A GENERALIZATION OF THE SHAPLEY-ICHIISHI RESULT AND ITS APPLICATION TO CONVEX GAMES

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Abstract

In this paper we present a generalization of the Shapley-Ichiishi result for convex games to the class of all exact games. Then we discuss two applications to the class of convex games. First we show that it can indeed be used to give an alternative proof of the Shapley-Ichiishi result. Secondly we use our generalization to derive the minimal defining system of linear inequalities of the class of convex games.

1. Introduction.

Convex games are interesting from more than one perspective. First of all, results on convex games can be applied to a wide range of practical game models, such as unanimity games, bankruptcy games (O'Neill, 1982, Aumann and Maschler, 1985) and sequencing games (Curiel *et al.*, 1989).

From a theoretical perspective convex games are interesting because the core of a convex game has a particularly regular form as well as several other pleasant properties. For example Shapley (1971) showed that the core of a convex game is the unique stable set of that game and that it is the convex hull of the marginal vectors of that game. Ichiishi (1981) showed that the converse of

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the latter statement is also true: if the core of a game is equal to the convex hull of its marginal vectors, then the game is necessarily convex.

This two-way result of Shapley and Ichiishi –a game is convex if and only if its marginal vectors are core elements– is very useful because it enables us for example to check fairly easily whether a game is convex.

In this paper we will present a theorem that can be seen as a generalization of the Shapley-Ichiishi result. We will first explain the contents of our theorem, and then elaborate on its implications for the class of convex games.

THE GENERALIZATION First we will prove a generalization of the Shapley-Ichiishi result on the class of exact games. Within this class we can identify equivalence classes of games whose cores have the same combinatorial structure. This combinatorial structure can be expressed in terms of the coalitions that define extreme points of the core. In its turn this way to describe equivalence classes is a generalization of the way orderings of players define marginal vectors, the extreme points of the core.

We show that these equivalence classes (one of them is the class of convex games) can be given in terms of linear inequalities on the (polyhedral) class of all exact games. The proof of this statement is based on a geometrical interpretation of combinatorial equivalence.

APPLICATIONS TO CONVEX GAMES In the second part of the paper we will give two applications of this result to the class of convex games. The first application is the derivation of the Shapley-Ichiishi result. We show that the class of (exact) games associated via the combinatorial equivalence relation with a strictly exact game is precisely the collection of convex games. Since for this specific strictly convex game it can easily be seen that its core equals the convex hull of all its marginal vectors, we get an alternative proof of the Shapley-Ichiishi theorem.

A second application is the derivation of the minimal defining system of linear inequalities of the class of convex games. The set of linear inequalities we get from our generalization, when applied to the collection of convex games, is a strict subset of the system of linear inequalities that is normally used to define convexity. We will show that, in this specific setting, it is even the minimal system of linear inequalities: each inequality in this system is facet-defining. So, leaving it out would enlarge the set of games satisfying the remaining system of inequalities.

2. Cooperative games.

A transferable utility game, or TU-game, is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players of the game and v is a function that assigns to each coalition $S \subset N$ its value $v(S) \in \mathbb{R}$. The value $v(\phi)$ of the empty set is assumed to be equal to zero.

Throughout this paper we will keep the player set N fixed. So we will simplify our notation a bit and write v instead of (N, v) to denote a game. The sum game $v+w$ of two games v and w is defined by $(v+w)(S) := v(S)+w(S)$ and for a real number λ the game λv is defined by $(\lambda v)(S) := \lambda v(S)$. The collection of all TU-games with player set N, equipped with these operations, is a vector space.

One convenient basis for this vector space was introduced in Shapley (1953). Let T be a non-empty coalition in the player set N. The corresponding unanimity game u_T is defined as follows.

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Shapley (1953) proved that the collection of unanimity games forms a basis of the vector space of all games. In other words, given a game v, there exist unique coefficients $(\alpha_T)_{T \neq \phi}$ such that

$$v = \sum_{T \neq \phi} \alpha_T u_T.$$

Many classes of games, like airport games (Littlechild and Owen, 1973) and sequencing games (Curiel *et al.*, 1989), can be characterized through restrictions on these coefficients.

CORE OF A GAME AND EXACTNESS A vector $x \in \mathbb{R}^N$ will be called an allocation. The i^{th} coordinate x_i of the allocation x represents the payoff to player $i \in N$. For coalition $S \subset N$, the aggregate payoff $\sum_{i \in S} x_i$ is denoted by $x(S)$. An allocation x is called efficient if it distributes the value of the grand coalition among the players, i.e. if $x(N) = v(N)$. An efficient allocation x is called a core allocation if

$$x(S) \geq v(S) \text{ for all } S \subset N.$$

The set of core allocations, denoted by $C(v)$, is called the core of the game v. A game whose core is not empty is called balanced. A game v is called exact if for every coalition S there exists an allocation x in the core of v with $x(S) = v(S)$. Notice that for every balanced game v the game w defined by

$$w(S) := \min\{x(S) \mid x \text{ is an element of the core of } v\}$$

is the unique exact game that has the same core as v.

