Minimal manipulability: Unanimity and Non-dictatorship

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Abstract This paper is concerned with the number of profiles at which a nondictatorial social choice function is manipulable. For three or more alternatives the lower bound is derived when the social choice function is nondictatorial and unanimous. In the case of three alternatives the lower bound is also derived when the social choice function is nondictatorial and surjective. In both cases all social choice functions reaching that lower bound are characterized when there are at least three agents. In the case of two agents the characterized social choice functions are only a subset of the set of all social choice functions reaching the minimum.

1 Introduction

A well-known result of Gibbard (1973) and Satterthwaite (1975) shows that any surjective nondictatorial social choice function with more than two alternatives must be manipulable. However, little is known about the degree of manipulability of nondictatorial social choice functions. An investigation was pioneered by Kelly (1988), who gave the minimal number of manipulable profiles for social choice functions with three alternatives and two agents, and formulated several conjectures about the general case. This line of research was continued by Fristrup and Keiding (1998), who gave the minimal number of manipulable profiles for two agents and any number of alternatives. It was also conjectured in their paper, that there is hope that the social choice functions they use also give the minimum in the general case.

We prove their conjecture if surjectivity is replaced by unanimity, but show that it is not true with only surjectivity in the case of three alternatives.The minimally manipulable social choice functions given as examples in Fristrup and Keiding (1998) can be described as almost dictatorial social choice functions. We show that these form the set of all minimally manipulable nondictatorial and unanimous social choice functions in the case of three agents or more, and a subset of this set in the case of two agents. Example 4 in Kelly (1989) shows a unanimous two person social choice function hitting the minimum, but which is not almost dictatorial. In the case of three alternatives almost dictatorial social choice functions lead to $2n-1$ manipulable profiles where n is the number of agents and there are at least three agents. We present six social choice functions that are manipulable at n profiles. We characterize these social choice functions as all minimally manipulable social choice functions for three alternatives and more than two agents. For $n=2$ these social choice functions reach the minimum of 2 manipulable profiles, but there are many other such social choice functions that do this, again see e.g. example 4 in Kelly (1989). We mention that an optimization program we wrote shows that there are 135 social choice functions reaching the minimum of two manipulable profiles in the case of two agents and three alternatives.

Another surprising feature of the minimally manipulable social choice functions characterized in the three alternative surjectivity case is that they are anonymous. Considering the distribution of power among agents, anonymous social choice functions are rather far away from dictatorial or almost dictatorial social choice functions. However, we did not find a generalization of these social choice functions to more than three alternatives that beats almost dictatorial social choice functions in that case. In fact, in Maus et al. (2004) we show that any surjective and anonymous social choice function has more manipulable profiles than an almost dictatorial one if $m \geq 4$ and $n \geq m+2$, where m is the number of alternatives. So the conjecture of Fristrup and Keiding (1998) can still be true for more than three alternatives, and as mentioned above we show that it is true if one replaces surjectivity by unanimity.

First, Section 2 is on notation and model description. Section 3 introduces the concept of manipulation and some basics about it. After that Section 4 concerns well-known results on minimal manipulability. Section 5 contains the result for unanimous and nondictatorial social choice functions, and Section 6 contains the result for surjective and nondictatorial social choice functions with three alternatives. Finally a conclusion is given in Section 7. The appendix is on unanimous and nondictatorial social choice functions for the special three agents case.

2 Preliminaries

We denote the cardinality of a set S by $|S|$ and its powerset by 2^N.

Let A be a finite set of alternatives, $m := |A| \geq 3$, and N a finite set of agents, $n := |N| \geq 2$.

Let $t \subset A \times A$. We call t *complete* if for all $x, y \in A$ $(x, y) \in t$ or $(y, x) \in t$. Note that completeness of t implies $(x, x) \in t$ for all $x \in A$. We call t *transitive* if for all $x, y, z \in A$ $(x, y) \in t$ and $(y, z) \in t$ implies $(x, z) \in A$. We call t *antisymmetric* if for all $x, y \in A$ $(x, y) \in t$ and $(y, x) \in t$ implies that $x = y$.

A *preference* $t \subset A \times A$ is a *linear ordering* (complete, transitive, antisymmetric) on A. Let P denote the set of all preferences. Suppose that $A = \{x_1, x_2, \ldots x_m\}$. By completeness, transitivity and antisymmetry we can write conveniently

$$t = x_1 x_2 \ldots x_m$$

for the preference t such that $(x_i, x_j) \in t$ if and only if $i \geq j$, $i, j \in \{1, 2, \ldots, m\}$,

$$t = \ldots x \ldots y \ldots$$

if we want to express only that x is strictly preferred to y, and

$$t = x \ldots$$

if we want to express only that x is preferred to all other alternatives. Furthermore we use

$$\ldots x \ldots y \ldots z \ldots, \quad xy \ldots \text{ and } xyz.$$

whose meanings can be easily deduced.

A *profile* p is a map from N to P. Let P^N denote the set of all these maps. Thus, a profile assigns to every agent i a preference $p(i)$ over the alternatives. For a nonempty subset S of N we denote by $p|_S$ the restriction of the map p to the domain S. By $(p|_{N-S}, t^S)$ we denote the profile q such that

$$q(i) := \begin{cases} p(i) & \text{for all } i \in N - S, \\ t & \text{for all } i \in S. \end{cases}$$

In the particular case $S = \{i\}$ we write (p_{-i}, t) instead of $(p_{N-\{i\}}, t^{\{i\}})$, and if $p(N - \{i\}) = \{\widetilde{t}\}$ we write $(\widetilde{t}^{N-\{i\}}, t)$. For a profile $p \in P^N$ and alternatives $x, y \in A$ let $S^{(x,y)}(p) := \{i \in N \mid (x, y) \in p(i)\}$ be the set of all agents that prefer x to y.

For two profiles $p, q \in P^N$ we define the *distance* between p and q by $dist(p, q) := |\{i \in N \mid p(i) \neq q(i)\}|$. A finite sequence $r^0, \ldots, r^l \in P$ such that $f(r^0) = p$, $f(r^l) = q$, and $dist(r^i, r^{i+1}) = 1$ for all $i = 0, \ldots, l-1$ is called a *path* from p to q. A path $r^0, \ldots, r^l$ such that $l = dist(p, q)$ is called a *shortest* path from p to q.

For a profile p and a set of profiles Q we define the *distance* between p and Q by $dist(p, Q) := min_{q \in Q} dist(p, q)$.

A *social choice function* is a function $f : P^N \to A$. Hence, a social choice function selects a unique alternative $f(p)$ at every profile p.

A social choice function is called *surjective* if any alternative in A is chosen at least once, i.e. if $f(P^N) = A$. In literature this is also known as citizen-sovereignity. A social choice function is called *unanimous* if $f(p) = x$ for all profiles $p \in P^N$ such that $p(i) = x \ldots$ for all $i \in N$, i.e. for all profiles in $top(x) := \{p \in P^N \mid p(i) = x \ldots \text{ for all } i \in N\}$. Clearly, unanimity is stronger than surjectivity. Throughout this paper we assume that any social choice function is at least surjective.

For a permutation σ of N and a profile $p \in P^N$ let $p \circ \sigma$ be the profile given by $(p \circ \sigma)(i) := p(\sigma(i))$ for all $i \in N$. A social choice function is called *anonymous* if $f(p) = f(p \circ \sigma)$ for all permutations σ of N. Thus, anonymous social choice functions are symmetric in the arguments. In a sense they treat agents equally.

In contrast to anonymity, the following dictatorial social choice functions $dict_d$ respect only the preference of one single agent $d \in N$, the *dictator*. For any profile p $dict_d$ is defined by

$$dict_d(p) := x$$

where x is such that $p(d) = x \ldots$. So, $dict_d(p)$ is the most preferred alternative of agent d in $p(d)$. A social choice function f is called *nondictatorial* if there is no agent d such that $f = dict_d$.

