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Arrow's Theorem for One-Dimensional Single-Peaked Preferences*

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Abstract

In one-dimensional environments with single-peaked preferences we consider social welfare functions satisfying Arrow's requirements, i.e. *weak Pareto* and *independence of irrelevant alternatives*. When the policy space is a one-dimensional continuum such a welfare function is determined by a collection of 2^N strictly quasi-concave preferences and a tie-breaking rule. As a corollary we obtain that when the number of voters is odd, simple majority voting is transitive if and only if each voter's preference is strictly quasi-concave.

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1 Introduction

A social welfare function is a procedure for aggregating profiles of individual preferences into social orderings. Arrow's theorem shows that it is impossible for a social welfare function to satisfy *weak Pareto* (if all individuals strictly prefer one alternative to another, then so does society), *independence of irrelevant alternatives* (the social ranking of two alternatives only depends on the individual rankings of these two alternatives), and *non-dictatorship* when the preference domain is unrestricted. When the set of alternatives is structured, the assumption of unrestricted domain might be unreasonable. Although the assumption of unrestricted domain is unreasonable in economic environments, it has been shown that Arrow's theorem remains valid in most of these environments.¹

There is one well-known economic environment in which Arrow's requirements are consistent. In one-dimensional environments with single-peaked preferences, alternatives are points in a Euclidean space of issue positions and individual preferences are continuous, quasi-concave, and have bliss points. If the number of voters is odd, then simple majority voting is transitive and satisfies *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity* (Black, 1948; Arrow, 1951, 1963). Simple majority voting is one example of an Arrovian welfare function belonging to the following class: if the number of voters is n , fix $n - 1$ preferences of constant voters, and apply to each profile of individual preferences majority voting over this profile and the $n - 1$ fixed voters. However, a characterization of all welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* was missing. Our paper fills this gap. The class of welfare functions described above, in which the preferences of the fixed voters are strictly quasi-concave, is characterized by *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity* up to a tie-breaking rule. If *anonymity* is dropped, then instead of $n - 1$ fixed preferences we need 2^N fixed strictly quasi-concave preferences.

A key step in the proof of Arrow's theorem is that decisiveness of a coalition

¹An excellent review of the literature is Le Breton and Weymark (1996,2000).

spreads from one pair of alternatives to all pairs of alternatives. In proving the previous fact we need to assume that each individual's preference domain is unrestricted. This is the difference between one-dimensional single-peaked environments and environments with an unrestricted domain. We show that in one-dimensional single-peaked environments decisiveness of a coalition spreads in the following way: if a coalition is decisive over “ a preferred to b ” (where $a, b \in \mathbb{R}$ are such that $a < b$), then it is decisive over “ a preferred to c ” for all $b < c$ and over “ c preferred to b ” for all $a < c < b$. This implies that for each coalition S , there is a point $x_S \in [-\infty, +\infty]$ such that S is decisive over “ a preferred to b ” for all $x_S < a < b$ and $N \setminus S$ is decisive over “ c preferred to d ” for all $d < c < x_S$. Therefore, when the peaks of S converge to $-\infty$ and the peaks of $N \setminus S$ to $+\infty$, then an Arrovian welfare function chooses a strictly quasi-concave ordering with quasi bliss point x_S . The collection of 2^N strictly quasi-concave preferences characterizes an Arrovian welfare function up to some tie-breaking rule. Here, the key is to show that if the ranking over two alternatives is not determined, then this does not cause any intransitivities, no matter what ranking we choose between these two alternatives. Of course, this is only possible if the two alternatives belong to opposite sides of the quasi bliss point of the social ordering and could form an indifference class.

Our characterization is one of the few positive results in Arrovian social choice. A corollary of our result is that simple majority voting is transitive if and only if individual preferences are strictly quasi-concave.

The paper is organized as follows. In Section 2 we introduce our notation and the main definitions. In Section 3 we characterize the welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* if the policy space is one-dimensional. Section 4 concludes.

2 Notation and Definitions

We use the same terminology and notation as Le Breton and Weymark (2000). Let $N \equiv \{1, 2, \dots, n\}$ denote a finite set of agents with $n \geq 2$, let \mathbb{R} denote the set of real numbers, and let A denote a set of alternatives. Each point in \mathbb{R} identifies the changes in the level of a certain policy, for example public spending on police or health care. In an election each element of \mathbb{R} represents a candidate's political ideology on a left-right spectrum. Throughout we will assume that $A = \mathbb{R}$, although all our results apply to any environment where the set of alternatives is an interval.

Let \mathcal{W} denote the set of all complete and transitive relations over A . An element of \mathcal{W} is called a *weak ordering* over A . Given $R_i \in \mathcal{W}$, the corresponding strict relation, P_i , and the indifference relation, I_i , are defined as follows: for all $a, b \in A$, (i) $aP_ib \Leftrightarrow \neg bR_ia$, and (ii) $aI_ib \Leftrightarrow aR_ib$ and bR_ia . Note that if $\neg aR_ib$, then by completeness of R_i , aR_ib . Hence, $\neg bR_ia$ is enough to describe the strict relation P_i .

Given $i \in N$, a weak ordering $R_i \in \mathcal{W}$ is *single-peaked* if there exists a point $p(R_i) \in \mathbb{R}$, called the peak of R_i (or the bliss point of R_i), such that for all $a, b \in \mathbb{R}$, if $a < b \leq p(R_i)$ or $p(R_i) \leq b < a$, then bP_ia . Let \mathcal{R} denote the set of all single-peaked preferences over \mathbb{R} . Each agent $i \in N$ has a single-peaked preference relation over \mathbb{R} . A single-peaked preference $R_i \in \mathcal{R}$ is *symmetric* if for all $a, b \in \mathbb{R}$, $aR_ib \Leftrightarrow |a - p(R_i)| \leq |b - p(R_i)|$.

A (*preference*) *profile* is a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. A (*social*) *welfare function* associates with each profile a weak ordering over A . Formally, a welfare function is a mapping $f : \mathcal{R}^N \rightarrow \mathcal{W}$ such that for all $R \in \mathcal{R}^N$, $f(R) \in \mathcal{W}$. We call $f(R)$ the *social ordering* (that f associates with R). Note that $f(R)$ need not belong to the individual preference domain \mathcal{R} . Other authors impose the restriction that any social ordering belongs to each individual's preference domain (for example, Peters, van der Stel, and Storcken, 1992, and Bossert and Weymark, 1993).

Arrow's requirements are as follows. The first axiom says that if all agents strictly prefer a to b , then a should also be socially strictly preferred to b .

Weak Pareto: For all $R \in \mathcal{R}^N$ and all $a, b \in A$, if for all $i \in N$, aP_ib , then $\neg bf(R)a$.

Given $R \in \mathcal{R}^N$, $X \subseteq A$, and $j \in N$, let $R_j|X$ denote the restriction of R_j to X , and $R|X \equiv (R_i|X)_{i \in N}$. The second axiom says that the social ordering of two alternatives only depends on the profile of individual preferences restricted to these two alternatives.

