Pareto Improving Price Regulation When the Asset Market Is Incomplete [1]

P.J.J. Herings [2] H.M. Polemarchakis [3]

April, 2000

[1]This text presents results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by its authors. The Commission of the European Communities provided additional support through the Human Capital and Mobility grant ERBCHRXCT940458. The research of Herings was made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences and a grant of the Netherlands Organization for Scientific Research (NWO). While this paper was being written, Herings enjoyed the generous hospitality of the Cowles Foundation and Polemarchakis of the Cowles Foundation, the Faculty of Economics and Politics of the University of Cambridge and the Indira Gandhi Institute for Development Research. John Geanakoplos and Hamid Sabourian made comments on earlier drafts of the paper that were very helpful. We would like to thank two anonymous referees, and in particular an associate editor, for helpful and insightful comments.

[2]Department of Economics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands. E-mail: P.Herings@algec.unimaas.nl

[3]CORE, Université Catholique de Louvain, B-1348, Louvain-la-Neuve, Belgium.

Abstract

When the asset market is incomplete, competitive equilibria are constrained suboptimal, which provides scope for Pareto improving interventions. Price regulation can be such a Pareto improving policy, even when the welfare effects of rationing are taken into account. An appealing aspect of price regulation is that it operates anonymously on market variables. The welfare analysis of price regulation calls for an extension of the equilibrium theory of incomplete markets to fix-price equilibria.

Fix-price equilibria exist under standard assumptions. There are robust examples, however, for which at regulated prices close to competitive prices, there are no fix-price equilibria close to competitive equilibria. We provide necessary and sufficient conditions for the local uniqueness of fix-price equilibria, and show that under these conditions Pareto improving price regulation is generically possible.

Key words: incomplete asset market, fix-price equilibria, Pareto improvement.

JEL classification numbers: D45, D52, D60.

1 Introduction

One of the major accomplishments of economic theory is a rigorous proof of the Pareto optimality of competitive equilibrium allocations. A crucial assumption to get such a result is that asset markets are complete. When the asset market is incomplete, competitive equilibrium allocations generically fail to satisfy the criterion of Pareto optimality. Completing the asset markets does not necessarily lead to Pareto improvements. Financial innovation may lead to a Pareto deterioration as is shown by the example of Hart (1975). Conditions for Pareto improving financial innovation to be possible are rather restrictive, see Elul (1995), Hara (1997), and Cass and Citanna (1998).

When the asset market is incomplete, competitive equilibrium allocations are generically not even constrained suboptimal, a criterion of optimality that recognizes the incompleteness of the asset market. As has been shown in Geanakoplos and Polemarchakis (1986), there exist reallocations of asset portfolios that yield Pareto improvements in welfare after prices in spot commodity markets adjust to attain equilibrium.

The failure of constrained optimality casts doubt on the desirability of non-intervention with competitive markets, such as the laissez faire policy in international trade. Nevertheless, the empirical content of portfolio reallocation is rather meager. Apart from informational requirements, the heterogeneity of individuals and the requirement of anonymity may interfere with improving interventions, see also Citanna, Kajii and Villanacci (1998) and Kajii (1994).

In this paper we investigate an alternative to the reallocation of asset portfolios, the direct regulation of prices in spot commodity markets. An intervention in spot market prices is not an intervention in individual choice variables but in market variables. As such it satisfies the requirement of anonymity. Interventions in the price mechanism are frequently observed. Nguyen and Whalley (1986) make the same observation, stating "Price controls have been employed by governments all over the world, during war and peace, in response to all manners of threats (both real and imaginary), and in all ages."

Price regulation seems odd when viewed from a traditional efficiency perspective. We show that Pareto improving price regulation is possible when asset markets are incomplete. Moreover, the deviation of prices from their competitive equilibrium values can be chosen independently of the state of the world [1]. This makes price regulation comparable to the reallocation of portfolios carried out before the resolution of uncertainty.

Direct antecedants of our result are the argument in Polemarchakis (1979), which showed that fixed wages that need not match shocks in productivity may yield higher expected utility in spite of the loss of output in an economy of overlapping generations; and the argument in Drèze and Gollier (1993), which employed the capital asset pricing model to determine optimal schedules of wages that differ from the marginal productivity of labor. An example of Pareto improving price regulation was developed in Kalmus (1997).

[1] John Geanakoplos insisted on this point.

To address the issue of Pareto improving price regulation, we need an equilibrium notion that allows for trading at non-competitive prices, while maintaining the scenario of frictionless markets which characterizes competitive equilibria with incomplete markets. The equilibrium notion used is an extension of the fix-price equilibrium of Drèze (1975) to the incomplete markets set-up. Such equilibria are shown to exist in Section 3.

In Section 4, we study the local behavior of fix-price equilibria in the neighborhood of competitive ones. Despite the equilibrium existence, the behavior of fix-price equilibria in the neighborhood of competitive equilibria is particularly complicated. There are robust examples for which at regulated prices close to competitive prices, there are no fix-price equilibria close to competitive equilibria. We provide necessary and sufficient conditions for local uniqueness of fix-price equilibria. The properties of the equilibrium manifold imply that these conditions are weaker than the requirement of uniqueness of fix-price equilibria for prices in the neighborhood of competitive equilibria. The welfare implications of price regulation in the neighborhood of competitive equilibria are derived.

Section 5 shows that Pareto improving price regulation is possible when the requirement of local uniqueness is satisfied. The conditions under which this result holds, that the number of instruments (commodities) exceeds the number of objectives (individuals), implies that the result complements the one of Geanakoplos and Polemarchakis (1986). Section 6 illustrates the results by means of an example.

2 The Economy

The economy is the standard two-period general equilibrium model with incomplete asset markets and numeraire assets. Transactions occur in assets before and in commodities after the state of nature is known. An economy $\mathcal{E} = ((\mathcal{X}^i, u^i, e^i)_{i\in\mathcal{I}}, R)$ consists of consumption sets $\mathcal{X}^i$, utility functions u^i and endowments e^i for all individuals $i \in \mathcal{I}$, and an asset return matrix R.

States of the world are $s \in \mathcal{S} = \{1, \ldots, S\}$ and commodities are $l \in \mathcal{L} = \{1, \ldots, L+1\}$. At state s, commodity $(L+1, s)$ is assumed to be a numeraire commodity. Assets are $a \in \mathcal{A} = \{1, \ldots, A+1\}$. Asset $A+1$ is assumed to be a numeraire asset. The payoffs of assets are denominated in the numeraire commodity, $(L+1, s)$, in every state of the world.

The economy satisfies the following assumptions.

A1. For every individual i, the consumption set is $\mathcal{X}^i = \mathbb{R}^{(L+1)S}_{++}$.

A2. For every individual i, the utility function is twice continuously differentiable, $\partial u^i \gg 0$, $\partial^2 u^i$ is negative definite on $(\partial u^i)^\perp$, and satisfies the boundary condition, for every $x^i \in \mathcal{X}^i$, the closure of the set $\{\overline{x}^i \in \mathcal{X}^i \mid u^i(\overline{x}^i) \geq u^i(x^i)\}$ is contained in $\mathbb{R}^{(L+1)S}_{++}$.

A3. For every individual i, the endowment is strictly positive, $e^i \in \mathcal{X}^i$.

A4. The asset return matrix has full column rank. The numeraire asset has positive payoff, $R_{\cdot A+1} > 0$.

We want to analyse the allocation that would result for any given terms of trade, that is at any given prices of commodities and assets. Prices of commodities across states of the world are $p = (p_1, \ldots, p_S)$. The price of the numeraire commodity in state of the world s is $p_{L+1,s} = 1$. The domain of prices of commodities is $\mathcal{P} = \{p \in \mathbb{R}^{(L+1)S}_{++} : p_{L+1,s} = 1, s \in \mathcal{S}\}$. Prices of assets are $q = (q_1, \ldots, q_{A+1})$. The price of the numeraire asset is $q_{A+1} = 1$. The domain of prices of assets is $\mathcal{Q} = \{q \in \mathbb{R}^{A+1} : q_{A+1} = 1\}$.

On several occasions we want to truncate prices of commodities and prices of assets by deleting the numeraires. Commodities (assets) other than the numeraire are $\check{\mathcal{L}} = \{1, \ldots, L\}$ ($\check{\mathcal{A}} = \{1, \ldots, A\}$). The domain of prices of commodities (assets) other than the numeraire is $\check{\mathcal{P}} = \mathbb{R}^{LS}_{++}$ ($\check{\mathcal{Q}} = \mathbb{R}^{A}$).

At arbitrary terms of trade, a competitive equilibrium is typically ruled out. In commodities and assets other than the numeraire, endogenously determined rationing on net trades serves to attain market clearing. To keep the presentation as simple as possible, rationing is assumed to be uniform across individuals. Rationing in the supply (demand) of commodities other than the numeraire is $\underline{z} \in -\mathbb{R}^{LS}_{+}$ ($\overline{z} \in \mathbb{R}^{LS}_{+}$). Rationing in the supply (demand) of assets other than the numeraire is $\underline{y} \in -\mathbb{R}^{A}_{+}$ ($\overline{y} \in \mathbb{R}^{A}_{+}$).

At prices and rationing scheme $(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, the budget set of individual i is

$$\begin{aligned} \beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) = \Big\{(x, y) \in \mathcal{X}^i \times \mathbb{R}^{A+1} \ : \ & qy \leq 0, \\ & p_s(x_s - e^i_s) \leq R_s.y, \quad s \in \mathcal{S}, \\ & \underline{z}_{l,s} \leq x_{l,s} - e^i_{l,s} \leq \overline{z}_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ & \underline{y}_a \leq y \leq \overline{y}_a, \quad a \in \check{\mathcal{A}}\}. \end{aligned}$$

The optimization problem of the individual is to choose a utility maximizing consumption bundle and asset portfolio, denoted $d^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, in his budget set. Despite the differentiability assumptions on primitives, the rationing constraints cause non-differentiabilities for the demand function.

At given prices and rationing scheme, an individual is effectively rationed in his supply (demand) for a commodity or an asset if he could increase his utility when the rationing scheme in the supply (demand) of that commodity or asset is removed. There is effective supply (demand) rationing in the market for a commodity or an asset if at least one individual is effectively rationed in his supply (demand) for this commodity or asset. At a competitive equilibrium the prices and rationing scheme are such that there is neither effective supply rationing nor effective demand rationing in the market for any commodity or asset. This makes the competitive equilibrium a special case of a fix-price equilibrium.