CONVEX GAMES Now we focus on the class of convex games. A game v is called convex if it satisfies the inequalities

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \qquad (*)$$

for all coalitions S and T. Notice that, with respect to the game v, these inequalities are linear. Hence, the set of convex games is polyhedral. A game that, whenever possible, strictly satisfies the inequalities $(*)$ (that is, whenever S is not a subset of T and T is not a subset of S) is called strictly convex. A game is strictly convex if and only if it is an element of the interior of the polyhedral set of convex games.

One of the main results on convex games is the Shapley-Ichiishi theorem. It can be stated as follows. Let v be a TU-game. Consider for a given permutation σ of the player set N the corresponding marginal vector $m^\sigma(v)$ defined by

$$m^\sigma(v)_{\sigma(k)} := v(S_k^\sigma) - v(S_{k-1}^\sigma)$$

for each player $k \in N$, where S_k^σ is defined to be

$$S_k^\sigma := \{\sigma(j) \mid j \leq k\}$$

for each $0 \leq k \leq n$. The result of Shapley (1971) and Ichiishi (1981) is

Theorem 1. *The game v is convex if an only if for each permutation σ the corresponding marginal vector $m^\sigma(v)$ is an element of the core of v.*

It is straightforward to show that in this case the collection of all extreme points of the core coincides with the collection of marginal vectors. The observation that for a convex game all marginal vectors are core elements was first made by Shapley (1971). The converse statement is due to Ichiishi (1981).

One other issue in this paper is the inequalities needed to define the class of convex games. Ichiishi (1981) showed that a game v is convex if and only if it satisfies the increasing marginal contributions (imc) inequalities, the inequalities of the form

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S) \qquad (**)$$

for all $S \subset T$ and $i \notin T$. So, not all inequalities in $(*)$ are needed. We will show that even the above system $(**)$ still has some redundancy.

3. The generalization.

First we will present a generalization of the Shapley-Ichiishi result to general classes of exact TU-games.

In this section we restrict ourselves to the class of exact games. Let v be an exact TU-game and let x be an allocation. A coalition S with $x(S) = v(S)$ is called tight at x in the game v. The collection of coalitions that are tight at x in v is denoted by $T(v, x)$. Furthermore, a core allocation x is called extreme if for any two core allocations y and z of v the equality $x = \frac{1}{2}y + \frac{1}{2}z$ implies that $x = y = z$ [1]. Define

$$\mathcal{T}(v) := \{T(v, x) \mid x \text{ is an extreme point of the core of } v\}.$$

[1] In other words, $\{x\}$ is a face of the core of v.

We say that two games v and w are equivalent if

$$\mathcal{T}(v) = \mathcal{T}(w).$$

Remark. The set $\mathcal{T}(v)$ fully captures the combinatorial structure of the core of v in the following sense. Let F be a face of the core of v. A coalition S is said to be tight at F if

$$x(S) = v(S)$$

holds for all allocations x in F. The set of tight coalitions at F is denoted by $T(F)$ (2). The collection

$$\{T(F) \mid F \text{ is a face of the core of } v\}$$

is called the combinatorial structure of v. Now two games are called combinatorially equivalent if they generate the same combinatorial structure. It is not difficult to show that equivalence and combinatorial equivalence are identical notions. ◁

In order to characterize the closure of an equivalence class of the game v, we say that the game w is a limiting case of v if for every extreme point x of the core of v there exists an extreme point y of the core of w such that $T(v, x)$ is a subset of $T(w, y)$.

The collection of limiting cases of v is denoted by $L(v)$. Note that equivalent games are limiting cases of each other. Note also that, by the Shapley-Ichiishi theorem, the collection of limiting cases of a strictly convex game is precisely the collection of convex games.

We will show that $L(v)$ is a polyhedral cone. In particular, we will describe how, given a game v, we can derive a defining system of linear inequalities for the polyhedral cone $L(v)$.

Let x be an extreme point of the core of v. A minimal characterizing set of x in v is a subset B of $T(v, x)$ such that the unique solution to the system

$$y(S) = v(S)$$

for all S in B is x, and such that no subset of B has this property. Equivalently, B is a basis of $\mathbb{R}^n$ (3). Since x is an extreme point of the core of v, we know that $T(v, x)$ spans $\mathbb{R}^N$, and hence contains at least one minimal characterizing set of x in v.

Now, given the (exact) game v, we construct a collection of linear (in)equalities on the set of games as follows. Take two –not necessarily distinct– extreme points x and y of the core of v. Suppose

(2) The face F is characterized by $T(F)$ in the sense that F is precisely the collection of core allocations x for which $x(S) = v(S)$ holds for all S in $T(F)$.