Independence of Irrelevant Alternatives: For all $R, \bar{R} \in \mathcal{R}^N$ and all $a, b \in A$, if $R|\{a, b\} = \bar{R}|\{a, b\}$, then $f(R)|\{a, b\} = f(\bar{R})|\{a, b\}$.

A welfare function is *Arrovian* if it satisfies *weak Pareto* and *independence of irrelevant alternatives*. A welfare function is *dictatorial* if there exists some agent such that for each profile the social strict preference relation respects the strict preference relation of this agent.

Non-Dictatorship: There exists no $i \in N$ such that for all $R \in \mathcal{R}^N$ and all $a, b \in A$, if aP_ib , then $\neg bf(R)a$.

A welfare function treats individuals symmetrically if for all permutations of individuals' preferences, the social ordering remains unchanged.

Anonymity: For all $R \in \mathcal{R}^N$ and all permutations σ of N , $f(\sigma(R)) = f(R)$.

3 One-Dimensional Policy Spaces

We will give a complete characterization of all welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives* for one-dimensional single-peaked environments. In showing this characterization we proceed as follows:

First, we describe a class of weak orderings which are “almost” single-peaked, namely the class of strictly quasi-concave preferences. These weak orderings will be important for any Arrovian welfare function.

Second, we show how decisiveness spreads for Arrovian welfare functions from one pair of alternatives to other pairs of alternatives.

Third, we enlarge the domain of single-peaked preferences by adding the \leq -relation and the \geq -relation. This has no effect on an Arrovian welfare function but allows us to consider social orderings at maximal conflicts where each agent’s preference is either the \leq -relation or the \geq -relation. Any such social ordering belongs to the class of strictly quasi-concave orderings. Then we show that for any profile of single-peaked preferences the social ordering is strictly quasi-concave.

Finally, we show that the social orderings at maximal conflicts determine any Arrovian welfare function up to a tie-breaking rule.

3.1 Strictly Quasi-Concave Preferences

A weak ordering $R_i \in \mathcal{W}$ is *strictly quasi-concave*, if for all $a, b, c \in \mathbb{R}$ such that $a < b < c$, we have $\neg aR_ib$ or $\neg cR_ib$. In other words, b is never a worst alternative in $\{a, b, c\}$ and the restriction of R_i to $\{a, b, c\}$ is single-peaked. Let \mathcal{C} denote the set of all strictly quasi-concave orderings over \mathbb{R} .

Lemma 3.1 *Let $R_i \in \mathcal{W}$. The weak ordering R_i is strictly quasi-concave if and only if there exists a quasi bliss point $p(R_i) \in \mathbb{R} \cup \{-\infty, +\infty\}$ (for convenience we use the same notation as for peaks) such that (i) for all $a, b \in \mathbb{R}$, if $a < b < p(R_i)$ or $p(R_i) < b < a$, then bP_ia , and (ii) if $p(R_i) \in \mathbb{R}$, then either (for all $x \in]-\infty, p(R_i)[$, $p(R_i)P_ix$), or (for all $x \in]p(R_i), +\infty[$, $p(R_i)P_ix$).*

Proof. It is easy to check that R_i is strictly quasi-concave if (i) and (ii) hold.

Let R_i be strictly quasi-concave. For all $x \in \mathbb{R}$, let $B(x, R_i) \equiv \{y \in \mathbb{R} \mid xR_iy\}$. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{R} such that for all $k \in \mathbb{N}$, if $B(x_k, R_i) \neq \mathbb{R}$, then $B(x_k, R_i) \subsetneq$

$B(x_{k+1}, R_i)$, and for all $y \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $y \in B(x_k, R_i)$. Without loss of generality, let $(x_k)_{k \in \mathbb{N}}$ converge to $p(R_i)$ (otherwise we choose a convergent subsequence of $(x_k)_{k \in \mathbb{N}}$). We show (i) and (ii).

Let $a, b \in \mathbb{R}$ be such that $a < b < p(R_i)$ or $p(R_i) < b < a$. Then there exists $k \in \mathbb{N}$ such that $b \in B(x_k, R_i)$ and $|x_k - p(R_i)| < |b - p(R_i)|$. Thus, $x_k R_i b$ and $a < b < x_k$ or $x_k < b < a$. Because R_i is strictly quasi-concave, $b P_i a$, which is (i).

Suppose that $p(R_i) \in \mathbb{R}$ but (ii) does not hold. Then for some $a, b \in \mathbb{R}$ we have $a < p(R_i) < b$, $a R_i p(R_i)$, and $b R_i p(R_i)$. Obviously, this is in contradiction to strict quasi-concavity of R_i . \square

Obviously, $\mathcal{R} \subseteq \mathcal{C}$. However, a strictly quasi-concave preference R_i may not be single-peaked, even if $p(R_i) \in \mathbb{R}$. For example, let $u : \mathbb{R} \rightarrow \mathbb{R}$ be such that for all $x \in \mathbb{R}$, $u(x) \equiv \frac{1}{x}$ if $x > 0$ and $u(x) = x$ if $x \leq 0$. Let $R_u \in \mathcal{W}$ be such that for all $a, b \in \mathbb{R}$, $a R_u b \Leftrightarrow u(a) \geq u(b)$. Then $p(R_u) = 0$, for all $x > 0$, $x P_u p(R_u)$, and $R_u \in \mathcal{C} \setminus \mathcal{R}$.

3.2 Decisiveness

Given $S \subseteq N$ and $a, b \in A$, we say that S is *semi-decisive* over “ a preferred to b ” if there exists $R \in \mathcal{R}^N$ such that (i) for all $i \in S$, $a P_i b$, (ii) for all $i \in N \setminus S$, $b P_i a$, and (iii) $a f(R) b$.² Let $D^s(a, b)$ denote the set of all coalitions that are semi-decisive over “ a preferred to b ”. Given $a, b \in A$ and $R \in \mathcal{R}$, if $\neg b f(R) a$ whenever $a P_i b$ for all $i \in S$, then S is said to be *decisive* over “ a preferred to b ”. Let $D(a, b)$ denote the set of all coalitions that are decisive over “ a preferred to b ”.

One of the important steps in Arrow’s impossibility theorem is that whenever a coalition $S \subseteq N$ is decisive over “ a preferred to b ”, then it is also decisive over “ a preferred to c ” and “ c preferred to b ” for any other alternative c . Therefore, every

²The literature on Arrowian social choice often refers to our definition as “a coalition is semi-decisive over the pair (a, b) ”.

coalition is either decisive over every pair of alternatives or over no pair.

Because here individual preferences are restricted to be single-peaked and the policy space is one-dimensional, decisiveness of a coalition does not spread over all pairs of alternatives as in Arrow's original theorem. Decisiveness of a coalition expands in a weaker form. Given $a, b \in \mathbb{R}$ such that $a < b$, if S is semi-decisive over “ a preferred to b ”, then S is decisive over “ a preferred to c ” for all $c \in]b, +\infty[$ and S is decisive over “ c preferred to b ” for all $c \in]a, b[$.