Definition 2.1 (Fix-price equilibrium) *A fix-price equilibrium for the economy $\mathcal{E}$ at prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$ is a pair $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$ such that*

1. *for every individual,* $(x^{i*}, y^{i*}) \in d^i(p, q, \underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$,
2. $\sum_{i=1}^{I} x^{i*} = \sum_{i=1}^{I} e^i$ *and* $\sum_{i=1}^{I} y^{i*} = 0$,

3. *for every $l \in \check{\mathcal{L}}$, if for some i' $x^{i'*}_{l,s} - e^{i'}_{l,s} = \underline{z}^*_{l,s}$, then for all $i \in \mathcal{I}$ $x^{i*}_{l,s} - e^i_{l,s} < \overline{z}^*_{l,s}$, while if for some i' $x^{i'*}_{l,s} - e^{i'}_{l,s} = \overline{z}^*_{l,s}$ then for all $i \in \mathcal{I}$ $x^{i*}_{l,s} - e^i_{l,s} > \underline{z}^*_{l,s}$, and*
4. *for every $a \in \check{\mathcal{A}}$, if for some i' $y^{i'*}_a = \underline{y}^*_a$, then for all $i \in \mathcal{I}$ $y^{i*}_a < \overline{y}^*_a$, while if for some i' $y^{i'*}_a = \overline{y}^*_a$, then for all $i \in \mathcal{I}$ $y^{i*}_a > \underline{y}^*_a$.*

Conditions 1 and 2 are the usual optimization and market clearing conditions. Conditions 3 and 4, together with the convexity of the consumption sets and the quasi-concavity of the utility functions of individuals, imply that there is no effective rationing, simultaneously, on both sides of a market. This expresses that we do not depart from the scenario of frictionless markets that characterizes competitive equilibria with incomplete markets. Markets are still transparent in the sense that it is not possible to find a buyer and a seller in a single market that could benefit from mutual exchange against the numeraire.

3 The Existence of Fix-price Equilibria

A fairly straightforward proof of the existence of a fix-price equilibrium at prices (p, q) can be given under A1-A4. Let $\widetilde{\mathcal{X}}^i$ be a compact, convex subset of $\mathcal{X}^i$ that contains the aggregate initial endowment in the interior. The assumptions on utility functions and on the asset return matrix imply that all $S+1$ budget inequalities in the definition of the budget set hold with equality at the optimal choice of an individual. The rationing inequalities do not necessarily hold with equality. The budget set related to $\widetilde{\mathcal{X}}^i$ with all budget inequalities required to hold with equality is denoted $\widetilde{\beta}^i$ and the corresponding demand function $\widetilde{d}^i$. Since prices are fixed at (p, q), they are omitted in the notation.

Lemma 3.1 *If $\mathcal{E}$ satisfies A1-A4, then $\widetilde{d}^i$, $i \in \mathcal{I}$, is continuous.*

Proof Let $(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n)$ be a sequence that converges to $(\underline{z}, \overline{z}, \underline{y}, \overline{y})$. Then $(\widetilde{d}^i(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n)) : n = 1, \ldots)$ has a convergent subsequence, with limit $(\widehat{x}, \widehat{y}) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$.

Suppose there exists $(\widetilde{x}, \widetilde{y}) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, such that $u^i(\widetilde{x}) > u^i(\widehat{x})$. Let $\widetilde{\mathcal{L}}_-$, $\widetilde{\mathcal{L}}_+$, $\widetilde{\mathcal{A}}_-$, and $\widetilde{\mathcal{A}}_+$, denote the sets of non-numeraire commodities and non-numeraire assets for which $\widetilde{x}_{l,s} - e^i_{l,s}$ is negative, positive, $\widetilde{y}_a$ is negative, and positive, respectively. For

$$\lambda^n = \min\left\{1, \frac{\underline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l,s) \in \widetilde{\mathcal{L}}_-, \frac{\overline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l,s) \in \widetilde{\mathcal{L}}_+, \frac{\underline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_-, \frac{\overline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_+\right\},$$

$\widetilde{x}^n = e^i + \lambda^n(\widetilde{x} - e^i)$, $n = 1, \ldots$, and $\widetilde{y}^n = \lambda^n \widetilde{y}$, $n = 1, \ldots$, it can be verified that $(\widetilde{x}^n, \widetilde{y}^n) \in \widetilde{\beta}^i(\underline{z}^n, \overline{z}^n, \underline{y}^n, \overline{y}^n)$. Evidently, $\lim_{n\to\infty} \lambda^n = 1$, and $\lim_{n\to\infty}(\widetilde{x}^n, \widetilde{y}^n) = (\widetilde{x}, \widetilde{y})$. By the continuity of u^i, $\widetilde{x}^n$ is strictly preferred to the consumption bundle in $\widetilde{d}^i(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n)$, a contradiction. $\square$

Since there is no rationing in the market of the numeraire asset nor in the market of the numeraire commodities, the argument for equilibrium existence is not trivial.

Proposition 3.2 *If $\mathcal{E}$ satisfies A1-A4, then a fix-price equilibrium exists at all prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$.*

Proof If $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$ is a fix-price equilibrium of $\mathcal{E}$ at prices (p, q), then $x^{*i'}_{l,s} < \sum_{i=1}^{I} e^i_{l,s} + \varepsilon$, with ε some fixed positive number. Since R has full column rank, this implies that there is $\alpha > 0$ such that $\|y^{*i}\|_\infty < \alpha$ for any y^{*i} consistent with a fix-price equilibrium at prices (p, q).

The functions $(\underline{z}, \overline{z}) : \mathcal{C}^{LS} \to -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+$ and $(\underline{y}, \overline{y}) : \mathcal{C}^{A} \to -\mathbb{R}^{A}_+ \times \mathbb{R}^{A}_+$, where $\mathcal{C}^K = \{r \in \mathbb{R}^K : 0 \leq r_k \leq 1\}$ denotes the unit cube of dimension K, are defined by

$$\begin{aligned}
&\underline{z}_{l,s}(r) = -\min\{2r_{l,s}(\textstyle\sum_{i=1}^{I} e^i_{l,s} + \varepsilon), \sum_{i=1}^{I} e^i_{l,s} + \varepsilon\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},\\
&\overline{z}_{l,s}(r) = \min\{(2 - 2r_{l,s})(\textstyle\sum_{i=1}^{I} e^i_{l,s} + \varepsilon), \sum_{i=1}^{I} e^i_{l,s} + \varepsilon\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},\\
&\underline{y}_a(\rho) = -\min\{2\rho_a\alpha, \alpha\}, \quad a \in \check{\mathcal{A}},\\
&\overline{y}_a(\rho) = \min\{(2 - 2\rho_a)\alpha, \alpha\}, \quad a \in \check{\mathcal{A}}.
\end{aligned}$$

We define the excess demand function $\tilde{z} : \mathcal{C}^{LS} \times \mathcal{C}^{A} \to \mathbb{R}^{LS} \times \mathbb{R}^{A}$ by

$$\begin{aligned}
&\tilde{z}_{l,s}(r,\rho) = \textstyle\sum_{i=1}^{I} \tilde{d}^i_{l,s}(\underline{z}(r), \overline{z}(r), \underline{y}(\rho), \overline{y}(\rho)) - \sum_{i=1}^{I} e^i_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times S\\
&\tilde{z}_a(r,\rho) = \textstyle\sum_{i=1}^{I} \tilde{d}^i_a(\underline{z}(r), \overline{z}(r), \underline{y}(\rho), \overline{y}(\rho)), \quad a \in \check{\mathcal{A}}.
\end{aligned}$$

If $(r^*, \rho^*) \in \mathcal{C}^{LS} \times \mathcal{C}^{A}$ is such that $\tilde{z}(r^*, \rho^*) = 0$, then $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$, where $(x^{*i}, y^{*i}) = \tilde{d}^i(\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$, $i \in \mathcal{I}$, $(\underline{z}^*, \overline{z}^*) = (\underline{z}(r^*), \overline{z}(r^*))$, $(\underline{y}^*, \overline{y}^*) = (\underline{y}(r^*), \overline{y}(r^*))$, is a fix-price equilibrium. It is obvious that Conditions 1 and 2 of Definition 1 are satisfied for non-numeraire commodities and assets. Using the budget equalities gives Conditions 1 and 2 for numeraire commodities and assets. The construction of the functions $(\underline{z}, \overline{z})$ and $(\underline{y}, \overline{y})$ takes care of Conditions 3 and 4.

The set $\tilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^{A})$ is compact. Let the set $\mathcal{ZY}$ be a compact, convex set that contains $\tilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^{A})$. The correspondence $\mu : \mathcal{ZY} \to \mathcal{C}^{LS} \times \mathcal{C}^{A}$ is defined by

$$\mu(z, y) = \arg\max\{\textstyle\sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} r_{l,s} z_{l,s} + \sum_{a \in \check{\mathcal{A}}} \rho_a y_a : r \in \mathcal{C}^{LS}, \rho \in \mathcal{C}^{A}\}.$$

The correspondence $\varphi : \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^{A} \to \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^{A}$ is defined by $\varphi(z, y, r, \rho) = \{\tilde{z}(r, \rho)\} \times \mu(z, y)$. It is a non-empty, compact, convex valued, upper hemi-continuous correspondence defined on a non-empty, compact, convex set. By Kakutani's fixed point theorem, φ has a fixed point, say (z^*, y^*, r^*, ρ^*).

If, for some $a \in \check{\mathcal{A}}$, $y^*_a < 0$, then, by the definition of μ, $\rho^*_a = 0$, so $y^*_a \geq 0$, a contradiction. If, for some $a \in \check{\mathcal{A}}$, $y^*_a > 0$, then, by the definition of μ, $\rho^*_a = 1$, so $y^*_a \leq 0$, a contradiction. Consequently, $y^*_a = 0$, for all $a \in \check{\mathcal{A}}$. Moreover, $y^*_{A+1} = -\sum_{a \in \check{\mathcal{A}}} q_a y^*_a = 0$.

If, for some $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z^*_{l,s} < 0$, then, by the definition of μ, $r^*_{l,s} = 0$, so $z^*_{l,s} \geq 0$, a contradiction. If, for some $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z^*_{l,s} > 0$, then, by the definition of μ, $r^*_{l,s} = 1$, so $z^*_{l,s} \leq 0$, a contradiction. Consequently, $z^*_{l,s} = 0$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$. Moreover, for every $s \in \mathcal{S}$, $z^*_{L+1,s} = -\sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} p_{l,s} z^*_{l,s} + R_s . y^* = 0$.

It follows that $0 \in \tilde{z}(r^*, \rho^*)$, so a fix-price equilibrium at prices (p, q) exists. $\square$

4 Local Comparative Statics

The state of markets at a fix-price equilibrium can be described by a sign vector

$$r = (r_{1,1}, \ldots, r_{L,S}, r_1, \ldots, r_A).$$

If there is effective supply rationing in the market for a commodity or an asset, the associated component of the sign vector is -1, if there is effective demand rationing it is +1, and if there is no effective rationing it is 0.

For a sign vector r, the set $\mathcal{PQ}(r)$ is the set of prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, for which there exists a fix-price equilibrium at prices (p, q) with state of the markets r. For prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, the set of fix-price equilibrium allocations is $\mathcal{D}(p, q)$, and, for a sign vector r, the set of fix-price equilibrium allocations with state of the markets r is $\mathcal{D}(p, q, r)$. In the following, $\mathcal{N}_\alpha$ denotes a neighborhood of α.