(3) This is a slight abuse of terminology. Officially we should say that $\{e_S \mid S \in B\}$ is a basis of $\mathbb{R}^N$.

that, in v, we have a minimal characterizing set B for x and a minimal characterizing set C for y such that $|B \cap C| = n - 1$ (generically, this will imply that x and y are neighboring extreme points). Take the unique coalition T in C that is not an element of B. Since B is a basis for $\mathbb{R}^n$ we can find (uniquely determined) real numbers $\lambda(S)$ for $S \in B$ such that

$$e_T = \sum_{S \in B} \lambda(S) e_S,$$

where $e_S \in \mathbb{R}^N$ denotes the characteristic vector of the coalition S. The linear inequality generated by these choices is

$$w(T) \leq \sum_{S \in B} \lambda(S) w(S).$$

The collection of inequalities thus generated is denoted by $I(v)$.

Lemma 1. *The inequalities in $I(v)$ imply that*

$$w(T) = \sum_{S \in B} \lambda(S) w(S)$$

in case $x = y$ (or, equivalently, if T is an element of $T(v, x)$).

Proof Let x, y, B, C and T be as above, with the additional assumption that $x = y$. We will show that the inequality

$$w(T) \geq \sum_{S \in B} \lambda(S) w(S)$$

is implied by the inequalities in $I(v)$. Let U be the unique coalition that is an element of B, but not of C. Consider the expression

$$e_T = \sum_{S \in B} \lambda(S) e_S = \lambda(U) e_U + \sum_{\substack{S \in B \\ S \neq U}} \lambda(S) e_S$$

Since C is a basis and $B \setminus \{U\}$ is a subset of C, at least we know that $\lambda(U) \neq 0$. Now, if $\lambda(U) > 0$, we have that

$$e_U = \frac{1}{\lambda(U)} e_T - \sum_{\substack{S \in B \\ S \neq U}} \frac{\lambda(S)}{\lambda(U)} e_S = \frac{1}{\lambda(U)} e_T - \sum_{\substack{S \in C \\ S \neq T}} \frac{\lambda(S)}{\lambda(U)} e_S$$

which generates the inequality

$$w(U) \leq \frac{1}{\lambda(U)} w(T) - \sum_{\substack{S \in C \\ S \neq T}} \frac{\lambda(S)}{\lambda(U)} w(S).$$

Using the fact that $\lambda(U) > 0$, this can be rewritten to

$$w(T) \geq \sum_{S \in B} \lambda(S) w(S).$$

So, assume that $\lambda(U) < 0$. In this case the equality

$$e_T = \lambda(U)e_U + \sum_{\substack{S \in B \\ S \neq U}} \lambda(S)e_S$$

implies that there is at least one coalition V in B with $V \neq U$ and $\lambda(V) > 0$. Now notice that the collection $D := (B \setminus \{V\}) \cup \{T\}$ must be a basis [4]. Consequently, D determines the extreme point x, while clearly $|B \cap D| = n - 1$. Thus the equation

$$e_V = \frac{1}{\lambda(V)} e_T - \sum_{\substack{S \in B \\ S \neq V}} \frac{\lambda(S)}{\lambda(V)} e_S$$

generates the inequality

$$w(V) \leq \frac{1}{\lambda(V)} w(T) - \sum_{\substack{S \in B \\ S \neq V}} \frac{\lambda(S)}{\lambda(V)} w(S).$$

This can be rewritten to

$$w(T) \geq \sum_{S \in B} \lambda(S) w(S),$$

which shows our claim. ◁

In the proof of the main theorem in this section we need one more lemma.

Lemma 2. *Let v be an exact game and let w be a game that satisfies all inequalities in $I(v)$. Let y be an extreme point in the core of v and let B be a basis in $T(v, y)$. Let z be the unique solution to the system of equalities*

$$x(S) = w(S) \qquad \textit{for all } S \in B.$$

Then $T(v, y)$ is a subset of $T(w, z)$.

Proof. Take a coalition T in $T(v, y)$. We will show that T is included in $T(w, z)$. First notice that $z(T) = w(T)$ holds when T is an element of B by the definition of z. So, suppose that T is an element of $T(v, y)$ that is not included in B. Take a minimal characterizing set C of y in v that includes T and such that T is the unique element of C that is not an element of B. Write

$$e_T = \sum_{S \in B} \lambda(S) e_S.$$

Since B and C define the same extreme point in v, by lemma 1 the game w automatically satisfies the equality

$$w(T) = \sum_{S \in B} \lambda(S) w(S).$$

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[4] This follows from the facts that D has exactly n elements and that D is necessarily independent. If it were not independent, then either B could not be a basis, or we would have found a way to write e_T as a linear combination of the elements e_S with $S \in B$ without using e_V, which would violate the uniqueness of the coefficients.

Hence,
$$\begin{aligned} z(T) = \langle z, e_T \rangle &= \sum_{S \in B} \lambda(S) z(S) \\ &= \sum_{S \in B} \lambda(S) w(S) \\ &= w(T) \end{aligned}$$

which proves our claim. ◁

This enables us to prove the main theorem.