Lemma 3.2 *Let $f : \mathcal{R}^N \rightarrow \mathcal{W}$ be a welfare function satisfying weak Pareto and independence of irrelevant alternatives. Let $S \subseteq N$ and $a, b, c \in \mathbb{R}$ be such that $a < b < c$ or $a > b > c$.*

(i) *If $S \in D^s(a, b)$, then $S \in D(a, c)$.*

(ii) *If $S \in D^s(a, c)$, then $S \in D(b, c)$.*

Proof. First, we show (i). Let $R \in \mathcal{R}^N$ be such that for all $i \in S$, $aP_i c$. We have to show that $\neg cf(R)a$. Let $\tilde{R} \in \mathcal{R}^N$ be such that

(a) for all $i \in N$, $\tilde{R}_i|\{a, c\} = R_i|\{a, c\}$;

(b) for all $i \in S$, $a\tilde{P}_i b$ and $b\tilde{P}_i c$; and

(c) for all $i \in N \setminus S$, $b\tilde{P}_i a$ and $b\tilde{P}_i c$.

It is easy to check that we can find $\tilde{R} \in \mathcal{R}^N$ such that (a) to (c) are satisfied (for all $i \in N$, we can even choose \tilde{R}_i to be symmetric). By (b) and (c), for all $i \in N$, $b\tilde{P}_i c$. Thus, by *weak Pareto*, $\neg cf(\tilde{R})b$. Because $S \in D^s(a, b)$ and f satisfies *independence of irrelevant alternatives*, (b) and (c) imply $a\tilde{P}_i b$. Thus, by transitivity of $f(\tilde{R})$, $\neg cf(\tilde{R})a$. Hence, by (a) and *independence of irrelevant alternatives*, $\neg cf(R)a$ and $S \in D(a, c)$, the desired conclusion.

Second, we show (ii). Let $R \in \mathcal{R}^N$ be such that for all $i \in S$, $bP_i c$. We have to show that $\neg cf(R)b$. Let $\tilde{R} \in \mathcal{R}^N$ be such that

- (a) for all $i \in N$, $\tilde{R}_i|\{b, c\} = R_i|\{b, c\}$,
- (b) for all $i \in S$, $b\tilde{P}_i a$ and $a\tilde{P}_i c$, and
- (c) for all $i \in N \setminus S$, $b\tilde{P}_i a$ and $c\tilde{P}_i a$.

(Again, for all $i \in N$, \tilde{R}_i can be chosen to be symmetric.) By *weak Pareto*, $\neg af(\tilde{R})b$. Because $S \in D^s(a, c)$, we have $af(\tilde{R})c$. Thus, by transitivity of $f(\tilde{R})$, $\neg cf(\tilde{R})b$. By (a) and *independence of irrelevant alternatives*, $\neg cf(R)b$. Hence, $S \in D(b, c)$, the desired conclusion. \square

3.3 Social Orderings at Maximal Conflicts

In avoiding limits of profiles we add two non-single-peaked preferences to the set \mathcal{R} . Obviously, \geq and \leq are preferences over \mathbb{R} , where \geq has a bliss point at $+\infty$ and \leq at $-\infty$.

Let $\tilde{\mathcal{R}} \equiv \mathcal{R} \cup \{\geq, \leq\}$. The following lemma shows that the addition of \geq and \leq to \mathcal{R} has no influence on a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*.

Lemma 3.3 *Let $f : \mathcal{R}^N \rightarrow \mathcal{W}$ be a welfare function satisfying weak Pareto and independence of irrelevant alternatives. Define $\tilde{f} : \tilde{\mathcal{R}}^N \rightarrow \mathcal{W}$ as follows: for all $R \in \tilde{\mathcal{R}}^N$ and all $a, b \in \mathbb{R}$, take $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}|\{a, b\} = R|\{a, b\}$ and set*

$$\tilde{f}(R)|\{a, b\} \equiv f(\bar{R})|\{a, b\}.$$

Then \tilde{f} is a well-defined welfare function satisfying weak Pareto and independence of irrelevant alternatives. Moreover, $\tilde{f}|\mathcal{R}^N = f$.

Proof. Let $R \in \tilde{\mathcal{R}}^N$. Because f satisfies *independence of irrelevant alternatives*, for all $a, b \in \mathbb{R}$, $\tilde{f}(R)|\{a, b\}$ is well-defined. Thus, \tilde{f} satisfies *weak Pareto* and *independence of irrelevant alternatives*. Obviously, $\tilde{f}(R)$ is complete. It remains to show that $\tilde{f}(R)$ is transitive. Let $a, b, c \in \mathbb{R}$ be such that $a \leq b \leq c$. Let $\bar{R} \in \mathcal{R}^N$ be such that

for all $i \in N$, (i) if $R_i = \leq$, then $p(\bar{R}_i) = a$, (ii) if $R_i \in \mathcal{R}$, then $\bar{R}_i = R_i$, and (iii) if $R_i = \geq$, then $p(\bar{R}_i) = c$. By definition, $\tilde{f}(R)|\{a, b, c\} = f(\bar{R})|\{a, b, c\}$ and $\tilde{f}(R)$ is transitive. \square

Given a coalition $S \subseteq N$, we consider the profile where the members belonging to S announce \leq and the other agents announce \geq . Clearly, the coalitions S and $N \setminus S$ disagree on every pair of different alternatives. We show that the social orderings at these profiles of maximal conflicts are strictly quasi-concave. They also satisfy a certain monotonicity property and determine for each profile the social ordering up to some tie-breaking when social preference can be chosen to be arbitrary.

Throughout the remaining part of Subsection 3.3, let f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. For all $S \subseteq N$, let $(\leq_S, \geq_{N \setminus S})$ be the profile $R \in \tilde{\mathcal{R}}^N$ such that for all $i \in S$, $R_i = \leq$, and for all $i \in N \setminus S$, $R_i = \geq$. Then for all $S \subseteq N$, let $Q^S \equiv \tilde{f}(\leq_S, \geq_{N \setminus S})$. Let $Q \equiv \{Q^S \mid S \subseteq N\}$. We call Q a *collection of calibration relations*. We will show that Q completely determines f (up to some tie-breaking). By Lemma 3.3, \tilde{f} satisfies *weak Pareto*. Thus, $Q^\emptyset = \geq$ and $Q^N = \leq$. The following condition is a translation of Lemma 3.2 to collections of calibration relations.

Definition 3.1 *Let Q be a collection of calibration relations and $a, b, c \in \mathbb{R}$ be such that $a < b < c$. The collection Q is (inclusive) monotone if for all $S, T \subseteq N$ such that $S \subseteq T$ the following holds:*

- (a) if $aQ^S b$, then $\neg cQ^T a$;
- (b) if $aQ^S c$, then $\neg cQ^T b$;
- (c) if $cQ^T b$, then $\neg aQ^S c$; and
- (d) if $cQ^T a$, then $\neg aQ^S b$.