Definition 4.1 (Local uniqueness) *Let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium of $\mathcal{E}$. The allocation (x^*, y^*) is locally unique as a fix-price equilibrium allocation if there exists a neighborhood $\overline{\mathcal{N}}_{x^*,y^*}$ such that for every $\mathcal{N}_{x^*,y^*} \subset \overline{\mathcal{N}}_{x^*,y^*}$ there exists a neighborhood $\mathcal{N}_{p^*,q^*}$ with $\mathcal{D}(p, q) \cap \mathcal{N}_{x^*,y^*}$ a singleton for every $(p, q) \in \mathcal{N}_{p^*,q^*}$.*

If a competitive equilibrium allocation is locally unique as a fix-price equilibrium allocation, then, for prices close to competitive equilibrium prices, there is exactly one fix-price equilibrium allocation close to the competitive allocation.

For a locally unique competitive equilibrium allocation, for each sign vector r, we define the function $(\widehat{x}^r, \widehat{y}^r) : \overline{\mathcal{N}}_{p^*,q^*} \cap \mathcal{PQ}(r) \to \mathbb{R}^{I(L+1)S+I(A+1)}$ by associating the unique fix-price equilibrium allocation in $\overline{\mathcal{N}}_{x^*,y^*} \cap \mathcal{D}(p, q, r)$ to (p, q).

Comparative statics require a differentiable form of local uniqueness.

Definition 4.2 (Differentiable local uniqueness) *Let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium of $\mathcal{E}$. The allocation (x^*, y^*) is differentiably locally unique as a fix-price equilibrium allocation if it is locally unique and there is a neighborhood $\mathcal{N}_{p^*,q^*}$ such that, for every sign vector r, the function $(\widehat{x}^r, \widehat{y}^r)_{|\mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r)}$ is differentiable* [2].

Laroque and Polemarchakis (1978) prove for a complete asset market that, generically, the set of fix-price equilibrium allocations can be represented by a finite number of continuously differentiable functions of prices. Nevertheless, the results in Laroque (1978) and the examples in Madden (1982) show that competitive equilibria need not be locally unique as fix-price equilibria. Even though fix-price equilibrium allocations exist for all prices, there may be robust local non-existence, and therefore local non-uniqueness as a fix-price equilibrium, at competitive prices. The equilibrium manifold has a particularly complicated structure at competitive prices. We analyse the local comparative statics of

[2] A function with domain a subset of Euclidean space which is not necessarily open is differentiable if it has a differentiable extension to an open neighborhood of its domain of definition.

fix-price equilibria in the neighborhood of a competitive price system. This analysis follows Laroque (1978, 1981) for economies with a complete asset market and leads to necessary and sufficient conditions for differentiable local uniqueness.

Consider the optimization problem an individual faces when determining his demand. The Lagrange multipliers corresponding to the rationing constraints in the markets for commodities (assets) are denoted π (ρ). The individual optimization problem leads us to study a modified demand function, $\widehat{d}^i$. At prices and Lagrange multipliers (p, q, π, ρ), $\widehat{d}^i$ is defined by the solution to the optimization problem

$$\begin{array}{ll} \max & u^i(x) - \sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} \pi_{l,s} x_{l,s} - \sum_{a\in\check{\mathcal{A}}} \rho_a y_a, \\ & \\ \text{s.t.} & qy \leq 0, \\ & p_s(x_s - e^i_s) \leq R_s.y, \quad s \in \mathcal{S}. \end{array}$$

The set of (p, q, π, ρ) on which each individual optimization problem has a solution is denoted $\mathcal{N}$. It is easily verified that $\mathcal{N}$ is a neighborhood of $(p^*, q^*, 0, 0)$, whenever (p^*, q^*) are competitive equilibrium prices.

Lemma 4.3 *If $\mathcal{E}$ satisfies A1-A4, then $\widehat{d}^i$, $i \in \mathcal{I}$, is continuously differentiable on $\mathcal{N}$.*

Proof It follows from a standard application of the implicit function theorem. □

At a competitive equilibrium $((p^*, q^*), (x^*, y^*))$, $z^-_{l,s}$, $z^+_{l,s}$, y^-_a and y^+_a, defined by

$$\begin{array}{lll} z^-_{l,s} = \min_{i\in\mathcal{I}} x^{i*}_{l,s} - e^i_{l,s}, & z^+_{l,s} = \max_{i\in\mathcal{I}} x^{i*}_{l,s} - e^i_{l,s}, & (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ y^-_a = \min_{i\in\mathcal{I}} y^{i*}_a, & y^+_a = \max_{i\in\mathcal{I}} y^{i*}_a, & a \in \check{\mathcal{A}}, \end{array}$$

determine the minimal and the maximal excess demands on both the spot and the asset markets. If

$$\begin{array}{lll} \underline{\mathcal{I}}_{l,s} = \{i \in \mathcal{I} : x^{i*}_{l,s} - e^i_{l,s} = z^-_{l,s}\}, & \overline{\mathcal{I}}_{l,s} = \{i \in \mathcal{I} : x^{i*}_{l,s} - e^i_{l,s} = z^+_{l,s}\}, & (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ \underline{\mathcal{I}}_a = \{i \in \mathcal{I} : y^{i*}_a = y^-_a\}, & \overline{\mathcal{I}}_a = \{i \in \mathcal{I} : y^{i*}_a = y^+_a\}, & a \in \check{\mathcal{A}}, \end{array}$$

then in a neighborhood of the competitive equilibrium, only individuals in $\underline{\mathcal{I}}_{l,s}$ ($\overline{\mathcal{I}}_{l,s}$) may be rationed on supply (demand) in the spot market (l, s), and only individuals in $\underline{\mathcal{I}}_a$ ($\overline{\mathcal{I}}_a$) on supply (demand) in the asset market a.

Lemma 4.4 *Let $((\mathcal{X}^i, u^i)_{i\in\mathcal{I}}, R)$ satisfy A1, A2 and A4. For an open set of endowments with full Lebesgue measure $\Omega \subset \mathbb{R}^{I(L+1)S}_{++}$, for any competitive equilibrium $((p^*, q^*), (x^*, y^*))$ of $\mathcal{E}$, $\mid \underline{\mathcal{I}}_{l,s} \mid = \mid \overline{\mathcal{I}}_{l,s} \mid = 1$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $\mid \underline{\mathcal{I}}_a \mid = \mid \overline{\mathcal{I}}_a \mid = 1$, $a \in \check{\mathcal{A}}$.*

Proof It follows from a standard transversality argument. □

There is a generic set of economies for which there is exactly one individual in each market with the minimal excess demand and exactly one individual with the maximal excess demand. For the remainder of this section, we consider an economy $\mathcal{E}$ with endowments in the set Ω and study the local structure of the set fix-price equilibria in the neighborhood of a competitive equilibrium $((p^*, q^*), (x^*, y^*))$ of $\mathcal{E}$.

For every individual, the function $c^i : \mathbb{R}^{LS} \times \mathbb{R}^A \to \mathbb{R}^{LS} \times \mathbb{R}^A$ is defined by

$$c^i_{l,s}(\pi, \rho) = \begin{cases} \pi_{l,s}, & \text{if } \pi_{l,s} \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_{l,s}, \text{ or } \pi_{l,s} \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_{l,s}, \\ 0, & \text{otherwise,} \end{cases}$$
$$c^i_a(\pi, \rho) = \begin{cases} \rho_a, & \text{if } \rho_a \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_a, \text{ or } \rho_a \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_a, \\ 0, & \text{otherwise.} \end{cases}$$

The function c relates the Lagrange multipliers (π, ρ) to the fix-price equilibria in the neighborhood of the competitive equilibrium. The aggregate modified excess demand function for commodities and assets other than the numeraire is $\widehat{z} : \mathcal{N} \to \mathbb{R}^{LS+A}$ defined by

$$\widehat{z}_{l,s}(p, q, \pi, \rho) = \textstyle\sum_{i \in \mathcal{I}} \widehat{d}^i_{l,s}(p, q, c^i(\pi, \rho)) - \sum_{i \in \mathcal{I}} e^i, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$
$$\widehat{z}_a(p, q, \pi, \rho) = \textstyle\sum_{i \in \mathcal{I}} \widehat{d}^i_a(p, q, c^i(\pi, \rho)), \quad a \in \check{\mathcal{A}}.$$

It is sufficient to restrict attention to the zero points of $\widehat{z}$ to analyze fix-price equilibria in the neighborhood of the competitive equilibrium. Choose neighborhoods $\mathcal{N}^i_{x^{i*}, y^{i*}}$ such that for every $(x, y) \in \mathcal{N}_{x^*, y^*} = \times_{i \in \mathcal{I}} \mathcal{N}^i_{x^{i*}, y^{i*}}$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, for all $a \in \check{\mathcal{A}}$,

$$x^{i'}_{l,s} - e^{i'}_{l,s} < 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} < x^i_{l,s} - e^i_{l,s}, \quad i \neq i', i' \in \underline{\mathcal{I}}_{l,s}$$
$$x^{i'}_{l,s} - e^{i'}_{l,s} > 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} > x^i_{l,s} - e^i_{l,s}, \quad i \neq i', i' \in \overline{\mathcal{I}}_{l,s}$$
$$y^{i'}_a < 0 \text{ and } y^{i'}_a < y^i_a, \quad i \neq i', i' \in \underline{\mathcal{I}}_a$$
$$y^{i'}_a > 0 \text{ and } y^{i'}_a > y^i_a, \quad i \neq i', i' \in \overline{\mathcal{I}}_a.$$

Lemma 4.5 *Let $\mathcal{E}$ satisfy A1-A4 with endowments in Ω, and let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium. Consider some $(x, y) \in \mathcal{N}_{x^*, y^*}$. Then $(x, y) \in \mathcal{D}(p, q)$ if and only if there is $(p, q, \pi, \rho) \in \mathcal{N}$ such that $\widehat{d}^i(p, q, c^i(\pi, \rho)) = (x^i, y^i)$, $i \in \mathcal{I}$, and $\widehat{z}(p, q, \pi, \rho) = (0, 0)$.*

Proof It follows from the first order conditions for a fix-price equilibrium and the first order conditions for the solution to the individual optimization problems leading to $\widehat{d}^i$. $\square$

The function $\widehat{z}$ is Lipschitz continuous because of the differentiability of the functions $\widehat{d}^i$ and the Lipschitz continuity of the functions c^i. It is differentiable at each $(p, q, \pi, \rho) \in \mathcal{N}$ where all components of π and ρ are non-zero. For each sign vector r without zero components, we define

$$\mathcal{N}^r = \{(p, q, \pi, \rho) \in \mathcal{N} : \pi_{l,s} r_{l,s} > 0, (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \ \rho_a r_a > 0, a \in \check{\mathcal{A}}\}.$$

The function $\widehat{z}$ is differentiable on $\mathcal{N}^r$. The limit of its Jacobian, $\lim_{n \to \infty} \partial \widehat{z}(p^n, q^n, \pi^n, \rho^n)$, along a sequence $((p^n, q^n, \pi^n, \rho^n) \in \mathcal{N}^r : n = 1, \ldots)$ that converges to $(p^*, q^*, 0, 0)$ exists and is denoted $\partial \widehat{z}^r(p^*, q^*, 0, 0)$. It holds that

$$\partial_{\check{p}, \check{q}} \widehat{z}^r_{l,s}(p^*, q^*, 0, 0) = \textstyle\sum_{i \in \mathcal{I}} \partial_{\check{p}, \check{q}} \widehat{d}^i_{l,s}(p^*, q^*, 0, 0) = \partial_{\check{p}, \check{q}} z_{l,s}(p^*, q^*),$$
$$\partial_{\check{p}, \check{q}} \widehat{z}^r_a(p^*, q^*, 0, 0) = \textstyle\sum_{i \in \mathcal{I}} \partial_{\check{p}, \check{q}} \widehat{d}^i_a(p^*, q^*, 0, 0) = \partial_{\check{p}, \check{q}} z_a(p^*, q^*),$$

where $z(p, q)$ denotes the unconstrained total excess demand function for commodities and assets other than the numeraires at prices (p, q). It follows that the Jacobian with respect to $(\check{p}, \check{q})$ is independent of r at a competitive equilibrium.