3 Manipulation of social choice functions

We are interested in strategic behaviour of individuals when facing cooperative decision-making as captured by social choice functions. This is formalised by the following definitions. Let f be a social choice function.

A social choice function is said to be *intermediate manipulable* at a profile p (by coalition $S \subseteq N$, $S \neq \emptyset$), if $p(S) = \{t\}$ for some $t \in P$, and there is an S-*deviation* q, i.e. $q|_{N-S} = p|_{N-S}$ and $q(S) = \{\widetilde{t}\}$, such that $(f(p), f(q)) \notin t$. We call p *intermediate manipulable towards* q. Let

$$IM_f := \{p \in P^N \mid f \text{ is intermediate manipulable at profile } p\}.$$

An S-deviation where $S = \{i\}$, $i \in N$, is called an *i-deviation.* If it is not important which set of agent deviates from p to q we call q a *deviation* from p. A profile p is called *(individually) manipulable* (under f) if there is an agent that is better off by being dishonest about his preference, i.e. if there is an $i \in N$, and an i-deviation q such that

$$(f(p), f(q)) \notin p(i).$$

We call *p manipulable towards q*. Let

$$M_f := \{p \in P^N \mid p \text{ is manipulable under } f\}.$$

A social choice function is called *strategy-proof* if $M_f = \emptyset$, otherwise it is said to be (individually) *manipulable.* Note that $M_f = \emptyset$ for all dictatorial rules f. The prominence of the dictatorial rules arises from the following impossibility result due to Gibbard (1973) and Satterthwaite (1975).

Theorem 1 *Let A be a finite set of alternatives, $|A| \geq 3$. Let $f : P^N \to A$ be a nondictatorial surjective social choice function. Then*

$$|M_f| \geq 1.$$

We show some useful connections between intermediate manipulability and manipulability. Clearly, $M_f \subseteq IM_f$. If $p \in IM_f - M_f$ is intermediate manipulable towards q, the following (standard) lemma holds.

Lemma 2 *Let $f : P^N \to A$. Let p be intermediate manipulable towards q, but not manipulable. Let $r^0, \ldots, r^l$ be a shortest path from p to q. Then there is a $k \in \{1, \ldots, l-1\}$ such that r^k is manipulable towards r^{k+1}.*

Proof. Let $D = \{i \in N \mid p(i) \neq q(i)\}$. Since f is intermediate manipulable at p by D towards q, we have $p(D) = \{t\}$ for some $t \in P$, so $p = (t^D, p|_{N-D})$ and $(f(p), f(q)) \notin t$. As $r^0, \ldots, r^l$ is a shortest path from p to q, r^{k+1} is a j-deviation from r^k for every $k \in \{0, \ldots, l\}$ for some agent $j \in D$, and $r^k(j) = t$. By the transitivity of t and $(f(r^0), f(r^l)) \notin t = r^k(j)$, there must be at least one $k \in \{0, \ldots, l-1\}$ such that $(f(r^k), f(r^{k+1})) \notin t$. Then r^k is manipulable towards r^{k+1}. As $r^0 \notin M_f$, $k \geq 1$. ■

Each set of disjoint shortest paths from p to q contains at most $dist(p, q)$ elements. So we can state the following corollary to Lemma 2.

Corollary 3 *Let $f : P^N \to A$. Let p be intermediate manipulable towards q, but not manipulable. Then there are at least $dist(p, q)$ profiles in M_f, each of them on a shortest path from p to q.*

Let $p, q \in IM_f - M_f$ be such that p is intermediate manipulable towards q and vice versa. Suppose that there is a shortest path from p to q that contains only one manipulable profile. Then we can state the following about the preferences of the manipulating coalition S in p and q.

Lemma 4 *Let $p, q \in IM_f - M_f$ be such that p is intermediate manipulable by S towards q and q is intermediate manipulable by S towards p. Suppose that there is a shortest path $r^0, \ldots, r^l$ from p to q such that $\{r^0, \ldots, r^l\} \cap M_f = \{r^k\}$. Let $x := f(p)$, $y := f(q)$, $z := f(r^k)$. Then $z \in A - \{x, y\}$ and $p(S) = \{t\}, q(S) = \{\widetilde{t}\}$, are such that $t = \ldots y \ldots x \ldots z \ldots$ and $\widetilde{t} = \ldots x \ldots y \ldots z \ldots$. In particular, if $m = 3$ then $t = yxz, \widetilde{t} = xyz$, and $f(r^i) = x$ for all $i \in \{0, \ldots, k-1\}$, $f(r^i) = y$ for all $i \in \{k+1, \ldots, l\}$.*

Proof. Note that $r^0, \ldots, r^l$ is a shortest path from p to q and $r^l, \ldots, r^0$ is a shortest path from q to p. By the intermediate manipulability of p towards q and q towards p we have $t = \ldots y \ldots x \ldots$ and $\widetilde{t} = \ldots x \ldots y \ldots, x \neq y$. As $\{r^0, \ldots, r^l\} \cap M_f = \{r^k\}$, we have $t = \ldots f(r^i) \ldots f(r^{i+1}) \ldots$ for all $i \in \{0, \ldots, k-1\}$. As $f(r^0) = f(p) = x$ and $t = \ldots y \ldots x \ldots$ this implies that $f(r^{i+1}) \neq y$ for all $i \in \{0, \ldots, k-1\}$. Likewise $f(r^{i-1}) \neq x$ for all $i \in \{k+1, \ldots, n\}$. So, $z = f(r^k) \in A - \{x, y\}$. Now, if $t = \ldots z \ldots x \ldots$ then $\{r^0, \ldots, r^{k-1}\} \cap M_f \neq \emptyset$, a contradiction.So $t = \ldots y \ldots x \ldots z \ldots$. Likewise $\widetilde{t} = \ldots x \ldots y \ldots z$. In particular if $m = 3$ then $t = yxz, \widetilde{t} = xyz$. Then $\{r^0, \ldots, r^{k-1}, r^{k+1}, \ldots, r^n\} \cap M_f = \emptyset$ implies that $f(\{r^0, \ldots, r^{k-1}\}) = x$ and $f(\{r^{k+1}, \ldots, r^n\}) = y$. ■

Note that any intermediate manipulable profile of a unanimous social choice function must contain at least two different preferences. We show that furthermore any $p \in IM_f$ that is intermediate manipulable by a coalition $S \subseteq N$ such that $p = (t^S, \widetilde{t}^{N-S})$ can be manipulated by using a preference $\overline{t} \notin \{t, \widetilde{t}\}$.

Lemma 5 *Let $f : P^N \to A$ be unanimous and let $(t^S, \widetilde{t}^{N-S})$ be intermediate manipulable by S. Then there is a preference $\overline{t} \notin \{t, \widetilde{t}\}$ such that $(f(t^S, \widetilde{t}^{N-S}), f(\overline{t}^S, \widetilde{t}^{N-S})) \notin t$.*

Proof. By the definition of intermediate manipulability there is a $\overline{t} \neq t$ such that $(f(t^S, \widetilde{t}^{N-S}), f(\overline{t}^S, \widetilde{t}^{N-S})) \notin t$. If $\overline{t} \neq \widetilde{t}$ we are done, so suppose that $\overline{t} = \widetilde{t}$. Then $((f(t^S, \widetilde{t}^{N-S}), f(\widetilde{t}^N)) \notin t$. Let $\overline{\overline{t}} \neq \overline{t}$ be a preference with the same top as $\widetilde{t}$. Such a preference exists if $m \geq 3$. Then $f(\overline{\overline{t}}^S, \widetilde{t}^{N-S}) = f(\widetilde{t}^N)$ by unanimity, and thus the coalition S can also manipulate $(t^S, \widetilde{t}^{N-S})$ towards $(\overline{\overline{t}}^S, \widetilde{t}^{N-S})$, $\overline{\overline{t}} \notin \{t, \widetilde{t}\}$. ■

4 Minimal manipulability of social choice functions

Let F be a nonempty set of social choice functions. For any $f \in F$ we measure its manipulability by the number of manipulable profiles $|M_f|$ that f has. We call $f^* \in F$ *minimally manipulable in* F if $|M_{f^*}| \leq |M_f|$ for all $f \in F$. Let F^* be the set of all minimally manipulable social choice functions in F. If $f^* \in F^*$ then $m_F := |M_{f^*}|$ is a lower bound on the number of manipulable profiles that any social choice function in F has.