The following lemma specifies some properties of the collection Q .

Lemma 3.4 (i) *The collection Q is (inclusive) monotone.*

(ii) *For all $S \subseteq N$, $Q^S \in \mathcal{C}$.*

(iii) *For all $S, T \subseteq N$, if $S \subseteq T$, then $p(Q^T) \leq p(Q^S)$.*

Proof. (i) By definition, $Q^S = \tilde{f}(\leq_S, \geq_{N \setminus S})$ and $Q^T = \tilde{f}(\leq_T, \geq_{N \setminus T})$. If aQ^Sb , then $S \in D^s(a, b)$. Thus, by Lemma 3.2, $S \in D(a, c)$. Hence, by $S \subseteq T$, we have $\neg cQ^T a$ and (a) of Definition 3.1 holds. Statements (b), (c), and (d) of Definition 3.1 follow similarly from Lemma 3.2.

(ii) Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$. Suppose that aQ^Sb and cQ^Sb . Since Q is (inclusive) monotone, (a) of Definition 3.1 implies $\neg cQ^S a$, and (c) of Definition 3.1 implies $\neg aQ^S c$, which contradicts completeness of Q^S .

(iii) Suppose that $p(Q^T) > p(Q^S)$. Let $a, b, c \in]p(Q^S), p(Q^T)[$ be such that $a < b < c$. Then aQ^Sb and $S \in D^s(a, b)$. Thus, by Lemma 3.2, $S \in D(a, c)$. By $S \subseteq T$, $aQ^T c$. This is a contradiction to $a < c < p(Q^T)$ and $Q^T \in \mathcal{C}$. \square

Recall that a strictly quasi-concave preference $R_i \in \mathcal{C}$ is either equal to \leq or \geq , or there exists a *quasi bliss point* $p(R_i) \in \mathbb{R}$ such that (i) for all $a, b \in \mathbb{R}$, if $a < b < p(R_i)$ or $p(R_i) < b < a$, then $bP_i a$, and (ii) either (for all $x \in]-\infty, p(R_i)[$, $p(R_i)P_i x$), or (for all $x \in]p(R_i), +\infty[$, $p(R_i)P_i x$). We use the convention that $p(\leq) \equiv -\infty$ and $p(\geq) \equiv +\infty$.

Next, we show that for each profile the social ordering is strictly quasi-concave.

Let $R \in \mathcal{R}^N$ be such that $p(R_{i_1}) \leq p(R_{i_2}) \leq \dots \leq p(R_{i_n})$. Let $S_0 \equiv \emptyset$ and for all $t \in \{1, 2, \dots, n\}$, $S_t \equiv \{i_1, i_2, \dots, i_t\}$. Thus, $S_n = N$. The *median* of $M(R) = \{p(R_{i_1}), p(R_{i_2}), \dots, p(R_{i_n}), p(Q^{S_0}), p(Q^{S_1}), \dots, p(Q^{S_n})\}$ is the number $m(R) \in \mathbb{R}$ such that (i) $m(R) \in M(R)$, (ii) at least $n+1$ elements of $M(R)$ are smaller than or equal to $m(R)$, and (iii) at least $n+1$ elements of $M(R)$ are greater than or equal to $m(R)$.

Theorem 3.1 *For all $R \in \mathcal{R}^N$, $f(R)$ is strictly quasi-concave with quasi bliss point $m(R)$. Furthermore, if $m(R) \notin \{p(Q^S) \mid S \subseteq N\}$, then $f(R)$ is single-peaked.*

Proof. Let $R \in \mathcal{R}^N$. Without loss of generality, suppose $p(R_1) \leq p(R_2) \leq \dots \leq p(R_n)$. By *independence of irrelevant alternatives*, we have (i) $f(R) \parallel [-\infty, p(R_1)] = Q^\emptyset \parallel [-\infty, p(R_1)] = \geq \parallel [-\infty, p(R_1)]$, (ii) for all $t \in \{1, 2, \dots, n-1\}$,

$$f(R) \parallel [p(R_t), p(R_{t+1})] = Q^{S_t} \parallel [p(R_t), p(R_{t+1})], \quad (1)$$

and (iii) $f(R) \parallel [p(R_n), +\infty[= Q^N \parallel [p(R_n), +\infty[= \leq \parallel [p(R_n), +\infty[$.

Because $p(Q^\emptyset) = +\infty$ and $p(Q^N) = -\infty$, we have $m(R) \in [p(R_1), p(R_n)]$. Let $a, b \in \mathbb{R}$ be such that $m(R) < a < b$. We show that $\neg bf(R)a$. If $a \geq p(R_n)$, then the assertion follows from $f(R) \parallel [p(R_n), +\infty[= \leq \parallel [p(R_n), +\infty[$. Let $t' \in \{1, 2, \dots, n-1\}$ be such that $a \in [p(R_{t'}), p(R_{t'+1})[$ and $c \in]a, p(R_{t'+1})[\cap]a, b[$. We claim that $p(Q^{S_{t'}}) \leq m(R)$. If $p(Q^{S_{t'}}) > m(R)$, then by Lemma 3.4 at least $n+1$ numbers are greater than $m(R)$, namely $\{p(Q^{S_0}), p(Q^{S_1}), \dots, p(Q^{S_{t'}}), p(R_{t'+1}), p(R_{t'+2}), \dots, p(R_n)\}$, which contradicts the median property of $m(R)$. Thus, $p(Q^{S_{t'}}) \leq m(R) < a$. By (1), $f(R) \parallel [p(R_{t'}), p(R_{t'+1})] = Q^{S_{t'}} \parallel [p(R_{t'}), p(R_{t'+1})]$. Hence, $\neg cf(R)a$. Thus, $S_{t'} \in D^s(a, c)$. By Lemma 3.2 and $a < c < b$, $S_{t'} \in D(a, b)$. Hence, $\neg bf(R)a$, the desired conclusion. Similarly it can be shown that for all $a, b \in \mathbb{R}$, if $b < a < m(R)$, then $\neg bf(R)a$.

Let $t'' \in \{1, 2, \dots, n-1\}$ be such that $m(R) \in [p(R_{t''}), p(R_{t''+1})]$. If $m(R) \in]p(R_{t''}), p(R_{t''+1})[$, then $f(R) \parallel [p(R_{t''}), p(R_{t''+1})] = Q^{S_{t''}} \parallel [p(R_{t''}), p(R_{t''+1})]$ implies $p(f(R)) = m(R) = p(Q^{S_{t''}})$ and $f(R) \in \mathcal{C}$. If $m(R) \in \{p(R_1), p(R_n)\}$, then (i) or (iii) implies $f(R) \in \mathcal{C}$.