Proposition 4.6 *Let $\mathcal{E}$ satisfy A1-A4 with endowments in Ω, and let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium such that $\partial z(p^*, q^*)$ is of full rank. For each sign vector r without zero components, the tangent cone at (p^*, q^*) to the set of price systems having a local fix-price equilibrium with state of the markets r is*

$$\{(p, q) \in \mathcal{P} \times \mathcal{Q} : \ (\check{p}, \check{q}) = (\partial z(p^*, q^*))^{-1} \partial_{\pi,\rho} \widehat{z}^r(p^*, q^*, 0, 0)(\pi, \rho),$$
$$\pi_{l,s} r_{l,s} > 0, (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \ \rho_a r_a > 0, a \in \check{\mathcal{A}}\}.$$

Proof The restriction of $\widehat{z}$ to $\mathcal{N}^r$ extends to a differentiable function $\widetilde{z} : \mathcal{N} \to \mathbb{R}^{LS+A}$ as follows. For $i \in \mathcal{I}$, the function $\widetilde{c}^i$ is defined by $\widetilde{c}^i_{l,s}(\pi, \rho) = \pi_{l,s}$ if $i \in \underline{\mathcal{I}}_{l,s}$, $r_{l,s} = -1$, or $i \in \overline{\mathcal{I}}_{l,s}$, $r_{l,s} = +1$, $\widetilde{c}^i_{l,s}(\pi, \rho) = 0$ otherwise, and $\widetilde{c}^i_a(\pi, \rho) = \rho_a$ if $i \in \underline{\mathcal{I}}_a$, $r_a = -1$, or $i \in \overline{\mathcal{I}}_a$, $r_a = +1$, and $\widetilde{c}^i_a(\pi, \rho) = 0$ otherwise. The function $\widetilde{z}$ is defined as $\widehat{z}$ with c replaced by $\widetilde{c}$. Since $\partial z(p^*, q^*)$ is of full rank, it follows by the implicit function theorem that the solution to $\widetilde{z}(p, q, \pi, \rho) = (0, 0)$ determines p and q as a function of π and ρ in a neighborhood of $(0, 0)$. The derivative of this function at $(0, 0)$ with respect to π and ρ is given by $(\partial z(p^*, q^*))^{-1} \partial_{\pi,\rho} \widetilde{z}(p^*, q^*, 0, 0)$. The expression in the proposition follows immediately if one takes into account that only π's and ρ's satisfying $\pi_{l,s} r_{l,s} > 0$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $\rho_a r_a > 0$, $a \in \check{\mathcal{A}}$, should be considered. $\square$

Proposition 2 in Geanakoplos and Polemarchakis (1986) shows that the assumption that $\partial z(p^*, q^*)$ has full rank at every competitive equilibrium holds generically in initial endowments. Proposition 4.6 characterizes the tangent cones to the regions in the price space having a fix-price equilibrium with state of the markets r in the neighborhood of a competitive equilibrium. It guarantees neither that the closures of these tangent cones cover the price space nor that the tangent cones are full-dimensional nor that the tangent cones do not intersect. If this were the case, local uniqueness would result.

In general, an increase in a price causes a different individual to be rationed as a decrease in a price. Since $\partial_{\pi,\rho} \widehat{z}^r$, and therefore the tangent cone, depends on $\partial_{\pi,\rho} \widehat{d}^i$ for the individual i that is rationed, the fact that the tangent cones need not fit nicely together does not come as a surprice. In abstract terms, the fact that different individuals get rationed at different prices in the neighborhood of a competitive equilibrium, creates non-differentiabilities in the function $\widehat{z}$ at competitive prices. At a point of non-differentiability, the implicit function theorem need not apply, and local uniqueness may fail.

The generalized Jacobian of a Lipschitz continuous function f at a point x is the convex hull of all matrices that are the limits of the sequence $(\partial f(x^n) : n = 1, \ldots)$, where $(x^n : n = 1, \ldots)$ is a convergent sequence with $\lim_{n \to \infty} x^n = x$ and f is differentiable at $x^n, n = 1, \ldots$.

If a function f is Lipschitz continuous, $f(\widehat{x}, \widehat{y}) = 0$, and every matrix M in $\partial_x f(\widehat{x}, \widehat{y})$ has full rank, then there exist a neighborhood $\mathcal{N}_{\widehat{x}, \widehat{y}}$, a neighborhood $\mathcal{N}_{\widehat{y}}$, and a Lipschitz continuous function g on $\mathcal{N}_{\widehat{y}}$ such that $(x, y) \in \mathcal{N}_{\widehat{x}, \widehat{y}}$ and $f(x, y) = 0$ if and only if $y \in \mathcal{N}_{\widehat{y}}$ and $x = g(y)$.

Proposition 4.7 *Let $\mathcal{E}$ satisfy A1-A4 with endowments in Ω, and let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium. If the determinants of the matrices $\partial_{\pi,\rho}\widehat{z}^r(p^*, q^*, 0, 0)$, with r sign vectors without zero components, are either all equal to -1 or all equal to $+1$, then the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.*

Proof The argument is similar to the one in the proof of Theorem 1, Laroque (1981). □

There are utility functions and asset return matrices such that the set of endowments, for which all determinants in Proposition 4.7 have the same sign, has full Lebesgue measure. Consider an economy with an arbitrary number of individuals, three states of the world, two commodities and two assets. The utility functions have an additively separable representation $u^i = \sum_{s \in \mathcal{S}} \pi_s u_s^i$ with

$$u_s^i(x_s) = \alpha^i \ln x_{1,s} + (1 - \alpha^i) x_{2,s}, \quad 0 < \alpha^i < 1,$$

and a uniform probability measure π over the states of the world. The payoffs of the assets are $R_{\cdot 1} = (1, 0, 0)'$, and $R_{\cdot 2} = (0, 1, 0)'$. Endowments are chosen such that $|\, \underline{\mathcal{I}}_{l,s} \,|=|\, \overline{\mathcal{I}}_{l,s} \,|= 1$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $|\, \underline{\mathcal{I}}_a \,|=|\, \overline{\mathcal{I}}_a \,|= 1$, $a \in \check{\mathcal{A}}$, so they belong to a set of full Lebesgue measure by Lemma 4.4.

Let (p^*, q^*) be competitive equilibrium prices in this economy. All partial derivatives are evaluated at $(p^*, q^*, 0, 0)$. It holds that $\partial_{\pi_{1,s}}\widehat{z}^r = \partial_{\pi_{1,s}}\widehat{d}^{i(1,s)}$, where $\{i(1, s)\} = \underline{I}_{1,s}$ if $r_{1,s} = -1$, and $\{i(1, s)\} = \overline{I}_{1,s}$ if $r_{1,s} = +1$. An increase in $\pi_{1,s}$ corresponds to the introduction of demand rationing or the disappearance of supply rationing on commodity $(1, s)$, which decreases the demand for commodity $(1, s)$, so $\partial_{\pi_{1,s}}\widehat{z}^r_{1,s}$ is negative. The change in income spent on commodity $(1, s)$ equals $p^*_{1,s}\partial_{\pi_{1,s}}\widehat{z}^r_{1,s}$. The individual $i(1)$ is the one affected by rationing in the asset market, so $\{i(1)\} = \underline{I}_1$ if $r_1 = -1$, and $\{i(1)\} = \overline{I}_1$ if $r_1 = +1$. Using the properties of the Cobb-Douglas utility function it follows that

$$\begin{array}{lll}
\partial_{\pi_{1,1}}\widehat{d}^{i(1,1)}_{1,2} = \frac{-\alpha_1^{i(1,1)} p^*_{1,1} q^*_1 \partial_{\pi_{1,1}}\widehat{z}^r_{1,1}}{p^*_{1,2} q^*_2 (2-\alpha_1^{i(1,1)})}, & \partial_{\pi_{1,1}}\widehat{d}^{i(1,1)}_{1,3} = 0, & \partial_{\pi_{1,1}}\widehat{d}^{i(1,1)}_{1} = \frac{p^*_{1,1}\partial_{\pi_{1,1}}\widehat{z}^r_{1,1}}{(2-\alpha_1^{i(1,1)})}, \\
\partial_{\pi_{1,2}}\widehat{d}^{i(1,2)}_{1,1} = \frac{-\alpha_1^{i(1,2)} p^*_{1,2} q^*_1 \partial_{\pi_{1,2}}\widehat{z}^r_{1,2}}{p^*_{1,1} q^*_2 (2-\alpha_1^{i(1,2)})}, & \partial_{\pi_{1,2}}\widehat{d}^{i(1,2)}_{1,3} = 0, & \partial_{\pi_{1,2}}\widehat{d}^{i(1,2)}_{1} = \frac{-p^*_{1,2} q^*_2 \partial_{\pi_{1,2}}\widehat{z}^r_{1,2}}{q^*_1 (2-\alpha_1^{i(1,2)})}, \\
\partial_{\pi_{1,3}}\widehat{d}^{i(1,3)}_{1,1} = 0, & \partial_{\pi_{1,3}}\widehat{d}^{i(1,3)}_{1,2} = 0, & \partial_{\pi_{1,3}}\widehat{d}^{i(1,3)}_{1} = 0, \\
\partial_{\rho_1}\widehat{d}^{i(1)}_{1,1} = \frac{\alpha_1^{i(1)}\partial_{\rho_1}\widehat{z}^r_1}{p^*_{1,1}}, & \partial_{\rho_1}\widehat{d}^{i(1)}_{1,2} = \frac{-\alpha_1^{i(1)} q^*_1 \partial_{\rho_1}\widehat{z}^r_1}{p^*_{1,2} q^*_2}, & \partial_{\rho_1}\widehat{d}^{i(1)}_{1,3} = 0.
\end{array}$$