Let F be the set of surjective nondictatorial social choice functions for fixed $m \geq 3, n \geq 2$. Then the result of Gibbard (1973) and Satterthwaite (1975) says that $m_F \geq 1$, but it does not say what the actual minimum is. This question has been solved by Kelly (1988) for two agents and three alternatives, and by Fristrup and Keiding (1998) for two agents and any number of alternatives larger than three. For reference we summarize their results in the following theorem.

Theorem 6 *Let $n = 2$. Then*

$$m_F = \begin{cases} 2 & \text{if } m = 3, \\ \frac{m!}{2} & \text{if } m \geq 4. \end{cases}$$

For the general case of $n \geq 2$, Fristrup and Keiding (1998) conjecture that $m_F = (n-1)(\frac{m!}{2} - 1) + 1$ if $(n, m) \neq (2, 3)$. We describe a class of functions that attain this number if $(n, m) \neq (2, 3)$. A social choice function f is called *almost dictatorial* if there is a profile $\overline{p} \in P^N$, an alternative $x \in A$, and an agent $d \in N$ such that $(x, dict_d(\overline{p})) \in \overline{p}(i)$ for all $i \in N - \{d\}$, and

$$f(p) = \begin{cases} dict_d(p) & \text{if } p \neq \overline{p} \\ x & \text{if } p = \overline{p}. \end{cases} \tag{1}$$

Proposition 7 *Let f be an almost dictatorial social choice function and let $\overline{p}, x$ be such that equation 1 holds. Then*

$$M_f = \{(\overline{p}_{-i}, t) \mid i \in N - \{d\}, (x, dict_d(\overline{p})) \in t, t \neq \overline{p}(i)\} \cup \{\overline{p}\}$$

and thus $|M_f| = (n-1)\left(\frac{m!}{2} - 1\right) + 1$.

Proof. Suppose that p and q are i-deviations such that $(f(p), f(q)) \notin p(i)$. This implies that if $i = d$ then $p = \overline{p}$ and if $i \neq d$ then $q = \overline{p}$. We show first that $\overline{p}$ is manipulable and treat then the case where $q = \overline{p}$. If agent d deviates to q by changing his preference in $\overline{p}$ to another preference with the same most preferred outcome we have $(f(q), f(\overline{p})) \in \overline{p}(d)$ since agent d is a dictator at q. Thus $\overline{p}$ is manipulable. Suppose that $i \in N - \{d\}$, i.e. $q = \overline{p}$. Let $p = (\overline{p}_{-i}, t), t \neq \overline{p}(i)$, be an i-deviation of q such that $(f(q), f(p)) \in p(i)$. Thus $(x, dict_d(\overline{p})) \in t$. As $(x, dict_d(\overline{p})) \in \overline{p}(i)$ there are $\frac{m!}{2} - 1$ such preferences t for every agent $i \in N - \{d\}$. This proves that $M_f = \{(\overline{p}_{-i}, t) \mid i \in N - \{d\}, (x, dict_d(\overline{p})) \in t, t \neq \overline{p}(i)\} \cup \{\overline{p}\}$. ■

We show that the conjecture of Fristrup and Keiding cannot be true for $m = 3$. Consider the following social choice functions if $m = 3$. Let $A = \{a, b, c\}$, $t = xyz \in P$. Let $m^t : P^N \to \{a, b, c\}$ be the social choice function given by

$$m^t(p) := \begin{cases} x & \text{if } S^{(x,y)}(p) = S^{(y,z)}(p) = N, \\ y & \text{if } S^{(y,z)}(p) = N \text{ and } S^{(x,y)}(p) \neq N, \\ z & \text{if } S^{(z,y)}(p) \neq \emptyset. \end{cases}$$

Observe that m^t is surjective but not unanimous. Then we have the following proposition.

Proposition 8 *Let $t = xyz$. Then $M_{m^t} = \{(xyz^{N-\{i\}}, xzy) \mid i \in N\}$ and thus $|M_{m^t}| = n$.*

Proof. No agent will want to manipulate when $f(p) = x$. Suppose that $f(p) = y$. Then $(y, z) \in p(i)$ for all $i \in N$, so no agent can manipulate to a profile q where $f(q) = z$. Furthermore there is an agent $i \in N$ such that $(x, y) \notin p(i)$. This agent has no incentive to manipulate to the profile t^N, which is the only profile q where $f(q) = x$. On the other hand, as $p(i) \neq t$, this agent would have to change his preference in order to manipulate to the profile t^N. So f is strategy-proof at all profiles $p \in P^N$ where $f(p) \in \{x, y\}$. Suppose that $f(p) = z$. Then there is an agent $i \in N$ such that $(z, y) \in p(i)$. This agent has no incentive to manipulate to a profile q where $f(q) = y$. Again, on the other hand, as $(y, z) \in q(i)$ for such profiles, this agent would have to change his preference in order to manipulate to such a profile. The only place left where manipulations can occur is from p to t^N. If p and t^N are i-deviations we have that $p = (t^{N-\{i\}}, p(i))$. If agent i has an incentive to manipulate to t^N we must have $(x, z) \in p(i)$ and as $f(p) = z$ we must have $(z, y) \in p(i)$. Hence, $M_{m^t} = \{(t^{N-\{i\}}, xzy) \mid i \in N\}$ and $|M_{m^t}| = n$. ■

Note that $n < (n-1)(\frac{m!}{2} - 1) + 1 = 2n - 1$ for all $n \geq 2$. Hence, the conjecture of Fristrup and Keiding cannot be true for $m = 3$. We will show however that it is true if we replace F by the set of unanimous nondictatorial social choice functions $G \subset F$, i.e., we show that $m_G = (n-1)(\frac{m!}{2} - 1) + 1$.

Moreover, we will show that then for $n \geq 3$ $g \in G^*$ if and only if g is almost dictatorial.

For the case $m = 3, n \geq 3$, we have thus $m_F \leq n < m_G = 2n - 1$. This will allow us to show that $m_F = n$ in this case, and that $f \in F^*$ for $n \geq 3$, if and only if $f = m^t$ for some $t \in P$.

5 Minimal manipulability with unanimity

Let $n \geq 3$. To show that $m_G = (n-1)(\frac{m!}{2} - 1) + 1$ we will use the results of Kelly (1988) and Fristrup and Keiding (1998), summarized in Theorem 6, applied to two agent social choice functions derived from $f \in G$. The first step is to embed the domain of two agent social choice functions into P^N, depending on some $S \in 2^N - \{\emptyset, N\}$. This is achieved by the map $\Pi_S : P^{\{1,2\}} \to P^N$ given by $\Pi_S(r) := (r(1)^S, r(2)^{N-S})$ for all $r \in P^{\{1,2\}}$. Now, for $f \in G$, define $f_S := f \circ \Pi_S$, then f_S is a unanimous two agent social choice function.