Suppose that $m(R) = p(R_{t''})$ and $p(R_{t''}) \notin \{p(R_1), p(R_n)\}$. Let $S \equiv \{i \in N \mid p(R_i) < m(R)\}$ and $T \equiv \{i \in N \mid p(R_i) \leq m(R)\}$. Then $S \neq \emptyset$, $S \subsetneq T$, and $T \neq N$. Thus, $]p(R_{|S|}), p(R_{|S|+1})[\neq \emptyset$ and $]p(R_{|T|}), p(R_{|T|+1})[\neq \emptyset$. Similar arguments as above imply that $p(Q^T) \leq m(R) \leq p(Q^S)$. If $p(Q^T) < m(R)$, then by (1), for all $x \in]p(R_{|T|}), p(R_{|T|+1})[$, $\neg xf(R)m(R)$, and $f(R) \in \mathcal{C}$. If $p(Q^S) > m(R)$, then by (1), for all $x \in]p(R_{|S|}), p(R_{|S|+1})[$, $\neg xf(R)m(R)$, and $f(R) \in \mathcal{C}$. Suppose that $p(Q^T) = m(R) = p(Q^S)$. By (1), $f(R) \parallel [p(R_{|S|}), p(R_{|S|+1})] = Q^S \parallel [p(R_{|S|}), p(R_{|S|+1})]$ and $f(R) \parallel [p(R_{|T|}), p(R_{|T|+1})] = Q^T \parallel [p(R_{|T|}), p(R_{|T|+1})]$. If $f(R) \notin \mathcal{C}$, then for some

$a \in]p(R_{|S|}), p(R_{|S|+1})[, af(R)m(R)$, and for some $b \in]p(R_{|T|}), p(R_{|T|+1})[, bf(R)m(R)$. Thus, $S \in D^s(a, m(R))$ and $N \setminus T \in D^s(b, m(R))$. Hence, by Lemma 3.2, $S \in D(a, b)$ and $N \setminus T \in D(a, b)$. Therefore, $\neg af(R)b$ and $\neg bf(R)a$, which contradicts completeness of $f(R)$. Hence, $f(R) \in \mathcal{C}$.

If $m(R) \notin \{p(Q^{S'}) \mid S' \subseteq N\}$, then $p(Q^T) < m(R) < p(Q^S)$. Hence, by (1), $f(R) \in \mathcal{R}$. \square

Remark 3.1 *If for all $S \subseteq N$, $Q^S \in \{\leq, \geq\}$, then for all $R \in \mathcal{R}^N$, $f(R)$ is a single-peaked preference relation.* \triangleleft

By Theorem 3.1, for all $R \in \mathcal{R}^N$, the collection Q determines the social ordering $f(R)$ restricted to $] - \infty, m(R)[$ and to $]m(R), +\infty[$ (and either on $] - \infty, m(R)[$ or $]m(R), +\infty[$). However, the quasi bliss point $m(R)$ needs not be the socially most preferred alternative. Next we show that Q also determines the ranking of $f(R)$ over all pairs (up to some arbitrary tie-breaking).

Lemma 3.5 *Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a < b$. Let $S \equiv \{i \in N \mid aP_i b\}$, $U \equiv \{i \in N \mid bP_i a\}$, and $T \equiv N \setminus (S \cup U)$. Then the following holds.*

- (i) *If for some $x \in]a, b[, aQ^S x$, or for some $x \in] - \infty, a[, xQ^S b$, then $\neg bf(R)a$.*
- (ii) *If for some $x \in]a, b[, bQ^{S \cup T} x$, or for some $x \in]b, +\infty[, xQ^{S \cup T} a$, then $\neg af(R)b$.*
- (iii) *If the presumptions of (i) and (ii) do not hold, then for all $x \in]a, b[$, we have $\neg af(R)x$ and $\neg bf(R)x$, and for all $x \in] - \infty, a[\cup]b, +\infty[$, we have $\neg xf(R)a$ and $\neg xf(R)b$.*

Proof. By definition, for all $i \in T$, $aI_i b$.

First, we show (i). If for some $x \in]a, b[, aQ^S x$, then $S \in D^s(a, x)$. Thus, by Lemma 3.2 and $a < x < b$, $S \in D(a, b)$ and $\neg bf(R)a$. If for some $x \in] - \infty, a[, xQ^S b$, then $S \in D^s(x, b)$. Thus, by Lemma 3.2 and $x < a < b$, $S \in D(a, b)$ and $\neg bf(R)a$.

Second, we show (ii). If for some $x \in]a, b[, bQ^{S \cup T} x$, then $U \in D^s(b, x)$. Thus, by Lemma 3.2 and $b > x > a$, $U \in D(b, a)$ and $\neg af(R)b$. If for some $x \in]b, +\infty[$,

$xQ^{S \cup T}a$, then $U \in D^s(x, a)$. Thus, by Lemma 3.2 and $x > b > a$, $U \in D(b, a)$ and $\neg af(R)b$.

Third, we show (iii). Because the presumptions of (i) and (ii) do not hold, we have

- (a) for all $x \in]-\infty, a[$, $\neg xQ^Sb$;
- (b) for all $x \in]a, b[$, $\neg aQ^Sx$ and $\neg bQ^{S \cup T}x$; and
- (c) for all $x \in]b, +\infty[$, $\neg xQ^{S \cup T}a$.

Let $x \in]-\infty, a[$. We want to show that $\neg xf(R)a$ and $\neg xf(R)b$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in T \cup U$, $aP_i x$ and $bP_i x$. Let $y \in]x, a[$. By (a), $bQ^S y$. Thus, $T \cup U \in D^s(b, y)$. By Lemma 3.2 and $b > y > x$, $T \cup U \in D(b, x)$. Hence, $\neg xf(R)b$. By Theorem 3.1, $f(R) \in \mathcal{C}$. Thus, from $x < a < b$ and $\neg xf(R)b$ we obtain $\neg xf(R)a$.

Let $x \in]a, b[$. We want to show that $\neg af(R)x$ and $\neg bf(R)x$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in S$, $xP_i b$, for all $i \in U$, $xP_i a$, and for all $i \in T$, $xP_i a$ and $xP_i b$. Let $y \in]x, b[$. By (b), $yQ^S a$ and $T \cup U \in D^s(y, a)$. Thus, by Lemma 3.2 and $y > x > a$, $T \cup U \in D(x, a)$ and $\neg af(R)x$. Let $z \in]a, x[$. By (b), $zQ^{S \cup T} b$ and $S \cup T \in D^s(z, b)$. Thus, by Lemma 3.2 and $z < x < b$, $S \cup T \in D(x, b)$ and $\neg bf(R)x$.

Let $x \in]b, +\infty[$. We want to show that $\neg xf(R)a$ and $\neg xf(R)b$. Because for all $i \in N$, R_i is single-peaked, we have for all $i \in S \cup T$, $aP_i x$ and $bP_i x$. Let $y \in]b, x[$. By (c), $aQ^{S \cup T} y$. Thus, $S \cup T \in D^s(a, y)$. By Lemma 3.2 and $a < y < x$, $S \cup T \in D(a, x)$. Hence, $\neg xf(R)a$. By Theorem 3.1, $f(R) \in \mathcal{C}$. Thus, from $a < b < x$ and $\neg xf(R)a$ we obtain $\neg xf(R)b$. \square

In (i) and (ii) of Lemma 3.5, the ranking of $f(R)$ over $\{a, b\}$ is uniquely determined by Q . If neither (i) nor (ii) hold, then the ranking of $f(R)$ over $\{a, b\}$ can be taken arbitrarily without causing intransitivities.