The sign of the determinant of $\partial_{\pi,\rho}\widehat{z}^r$ does not change by premultiplying it by the strictly positive row vector $(p^*_{1,1}q^*_1, p^*_{1,2}q^*_2, 1, q^*_1)$ and postmultiplying it by the strictly positive column vector $((2-\alpha_1^{i(1,1)})/-p^*_{1,1}q^*_1\partial_{\pi_{1,1}}\widehat{z}^r_{1,1}, (2-\alpha_1^{i(1,2)})/-p^*_{1,2}q^*_2\partial_{\pi_{1,2}}\widehat{z}^r_{1,2}, 1/-\widehat{z}^r_{1,3}, 1/-q^*_1\partial_{\rho_1}\widehat{z}^r_1)'$. The resulting matrix is given by

$$\begin{bmatrix}
\alpha_1^{i(1,1)} - 2 & \alpha_1^{i(1,2)} & 0 & -\alpha_1^{i(1)} \\
\alpha_1^{i(1,1)} & \alpha_2^{i(1,2)} - 2 & 0 & \alpha_1^{i(1)} \\
0 & 0 & -1 & 0 \\
-1 & 1 & 0 & -1
\end{bmatrix}$$

and its determinant equals

$$(4 - 2\alpha_1^{i(1,1)} - 2\alpha_1^{i(1,2)})(1 - \alpha_1^{i(1)}) > 0.$$

The determinant of $\partial_{\pi,\rho}\widehat{z}^r$ is positive, irrespective of the sign vector r. It follows by Proposition 4.7 that the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.

As in Laroque (1981), whenever there are two sign vectors without zero components r^1 and r^2 such that the determinants of $\partial_{\pi,\rho}\widehat{z}^{r^1}(p^*, q^*, 0, 0)$ and $\partial_{\pi,\rho}\widehat{z}^{r^2}(p^*, q^*, 0, 0)$ have opposite signs and $\partial z(p^*, q^*)$ has full rank, then for every neighborhood $\mathcal{N}_{x^*,y^*}$ there exists for every neighborhood $\mathcal{N}_{p^*,q^*}$ a price system $(p, q) \in \mathcal{N}_{p^*,q^*}$ with at least two fix-price equilibrium allocations in $\mathcal{N}_{x^*,y^*}$. The conditions in Proposition 4.7 are almost necessary.

Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement. It is less demanding than the requirement of uniqueness of fix-price equilibrium allocations at prices in a neighborhood of competitive prices. It is an open question whether the interior of the set of endowments for which all competitive equilibrium allocations of the economy are differentiably locally unique as fix-price equilibrium allocations can be empty. The set of initial endowments for which the differentiable local uniqueness property holds, is denoted Ω^*. In the sequel we restrict attention to endowments in Ω^*.

The function $(\widehat{x}, \widehat{y}) : \mathcal{N}_{p^*,q^*} \to \mathbb{R}^{I(L+1)S+I(A+1)}$ associates the unique fix-price equilibrium allocation in $\mathcal{N}_{x^*,y^*}$ to $(p, q) \in \mathcal{N}_{p^*,q^*}$. The indirect utility function of an individual at a locally unique fix-price equilibrium is defined by

$$v^i(p, q) = u^i(\widehat{x}^i(p, q)), \quad (p, q) \in \mathcal{N}_{p^*,q^*}.$$

Proposition 4.8 *Let $\mathcal{E}$ satisfy A1-A4 with endowments in Ω^*, and let $((p^*, q^*), (x^*, y^*))$ be a competitive equilibrium. The indirect utility function $v^i : \mathcal{N}_{p^*,q^*} \to \mathbb{R}$ is differentiable and*

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i(x^{i*})(x^{i*}_{l,s} - e^i_{l,s}), \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Proof For every sign vector r, the restriction of v^i to $\mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r)$, denoted v^{i^r}, is differentiable. From the differentiation of the budget constraints

$$q\widehat{y}^{i^r}(p, q) = 0 \text{ and } p_s(\widehat{x}^{i^r}_s(p, q) - e^i_s) = R_{s\cdot}\widehat{y}^{i^r}(p, q), \quad s \in \mathcal{S},$$

with respect to $p_{\overline{l},\overline{s}}$, and the first order conditions for individual optimization at a competitive equilibrium,

$$\partial_{x^i_{l,s}} u^i(x^{i*}) = \partial_{x^i_{L+1,s}} u^i(x^{i*})p^*_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$
$$\sum_{s\in\mathcal{S}} \partial_{x^i_{L+1,s}} u^i(x^{i*})R_{s\cdot} = \mu^i q^*, \quad \text{for some } \mu^i > 0,$$

it follows that

$$\partial_{p_{\overline{l},\overline{s}}} v^{i^r}(p^*, q^*) = -\partial_{x^i_{L+1,\overline{s}}} u^i(x^{i*})(x^{i*}_{\overline{l},\overline{s}} - e^i_{\overline{l},\overline{s}}).$$

Since the derivative is independent of the sign vector r, the result follows. □

The effect of a change in the spot market price of commodity $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ is equal to minus the marginal utility of the numeraire commodity in state s multiplied by the excess demand of commodity (l, s) at the competitive equilibrium. Proposition 4.8 implies that the indirect welfare effects of a change in prices, generated by the induced change in the rationing constraints and agents' choices, equal zero.

5 Pareto Improving Price Regulation

Price regulation can Pareto improve on a competitive equilibrium $((p^*, q^*), (x^*, y^*))$ if there exist prices of commodities p such that a fix-price equilibrium of commodities at prices of commodities and assets (p, q^*) Pareto dominates the allocation x^*. The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations at prices (p, q^*) is circumvented by considering local variations at competitive equilibrium allocations that are differentiably locally unique as fix-price equilibria.

Definition 5.1 (Pareto improving price regulation) *A competitive equilibrium* $((p^*, q^*), (x^*, y^*))$ *can be Pareto improved by price regulation if it is differentiably locally unique as fix-price equilibrium and there exists an infinitesimal variation in commodity prices* $d\check{p}$ *such that* $\sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} \partial_{p_{l,s}} v^i(p^*, q^*) dp_{l,s} > 0,\ i \in \mathcal{I}$.

The competitive equilibrium can be Pareto improved by uniform price regulation if it can be Pareto improved by a price regulation with $d\check{p}_s = d\check{p}_{s'},\ s, s' \in \mathcal{S}$.

Pareto improvement by price regulation is possible only if the asset market is incomplete. Another necessary requirement is that the economy allows for heterogeneous individuals. This is summarized in the following assumption.

A5. $A + 1 < S$ and $I > 1$.

The function φ is defined by

$$\varphi(x, \widetilde{\lambda}, \widetilde{p}, e) = \begin{pmatrix} \partial u^i(x^i) - \widetilde{\lambda}^i \widetilde{p}, \quad i \in \mathcal{I} \\ \sum_{s\in\mathcal{S}} \widetilde{p}_s(x^i_s - e^i_s), \quad i \in \mathcal{I} \\ \sum_{i\in\mathcal{I}} (x^i_{l,s} - e^i_{l,s}), \quad (l, s) \in \mathcal{L} \times \mathcal{S} \setminus \{(L+1, S)\} \\ \sum_{s\in\mathcal{S}} n_s \widetilde{p}_s(x^i_s - e^i_s), \quad i \in \mathcal{I} \setminus \{1\} \end{pmatrix},$$

where the Lagrangian multiplier $\widetilde{\lambda}^i \in \mathbb{R}$ does not vary with the state of the world, the prices of commodities $\widetilde{p} \in \mathbb{R}^{(L+1)S-1}_{++} \times \{1\}$ are discounted prices, with only the price of commodity $(L + 1, S)$ normalized to 1, and $n \neq 0$ is a fixed vector such that $nR = 0$. Consider the standard reformulation of the incomplete markets model in discounted prices that utilizes the Cass trick. The first individual is assumed to be unconstrained, so his marginal utility at an optimal choice is proportional to the price system. Pareto optimality implies that the marginal utility vectors of all agents should be proportional to the price system. The function φ is completed by specifying budget constraints and market clearing conditions, and one condition for every individual but the first that recognizes the incompleteness of markets: $\sum_{s \in \mathcal{S}} n_s \tilde{p}_s (x^i_s - e^i_s) = 0$. The existence of $n \neq 0$ such that $nR = 0$ follows from market incompleteness. It follows that the function φ vanishes at a Pareto optimal competitive equilibrium.

We use the following as a general notation. For a function f that depends on a vector of variables α and on endowments e, $f_e(\alpha)$ denotes the function that results from fixing e. For instance, $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = \varphi(x, \tilde{\lambda}, \tilde{p}, e)$.

Lemma 5.2 *Let $\mathcal{E}$ satisfy A1, A2, A4 and A5. For an open set of endowments with full Lebesgue measure in $\mathbb{R}^{I(L+1)S}_{++}$, competitive equilibrium allocations are not Pareto optimal.*

Proof A necessary condition for x to be a Pareto optimal competitive equilibrium allocation for an economy e is that $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = 0$. Since the dimension of the domain of φ_e is lower than the dimension of the range, whenever φ_e is transverse to 0, a solution to $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = 0$ does not exist. By a standard argument, φ is transverse to 0. By the transversal density theorem, the set of economies for which φ_e is transverse to 0 has full Lebesgue measure. By a standard argument, this set can be chosen to be open. □

The function $\psi : \Xi \times \Omega^* \to \mathbb{R}^N$ is defined by

$$\psi(\xi, e) = \begin{pmatrix} \partial_{x^i_s} u^i(x^i) - \lambda^i_s p_s, \quad i \in \mathcal{I}, s \in \mathcal{S} \\ p_s(x^i_s - e^i_s) - R_{s\cdot} y^i, \quad i \in \mathcal{I}, s \in \mathcal{S} \\ \lambda^i R - \mu^i q, \quad i \in \mathcal{I} \\ \sum_{i \in \mathcal{I}} (x^i_{l,s} - e^i_{l,s}), \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \sum_{i \in \mathcal{I}} y^i_a, \quad a \in \check{\mathcal{A}} \\ q y^i, \quad i \in \mathcal{I} \end{pmatrix},$$

where $\xi = (x, \lambda, y, \mu, \check{p}, \check{q})$ and $\Xi = \mathbb{R}^{I(L+1)S}_{++} \times \mathbb{R}^{IS}_{++} \times \mathbb{R}^{I(A+1)} \times \mathbb{R}^I \times \check{P} \times \check{Q}$. The dimension of Ξ is denoted by N. When ξ^* is consistent with a competitive equilibrium, it is necessarily the case that $\psi_e(\xi^*) = 0$.