Theorem 2. *A game w is a limiting case of v if and only if it satisfies the inequalities in $I(v)$.*

Proof. We shall first prove the easy direction. Let w be a limiting case of v. We will show that w satisfies the inequalities in $I(v)$. To this end, take two neighboring extreme points x and y of the core of v and let B and C be two minimal characterizing sets in v for x and y respectively such that $|B \cap C| = n - 1$. Take the unique coalition T in C that is not an element of B and take real numbers $\lambda(S)$ for $S \in B$ such that
$$e_T = \sum_{S \in B} \lambda(S) e_S.$$
We will show that w satisfies the corresponding inequality.

To this end, let a be an extreme point of the core of w such that $T(v, x)$ is a subset of $T(w, a)$. This can be done because w is a limiting case of v. Then we have
$$\begin{aligned} w(T) \le a(T) = \langle a, e_T \rangle &= \sum_{S \in B} \lambda(S) \langle a, e_S \rangle \\ &= \sum_{S \in B} \lambda(S) a(S) \\ &= \sum_{S \in B} \lambda(S) w(S) \end{aligned}$$
where the inequality follows from the fact that a is a core element of w and for the last equality we use the fact that $a(S) = w(S)$ for all coalitions S in
$$B \subset T(v, x) \subset T(w, a).$$

Now suppose conversely that a game w satisfies all inequalities in $I(v)$. We will show that w is a limiting case of v. In order to show this, take an extreme point y in the core of v. Take a basis B in $T(v, y)$. Since B is a basis, we know that the system of equalities
$$x(S) = w(S) \qquad \text{for all } S \in B$$
has a unique solution, say z. By lemma 2 we already know that $T(v, y)$ is a subset of $T(w, z)$. So, we only need to show that z is a core element of w.

In order to do that, first notice that z is efficient because the grand coalition $N \in T(v, y)$ is also an element of $T(w, z)$ by lemma 2, and hence $z(N) = w(N)$.

Now take an arbitrary coalition T. We will show that $z(T) \geq w(T)$. Consider the following linear program

$$\begin{array}{ll} & \min\langle e_T, x\rangle \\ \text{s.t.} & x \in C(v). \end{array}$$

Solving this linear program by means of the simplex algorithm using starting point $y^1 = y$ and basis $B_1 = B$ for the initialization yields a sequence

$$y^1, \ldots, y^k$$

of extreme points of the core of v together with a basis B_m for each point y^m. Moreover, since v is assumed to be exact, we know that $y^k(T) = v(T)$. And from the properties of the simplex algorithm we know that

$$|B_{m+1} \cap B_m| = n - 1$$

and that

$$\langle e_T, y^{m+1} - y^m\rangle \leq 0.$$

Now compute for each B_m the unique solution z^m to the system

$$x(S) = w(S) \qquad \text{for all } S \in B_m$$

of linear equalities. First we will show that there exists a $\mu_{m+1} \geq 0$ such that

$$z^{m+1} - z^m = \mu_{m+1}(y^{m+1} - y^m).$$

If $y^{m+1} = y^m$, then by lemma 2 we know that also $z^{m+1} = z^m$. So, in this case we can take $\mu_{m+1} = 0$. Now assume that y^{m+1} is not equal to y^m. Write

$$z(\mu) = z^m + \mu(y^{m+1} - y^m).$$

Let T_{m+1} be the unique coalition in $B_{m+1} \setminus B_m$. Take real numbers $\lambda_m(S)$ for $S \in B_m$ such that

$$e_{T_{m+1}} = \sum_{S \in B_m} \lambda_m(S) e_S.$$

By assumption, the game w satisfies the inequality

$$w(T_{m+1}) \leq \sum_{S \in B_m} \lambda_m(S) w(S).$$

Thus,

$$\begin{aligned} z^m(T_{m+1}) &= \sum_{S \in B_m} \lambda_m(S) z^m(S) \\ &= \sum_{S \in B_m} \lambda_m(S) w(S) \geq w(T_{m+1}). \end{aligned}$$

Furthermore, since y^{m+1} is not equal to y^m, we know that T_{m+1} is not tight on y^m. Thus, $\langle e_{T_{m+1}}, y^{m+1} - y^m \rangle < 0$, and this in its turn implies that

$$\mu_{m+1} := \max\{\mu \mid z(\mu)(T_{m+1}) \geq w(T_{m+1})\}$$

exists and is not negative. We will show that $z(\mu_{m+1}) = z^{m+1}$. To this end, notice that all coalitions S in $B_m \cap B_{m+1}$ are tight on all $z(\mu)$ in the game w, because they are tight on z^m in w, and on both y^m and y^{m+1} in v. Moreover, T_{m+1} is tight on $z(\mu_{m+1})$ in w by the definition of μ_{m+1}. So, all coalitions in B_{m+1} are tight on $z(\mu_{m+1})$ in w. Hence, $z(\mu_{m+1})$ coincides with z^{m+1}.