Clearly, any profile in $\Pi_S(M_{f_S})$ is in IM_f. Let $M_S^2 := \Pi_S(M_{f_S}) \cap M_f$. We make the following important observation, which holds by unanimity.

Lemma 9 *If $S, T \in 2^N - \{\emptyset, N\}, S \neq T$, are such that $M_S^2 \cap M_T^2 \neq \emptyset$, then $T = N - S$.*

Proof. Let $p \in M_S^2 \cap M_T^2$. Then $p = (t_1^S, \widetilde{t}_1^{N-S}) = (t_2^T, \widetilde{t}_2^{N-T})$ for some $t_i, \widetilde{t}_i \in P$. If $T \neq N - S$ then, since $T \neq S$, this implies that $p = t^N$ for some $t \in P$. But by unanimity such p are not intermediate manipulable. Hence, $T = N - S$. ■

Let $\mathcal{S} := \{\{1, i\} \mid i \in N\}$. Then, if $S, T \in \mathcal{S}$ are such that $S \neq T$, also $S \neq N - T$. We will show that for all $f \in G$

$$\begin{aligned} |M_f| &\geq \begin{cases} 2n & \text{if } m = 3 \\ (n-1)\frac{m!}{2} & \text{if } m \geq 4 \end{cases} \\ &> (n-1)(\frac{m!}{2} - 1) + 1, \end{aligned}$$

if f_S is nondictatorial for all $S \in \mathcal{S}$. The simplest case is covered by the following lemma.

Lemma 10 *Suppose that all $f_S, S \in \mathcal{S}$, are nondictatorial and that $\Pi_S(M_{f_S}) \subseteq M_f$ for all $S \in \mathcal{S}$. Then*

$$|M_f| \geq \begin{cases} 2n & \textit{if } m = 3, \\ \frac{m!}{2} n & \textit{if } m \geq 4. \end{cases}$$

Proof. In this case, for all $S \in \mathcal{S}$, $M_S^2 = \Pi_S(M_{f_S}) \cap M_f = \Pi_S(M_{f_S})$, and thus by Theorem 6

$$|M_S^2| = |M_{f_S}| \geq \begin{cases} 2 & \text{if } m = 3, \\ \frac{m!}{2} & \text{if } m \geq 4. \end{cases}$$

By Lemma 9, $M_S^2 \cap M_T^2 = \emptyset$ for all pairwise different $S, T \in \mathcal{S}$. Since, $|\mathcal{S}| = n$ this implies the lemma. ■

Suppose that we cannot use Lemma 10, i.e. there are $S \in \mathcal{S}$ such that $\Pi_S(M_{f_S}) - M_f \neq \emptyset$. If there is only one such S we can show the following. Note that $r \notin M_S^2$ for all $r \in P^N$ such that $|r(N)| \neq 2$.

Lemma 11 *Suppose that all $f_S, S \in \mathcal{S}$, are nondictatorial and that there is a $T \in \mathcal{S}$ such that $\Pi_S(M_{f_S}) \subseteq M_f$ for all $S \in \mathcal{S} - \{T\}$, $\Pi_T(M_{f_T}) - M_f \neq \emptyset$. Then*

$$|M_f| \geq \begin{cases} 2n & \text{if } m = 3, \\ \frac{m!}{2}(n-1) & \text{if } m \geq 4. \end{cases}$$

Proof. For $m \geq 4$ this follows in the same way as in the proof of Lemma 10. Suppose that $m = 3$. There is a $p \in \Pi_T(M_{f_T}) - M_f$. Then p is intermediate manipulable towards some $q \in P^N$, and by Lemma 5 we can assume that $|p(N) \cup q(N)| = 3$. By Corollary 3 there are at least $dist(p, q) \geq 2$ (otherwise $p \in M_f$) manipulable profiles $r \notin \{p, q\}$ on a shortest path from p to q, say r_1, r_2. As $|p(N) \cup q(N)| = 3$ and $|p(N)| = |q(N)| = 2$ we must have $|r_1(N)| = |r_2(N)| = 3$. But then $\{r_1, r_2\} \cap M_S^2 = \emptyset$ for all $S \in \mathcal{S}$. By Theorem 6, $|M_S^2| \geq 2$ for all $S \in \mathcal{S} - \{T\}$. Hence, by Lemma 9, $|M_f| \geq |\{r_1, r_2\}| + \sum_{S \in \mathcal{S}} |M_S^2| = 2n$. ■

If there is more than one $S \in \mathcal{S}$ such that $\Pi_S(M_{f_S}) - M_f \neq \emptyset$ the manipulable profiles that we find by the intermediate manipulability of the $p \in \Pi_S(M_{f_S}) - M_f$ can be the same. The following lemma combines Lemma 5 and Corollary 3 to ensure that we are then still able to find sufficiently many manipulable different profiles $r \in P^N$ such that $|r(N)| = 3$ if there are more than three agents.

Lemma 12 *Let $S, T \in \mathcal{S}$ and let $n \geq 3$. Suppose that there are $p \in \Pi_S(M_{f_S}) - M_f$ and $\overline{p} \in \Pi_T(M_{f_T}) - M_f, p \neq \overline{p}$. Then there are $q \in \Pi_S(P^{\{1,2\}}), \overline{q} \in \Pi_T(P^{\{1,2\}})$, such that $|p(N) \cup q(N)| = |\overline{p}(N) \cup \overline{q}(N)| = 3$ and $\Pi_S^{-1}(p)$ is manipulable towards $\Pi_S^{-1}(q)$, $\Pi_S^{-1}(\overline{p})$ is manipulable towards $\Pi_S^{-1}(\overline{q})$. Suppose that there is an $r \notin \{p, \overline{p}\}$ that is on a shortest path from p to q and on a shortest path from $\overline{p}$ to $\overline{q}$. Then:*

1. *If $n \geq 4$ and $S, T \in \mathcal{S} - \{\{1\}\}$, then $S = T$, $p = \overline{q}$ and $\overline{p} = q$.*

2. *Suppose that $S = T$. Then $p = \overline{q}$ and $\overline{p} = q$. Furthermore, if $m = 3$ and there is a path between p and q that contains only one manipulable profile then p or $\overline{p}$ is intermediate manipulable by S and by $N - S$ and so $n \geq 4$. On the other hand, if $n = 3$, then all profiles on shortest paths between p and q are manipulable.*