The function $h : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{LS+1}$ is defined by

$$h(\xi, \alpha, e) = \begin{pmatrix} \sum_{i \in \mathcal{I}} \alpha^i \lambda^i_s (x^i_{l,s} - e^i_{l,s}), \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix}.$$

A competitive equilibrium can be Pareto improved by price regulation if the matrix of partial derivatives of the indirect utility functions with respect to prices has full rank[3].

[3] If the matrix of partial derivatives has full rank, it is possible to generate any desired marginal change in utilities by means of price regulation.

By Proposition 4.8, this matrix is guaranteed to have full rank if there is no solution to $\psi_e(\xi) = 0$ in combination with $h_e(\xi, \alpha) = 0$.

The function $\widetilde{\psi} : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{N+LS+1}$ is defined by

$$\widetilde{\psi}(\xi, \alpha, e) = \begin{pmatrix} \psi(\xi, e) \\ h(\xi, \alpha, e) \end{pmatrix}.$$

If $\widetilde{\psi}$ is transverse to 0, then it follows from the transversal density theorem that for a subset of endowments of full Lebesgue measure, $\widetilde{\psi}_e$ is transverse to 0. If $LS \geq I$, then the dimension of the range of $\widetilde{\psi}_e$ exceeds that of the domain. Transversality of $\widetilde{\psi}_e$ implies that there are no solutions to the associated system of equations. It is possible to Pareto improve all competitive equilibria by price regulation.

Proposition 5.3 *Let $\mathcal{E}$ satisfy A1, A2, A4 and A5. If $LS \geq I$, then for an open subset of endowments in Ω^* with full Lebesgue measure, all competitive equilibria of $\mathcal{E}$ can be Pareto improved by price regulation.*

Proof One fixes $(\bar{l}, \bar{s}) \in \check{\mathcal{L}} \times \mathcal{S}$ and Ω^{**}, an open subset of endowments in Ω^* of full Lebesgue measure, such that no competitive equilibrium of the associated economy $\mathcal{E}$ is Pareto optimal. The function $\widehat{\psi} : \Xi \times \Omega^{**} \to \mathbb{R}^{N+1}$ is defined by

$$\widehat{\psi}(\xi, e) = \begin{pmatrix} \psi(\xi, e) \\ \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}}(x^i_{\bar{l},s} - e^i_{\bar{l},s}) \end{pmatrix}.$$

We show that if $\widehat{\psi}(\xi, e) = 0$, then the matrix $\widehat{M}$ of partial derivatives of $\widehat{\psi}$ evaluated at (ξ, e) has full row rank: if $v'\widehat{M} = 0$, then $v = 0$. The components of v are denoted $v_{1,i,l,s}$, $i \in \mathcal{I}$, $(l, s) \in \mathcal{L} \times \mathcal{S}$, $v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$, $v_{3,i,a}$, $i \in \mathcal{I}$, $a \in \mathcal{A}$, $v_{4,l,s}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $v_{5,a}$, $a \in \check{\mathcal{A}}$, $v_{6,i}$, $i \in \mathcal{I}$, and v_9, according to the labelling of the equations defining $\widehat{\psi}$.

If v is such that $v'\widehat{M} = 0$, then $0 = v'\partial_{e^i_{L+1,s}} \widehat{\psi}(\xi, e) = -v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. It follows that, for $i \in \mathcal{I}$,

$$\begin{aligned} &0 = v'\partial_{e^i_{l,s}} \widehat{\psi}(\xi, e) = -v_{4,l,s}, \quad (l, s) \in (\check{\mathcal{L}} \setminus \{\bar{l}\}) \times \mathcal{S}, \\ &0 = v'\partial_{e^i_{\bar{l},s}} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},s} - v_9 \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} = 0, \quad s \in \mathcal{S} \setminus \{\bar{s}\}, \\ &0 = v'\partial_{e^i_{\bar{l},\bar{s}}} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},\bar{s}}. \end{aligned}$$

Consequently, if $v_{4,\bar{l},\hat{s}} = 0$ for some $\hat{s} \in \mathcal{S} \setminus \{\bar{s}\}$, then $v_9 = 0$ and $v_{4,\bar{l},s} = 0$, for all $s \in \mathcal{S} \setminus \{\bar{s}\}$. If, on the contrary, $v_{4,\bar{l},s} \neq 0$, for all $s \in \mathcal{S} \setminus \{\hat{s}\}$, then

$$\frac{\lambda^i_s}{\lambda^i_{\bar{s}}} = -\frac{v_{4,\bar{l},s}}{v_9} = \frac{\lambda^{i'}_s}{\lambda^{i'}_{\bar{s}}}, \quad i, i' \in \mathcal{I}, \ s \in \mathcal{S} \setminus \{\bar{s}\}.$$

Hence, for $i, i' \in \mathcal{I}$, for $s^1, s^2 \in \mathcal{S}$, $\lambda^i_{s^1}/\lambda^i_{s^2} = (\lambda^i_{s^1}/\lambda^i_{\bar{s}})(\lambda^i_{\bar{s}}/\lambda^i_{s^2}) = (\lambda^{i'}_{s^1}/\lambda^{i'}_{\bar{s}})(\lambda^{i'}_{\bar{s}}/\lambda^{i'}_{s^2}) = \lambda^{i'}_{s^1}/\lambda^{i'}_{s^2}$. The economy e has then a Pareto optimal competitive equilibrium induced by ξ, contradicting $e \in \Omega^{**}$. Consequently, $v_{4,\bar{l},s} = 0$, $s \in \mathcal{S} \setminus \{\bar{s}\}$, and $v_9 = 0$.

For $i \in \mathcal{I}$, for $(l,s) \in \mathcal{L} \times \mathcal{S}$,

$$0 = v' \partial_{x^i_{l,s}} \widehat{\psi}(\xi, e) = v'_{1,i,\cdot,\cdot} \partial_{x^i_{l,s}} \partial u^i(x^i).$$

It is possible to represent a utility function satisfying A2 by one with $\partial^2 u^i(x^i)$ negative definite on a bounded subset of the consumption set. Then it follows that $v_{1,i,\cdot,\cdot} = 0$. For $i \in \mathcal{I}$, $0 = v' \partial_{y^i_{A+1}} \widehat{\psi}(\xi, e) = v_{8,i}$. Also, for $a \in \check{\mathcal{A}}$, $0 = v' \partial_{y^i_a} \widehat{\psi}(\xi, e) = v_{5,a}$. Finally, $0 = v' \partial_{\lambda^i_s} \widehat{\psi}(\xi, e) = v'_{3,i,\cdot} R'_{s\cdot}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. Since R has full column rank it follows that $v_{3,i,a} = 0$, $i \in \mathcal{I}, a \in \mathcal{A}$.

Therefore, $v = 0$, $\widehat{M}$ has full row rank $N+1$, and $\widehat{\psi}$ is transverse to 0. The set of endowments such that $\widehat{\psi}_e$ is transverse to zero is denoted $\widehat{\Omega}_{\bar{l},\bar{s}}$. By the transversal density proposition, $\Omega^{**} \setminus \widehat{\Omega}_{\bar{l},\bar{s}}$ has Lebesgue measure zero. For $e \in \widehat{\Omega}_{\bar{l},\bar{s}}$, the dimension of the range of $\widehat{\psi}_e$ exceeds that of the domain, so $(\widehat{\psi}_e)^{-1}(\{0\}) = \emptyset$.

The set $\widehat{\Omega} = \cap_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} \widehat{\Omega}_{l,s}$ is of full Lebesgue measure and, by a standard argument, open. Redefine the function $\widetilde{\psi}$ such that endowments belong to $\Omega^* \cap \widehat{\Omega}$. For (ξ, α, e) such that $\widetilde{\psi}(\xi, \alpha, e) = 0$, $\widetilde{M}$ is the matrix of partial derivatives of $\widetilde{\psi}$ evaluated at (ξ, α, e).

Let v be such that $v' \widetilde{M} = 0$. The components of v are denoted by $v_{1,i,l,s}$, $v_{2,i,s}$, $v_{3,i,a}$, $v_{4,l,s}$, $v_{5,a}$, $v_{6,i}$, $v_{7,l,s}$, and v_8. Then, $0 = v' \partial_{e^i_{L+1,s}} \widetilde{\psi}(\xi, \alpha, e) = -v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. Hence,

$$0 = v' \partial_{e^i_{l,s}} \widetilde{\psi}(\xi, \alpha, e) = -v_{4,l,s} - \alpha^i \lambda^i_s v_{7,l,s}, \quad i \in \mathcal{I}, \ (l,s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Since $\sum_{i \in \mathcal{I}} (\alpha^i)^2 = 1$, there is i' such that $\alpha^{i'} \neq 0$. If there is $\bar{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'} \lambda^{i'}_{\bar{s}} - \alpha^i \lambda^i_{\bar{s}} = 0$, then, for any $l \in \check{\mathcal{L}}$,

$$\begin{aligned} 0 &= \textstyle\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \alpha^i \lambda^i_s (x^i_{l,s} - e^i_{l,s}) = \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\alpha^{i'} \lambda^{i'}_{\bar{s}}}{\lambda^i_{\bar{s}}} \lambda^i_s (x^i_{l,s} - e^i_{l,s}) \\ &= \alpha^{i'} \lambda^{i'}_{\bar{s}} \textstyle\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{l,s} - e^i_{l,s}). \end{aligned}$$

Since $\alpha^{i'} \neq 0$, $\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} (\lambda^i_s / \lambda^i_{\bar{s}})(x^i_{l,s} - e^i_{l,s}) = 0$, a contradiction since $e \in \widehat{\Omega}$. Consequently, for every $s \in \mathcal{S}$, there is $i \in \mathcal{I} \setminus \{i'\}$ such that $\alpha^{i'} \lambda^{i'}_s - \alpha^i \lambda^i_s \neq 0$. For $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, $(\alpha^{i'} \lambda^{i'}_s - \alpha^i \lambda^i_s) v_{7,l,s} = 0$, so $v_{7,l,s} = 0$, and, thus $v_{4,l,s} = 0$. Also, $0 = v' \partial_{\alpha^{i'}} \widetilde{\psi}(\xi, \alpha, e) = 2\alpha^{i'} v_8$, so, since $\alpha^{i'} \neq 0$, $v_8 = 0$. It follows as in the first part of the proof that $v_{1,i,l,s} = 0$, $i \in \mathcal{I}$, $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, that $v_{6,i} = 0$, $i \in \mathcal{I}$, that $v_{5,a} = 0$, $a \in \check{\mathcal{A}}$, and that $v_{3,i,a} = 0$, $i \in \mathcal{I}$, $a \in \mathcal{A}$.

Therefore, $\widetilde{M}$ has rank $N + LS + 1$ and $\widetilde{\psi}$ intersects 0 transversally. If $\widetilde{\Omega}$ is the set of economies such that $\widetilde{\psi}_e$ is transverse to 0, then $\Omega^* \setminus \widetilde{\Omega}$ has Lebesgue measure zero by the transversal density theorem. Openness follows by a standard argument. □

Generically, it is possible to make every individual better off by choosing appropriate price regulations on the spot markets when asset markets are incomplete. One needs at least as many instruments, LS, as individuals, I. Proposition 5.3 makes clear that this is all one needs. This is not the case in the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when $2L \leq I \leq L(S-1) + 1$.