Now we will show that $z(T) \geq w(T)$. First of all, notice that T is an element of $T(v, y^k)$ by construction. So, by lemma 2, $z^k(T) = w(T)$. Hence,

$$\begin{aligned} w(T) = z^k(T) = \langle e_T, z^k \rangle = \langle e_T, z^1 \rangle + \sum_{m=1}^{k-1} \langle e_T, z^{m+1} - z^m \rangle \\ = z^1(T) + \sum_{m=1}^{k-1} \mu_{m+1} \langle e_T, y^{m+1} - y^m \rangle \\ \leq z(T) \end{aligned}$$

where the final inequality follows from the two facts that for all $1 \leq m \leq k-1$ we have both $\mu_{m+1} \geq 0$ and $\langle e_T, y^{m+1} - y^m \rangle \leq 0$. This concludes the proof. $\triangleleft$

Remark. The defining system $I(v)$ of the polyhedral set $L(v)$ we derived in this section does not need to be minimal [(5)] for an arbitrary exact game. Consider for example the following TU-game with player set $N = \{1, 2, 3, 4\}$.

$$v(S) = \begin{cases} 0 & \text{if } S = (1) \text{ or } S = (4) \\ 2 & \text{if } S = (2) \text{ or } S = (3) \\ 5 & \text{if } S = (12) \text{ or } S = (13) \text{ or } S = (23) \\ 0 & \text{if } S = (14) \\ 2 & \text{if } S = (24) \text{ or } S = (34) \\ 5 & \text{if } |S| = 3 \text{ and } 4 \in S \\ 9 & \text{if } S = (123) \\ 11 & \text{if } S = N. \end{cases}$$

This is indeed an exact game. Each coalition is tight on either one of the following six extreme

(5) See the appendix for a precise definition of a minimal defining system.

points of the core of v (the ones in which player 4 is only a member of the tight coalition N)

$$a = (3, 4, 2, 2)$$

$$b = (4, 3, 2, 2)$$

$$c = (4, 2, 3, 2)$$

$$d = (3, 2, 4, 2)$$

$$e = (1, 4, 4, 2)$$

$$f = (0, 5, 5, 1)$$

or on at least one of the three (extreme) core elements $(0, 5, 6, 0)$, $(0, 6, 5, 0)$ and $(6, 3, 2, 0)$. Now, for the six extreme core elements above we can check that

$$T(v, a) = \{(3), (13), (123), N\}$$

$$T(v, b) = \{(3), (23), (123), N\}$$

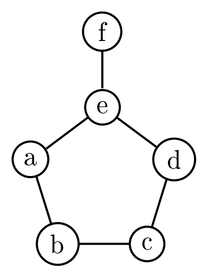
$$T(v, c) = \{(2), (23), (123), N\}$$

$$T(v, d) = \{(2), (12), (123), N\}$$

$$T(v, e) = \{(12), (13), (123), N\}$$

$$T(v, f) = \{(1), (12), (13), N\}$$

Thus, the graph of neighboring extreme points restricted to these six points looks as follows.



Each edge in this graph induces an inequality. For example the inequality $(f \to e)$ is generated by writing $e_{(123)}$ as a linear combination of the characteristic vectors corresponding to the elements in $T(v, f)$:

$$e_{(123)} = e_{(12)} + e_{(13)} - e_1.$$

The inequalities thus induced by the edges in this graph are

$(f \to e)$ $$w(123) \leq w(12) + w(13) - w(1)$$

$(e \to a)$ $$w(3) \leq w(123) - w(12)$$

$(a \to b)$ $$w(23) \leq w(123) + w(3) - w(13)$$

$(b \to c)$ $$w(2) \leq w(23) - w(3)$$

$(c \to d)$ $\qquad w(12) \leq w(123) - w(23) + w(2)$

$(d \to e)$ $\qquad w(13) \leq w(123) - w(2)$

Notice that the inequality $(e \to a)$ is implied by inequalities $(b \to c)$ and $(c \to d)$. Symmetrically, $(d \to e)$ is implied by $(a \to b)$ and $(b \to c)$. ◃

4. Application to convex games.

The previous result on arbitrary exact games has several implications in case the game in question happens to be convex [(6)]. It can for example be used to show a version of the Shapley-Ichiishi result. This is what we will do first in this section. We will also show how it can be used to derive a minimal defining system for the polyhedral set of convex games.

To this end, consider the game v^* defined by

$$v^*(S) := s^2$$

for each coalition, where $s = |S|$. It is easy to check that this is a convex game, and that

$$v^*(S) + v^*(T) = v^*(S \cap T) + v^*(S \cup T).$$

if and only if $S \subset T$ or $T \subset S$ [(7)]. Thus, the game v^* is strictly convex.

In order to apply our generalization to convex games, we need the following two observations concerning v^*.