3.4 The Characterization

In the previous subsection we identified a monotone collection Q of strictly quasi-concave orderings from a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Theorem 3.1 and Lemma 3.5 imply that the welfare function is completely determined by Q . These results reveal the characteristics of such a welfare function.

In formulating the characterization result, we need a precise definition of tie-breaking. Given $a, b \in \mathbb{R}$, let $\mathcal{W}_{\{a,b\}}$ denote the set of weak orderings over $\{a, b\}$. Obviously, only three orderings over $\{a, b\}$ are possible: a is strictly preferred to b , a and b are indifferent, and b is strictly preferred to a . A tie-breaking rule τ is a family of functions $\tau_{\{a,b\}}$, indexed by $\{a, b\}$ (where $a, b \in \mathbb{R}$ and $a \neq b$), from $\mathcal{W}_{\{a,b\}}^N$ to $\mathcal{W}_{\{a,b\}}$. The function $\tau_{\{a,b\}}$ assigns to each profile of orderings over $\{a, b\}$ an element in $\mathcal{W}_{\{a,b\}}$. Tie-breaking should not be read to be strict because the resulting ordering over $\{a, b\}$ may be indifference.

Fixed-Strictly-Quasi-Concave Welfare Function, f_τ^Q : Given a monotone collection Q of strictly quasi-concave orderings such that $Q^\emptyset = \geq$ and $Q^N = \leq$, and a tie-breaking rule τ , the *fixed-strictly-quasi-concave welfare function* f_τ^Q associated with Q and τ is defined as follows. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a \leq b$. Let $S \equiv \{i \in N \mid aP_i b\}$, $U \equiv \{i \in N \mid bP_i a\}$, and $T \equiv N \setminus (S \cup U)$. Then

- (α) if $a = b$, then $a f_\tau^Q(R) b$;
- (β) if $a < b$ and for some $x \in]a, b[$, $aQ^S x$, or for some $x \in]-\infty, a[$, $xQ^S b$, then $a f_\tau^Q(R) b$ and $\neg b f_\tau^Q(R) a$;
- (γ) if $a < b$ and for some $x \in]a, b[$, $bQ^{S \cup T} x$, or for some $x \in]b, +\infty[$, $xQ^{S \cup T} a$, then $b f_\tau^Q(R) a$ and $\neg a f_\tau^Q(R) b$; and
- (δ) if $a < b$ and (a) for all $x \in]-\infty, a[$, $\neg xQ^S b$, (b) for all $x \in]a, b[$, $\neg aQ^S x$ and $\neg bQ^{S \cup T} x$, and (c) for all $x \in]b, +\infty[$, $\neg xQ^{S \cup T} a$, then $f_\tau^Q(R)|\{a, b\} \equiv$

$$\tau_{\{a,b\}}(R|\{a,b\}).$$

◦

Lemma 3.6 *A fixed-strictly-quasi-concave welfare function is a well-defined welfare function satisfying weak Pareto and independence of irrelevant alternatives.*

Proof. Let Q be a monotone collection of strictly quasi-concave orderings and τ be a tie-breaking rule. Let f_τ^Q be defined as above.

First, we prove well-definedness of f_τ^Q . We have to show that (β) and (γ) exclude each other.

Let $a, b \in \mathbb{R}$ be such that (β) holds for a and b . Thus, $a < b$. If for some $x \in]a, b[$, aQ^Sx , then by strict quasi-concavity of Q^S and $a < x < b$, $\neg bQ^Sx$. Thus, by transitivity, $\neg bQ^Sa$. If for some $x \in]-\infty, a[$, xQ^Sb , then by strict quasi-concavity of Q^S and $x < a < b$, $\neg bQ^Sa$. Hence, in both cases we have $\neg bQ^Sa$ and aQ^Sb .

Because Q is monotone and aQ^Sb , by (i.ii) of Lemma 3.4 we have for all $y \in]a, b[$, $\neg bQ^{S \cup T}y$. Because Q is monotone and aQ^Sb , by (i.i) of Lemma 3.4 we have for all $y \in]b, +\infty[$, $\neg yQ^{S \cup T}a$. Hence, (γ) does not hold.

Second, we show that f_τ^Q is a welfare function. Let $R \in \mathcal{R}^N$. By definition, $f_\tau^Q(R)$ is complete. It remains to show that $f_\tau^Q(R)$ is transitive. Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$. If $\neg a f_\tau^Q(R)b$ and $\neg c f_\tau^Q(R)b$, then $f_\tau^Q(R)|\{a, b, c\}$ is transitive. Thus, in proving transitivity of $f_\tau^Q(R)$, it suffices to show the following two implications: if $a f_\tau^Q(R)b$, then $\neg c f_\tau^Q(R)a$; and if $c f_\tau^Q(R)b$, then $\neg a f_\tau^Q(R)c$.

We only prove the first implication. The second implication follows similarly. Let $a f_\tau^Q(R)b$. Then (β) or (δ) holds for a and b . Let $S \equiv \{i \in N \mid aP_i b\}$, $U \equiv \{i \in N \mid bP_i a\}$, $T \equiv N \setminus (S \cup U)$, and $S' \equiv \{i \in N \mid aP_i c\}$. Because $a < b < c$ and for all $i \in N$, R_i is single-peaked, we have $S \cup T \subseteq S'$.³ We distinguish two cases.

Case 1: $a f_\tau^Q(R)b$ because of (β) .

If for some $x \in]a, b[$, aQ^Sx , then, because Q is monotone and $S \subseteq S'$, for some $y \in]x, b[$, $aQ^{S'}y$. Since $a < b < c$, we have $y \in]a, c[$ and by (β) , $\neg c f_\tau^Q(R)a$. If

³Note that this implication remains true if for all $i \in N$, R_i is strictly quasi-concave.

for some $x \in]-\infty, a[$, $xQ^S b$, then, because Q is monotone and $S \subseteq S'$, for some $y \in]x, a[$, $yQ^{S'} b$. Thus, by strict quasi-concavity of $Q^{S'}$ and $y < a < b < c$, $yQ^{S'} c$. Hence, by (β) , $\neg c f_\tau^Q(R) a$.

Case 2: $a f_\tau^Q(R) b$ because of (δ) .

We show that (β) holds for a and c . Let $x \in]b, c[$. Then by (δ) , part (c), $\neg x Q^{S \cup T} a$. Thus, $a Q^{S \cup T} x$. Because Q is monotone and $S \cup T \subseteq S'$, for some $y \in]x, c[$, $a Q^{S'} y$. Thus, by $y \in]a, c[$ and (β) , $\neg c f_\tau^Q(R) a$, the desired conclusion.