Proof. The existence of such q and $\overline{q}$ is an immediate implication of $p \in \Pi_S(M_{f_S}) - M_f$ and $\overline{p} \in \Pi_T(M_{f_T}) - M_f$ and Lemma 5. Then $p = (t_p^S, \widetilde{t}_p^{N-S})$ and $\overline{p} = (t_{\overline{p}}^T, \widetilde{t}_{\overline{p}}^{N-T})$ and there are preferences $\overline{t}_p, \overline{t}_{\overline{p}} \in P$ such that $q \in \{(\overline{t}_p^S, \widetilde{t}_p^{N-S}), (t_p^S, \overline{t}_p^{N-S})\}$ and $\overline{q} \in \{(\overline{t}_{\overline{p}}^T, \widetilde{t}_{\overline{p}}^{N-T}), (t_{\overline{p}}^T, \overline{t}_{\overline{p}}^{N-T})\}$. By our assumptions there is an $r \notin \{p, \overline{p}\}$ that is on a shortest path from p to q and on a shortest path from $\overline{p}$ to $\overline{q}$. Then, either $r = (t_p^S, \widetilde{t}_p^{N-S-U_p}, \overline{t}_p^{U_p})$ for some $U_p \subsetneq N - S$, $1 \leq |U_p| \leq |N| - |S| - 1$, or $r = (t_p^{S-V_p}, \overline{t}_p^{V_p}, \widetilde{t}_p^{N-S})$ for some $V_p \subsetneq S$, $1 \leq |V_p| \leq |S| - 1$. Similarly, $r = (t_{\overline{p}}^T, \widetilde{t}_{\overline{p}}^{N-T-U_{\overline{p}}}, \overline{t}_{\overline{p}}^{U_{\overline{p}}})$ for some $U_{\overline{p}} \subsetneq N - T$, $1 \leq |U_{\overline{p}}| \leq |N| - |T| - 1$, or $r = (t_{\overline{p}}^{T-V_{\overline{p}}}, \overline{t}_{\overline{p}}^{V_{\overline{p}}}, \widetilde{t}_{\overline{p}}^{N-T})$ for some $V_{\overline{p}} \subsetneq T$, $1 \leq |V_{\overline{p}}| \leq |T| - 1$. So, $\{t_p, \widetilde{t}_p, \overline{t}_p\} = \{t_{\overline{p}}, \widetilde{t}_{\overline{p}}, \overline{t}_{\overline{p}}\}$. As $|\{t_p, \widetilde{t}_p, \overline{t}_p\}| = 3$ this implies that $S \in \{T, N - T - U_{\overline{p}}, U_{\overline{p}}\}$ or $S \in \{T - V_{\overline{p}}, V_{\overline{p}}, N - T\}$ or $N - S \in \{T, N - T - U_{\overline{p}}, U_{\overline{p}}\}$ or $N - S \in \{T - V_{\overline{p}}, V_{\overline{p}}, N - T\}$.

Proof of (1): Then $1 \in S$ and $1 \in T$ and $|S| = |T| = 2$ imply that only the first and the last case is possible, and both cases lead to $S = T$, $p = \overline{q}$ and $\overline{p} = q$ as $n \geq 4$.

Proof of (2): If $n \geq 4$ the first part of (2) follows by (1) if $S \in \mathcal{S} - \{\{1\}\}$. If $n \geq 4$ and $S = \{1\}$ or if $n = 3$ then $p = \overline{q}$ and $\overline{p} = q$ follows in a similar way using the extra assumption that $S = T$. Let $m = 3$. Let $x := f(p)$, $y := f(\overline{p})$ and $z := f(r)$. Then by Lemma 4, $p = (yxz^U, t^{N-U})$ and $\overline{p} = (xyz^U, t^{N-U})$ for some $U \in \{S, N - S\}$. Unanimity of f implies that $t \in \{zxy, zyx\}$. Without loss of generality $t = zyx$. Then p is intermediate manipulable towards (yxz^U, yzx^{N-U}) and towards $\overline{p}$, i.e. by S and $N - S$, and so $|S|, |N - S| \geq 2$ implying that $n \geq 4$.

On the other hand, if $n = 3$, then on any shortest path between p and q there is only one element, which is manipulable by Lemma 2. ■

Now, we can prove the following theorem.

Theorem 13 *Suppose that for $m \geq 4$ all $f_S, S \in \mathcal{S} - \{\{1\}\}$, are nondictatorial, and for $m = 3$ all $f_S, S \in \mathcal{S}$, are nondictatorial. Then*

$$|M_f| \geq \begin{cases} 2n & \text{if } m = 3, \\ \frac{m!}{2}(n-1) & \text{if } m \geq 4. \end{cases}$$

Proof. We distinguish two cases. If $n \geq 4$ we can use Lemma 12. This case is presented here. The case $n = 3$ is presented in an appendix because of its different and rather elementary technique, see Propositions 16 and 17.

So, let us assume that $n \geq 4$. For each $S \in \mathcal{S}$ and $p \in \Pi_S(M_{f_S}) - M_f$ let

$$\begin{aligned} Q(p) \ : \ &= \{q \in \Pi_S(P^{\{1,2\}}) \mid \Pi_S^{-1}(p) \text{ is manipulable towards } \Pi_S^{-1}(q) \\ &\quad \text{and } |p(N) \cup q(N)| = 3\}. \end{aligned}$$

By Lemma 5 $Q(p) \neq \emptyset$, and by Corollary 3 there are at least $dist(p,q) \geq 2$ manipulable profiles for each $q \in Q(p)$ on shortest paths between p and q. Let $R(p,q)$ be the set of all these manipulable profiles for given $p \in \Pi_S(M_{f_S}) - M_f, S \in \mathcal{S}$, and $q \in Q(p)$.

For all $S \in \mathcal{S}$ let

$$M_S^3 := \bigcup_{p \in \left(\Pi_S(M_{f_S}) - M_f\right)} \bigcup_{q \in Q(p)} R(p,q).$$

Let $S, T \in \mathcal{S} - \{\{1\}\}$ and $p \in \Pi_S(M_{f_S}) - M_f, q \in Q(p), \overline{p} \in \Pi_S(M_{f_S}) - M_f, \overline{q} \in Q(\overline{p})$. By part 1 of Lemma 12

$$R(p,q) \cap R(\overline{p}, \overline{q}) \neq \emptyset$$

implies that $S = T$ and $p = \overline{q}, \overline{p} = q$ if $n \geq 4$. Hence, if $S \neq T$, then

$$M_S^3 \cap M_T^3 = \emptyset,$$

and for each $S \in \mathcal{S}$ one $p \in \Pi_S(M_{f_S}) - M_f$ can be assumed to yield at least two new manipulable profiles in some $R(p,q), q \in Q(p)$, and thus

$$|M_S^3| \geq \frac{|\Pi_S(M_{f_S}) - M_f|}{2} * 2 = |\Pi_S(M_{f_S}) - M_f|,$$

for all $S \in \mathcal{S}$.

By definition $M_S^2 \cap M_T^3 = \emptyset$ for all $S, T \in \mathcal{S}$. Hence,

$$M_f \supseteq \bigcup_{S \in \mathcal{S}, S \neq \{1\}} M_S^2 \cup \bigcup_{S \in \mathcal{S}, S \neq \{1\}} M_S^3$$

and all these sets are pairwise disjoint, so

$$\begin{aligned} |M_f| \ &\geq \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^2| + \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^3| \\ &\geq \sum_{S \in \mathcal{S}, S \neq \{1\}} |\Pi_S(M_{f_S}) \cap M_f| + \sum_{S \in \mathcal{S}, S \neq \{1\}} |\Pi_S(M_{f_S}) - M_f| \\ &= \sum_{S \in \mathcal{S}, S \neq \{1\}} |\Pi_S(M_{f_S})| = \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_{f_S}|. \end{aligned}$$

If $m \geq 4$ this proves the theorem, as $|M_{f_S}| \geq \frac{m!}{2}$ for all $S \in \mathcal{S}$ by Theorem 6.