Lemma 3. *An allocation is an extreme point of the core of v^* if and only if it is a marginal vector of v^*.*

Proof. Let $m^\sigma(v^*)$ be a marginal vector of v^*. Assume w.l.o.g. that σ is the identity. Thus, the payoff to player i equals $i^2 - (i-1)^2 = 2i - 1$, and for a (non-empty) coalition S with $|S| = s$ we get that

$$m^\sigma(v^*)(S) = \sum_{i \in S}(2i-1) \geq \sum_{i=1}^{s}(2i-1) = 2 \cdot \frac{1}{2}s(s+1) - s = s^2 = v^*(S).$$

Conversely, assume that x is an extreme element of the core of v^*. Since $T(v^*, x)$ spans $\mathbb{R}^N$, it suffices to show that, for any two coalitions S and T that are tight on x in v^* we have either $S \subset T$ or $T \subset S$. To this end, notice that

$$v^*(S \cap T) + v^*(S \cup T) \leq x(S \cap T) + x(S \cup T) = x(S) + x(T) = v^*(S) + v^*(T).$$

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(6) Notice that convex games are indeed exact.

(7) This can be seen as follows. Write $a = |S \setminus T|$, $b = |T \setminus S|$ and $k = |S \cap T|$. Then $v(S) + v(T) = (a+k)^2 + (b+k)^2$ and $v(S \cup T) + v(S \cap T) = (a+k+b)^2 + k^2$ which happens to be greater or equal to the previous expression. Strict inequality occurs if and only if $ab > 0$, in other words if $a > 0$ and $b > 0$.

So, by the convexity of v^*,

$$v^*(S \cap T) + v^*(S \cup T) = v^*(S) + v^*(T)$$

which implies that $S \subset T$ or $T \subset S$ because v^* is strictly convex. $\triangleleft$

Theorem 3. *Let v^* be the strictly convex game defined above. Then the system $I(v^*)$ is precisely the system of inequalities*

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S) \qquad (***)$$

for each coalition S and players i and j not in S.

Proof. It is straightforward to show for each marginal vector $m^\sigma(v^*)$ that

$$T(v^*, m^\sigma(v^*)) = \{S_k^\sigma \mid k \in N\}.$$

Thus, each $T(v^*, m^\sigma(v^*))$ is the unique minimal characterizing set of $m^\sigma(v^*)$ in v^*. This enables us to derive the system $I(v^*)$ for this particular game v^*.

In order to do this, let $m^\sigma(v^*)$ and $m^\tau(v^*)$ be two extreme points of v^* with

$$|T(v^*, m^\sigma(v^*)) \cap T(v^*, m^\tau(v^*))| = n - 1.$$

Let T be the unique coalition in $T(v^*, m^\tau(v^*))$ that is not an element of $T(v^*, m^\sigma(v^*))$. It is not so difficult to show that there is a coalition S and two players i and j not in S such that the coalitions S, $S \cup \{i\}$ and $S \cup \{i, j\}$ are all three elements of $T(v^*, m^\sigma(v^*))$, while

$$T = S \cup \{j\}.$$

Thus, from the fact that

$$e_T = e_{S \cup \{j\}} = e_{S \cup \{i,j\}} - e_{S \cup \{i\}} + e_S$$

we get that the inequality in $I(v^*)$ generated by these choices is exactly

$$w(S \cup \{j\}) \leq w(S \cup \{i, j\}) - w(S \cup \{i\}) + w(S)$$

which can be rewritten to

$$w(S \cup \{i, j\}) - w(S \cup \{j\}) \geq w(S \cup \{i\}) - w(S)$$

Conversely, each inequality of this form can be obtained by an appropriate choice of σ and τ. Thus, $I(v^*)$ is precisely our system $(***)$. $\triangleleft$

APPLICATION I. THE SHAPLEY-ICHIISHI RESULT The above observations concerning the game v^* can now be used to derive the result of Shapley and Ichiishi.

Theorem 4. *A game v is convex if and only if it is a limiting case of v^*.*

Proof. Suppose that v is convex. Observe that an inequality of the form

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S)$$

is simply a special case of an imc inequality for $T = S \cup \{j\}$. Hence, by theorem 3, v is a limiting case of v^*.

Suppose conversely that v is a limiting case of v^*. It is sufficient to prove the imc inequalities $(**)$. So, let S and T be two coalitions such that $S \subset T$ and let i be a player who is not a member of T. We will prove that

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

If $T = S$, the inequality is trivial, so assume that $T \neq S$. Write $T \setminus S = \{i_1, \ldots, i_m\}$. Repeated application of the inequalities given in theorem 3 yields

$$\begin{aligned} v(T \cup \{i\}) - v(T) &= v(S \cup \{i_1, \ldots, i_m\} \cup \{i\}) - v(S \cup \{i_1, \ldots, i_m\}) \\ &\geq v(S \cup \{i_1, \ldots, i_{m-1}\} \cup \{i\}) - v(S \cup \{i_1, \ldots, i_{m-1}\}) \\ &\ldots \\ &\geq v(S \cup \{i_1\} \cup \{i\}) - v(S \cup \{i_1\}) \\ &\geq v(S \cup \{i\}) - v(S). \end{aligned}$$

This concludes the proof. ◁

This theorem boils down to the result of Shapley and Ichiishi saying that a game is convex if and only if its core is the convex hull of all marginal vectors. This can be seen as follows.