Third, we show that f_τ^Q satisfies *weak Pareto* and *independence of irrelevant alternatives*. Because $Q^\emptyset = \geq$ and $Q^N = \leq$, it is easy to check that f_τ^Q satisfies *weak Pareto*. Obviously, by definition, f_τ^Q satisfies *independence of irrelevant alternatives*. \square

The following theorem characterizes the class of welfare functions satisfying Arrow's requirements on the domain of all single-peaked preferences over a one-dimensional policy space.

Theorem 3.2 *On the domain of single-peaked preferences, the class of fixed-strictly-quasi-concave welfare functions is characterized by weak Pareto and independence of irrelevant alternatives.*

Proof. From Lemma 3.6 it follows that a fixed-strictly-quasi-concave welfare function is a well-defined welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*.

Conversely, let f be a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Lemma 3.4 shows the existence of a monotone collection Q of strictly quasi-concave orderings. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$. From Lemma 3.5 it follows that $f(R)|\{a, b\}$ satisfies the definitions (α) , (β) , and (γ) . Finally, if (δ) holds, then define $\tau_{\{a, b\}}(R|\{a, b\}) \equiv f(R)|\{a, b\}$. Now it is obvious that f is a fixed-strictly-quasi-concave welfare function associated with Q and τ . \square

Theorem 3.2 remains valid if the set of alternatives is an interval and/or each individual's preference domain is the set of all symmetric single-peaked preferences. Since the set of symmetric single-peaked preferences is identical with the set of spatial preferences, Theorem 3.2 applies equally to spatial environments. Those environments play an important role in political economy and political science.

3.5 Majority Voting

In this subsection we explore the relation between fixed-strictly-quasi-concave welfare functions and majority voting. Under majority voting, for a given profile of preferences, one alternative is ranked above another if and only if a majority of voters weakly prefers the former alternative to the latter. If the number of individuals is odd, majority voting is a well-defined welfare function (Black, 1948; Arrow, 1951). However, if the number of individuals is even, then the social indifference relation may be intransitive. In resolving these intransitivities, we add an odd number of fixed strictly quasi-concave relations and determine the majority preference relation for the profile of individual's preferences and the fixed voter's preferences.

Let $O \equiv (O_0, O_1, \dots, O_n)$ denote a profile of $n + 1$ fixed voters' preferences such that $O_0 = \geq$, $O_n = \leq$, and for all $t \in \{1, \dots, n - 1\}$, $O_t \in \mathcal{C}$. Let $Q(O)$ denote the collection of strictly quasi-concave preferences associated with O , i.e. for all $S \subseteq N$, $Q^S = O_{|S|}$. We call O *strongly monotone* when for all $S, T \subseteq N$ such that $S \subseteq T$ and all $a, b \in \mathbb{R}$ such that $a < b$, (i) if $aO_{|S|}b$, then $aO_{|T|}b$; and (ii) if $bO_{|T|}a$, then $bO_{|S|}a$. Obviously, if O is strongly monotone, then $Q(O)$ is monotone. Given a strongly monotone profile O of $n + 1$ fixed voters, the *majority welfare function* g^O associated with O is defined as follows. For all $R \in \mathcal{R}^N$ and all $a, b \in \mathbb{R}$,

$$\begin{aligned} ag^O(R)b &\Leftrightarrow |\{i \in N \mid \neg bR_i a\}| \cup |\{t \in \{0, 1, \dots, n\} \mid \neg bO_t a\}| \\ &\geq |\{i \in N \mid \neg aR_i b\}| \cup |\{t \in \{0, 1, \dots, n\} \mid \neg aO_t b\}|. \end{aligned}$$

The majority tie breaking rule τ^O associated with O is defined as follows: for all $R \in \mathcal{R}^N$ and all $a, b \in \mathbb{R}$, $\tau_{\{a, b\}}^O(R|\{a, b\}) \equiv g^O(R)|\{a, b\}$.

If O is a strongly monotone profile of $n + 1$ fixed voters, then the majority welfare function g^O is the fixed-strictly-quasi-concave welfare function associated with $Q(O)$ and the majority tie-breaking rule τ^O .

Proposition 3.1 *If O is a strongly monotone profile of $n + 1$ fixed voters, then $g^O = f_{\tau^O}^{Q(O)}$.*

Proof. Let $R \in \mathcal{R}^N$ and $a, b \in \mathbb{R}$ be such that $a < b$. Let $S \equiv \{i \in N \mid aP_i b\}$, $U \equiv \{i \in N \mid bP_i a\}$, and $T \equiv N \setminus (S \cup U)$. It suffices to show that $f_{\tau^O}^{Q(O)}|\{a, b\} = g^O(R)|\{a, b\}$.

Let $a f_{\tau^O}^{Q(O)}(R) b$. If $a f_{\tau^O}^{Q(O)}(R) b$ because of (δ) , then by definition of τ^O , $a g^O(R) b$. If $a f_{\tau^O}^{Q(O)}(R) b$ because of (β) , then $\neg b O_{|S|} a$ and because O is strongly monotone, at least $n + 1$ voters strictly prefer a to b , namely $S \cup \{|S|, |S| + 1, \dots, |N|\}$. Thus, $\neg b g^O(R) a$. Similarly it can be shown that $b f_{\tau^O}^{Q(O)}(R) a$ implies $b g^O(R) a$.

Let $a g^O(R) b$. First, we show that if $\neg b O_{|S|} a$, then (γ) does not hold. Because O is strongly monotone, we have $\neg b O_{|S \cup T|} a$. Thus, by strict quasi-concavity of $O_{|S \cup T|}$, for all $x \in]a, b[$, $\neg b O_{|S \cup T|} x$, and for all $x \in]b, +\infty[$, $\neg x O_{|S \cup T|} a$, and (γ) does not hold. Thus, if $\neg b O_{|S|} a$, then (β) or (δ) holds for a and b and by definition, $a f_{\tau^O}^{Q(O)}(R) b$.

Similarly it can be shown if $b g^O(R) a$ and $\neg a O_{|S \cup T|} b$, then (β) does not hold and $b f_{\tau^O}^{Q(O)}(R) a$.

If $b O_{|S|} a$ and $a O_{|S \cup T|} b$, then (β) and (γ) do not hold and by (δ) , $f_{\tau^O}^{Q(O)}|\{a, b\} = g^O(R)|\{a, b\}$, the desired conclusion. \square

If f_{τ}^Q is an *anonymous* welfare function, then for all coalitions S , Q^S depends only on the cardinality of $|S|$. In particular, the profile $(Q_{\emptyset}, Q_{\{1\}}, Q_{\{1,2\}}, \dots, Q_N)$ is a profile of $n + 1$ fixed voters. However, majority welfare functions are not characterized by *weak Pareto*, *independence of irrelevant alternatives*, and *anonymity*. This is due to the fact that the tie breaking rule τ need not be the majority tie breaking rule associated with the profile of $n + 1$ fixed voters.