A competitive equilibrium can be Pareto improved by uniform price regulation if the matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation has full rank.

The function $k : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{L+1}$ is defined by

$$k(\xi, \alpha, e) = \begin{pmatrix} \sum_{s \in \mathcal{S}} h_{l,s}(x, \lambda, \alpha, e), & l \in \check{\mathcal{L}} \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 & \end{pmatrix}.$$

By Proposition 4.8, the matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation is guaranteed to have full rank if there is no solution to $\psi_e(\xi) = 0$ in combination with $k_e(\xi, \alpha) = 0$.

Proposition 5.4 *Let $\mathcal{E}$ satisfy A1, A2, A4 and A5. If $L \geq I$, then for an open subset of endowments in Ω^* with full Lebesgue measure, all competitive equilibria of $\mathcal{E}$ can be Pareto improved by uniform price regulation.*

Proof The argument follows that in the proof of Proposition 5.3. The equations related to h that characterize Pareto improving price regulation are replaced by the equations related to k that characterize Pareto improvements by uniform price regulation. This defines a function $\overline{\psi}$. The matrix $\overline{M}$ gives the partial derivatives of $\overline{\psi}$ evaluated at some (ξ, α, e) with $\overline{\psi}(\xi, \alpha, e) = 0$. If $v'\overline{M} = 0$, by considering the partial derivatives with respect to $e^i_{l,s}$, it follows that $v_{2,i,s} = 0$, $i \in \mathcal{I}, s \in \mathcal{S}$, and $v_{4,l,s} + \alpha^i \lambda^i_s v_{7,l} = 0$, $i \in \mathcal{I}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$. If i' is such that $\alpha^{i'} \neq 0$, and if $\overline{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'} \lambda^{i'}_{\overline{s}} - \alpha^i \lambda^i_{\overline{s}} = 0$, then

$$\begin{aligned} 0 &= \sum_{i \in \mathcal{I}} \alpha^i \sum_{s \in \mathcal{S}} \lambda^i_s (x^i_{l,s} - e^i_{l,s}) = \alpha^{i'} \lambda^{i'}_{\overline{s}} \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \tfrac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}) \\ &= \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \tfrac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}) = \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S} \setminus \{\overline{s}\}} \tfrac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}), \quad l \in \check{\mathcal{L}}, \end{aligned}$$

which contradicts $e \in \widehat{\Omega}$. It follows that $v_{4,l,s} = 0$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $v_{7,l} = 0$, $l \in \check{\mathcal{L}}$. The remainder of the proof follows the argument in the proof of Proposition 6. □

Uniform price regulation is effective when $L \geq I$, which reflects again that the number of instruments has to exceed the number of objectives. It complements the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when $2L \leq I \leq L(S - 1) + 1$.

6 An Example

Consider an economy with two individuals, three states of the world, two commodities, and two assets. The utility function of individual i has an additively separable representation, $u^i = \sum_{s \in \mathcal{S}} \pi_s u^i_s$, with state dependent cardinal utility

$$u^i_s(x_s) = \alpha^i_s \ln x_{1,s} + \beta^i_s x_{2,s}, \quad \alpha^i_s > 0, \beta^i_s > 0,$$

and a strictly positive probability measure $(\pi_1, \ldots, \pi_S)$ over the states of the world. The payoffs of assets are $R_{\cdot 1} = (0,1,1)'$, and $R_{\cdot 2} = (1,0,0)'$. The payoffs of assets allow for the following interpretation. Consumption at state of the world 1 is concurrent with the trade in assets, while the only asset available, traded against consumption, is an indexed bond with state-independent payoffs.

The parameters in the utility functions of individuals and their endowments are such that

$$\pi = \frac{\pi_1\beta_1^1}{\pi_2\beta_2^1+\pi_3\beta_3^1} = \frac{\pi_1\beta_1^2}{\pi_2\beta_2^2+\pi_3\beta_3^2},$$

and, for $\gamma_s^i = \alpha_s^i/\beta_s^i$,

$$\max\left\{-e_{2,s}^1 + \frac{\gamma_s^1 e_{1,s}^2 - \gamma_s^2 e_{1,s}^1}{e_{1,s}^1+e_{1,s}^2} : s=2,3, -\pi e_{2,1}^2 + \pi\frac{\gamma_1^2 e_{1,1}^1 - \gamma_1^1 e_{1,1}^2}{e_{1,1}^1+e_{1,1}^2}\right\}$$
$$\leq \min\left\{\pi e_{2,1}^1 + \pi\frac{\gamma_1^2 e_{1,1}^1 - \gamma_1^1 e_{1,1}^2}{e_{1,1}^1+e_{1,1}^2}, e_{2,s}^2 + \frac{\gamma_s^1 e_{1,s}^2 - \gamma_s^2 e_{1,s}^1}{e_{1,s}^1+e_{1,s}^2} : s=2,3\right\},$$

which eliminates equilibria at the boundaries of their consumption sets [4].

Fix-price equilibrium exists for all prices of commodities, p, and prices of assets $q = \frac{1}{\pi}$. We assume i to be the individual such that $\gamma_s^i/e_{1,s}^i \leq \gamma_s^{i'}/e_{1,s}^{i'}$ and consider four different cases: (i) $0 < p_s \leq \frac{\gamma_s^i}{e_{1,s}^i}$, (ii) $\frac{\gamma_s^i}{e_{1,s}^i} \leq p_s \leq \frac{\gamma_s^i+\gamma_s^{i'}}{e_{1,s}^i+e_{1,s}^{i'}}$, (iii) $\frac{\gamma_s^i+\gamma_s^{i'}}{e_{1,s}^i+e_{1_s}^{i'}} \leq p_s \leq \frac{\gamma_s^{i'}}{e_{1,s}^{i'}}$, and (iv) $\frac{\gamma_s^{i'}}{e_{1,s}^{i'}} \leq p_s$.

(i) If $0 < p_s \leq \gamma_s^i/e_{1,s}^i$, both individuals have an excess demand for commodity 1. Equilibria obtain for $\overline{z}_s^* = 0$, $x_{1,s}^{i*} = e_{1,s}^i$, $x_{1,s}^{i'*} = e_{1,s}^{i'}$, and $y^{i'*} = -y^{i*}$. At $s=1$, $x_{2,1}^i = e_{2,1}^i - (1/\pi)y^{i*}$, $x_{2,1}^{i'*} = e_{2,1}^{i'} + (1/\pi)y^{i*}$, $y^{i*} \leq \pi e_{2,1}^i$, and $y^{i*} \geq -\pi e_{2,1}^{i'}$. At $s=2$ or $s=3$, $x_{2,s}^{i*} = e_{2,s}^i + y^{i*}$, $x_{2,s}^{i'*} = e_{2,s}^{i'} - y^{i*}$, $y^{i*} \geq -e_{2,s}^i$, and $y^{i*} \leq e_{2,s}^{i'}$. The remaining parameters of the rationing scheme are set so as not to be binding. Owing to the linearity of utility in the amount consumed of the numeraire commodity in each state, the demand for the numeraire commodities is not uniquely determined in equilibrium. There is a trade-off between more consumption of the numeraire commodity in state 1 and an amount of consumption of the numeraire commodity in both states 2 and 3. This does not affect the utility levels reached.

(ii) If $\gamma_s^i/e_{1,s}^i \leq p_s \leq (\gamma_s^i+\gamma_s^{i'})/(e_{1,s}^i+e_{1,s}^{i'})$, there is aggregate excess demand for commodity 1, but individual i supplies the commodity, and trade takes place, with individual i' rationed on his demand of the commodity. Equilibria obtain for $\overline{z}_{1,s}^* = e_{1,s}^i - \gamma_s^i/p_s$, $x_{1,s}^{i*} = \gamma_s^i/p_s$, $x_{1,s}^{i'*} = e_{1,s}^{i'} + e_{1,s}^i - \gamma_s^i/p_s$, and $y^{i'*} = -y^{i*}$. At $s=1$, $x_{2,1}^{i*} = p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i - (1/\pi)y^{i*}$, $x_{2,1}^{i'*} = e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i + (1/\pi)y^{i*}$, $y^{i*} \leq \pi(p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i)$, and $y^{i*} \geq -\pi(e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i)$. At $s=2$ or $s=3$, $x_{2,s}^{i*} = p_s e_{1,s}^i + e_{2,s}^i - \gamma_s^i + y^{i*}$, $x_{2,s}^{i'*} = e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i - y^{i*}$, $y^{i*} \geq -p_s e_{1,s}^i - e_{2,s}^i + \gamma_s^i$,

[4] A possible choice of parameters is for instance

$$\begin{aligned}
&\pi_1 = 1, \pi_2 = \pi_3 = \tfrac{1}{2},\\
&\alpha_1^1 = \beta_1^1 = 1, \alpha_2^1 = \beta_2^1 = \tfrac{4}{3}, \alpha_3^1 = \beta_3^1 = \tfrac{2}{3},\\
&\alpha_1^2 = \beta_1^2 = 1, \alpha_2^2 = \beta_2^2 = \tfrac{2}{3}, \alpha_3^2 = \beta_3^2 = \tfrac{4}{3},\\
&e_1^1 = (1,1)', e_2^1 = (1,1)', e_3^1 = (2,1)',\\
&e_1^2 = (1,1)', e_2^2 = (2,1)', e_3^2 = (1,1)'.
\end{aligned}$$

and $y^{i*} \leq e^{i'}_{2,s} - p_s e^i_{1,s} + \gamma^i_s$. The remaining parameters of the rationing scheme are set so as not to be binding.

(iii) If $(\gamma^i_s + \gamma^{i'}_s)/(e^i_{1,s} + e^{i'}_{1,s}) \leq p_s \leq \gamma^{i'}_s / e^{i'}_{1,s}$, there is aggregate excess supply of commodity 1, and individual i supplies the commodity, rationed by the demand of individual i'. Equilibria obtain for $\underline{z}^*_{1,s} = e^{i'}_{1,s} - \gamma^{i'}_s/p_s$, $x^{i*}_{1,s} = e^i_{1,s} + e^{i'}_{1,s} - \gamma^{i'}_s/p_s$, $x^{i'*}_{1,s} = \gamma^{i'}_s/p_s$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x^{i*}_{2,1} = e^i_{2,1} - p_1 e^{i'}_{1,1} + \gamma^{i'}_1 - (1/\pi)y^{i*}$, $x^{i'*}_{2,1} = p_1 e^{i'}_{1,1} + e^{i'}_{2,1} - \gamma^{i'}_1 + (1/\pi)y^{i*}$, $y^{i*} \leq \pi(e^i_{2,1} - p_1 e^{i'}_{1,1} + \gamma^{i'}_1)$, and $y^{i*} \geq -\pi(p_1 e^{i'}_{1,1} + e^{i'}_{2,1} - \gamma^{i'}_1)$. At $s = 2$ or $s = 3$, $x^{i*}_{2,s} = e^i_{2,s} - p_s e^{i'}_{1,s} + \gamma^{i'}_s + y^{i*}$, $x^{i'*}_{2,s} = p_s e^{i'}_{1,s} + e^{i'}_{2,s} - \gamma^{i'}_s - y^{i*}$, $y^{i*} \geq -e^i_{2,s} + p_s e^{i'}_{1,s} - \gamma^{i'}_s$, and $y^{i*} \leq p_s e^{i'}_{1,s} + e^{i'}_{2,s} - \gamma^{i'}_s$. The remaining parameters of the rationing scheme are set so as not to be binding.