Corollary. (Shapley-Ichiishi) *A game v is convex if and only if all marginal vectors of v are elements of the core of v.*

Proof. The previous theorem states that a game v is convex if and only if it is a limiting case of v^*. But for the game v^* we know by lemma 3 that each extreme point of the core is a marginal vector and, conversely, that each marginal vector is an element of its core. Thus, a game v is a limiting case of v^* if and only if all marginal vectors $m^\sigma(v)$ are elements of the core of v. ◁

APPLICATION II. A MINIMAL DEFINING SYSTEM A second application is the derivation of the minimal defining system (see appendix) of the polyhedral set of convex games. In this section we will explain how this can be done.

First notice that the space of convex games is indeed a polyhedral set since the defining system of inequalities

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

is linear. And since the game v^* satisfies all inequalities strictly, unless $S \subset T$ or $T \subset S$, it is also of full dimension. So, according to proposition 1 in the appendix, there exists a unique minimal defining system for the polyhedral set of convex games.

However, the above defining system is not minimal. For example, the imc inequalities $(**)$ defined by Ichiishi are also sufficient. Still, also this system doesn't satisfy the minimality requirement of a minimal defining system, and there is some redundancy left. This is clear once we have the following immediate consequence of theorems 3 and 4.

Theorem 5. *A game v is convex if and only if it satisfies the set $I(v^*)$ of inequalities of the form*

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S) \qquad (***)$$

for each coalition S and players i and j not in S.

It is not known to us for precisely which polyhedral sets $L(v)$ of exact games the defining system $I(v)$ will be minimal. But now we will show that the system $I(v^*)$ is the minimal defining system of the set $L(v^*)$ of convex games. For that we need the following statement.

Theorem 6. *The following two conditions are equivalent:*

(a) The game v is convex.

(b) For all $i, j \in N, i \neq j$, and each coalition S that does not contain i and j:

$$\sum_{T \subset S} \alpha_{T \cup \{i,j\}} \geq 0,$$

where the numbers α_T are the unique unanimity coefficients defined in section 2.

Proof. By theorem 5 it suffices to prove the equivalence of $(***)$ and (b). Write $v = \sum_{T \neq \phi} \alpha_T u_T$. Let i and j be two players, and let S be a coalition that does not contain i and j. Then

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S)$$

$$\Leftrightarrow$$

$$[v(S \cup \{i,j\}) - v(S \cup \{j\})] - [v(S \cup \{i\}) - v(S)] \geq 0$$

$$\Leftrightarrow$$

$$\left[\sum_{T \subset S \cup \{i,j\}} \alpha_T - \sum_{T \subset S \cup \{j\}} \alpha_T\right] - \left[\sum_{T \subset S \cup \{i\}} \alpha_T - \sum_{T \subset S} \alpha_T\right] \geq 0\,,$$

$$\Leftrightarrow$$

$$\sum_{T \subset S \cup \{j\}} \alpha_{T \cup \{i\}} - \sum_{T \subset S} \alpha_{T \cup \{i\}} \geq 0$$

$$\Leftrightarrow$$

$$\sum_{T \subset S} \alpha_{T \cup \{i,j\}} \geq 0$$

which concludes the proof. ◁

This theorem enables us to prove that a further reduction of the number of inequalities is not possible. In order to see why this is so, let i and j be two players and let S be a coalition that does not contain i and j. In order to show that the condition

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S)$$

is not redundant we construct a game that violates exactly the convexity condition corresponding to the triplet (S,i,j), while still satisfying all other conditions. Consider the game $v = \sum_T \alpha_T u_T$ with

$$\alpha_T = \begin{cases} -1 & \text{if } T = S \cup \{i,j\} \\ 1 & \text{if } T = S \cup \{i\} \text{ or } T = S \cup \{j\} \text{ or } T \not\subset S \cup \{i,j\} \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum_{T \subset S} \alpha_{T \cup \{i,j\}} = -1 < 0$, so the condition for (S,i,j) is indeed violated.

In order to show that all other inequalities are satisfied, let k and l be two different players and let V be a coalition such that the triplet (V,k,l) does not equal the triplet (S,i,j). Consider the quantity

$$\sum_{T \subset V} \alpha_{T \cup \{k,l\}}.$$

If $\alpha_{S \cup \{i,j\}}$ does not appear in this summation the sum only concerns nonnegative terms and is hence nonnegative. If $\alpha_{S \cup \{i,j\}}$ does appear in the summation, discern two cases.