So, suppose that $m = 3$. For all $S \in \mathcal{S}$ by Theorem 6 $|M_{f_S}| \geq 2$. If $|M_S^2| = |\Pi_S(M_{f_S}) \cap M_f| = 1$ then there is a $p \in \Pi_S(M_{f_S}) - M_f$ and $q \in Q(p)$ and thus

$$|M_S^3| \geq |R(p,q)| \geq 2.$$

If $|M_S^2| = 0$ then $|\Pi_S(M_{f_S}) - M_f| \geq 2$. Let $p, \overline{p} \in \Pi_S(M_{f_S}) - M_f, p \neq \overline{p}$, $q \in Q(p), \overline{q} \in Q(\overline{p})$. Then, either

$$R(p,q) \cap R(\overline{p},\overline{q}) = \emptyset$$

and thus

$$|M_S^3| \geq |R(p,q)| + |R(\overline{p},\overline{q})| \geq 2 + 2 = 4,$$

or

$$R(p,q) \cap R(\overline{p},\overline{q}) \neq \emptyset$$

and without loss of generality part 2 of Lemma 12 yields that p is intermediate manipulable by S and $N - S$. So there are $q_S, q_{N-S} \in Q(p)$ such that p is intermediate manipulable towards q_S by S and intermediate manipulable towards q_{N-S} by $N - S$. Clearly, $R(p, q_S) \cap R(p, q_{N-S}) = \emptyset$ and thus

$$|M_S^3| \geq |R(p,q_S)| + |R(p,q_{N-S})| \geq 2 + 2 = 4.$$

Altogether, for all $S \in \mathcal{S}$ either

$$|M_S^2| = |\Pi_S(M_{f_S}) \cap M_f| = |\Pi_S(M_{f_S})| = |M_{f_S}| \geq 2,$$

or

$$|M_S^2| = 1 \text{ and } |M_S^3| \geq 2,$$

or

$$|M_S^2| = 0 \text{ and } |M_S^3| \geq 4.$$

We distinguish four cases, which show altogether that $|M_f| \geq 2n$.
<u>Case 1:</u> There are $U, V \in \mathcal{S} - \{\{1\}\}$, $U \neq V$, such that $|M_U^2| = |M_V^2| = 1$.

Then

$$\begin{aligned} |M_f| &\geq \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^2| + \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^3| \\ &= \left(\sum_{S \in \mathcal{S}, S \notin \{\{1\}, U, V\}} |M_S^2| + |M_S^3| \right) + |M_U^2| + |M_U^3| + |M_V^2| + |M_V^3| \\ &\geq \left(\sum_{S \in \mathcal{S}, S \notin \{\{1\}, U, V\}} 2 \right) + 1 + 2 + 1 + 2 = (n-3)2 + 6 = 2n. \end{aligned}$$

Case 2: There is a $U \in \mathcal{S} - \{\{1\}\}$ such that $|M_U^2| = 0$.
Then

$$\begin{aligned}
|M_f| &\geq \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^2| + \sum_{S \in \mathcal{S}, S \neq \{1\}} |M_S^3| \\
&= \left(\sum_{S \in \mathcal{S}, S \notin \{\{1\}, U\}} |M_S^2| + |M_S^3| \right) + |M_U^2| + |M_U^3| \\
&\geq \left(\sum_{S \in \mathcal{S}, S \notin \{\{1\}, U\}} 2 \right) + 0 + 4 = (n-2)2 + 4 = 2n.
\end{aligned}$$

Case 3: There is a $U \in \mathcal{S} - \{\{1\}\}$ such that $|M_S^2| \geq 2$ for all $S \in \mathcal{S} - \{\{1\}, U\}$.
Consider $M_{\{1\}}^2, M_{\{1\}}^3$. By definition also $M_{\{1\}}^2 \cup M_{\{1\}}^3 \subseteq M_f$.
Case 3.1: $|M_{\{1\}}^2| + |M_{\{1\}}^3| \geq 4$.
Then

$$\begin{aligned}
|M_f| &\geq |M_{\{1\}}^2| + |M_{\{1\}}^3| + \left(\sum_{S \in \mathcal{S} - \{\{1\}, U\}} |M_S^2| \right) \\
&\geq 4 + \left(\sum_{S \in \mathcal{S} - \{\{1\}, U\}} 2 \right) = 4 + 2(n-2) = 2n.
\end{aligned}$$

Note that in particular we have case 3.1 if $|M_{\{1\}}^2| = 0$.
Case 3.2: $|M_{\{1\}}^2| \geq 2$.
Then

$$\begin{aligned}
|M_f| &\geq |M_U^2| + |M_U^3| + \sum_{S \in \mathcal{S} - \{U\}} |M_S^2| \\
&\geq 2 + \left(\sum_{S \in \mathcal{S} - \{U\}} 2 \right) = 2 + (n-1)2 = 2n.
\end{aligned}$$

Case 3.3: $|M_{\{1\}}^2| = 1$ and $|M_{\{1\}}^3| = 2$.
Then

$$\begin{aligned}
|M_f| &\geq |M_{\{1\}}^2| + |M_{\{1\}}^3| + \left(\sum_{S \in \mathcal{S} - \{\{1\}, U\}} |M_S^2| \right) + |M_U^3 - M_{\{1\}}^3| + |M_U^2| \\
&\geq 1 + 2 + \left(\sum_{S \in \mathcal{S} - \{\{1\}, U\}} 2 \right) + |M_U^3 - M_{\{1\}}^3| + |M_U^2| \\
&= (n-1)2 + 1 + |M_U^3 - M_{\{1\}}^3| + |M_U^2| \geq 2n,
\end{aligned}$$

if $|M_U^2| \neq 0$ or $M_U^3 \subsetneq M_{\{1\}}^3$. But suppose that $|M_U^2| = 0$. Then $|M_U^3| \geq 4$ and so $M_U^3 \subsetneq M_{\{1\}}^3$, as $|M_{\{1\}}^3| = 2$. This finishes case 3.3, thus case 3, and proves the theorem. ■

Let $f \in G$. We use Theorem 13 to conclude that there must be a dictatorial $f_S, S \in \mathcal{S}$, if $|M_f| \leq (n-1)(\frac{m!}{2} - 1) + 1$. With the help of this f_S we can show that f must be almost dictatorial, and thus $m_G = |M_f| = (n-1)(\frac{m!}{2} - 1) + 1$.

Theorem 14 *For all $n \geq 3$, $m_G = (n-1)(\frac{m!}{2} - 1) + 1$ and $G^* = \{f \in G \mid f$ is almost dictatorial$\}$.*

Proof. We want to prove the theorem by induction. Our induction basis is given by Theorem 6 and our induction assumption is as follows. Let $n \geq 3$. Assume that it has been shown for all $k \in \{2, \ldots, n-1\}$ and k agent unanimous nondictatorial social choice functions g that

$$|M_g| \geq \begin{cases} (k-1)(\frac{m!}{2} - 1) + 1 & \text{if } (k,m) \neq (2,3), \\ 2 & \text{if } (k,m) = (2,3), \end{cases}$$

and that equality holds for $k \geq 3$ if and only if g is almost dictatorial.

Induction Step:

Let $f \in G$ be such that $|M_f| \leq (n-1)(\frac{m!}{2} - 1) + 1$. Under the induction assumption we show that then $|M_f| = (n-1)(\frac{m!}{2} - 1) + 1$ and that f is almost dictatorial. This proves the theorem by the induction principle.

By Theorem 13 there must be an $S \in \mathcal{S}$ such that f_S is dictatorial. If $m \geq 4$ this S has to be in $\mathcal{S} - \{\{1\}\}$. Let $D := S$ if agent 1 is the dictator in f_S, and $D := N - S$ if agent 2 is the dictator. Let $i \in N$ and $t \in P$. We define $(n-1)$-agent social choice functions $g_{i,t} : P^{N-\{i\}} \to A$ derived from f by $g_{i,t}(p) := f(p, t^{\{i\}}), p \in P^{N-\{i\}}$. Now, for all $i \in N - D \neq \emptyset$ and $t \in P$, $g_{i,t}$ is unanimous. If such a $g_{i,t}$ is nondictatorial, then by the induction assumption

$$|M_{g_{i,t}}| \geq \begin{cases} (n-2)(\frac{m!}{2} - 1) + 1 & \text{if } (n,m) \neq (3,3), \\ 2 & \text{if } (n,m) = (3,3). \end{cases}$$