(iv) If $\gamma^{i'}_s / e^{i'}_{1,s} \leq p_s$, both individuals supply commodity 1, are fully rationed on their supply of the commodity and no trade takes place. Fix-price equilibria obtain for $\underline{z}^*_{1,s} = 0$, $x^{i*}_{1,s} = e^i_{1,s}$, $x^{i'*}_{1,s} = e^{i'}_{1,s}$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x^{i*}_{2,1} = e^i_{2,1} - (1/\pi)y^{i*}$, $x^{i'*}_{2,1} = e^{i'}_{2,1} + (1/\pi)y^{i*}$, $y^{i*} \leq \pi e^i_{2,1}$, and $y^{i*} \geq -\pi e^{i'}_{2,1}$. At $s = 2$ or $s = 3$, $x^{i*}_{2,s} = e^i_{2,s} + y^{i*}$, $x^{i'*}_{2,s} = e^{i'}_{2,s} - y^{i*}$, $y^{i*} \geq -e^i_{2,s}$, and $y^{i*} \leq e^{i'}_{2,s}$. The remaining parameters of the rationing scheme are set so as not to be binding.

Competitive equilibrium prices are given by $p^*_s = \frac{\gamma^1_s + \gamma^2_s}{e^1_{1,s} + e^2_{1,s}}$, $s = 1, 2, 3$, and $q^* = \frac{1}{\pi}$. Those prices belong to the intersection of cases (ii) and (iii). The allocations described there qualify as competitive equilibrium allocations.

The utility attained by each individual at a fix-price equilibrium is unambiguously determined by the prices of commodities. At prices p, the utility of individual i at the fix-price equilibrium is $v^i(p) = \sum_{s \in \mathcal{S}} \pi_s v^i_s(p_s)$, where

$$
\begin{array}{ll}
\text{Case (i)} & v^i_s(p_s) = \alpha^i_s \ln e^i_{1,s} + \beta^i_s e^i_{2,s}, \\
& v^{i'}_s(p_s) = \alpha^{i'}_s \ln e^{i'}_{1,s} + \beta^{i'}_s e^{i'}_{2,s}, \\
\text{Case (ii)} & v^i_s(p_s) = \alpha^i_s \ln(\frac{\gamma^i_s}{p_s}) + \beta^i_s (p_s e^i_{1,s} + e^i_{2,s} - \gamma^i_s), \\
& v^{i'}_s(p_s) = \alpha^{i'}_s \ln(e^{i'}_{1,s} + e^i_{1,s} - \frac{\gamma^i_s}{p_s}) + \beta^{i'}_s (e^{i'}_{2,s} - p_s e^i_{1,s} + \gamma^i_s), \\
\text{Case (iii)} & v^i_s(p_s) = \alpha^i_s \ln(e^i_{1,s} + e^{i'}_{1,s} - \frac{\gamma^{i'}_s}{p_s}) + \beta^i_s (e^i_{2,s} - p_s e^{i'}_{1,s} + \gamma^{i'}_s), \\
& v^{i'}_s(p_s) = \alpha^{i'}_s \ln(\frac{\gamma^{i'}_s}{p_s}) + \beta^{i'}_s (p_s e^{i'}_{1,s} + e^{i'}_{2,s} - \gamma^{i'}_s), \\
\text{Case (iv)} & v^i_s(p_s) = \alpha^i_s \ln e^i_{1,s} + \beta^i_s e^i_{2,s}, \\
& v^i_s(p_s) = \alpha^{i'}_s \ln e^{i'}_{1,s} + \beta^{i'}_s e^{i'}_{2,s}.
\end{array}
$$

Substitution of the competitive equilibrium prices in either case (ii) or case (iii) gives the utility levels at the competitive equilibrium. The indirect utility function is differentiable at competitive prices which confirms Proposition 4.8. The derivative is given by

$$
\begin{aligned}
\partial_{p_s} v^i(p^*) &= \pi_s \beta^i_s \frac{\gamma^{i'}_s e^i_{1,s} - \gamma^i_s e^{i'}_{1,s}}{\gamma^i_s + \gamma^{i'}_s} = -\pi_s \beta^i_s (x^{i*}_s - e^i_s), \\
\partial_{p_s} v^{i'}(p^*) &= \pi_s \beta^{i'}_s \frac{\gamma^i_s e^{i'}_{1,s} - \gamma^{i'}_s e^i_{1,s}}{\gamma^{i'}_s + \gamma^i_s} = -\pi_s \beta^{i'}_s (x^{i'*}_s - e^{i'}_s).
\end{aligned}
$$

For $v_s = \pi_s(\gamma^2_s e^1_{1,s} - \gamma^1_s e^2_{1,s})/(\gamma^1_s + \gamma^2_s)$, it holds that

$$
V = \begin{pmatrix} \partial v^1(p^*) \\ \partial v^2(p^*) \end{pmatrix} = \begin{pmatrix} \beta^1_1 v_1 & \beta^1_2 v_2 & \beta^1_3 v_3 \\ -\beta^2_1 v_1 & -\beta^2_2 v_2 & -\beta^2_3 v_3 \end{pmatrix}.
$$

If the matrix V has full row rank, then price regulation can Pareto improve the competitive equilibrium allocation. If the ratios of the marginal utilities of income of the individuals are not the same across all states of the world, $\beta_1^1/\beta_1^2 \neq \beta_2^1/\beta_2^2$ or $\beta_3^1/\beta_3^2 \neq \beta_2^1/\beta_2^2$, for the matrix V to have full row rank it is sufficient that $v_s \neq 0$, for every state of the world. Since $v_s = 0$ if and only if $e_{1,s}^1/e_{1,s}^2 = \gamma_s^1/\gamma_s^2$, generically in the endowments of individuals it is possible to Pareto improve on the competitive allocation [5]. This is also the essence of Proposition 5.3. Only here, because of linear utility in the numeraire commodity, variations in endowments do not affect the marginal utilities of income at equilibrium and an ad hoc argument is required.

Since $L < I$, it is not always possible to Pareto improve on the competitive equilibrium by a uniform price regulation. A Pareto improvement by a uniform price regulation may fail if $\beta_1^1 v_1 + \beta_2^1 v_2 + \beta_3^1 v_3$ and $-\beta_1^2 v_1 - \beta_2^2 v_2 - \beta_3^2 v_3$ have opposite signs. This is by no means excluded.

7 Conclusion

Given any prices for commodities and assets, a competitive allocation of resources exists, but does in general involve endogenously determined amounts of rationing. Local comparative statics are complicated at competitive equilibrium prices. Arbitrarily small deviations from competitive prices may lead to discontinuous jumps in allocations and utilities. Necessary and sufficient conditions for local uniqueness of fix-price equilibria in the neighborhood of competitive equilibria are derived. Provided those conditions hold, price regulation offers opportunities for efficiency gains when asset markets are incomplete. This conclusion does not change when uniform price regulation is considered only.

A serious concern are the informational requirements needed to determine, even compute, improving interventions. In the case of price regulation they involve knowledge of marginal utilities of income and excess demands for commodities across states. The characterization in Geanakoplos and Polemarchakis (1990) and in Kübler and Polemarchakis (1999) are only first steps towards an analysis of the informational requirements of active policy.

[5] For the specification of parameters given in footnote 12,

$$V = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}.$$

Both individuals benefit if the price of commodity 1 in states 2 and 3 is fixed below its competitive equilibrium value. A Pareto improvement can even be achieved by a uniform price regulation, although this is not necessarily the case if $L < I$.

References

1. Cass, D., and A. Citanna (1998), "Pareto improving financial innovation in incomplete markets," *Economic Theory,* 11, 467-494.

2. Citanna, A., A. Kajii and A. Villanacci (1998), "Constrained suboptimality in incomplete markets: a general approach and two applications," *Economic Theory,* 11, 495-522.

3. Drèze, J.H. (1975), "Existence of an exchange equilibrium under price rigidities," *International Economic Review,* 16, 301-320.

4. Drèze, J.H., and C. Gollier (1993), "Risk sharing on the labour market," *European Economic Review,* 37, 1457-1482.

5. Elul, R. (1995), "Welfare effects of financial innovation in a general equilibrium model," *Journal of Economic Theory,* 65, 43-78.

6. Geanakoplos, J.D. and H.M. Polemarchakis (1986), "Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete," in W. P. Heller, R. M. Starr and D. A. Starrett (eds.), *Uncertainty, Information and Communication: Essays in Honor of K. J. Arrow, Vol. III,* Cambridge University Press, 65-96.

7. Geanakoplos, J.D. and H.M. Polemarchakis (1990), "Observability and optimality," *Journal of Mathematical Economics,* 19, 153-165.

8. Hara, C. (1997), "A sequence of Pareto - improving financial innovations," mimeo.

9. Hart, O.D. (1975), "On the optimality of equilibrium when the market structure is incomplete," *Journal of Economic Theory,* 11, 418-443.

10. Kajii, A. (1994), "Anonymity and optimality of competitive equilibria when assets are incomplete," *Journal of Economic Theory,* 64, 115-129.

11. Kalmus, P. (1997), "Pareto improving trade restrictions: an example," mimeo.

12. Kübler, F., and H.M. Polemarchakis (1999), "The identification of preferences from the equilibrium prices of commodities and assets," mimeo.

13. Laroque, G. (1978), "The fixed price equilibria: some results in local comparative statics," *Econometrica,* 46, 1127-1154.

14. Laroque, G. (1981), "On the local uniqueness of the fixed price equilibria," *Review of Economic Studies,* 48, 113-129.

15. Laroque, G. and H.M. Polemarchakis (1978), "On the structure of the set of fixed price equilibria," *Journal of Mathematical Economics,* 5, 53-69.

16. Madden, P. (1982), "Catastrophic Walrasian equilibrium from the non-Walrasian viewpoint: A three-good macroeconomic example," *Review of Economic Studies,* 49, 661-667.

17. Nguyen, T.T., and J. Whalley (1986), "Equilibrium under price controls with endogenous transactions costs," *Journal of Economic Theory,* 39, 290-300.

18. Polemarchakis, H.M. (1979), "Incomplete markets, price regulation, and welfare," *American Economic Review,* 69, 662-669.