lead to trade restrictions that also concern other agents' objects. By bundling fewer trade restrictions can be imposed and therefore we can interpret bundling-proofness as a much weaker requirement than strategy-proofness. $\diamond$

We consider now the additive preference domain: agent i's preferences are *additive* if there is a function $u_i : K \to \mathbb{R}$ ($u_i(\emptyset) = 0$) such that for all $T, T' \in 2^K$, $T\, R_i\, T'$ if and only if $\sum_{k \in T} u_i(k) \geq \sum_{k \in T'} u_i(k)$. We show that on this domain no rule is efficient, individually rational, and bundling-proof. Clearly, this result also holds on any preference domain that contains the domain of additive preferences.

Theorem 1. *For markets with additive preferences, no rule is efficient, individually rational, and bundling-proof.*

Proof. Let $N = \{1, 2\}$, $B = (B_1, B_2)$ such that $B_1 = (\{a\}, \{b\}, \{c\})$, $B_2 = (\{d\}, \{e\})$. Preference profile (R_1, R_2) has the following utility representation:

$u_1(a) = 1.5,$	$u_2(a) = 2.9,$
$u_1(b) = 1.3,$	$u_2(b) = 1.8,$
$u_1(c) = 1.1,$	$u_2(c) = 1.5,$
$u_1(d) = 3,$	$u_2(d) = 3,$
$u_1(e) = 0.01,$	$u_2(e) = 0.5.$

Since all objects have positive values, by *individual rationality* no agent will give up an object without receiving at least one other object from the other agent. By *individual rationality*, agent 1 is willing to swap either $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, or $\{b, c\}$ for $\{d\}$ or $\{d, e\}$ (no individually rational swap for agent 1 involves only object e). By *individual rationality*, agent 2 is willing to swap $\{d\}$ for $\{a, b, c\}$, $\{a, b\}$, $\{a, c\}$, or $\{b, c\}$ or swap $\{d, e\}$ for $\{a, b, c\}$, $\{a, b\}$, or $\{a, c\}$. Hence, the only *individually rational* allocations are $A^1 = (\{a, b, c\}, \{d, e\})$, $A^2 = (\{c, d\}, \{a, b, e\})$, $A^3 = (\{b, d\}, \{a, c, e\})$, $A^4 = (\{a, d\}, \{b, c, e\})$, $A^5 = (\{c, d, e\}, \{a, b\})$, and $A^6 = (\{b, d, e\}, \{a, c\})$. Allocation A^1 is not *efficient* (it is Pareto dominated by A^2). Among the remaining allocations, allocation A^4 gives the highest utility to agent 1.

Now assume that agent 1 bundles objects b and c, i.e., consider $(\hat{B}_1, B_2)$ where $\hat{B}_1 = (\{a\}, \{b, c\})$. Then the only *efficient* and *individually rational* allocation for $(\hat{B}_1, B_2)$ is $A^4 = (\{a, d\}), \{b, c, e\})$. Thus, by *bundling-proofness*, $\varphi(B) = A^4$.

Next, assume that agent 2 bundles objects d and e, i.e., consider $(B_1, \hat{B}_2)$ where $\hat{B}_2 = (\{d, e\})$. Then the only efficient and individually rational allocations for $(B_1, \hat{B}_2)$ are $A^5 = (\{c, d, e\}), \{a, b\})$ and $A^6 = (\{b, d, e\}, \{a, c\})$. If $\varphi_2(B_1, \hat{B}_2) = A^5$, then, in contradiction to bundling-proofness, $\{a, b\} = \varphi_2(B_1, \hat{B}_2)\, P_2\, \varphi_2(B_1, B_2) = \{b, c, e\}$. If $\varphi_2(B_1, \hat{B}_2) = A^6$, then, again in contradiction to bundling-proofness, $\{a, c\} = \varphi_2(B_1, \hat{B}_2)\, P_2\, \varphi_2(B_1, B_2) = \{b, c, e\}$.

Hence, for $n = 2$, efficiency, individual rationality, and bundling-proofness are incompatible. For $n > 2$, we simply add agents who prefer their endowments to any other set of objects. Since only agents 1 and 2 trade in this case, the incompatibility persists for $n > 2$. $\qquad\square$

Note that the example used in the proof of Theorem 1 is not only minimal with respect to the number of agents, but also with respect to the number of objects used.

4 Conclusion

The setup presented in this paper differs from the standard exchange model with heterogeneous indivisible goods (Papái, 2005; Atlamaz and Klaus, 2005) in that it allows for more structure in agents' endowments that implicitly expresses trade restrictions. Our impossibility result in this framework is in accordance with the non-existence of efficient, individually rational, and hiding-proof rules (Atlamaz and Klaus, 2005) and, more generally, with the non-existence of efficient, individually rational, and strategy-proof rules (Sönmez, 1999).

Since we assume that bundling is costless and objects do not change by bundling, agents can also decompose or regroup bundles in their endowments (Dimitrov and Haake, 2005). Note that when decomposing endowment, more feasible allocations are created. Hence, as demonstrated in the next example, decomposition may in fact be efficiency enhancing. Therefore, efficiency and "decomposition-proofness" are not compatible.

Example 1. Let $N = \{1, 2\}$ and $B = (B_1, B_2)$ such that $B_1 = (\{a, b\})$ and $B_2 = (\emptyset)$. Preferences are additive such that $u_1(a) = 1$, $u_1(b) = -1$, $u_2(a) = -1$, and $u_2(b) = 1$. The only feasible allocations are $(\{a, b\}, \emptyset)$ and $(\emptyset, \{a, b\})$ with final utilities $(0, 0)$.

Next assume that agent 1 decomposes B_1 into $B_1' = (\{a\}, \{b\})$. Thus, for $B' = (B_1', B_2)$ the additional feasible allocations are $(\{a\}, \{b\})$ and $(\{b\}, \{a\})$. The associated final utilities are $(1, 1)$ and $(-1, -1)$. Hence, the only feasible and efficient allocation is $(\{a\}, \{b\})$ with final utilities $(1, 1)$. $\square$

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