Furthermore, if such a $g_{i,t}$ is dictatorial then the dictator d must be in D. Let T_i be the set of $t \in P$ for which $g_{i,t}$ is dictatorial. Now, if $p \in M_{g_{i,t}}$ then $(p, t^{\{i\}}) \in M_f$ and clearly these profiles are different for different t. Hence, as $|M_f| \leq (n-1)(\frac{m!}{2} - 1) + 1$,

$$|T_i| \geq \begin{cases} |P| - 1 & \text{if } (n,m) \neq (3,3), \\ |P| - 2 & \text{if } (n,m) = (3,3), \end{cases}$$

as each $g_{i,t}, t \in P - T_i$, is nondictatorial and yields $|M_{g_{i,t}}|$ different manipulable profiles. We show that there is a $d \in D$ that is the dictator for all $g_{i,t}, t \in T_i$. Suppose to the contrary that there are $d_1, d_2 \in D, d_1 \neq d_2$, and $t_1, t_2 \in T_i$ such that g_{i,t_1} is dictatorial with dictator d_1 and g_{i,t_2} is dictatorial with dictator d_2. Then, agent i can choose a dictator to his advantage. Let p be a profile such that $(dict_{d_1}(p), dict_{d_2}(p)) \in p(i) = t_2$ or $(dict_{d_1}(p), dict_{d_2}(p)) \in p(i) = t_1$. Each such profile is manipulable by agent i, and there are

$$2\frac{m(m-1)}{2}(m-1)!^2(m!)^{n-3} = \left(2(m-1)(m-1)!(m!)^{n-3}\right)\frac{m!}{2} > n\frac{m!}{2} \;(n, m \geq 3)$$

such profiles, $(m-1)!^2(m!)^{n-3}$ ones for every of the $\frac{m(m-1)}{2}$ pairs $(x, y) \in p(i), x \neq y$, and a given $p(i) \in \{t_1, t_2\}, |\{t_1, t_2\}| = 2$. This contradicts $|M_f| \leq (n-1)(\frac{m!}{2} - 1) + 1$. Hence, every $g_{i,t}, t \in T_i$, is dictatorial with the same dictator $d \in D$.

As f is nondictatorial, there must be a $\overline{p} \in P^N$ such that $f(\overline{p}) \neq dict_d(\overline{p})$. Then, for all $t \in T_i$ such that $(dict_d(\overline{p}_{-i}, t), f(\overline{p})) \notin t$, $(\overline{p}_{-i}, t)$ is manipulable. For all $p \in P^N$ let

$$U(p) := \{t \in T_i \mid (dict_d(p_{-i}, t), f(p)) \notin t\},$$

then because of reflexivity of t $f(p) \neq dict_d(p_{-i}, t)$ and $(p_{-i}, t) \in M_f$ for all $t \in U(p)$. Furthermore, $|U(p)| \geq \frac{m!}{2} - |P - T_i| = |T_i| - \frac{m!}{2}$ for all p such that $f(p) \neq dict_d(p)$ and the manipulable profiles $(p_{-i}, t), (q_{-i}, t)$ that we find for such $p, q \in P^N$ are different if $p_{-i} \neq q_{-i}$. Hence,

$$(n-1)(\frac{m!}{2} - 1) + 1 \geq |M_f| \geq \sum_{t \in P - T_i} |M_{g_{i,t}}| + u\left(|T_i| - \frac{m!}{2}\right),$$

where u is the number of p having different p_{-i} and satisfying $f(p) \neq dict_d(p)$. If $(n, m) = (3, 3)$ this implies $5 \geq 2 * (6 - |T_i|) + u(|T_i| - 3)$ which, as $|T_i| \in \{4, 5\}$, shows that $u = 1$. On the other hand, if $(n, m) \neq (3, 3)$ we use $|T_i| = m! - 1$ and obtain $(n-1)(\frac{m!}{2} - 1) + 1 \geq (n-2)(\frac{m!}{2} - 1) + 1 + u(\frac{m!}{2} - 1)$, which also implies that $u = 1$.

Now, note that $u = 1$ implies that $f(p) = dict_d(p)$ for all $p \in P^N$ such that $p_{-i} \neq \overline{p}_{-i}$. We show that $g_{i,\overline{p}(i)}$ is almost dictatorial. Clearly, $\overline{p}$ is manipulable by agent d. Let $j \in N - \{i, d\}$. Then all $(\overline{p}_{-j}, t)$ such that $(dict_d(\overline{p}), f(\overline{p})) \notin t \neq \overline{p}(j)$ are manipulable. This yields

$$|M_{g_{i,\overline{p}(i)}}| \geq |N - \{i, d\}|(\frac{m!}{2} - 1) + 1 = (n-2)(\frac{m!}{2} - 1) + 1,$$

where equality holds if and only if $(f(\overline{p}), dict_d(\overline{p})) \in \overline{p}(j)$ for all $j \in N - \{i, d\}$. Hence, $g_{i,\overline{p}(i)}$ is almost dictatorial. If $(n, m) = (3, 3)$ this implies that $|T_i| = m! - 1$ as otherwise

$$5 \geq |M_f| \geq \sum_{t \in P - T_i} |M_{g_{i,t}}| \geq 3 + 3 = 6,$$

a contradiction.

Finally,

$$(n-1)(\frac{m!}{2} - 1) + 1 \geq |M_f| \geq |M_{g_{i,\overline{p}(i)}}| + |U(\overline{p})| \geq (n-2)(\frac{m!}{2} - 1) + 1$$

shows that $|U(\overline{p})| = (\frac{m!}{2} - 1)$ and so $(f(\overline{p}), dict_d(\overline{p})) \in \overline{p}(i)$ and f is almost dictatorial. ■

6 Minimal manipulability with three alternatives and surjectivity

Theorem 15 *Let $n \geq 3$, $m = 3$ and let F be the set of surjective nondictatorial social choice functions. Then $m_F = n$ and $F^* = \{m^t \mid t \in P\}$.*

Proof. Let $f \in F^*$. Since m^t is surjective and nondictatorial we have $|M_f| \leq |M_{m^t}| = n$. If f is unanimous then by Theorem 14 $|M_f| \geq 2(n-1) + 1 > n$, a contradiction. So f is not unanimous and there are $x \in A = \{a, b, c\}$ and $\widetilde{p} \in top(x)$ such that $f(\widetilde{p}) \neq x$. Fix such an x and $\widetilde{p}$, and let $\{y, z\} = A - \{x\}$.

Claim 1: $a \in f(top(a))$ for all $a \in A$.

To the contrary suppose $a \notin f(top(a))$. We deduce that $|M_f| > n$. As f is surjective there exist $p \in P^N$ such that $f(p) = a$. Then $a \notin f(top(a))$ implies that for some $j \in N$ we have $p(j) \notin \{abc, acb\}$. Without loss of generality $j = 1$. Furthermore, let p be chosen such that for all q, with $f(q) = a$ we have $dist(q, top(a)) \geq dist(p, top(a))$. For $i \geq 2$ let $\overline{r}^i := (t, p_{-i})$, where $t \in \{abc, acb\} - \{p(i)\}$ and let $\overline{\overline{r}}^i := (abc, \overline{r}^i_{-1})$. Furthermore let $\overline{r}^1 := (abc, p_{-1})$ and $\widehat{r}^1 := (acb, p_{-1})$. As for $i \geq 2$

$$dist(\overline{\overline{r}}^i, top(a)) < dist(\overline{r}^i, top(a)) \leq dist(p, top(a))$$

and

$$dist(\overline{r}^1, top(a)) = dist(\widehat{r}^1, top(a)) < dist(p, top(a))$$

we have $f(\overline{\overline{r}}^i) \neq a$ for $i \geq 2$, $f(\overline{r}^1) \neq a$ and $f(\widehat{r}^1) \neq a$. So $\overline{r}^1, \widehat{r}^1 \in M_f$ and $M_f \cap \{\overline{r}^i, \overline{\overline{r}}^i\} \neq \emptyset$ for all $i \geq 2$. Hence $|M_f| \geq n+1$, the desired contradiction, and claim 1 is proven.