If n is odd, for all $k \in \{0, 1, \dots, \frac{1}{2}(n-1)\}$, $O_k = \geq$, and for all $l \in \{\frac{1}{2}(n-1) + 1, \dots, n\}$, $O_l = \leq$, then Theorem 3.1 and Proposition 3.1 yield Black's celebrated median voter theorem saying that the median of the individual peaks is equal to the alternative that is top-ranked according to simple majority preference. Let g^m denote the simple majority welfare function. As an application of our results we show the following. If the number of voters is odd, the policy space is one-dimensional and the domain of preferences includes all spatial preferences, then simple majority voting is transitive if and only if each voter's preference relation is strictly quasi-concave. Let \mathcal{S} denote the set of all symmetric single-peaked preferences (or spatial preferences).

Theorem 3.3 *Let $|N|$ be odd and $|N| \neq 1$. The domain of strictly quasi-concave preferences \mathcal{C}^N is the unique maximal domain $\bar{\mathcal{R}}^N$ such that*

(i) $\mathcal{S}^N \subseteq \bar{\mathcal{R}}^N \subseteq \mathcal{W}^N$ and

(ii) simple majority voting is transitive on the domain $\bar{\mathcal{R}}^N$.

Proof. Obviously, $|N| \geq 3$. It is straightforward to adjust the definition of a fixed-strictly-quasi-concave welfare function to \mathcal{C}^N and to show that Lemma 3.6 remains true. Hence, by Proposition 3.1, simple majority voting is transitive on the domain \mathcal{C}^N .

Suppose that $\bar{\mathcal{R}}^N$ is a domain such that $\mathcal{S}^N \subseteq \bar{\mathcal{R}}^N$ and majority voting is transitive. Thus, g^m is a welfare function with domain $\bar{\mathcal{R}}^N$ satisfying *weak Pareto* and *independence of irrelevant alternatives*. Suppose that $\bar{\mathcal{R}} \setminus \mathcal{C} \neq \emptyset$. Let $R_0 \in \bar{\mathcal{R}} \setminus \mathcal{C}$. Define $f : \mathcal{S}^{N \setminus \{n\}} \rightarrow \mathcal{W}$ as follows: for all $R \in \mathcal{S}^{N \setminus \{n\}}$, $f(R) \equiv g^m(R, R_0)$. Because g^m is simple majority voting and $|N| \geq 3$, f satisfies *weak Pareto*. Thus, f is a welfare function satisfying *weak Pareto* and *independence of irrelevant alternatives*. Let $S \equiv \{1, \dots, \frac{1}{2}(|N| - 1)\}$. By definition of f and the fact that g^m is simple majority voting, we have $\tilde{f}(\leq_S, \geq_{N \setminus (S \cup \{n\})}) = R_0$. Because all lemmas hold on the domain \mathcal{S}^N and R_0 is not strictly quasi-concave, this is a contradiction to Lemma 3.4. \square

4 Conclusion

In this paper we solve a classical open problem in Arrovian social choice. For a one-dimensional policy space and the domain of single-peaked preferences, we characterize the social welfare functions satisfying *weak Pareto* and *independence of irrelevant alternatives*. As a corollary we obtain that majority voting is transitive if and only if the number of individuals is odd and individual preferences are strictly quasi-concave.

A parallel line of research considers social choice functions. In this case, the objective is to choose for each admissible set, called an agenda, the socially optimal alternatives in this set. *Arrow's choice axiom* requires that if for an admissible set and some admissible subset of it, some choices made at the former set belong to the smaller set, then the choices at the latter set are exactly the choices made at the former set that belong to the smaller set (Arrow, 1959).⁴ In spatial environments with one-dimensional policy spaces and agenda domain consisting of all compact intervals, the social choice functions satisfying *weak Pareto*, *Arrow's choice axiom*, and *independence of infeasible alternatives* have been investigated (Moulin, 1984; Ehlers, 2001). These papers obtain characterizations of the “generalized median rules” by these and additional axioms. A generalized median rule assigns to each compact interval and each profile the point belonging to the compact interval that is closest to the median of the peaks reported by the voters and 2^N fixed voters. The quasi bliss points of the 2^N fixed strictly quasi-concave preferences play the role of these fixed voters. The quasi bliss point of the social ordering chosen by an Arrovian welfare function for a profile of preferences is the outcome of the generalized median rule applied to this profile of quasi bliss points. However, the natural correspondence between generalized median rules and Arrovian welfare functions fails because the median may *not* be the socially most preferred alternative. The class of welfare functions underlying median rules is smaller than the class of Arrovian welfare functions: in the former the social

⁴Dutta, Jackson, and Le Breton (2001) consider Arrovian-type axioms in elections in which candidates have the possibility to withdraw. See also Ehlers and Weymark (2003).

ordering is single-peaked.

The same applies to the strand of literature on strategy-proof social choice rules when the set of alternatives is one-dimensional (Moulin, 1980; Border and Jordan, 1983; Kim and Roush, 1984; Barberà, Gul, and Stacchetti, 1993). For the case of a finite set of alternatives and unrestricted domain, Satterthwaite (1975) associates with each social choice rule satisfying strategy-proofness and unanimity an Arrovian welfare function. For his construction it is essential that any pair of alternatives can be moved to the top two positions in each agent's preference relation. It is obvious that this trick does not work in one-dimensional single-peaked environments.

Independent and parallel research by Sethuraman, Teo, and Vohra (2003,2006) considers Arrovian social welfare functions for a finite set of alternatives. Indifferences between two alternatives are excluded for both individual preferences and social orderings. Sethuraman, Teo, and Vohra (2003) establish an integer linear programming formulation for the problem of finding an Arrovian welfare function in this setup. For neutral Arrovian social welfare functions they find an easy existence test and for the domain of finite alternatives with single-peaked preferences they find that an Arrovian social welfare function is anonymous and monotonic if and only if it is simple majority voting with $n + 1$ fixed voters. Sethuraman, Teo, and Vohra (2006) generalize this result to any domain with no Condorcet triples and which contains an ordering and its inversion. Their works and ours are distinct in several aspects:

First, the last two characterization results only hold when individual and social indifferences are excluded. In our paper both individual indifferences and social indifferences are allowed and any Arrovian welfare function needs to be described with the help of a tie-breaking rule.

Second, with the exclusion of indifferences for single-peaked preferences and the finiteness of the set of alternatives, in their papers any social ordering of an anonymous Arrovian welfare function is single-peaked and the set of social orderings is a subset of the set of single-peaked preferences. For a continuum of alternatives with single-

peaked preferences this is no longer true. Social orderings are strictly quasi-concave and those orderings are not necessarily single-peaked.

Third, for a continuum of alternatives we cannot use their (finite) integer linear programming formulation for the problem of finding an Arrovian welfare function. Our proof techniques are in the spirit of Arrow's Theorem in the sense that we use only the basic requirements *weak Pareto* and *independence of irrelevant alternatives* and we determine exactly how decisiveness spreads from one pair of alternatives to other pairs of alternatives.

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