(a) If $V \neq S$ and $\{i,j\} = \{k,l\}$. The fact that $\alpha_{S \cup \{i,j\}}$ appears in the above sum implies that the set S is a subset of V. So, since $V \neq S$, we can choose $m \in V \setminus S$. Then $\alpha_{\{m\} \cup \{k,l\}} = 1$ appears in the sum too, compensating for $\alpha_{S \cup \{i,j\}} = -1$ and consequently yielding a nonnegative outcome.
(b) If $\{i,j\} \neq \{k,l\}$. Assume without loss of generality that i is not an element of $\{k,l\}$. Since $\alpha_{S \cup \{i,j\}} = -1$ appears in the sum, also $\alpha_{S \cup \{j\}} = 1$ appears in it, compensating the negative number and hence yielding a nonnegative outcome. ◁

A brief final remark on the complexity of our minimal test for convexity. There are $\binom{n}{2}$ ways to choose two different players i and j in N, and 2^{n-2} ways to choose a coalition S that doesn't contain players i and j. Thus we have $2^{n-2}\binom{n}{2}$ conditions of the form

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S).$$

Now $2^{n-2}\binom{n}{2} = \frac{2^n n(n-1)}{8}$ conditions seems to be quite a lot, until one realizes that the game itself is defined on its 2^n coalitions. In other words, the input size is not the number of players n, but the number of coalitions $x = 2^n$. Thus, testing $\frac{2^n n(n-1)}{8}$ conditions is only of complexity $\mathcal{O}(x(\log x)^2)$ while the original system $(*)$ has complexity $\mathcal{O}(x^2)$. The complexity of the imc system $(**)$ of Ichiishi is of order $\mathcal{O}(x^{\frac{3}{2}} \log x)$.

Appendix. Defining systems.

In this appendix we will explain in a general setting what the (minimal) defining system of a polyhedral set exactly is. All of our assertions in this section are well-known. We only repeat them here for convenience.

The dimension of a convex subset C of $\mathbb{R}^n$ is defined to be the dimension of the smallest affine subspace that contains C. It is denoted by $\dim(C)$. Let A be an $m \times n$ matrix and let b be an element of $\mathbb{R}^m$. The solution set

$$P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

of the system $Ax \geq b$ of linear inequalities is called a polyhedron. Evidently every polyhedron is convex.

A closed subset F of P is called a face of P if any x and y in P are elements of F if and only if $\frac{1}{2}x + \frac{1}{2}y$ is an element of F. Since P is convex, a face must also be convex. Thus each face has a dimension and a face with dimension $\dim(P) - 1$ is called a facet.

MINIMAL DEFINING SYSTEMS Now let P be a polyhedron in $\mathbb{R}^n$ of full dimension, meaning that $\dim(P) = n$. So, the dimension of each facet is equal to $n - 1$ in this case. Any $m \times n$ matrix A and vector b such that

$$P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

is called a defining system for P. Let A_i denote the i^{th} row of the matrix A. A defining system (A, b) is called minimal if for each and every $1 \leq i \leq m$ the collection of points x in $\mathbb{R}^n$ that satisfy

$$A_j x \geq b_j \qquad \text{for all } j \neq i$$

strictly includes P.

We can also talk about minimal defining systems in terms of subsets of the set of inequalities used to define P. To make this a little more precise, take a collection $I \subset \{1, \ldots, m\}$ of indices. Then the set of inequalities

$$A_i x \geq b_i \qquad \text{for all } i \in I$$

is a minimal defining system of P if and only if
(i) (sufficiency) the set of points x satisfying the inequalities

$$A_i x \geq b_i \qquad \text{for all } i \in I$$

is exactly P, and
(ii) (necessity) for each $j \in I$ there is a point x that violates the inequality $A_j x \geq b_j$, but satisfies $A_i x \geq b_i$ for all $i \in I$ with i not equal to j.

It can be shown that in some sense each full-dimensional polyhedron has a unique minimal defining system. We will explain why this should be so. We start with the existence of a minimal defining system. A row A_i of A defines a face of P in the following way. Define

$$F_i = \{x \in P \mid A_i x = b_i\}.$$

It can easily be shown that each F_i is a face of P.

Now let I be a maximal collection of indices i for which F_i is a facet and such that, for all i and j in I, $F_i \neq F_j$ whenever $i \neq j$. Then (here we need P to be of full dimension) we have the following result.

Proposition 1. *The collection of linear inequalities*

$$A_i x \geq b_i \qquad \text{for all } i \in I$$

is a minimal defining system for P.

The fact that all minimal defining systems are essentially identical can be seen in a couple of different ways. Here is one of them. With each $i \in I$ we can associate the half-space

$$H_i = \{x \in \mathbb{R}^n \mid A_i x \geq b_i\}.$$

Thus we get that

$$P = \bigcap_{i \in I} H_i,$$

because I is a defining system of P. And because of the minimality we end up with a polyhedron that strictly includes P if we leave out one or more half-spaces in this intersection.

It is not too difficult to show that these half-spaces are really unique in the formal sense. In other words, any finite intersection of half-spaces that results in P will include these ones.

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