Let $V := top(x) \cap M_f$. Claim 1 shows that there exist $q \in top(x)$ such that $f(q) = x$. Now consider a path from $\widetilde{p}$ to q through $top(x)$. On this path at least one profile is manipulable. So, $V \neq \emptyset$. Let $W := \{p \in top(x) \mid f(p) \neq x\}$. Obviously, $V \subseteq W$.

Claim 2: $|W| > n$.
To the contrary suppose $|W| \leq n$. Consider $\overline{f} : P^N \to A$ defined by

$$\overline{f}(q) := \begin{cases} x & q \in W \\ f(q) & \text{otherwise.} \end{cases}$$

We show that $|M_{\overline{f}}| < |M_f|$ and have a contradiction with minimal manipulability of f if $\overline{f}$ is surjective and nondictatorial.

- $\overline{f}$ is surjective By claim 1 $y \in f(top(y))$ for all $y \in A$. So $\overline{f}$ is surjective.

- $\overline{f}$ is nondictatorial Suppose to the contrary that $\overline{f}$ is dictatorial with dictator d. There is a $\widehat{p} \in W$ with $f(\widehat{p}) \neq x$. By the definition of $\overline{f}$ we have $f(p) = \overline{f}(p) = dict_d(p)$ for all $p \notin W$. For $i \in N - \{d\}$ consider i-deviations q of $\widehat{p}$ such that $(x, f(\widehat{p})) \notin q(i)$. Then f is manipulable at q by i towards $\widehat{p}$, as $q \notin W$ and thus $f(q) = dict_d(q) = dict_d(\widehat{p}) = x$. As there are three preferences t in P at which $(x, f(\widehat{p}), x) \notin t$ there are at least $2(n-1)$ of these q-profiles. Since, $3 \leq n$ we obtain $|M_f| \geq 2(n-1) > n$, a contradiction with the assumption that $|M_f| \leq n$. Thus f is nondictatorial.

- $|M_{\overline{f}}| < |M_f|$ By the assumption $|W| \leq n$, there is for all $q \in W$ an adjacent $r \in top(x)$, i.e. $dist(r, q) = 1$, such that $f(r) = x$. So f is manipulable at all $q \in W$. Therefore, $V = W$. We show that the transition from f to $\overline{f}$ repairs more manipulable profiles than it creates. All the manipulable profiles in V are not manipulable in $\overline{f}$ anymore. Let $q \in M_{\overline{f}} - M_f$ be a manipulable profile that was created by the transition. Then there is an agent i and a profile $p \in W$ such that p and q are i-deviations satisfying $(f(q), x) \notin q(i) \notin \{xyz, xzy\}$. This leads to the two possibilities $f(q) = z$ and $q(i) = yxz$ or $f(q) = y$ and $q(i) = zxy$. Then $q \notin M_f$ implies in the first case that $f(r) = z \neq x$ and in the second case that $f(r) = y \neq x$ for all i-deviations r from q. If $r(i) \neq p(i)$, r is also an i-deviation from p. Hence, for $t \in \{xyz, xzy\} - \{p(i)\}$ we have $f(p_{-i}, t) \neq x$. As $(p_{-i}, t) \in V = W$, $(p_{-i}, t) \neq p$, there are at most $(|V|-1)$ agents i such that there is a $q \in M_{\overline{f}} - M_f$ manipulable by i towards a profile p in W. Let m_i be the number of such profiles q for agent i. As $q(i) = yxz$ if $f(q) = z$ and $q(i) = zxy$ if $f(q) = y$, we have $m_i \leq 1$. So, $|M_{\overline{f}}| \leq |M_f| - |V| + (|V|-1)m_i \leq |M_f| - 1$. This proves claim 2.

Claim 3: $|V| = n$.

By claim 2 $|W| \geq n+1$. By claim 1 $top(x) - W \neq \emptyset$. So, there are $p, q \in P^N$ such that $p \in W$ and $q \in top(x) - W$. Hence, $f(p) \neq x$ and $f(q) = x$. Without loss of generality let $p \in W - V$, if such p does not exist we have $|M_f| \geq |V| \geq n+1$, a contradiction. As $p \notin M_f$ all profiles $r^1, r^2, \ldots, r^n$ adjacent to p in $top(x)$ are in W. Consider a path from r^j to q, say $r^j_1, \ldots, r^j_{k_j} \in top(x)$. Then $V \cap \{r^j_1, \ldots, r^j_{k_j - 1}\} \neq \emptyset$. So, if we can find disjoint paths for every j from r^j to q we have $|V| \geq n$. We show how to construct such paths. Without loss of generality there is a $k \in \mathbb{N}$ such that

$$\begin{aligned} p(i) &= q(i) \text{ for } n \geq i > k, \\ p(i) &\neq q(i) \text{ for } 1 \leq i \leq k, \end{aligned}$$

and the $r^1, r^2, \ldots, r^n$ are such that $r^j = (p_{-i}, t), t \in \{xyz, xzy\} - \{p(j)\}$ for all $j \in \{1, 2, \ldots, n\}$. Hence, $r^j(i) = p(i)$ if $i \neq j$, $r^j(j) = q(j)$ if $j \leq k$, and $r^j(j) \neq q(j) = p(j)$ if $j > k$. Let $1 \leq j_1 \leq n, 1 \leq j_2 < k$. For $j_1 \leq k, j_2 < k$, let

$$r^{j_1}_{j_2}(i) := \begin{cases} q(i) & \begin{array}{c} \text{if } j_1 < i \leq j_1 + j_2 \leq k \text{ or} \\ i \leq j_1 + j_2 - k \text{ and } j_1 + j_2 > k \end{array} \\ p(i) & \text{otherwise,} \end{cases}$$

and for $j_1 > k, j_2 \leq k$, let

$$r^{j_1}_{j_2}(i) := \begin{cases} q(i) & \text{if } i \leq j_2 \\ p(i) & \text{otherwise.} \end{cases}$$

Then these are disjoint paths $r^{j_1}, 1 \leq j_1 \leq n$, between p and q. So, $|V| \geq n$ and since $|V| \leq |M_f| \leq n$ this proves claim 3.

Now, $n = |V| = |M_f \cap top(x)| \leq |M_f| \leq n$. So, $M_f \subseteq top(x)$. By claim 3 $|W| \geq n+1$. So, there is a profile $p \in W - V$.

Claim 4: $f^{-1}(\{x\}) \subseteq top(x)$.

Suppose contrapositive without loss of generality that there is an $r \in P^N$ such that $f(r) = x$ and $r(n) \notin \{xyz, xzy\}$. Then

$$r^k(i) = \begin{cases} r(i) & \text{if } k < i \\ p(i) & \text{otherwise} \end{cases}$$

defines a path from $r = r^0$ to $r^n = p$ outside $top(x)$, i.e. $r^k \notin top(x)$ for all $k < n$. Since $M_f \cap \{r^0, r^1, \ldots, r^{n-1}\} \neq \emptyset$, we have a contradiction because $M_f \subseteq top(x)$. This proves claim 4.

Let $U := P^N - top(x)$. Then $U \cap M_f = \emptyset$ and $f(U) = \{y, z\}$.

Claim 5: For profiles $p \in P^N$ and $q \in U$ such that $f(p) = y$ and $S^{(y,z)}(p) \subseteq S^{(y,z)}(q) \neq \emptyset$, we have $f(q) = y$.

By claim 4 $x \notin f(U)$, so $f(q) \neq x$. Without loss of generality p, q are i-deviations and $q(i) \in \{yxz, yzx\}$. As $q \notin M_f$, $f(q) = y$. This proves claim 5.

Now fix $y, z \in A - \{x\}$. Let

$$\begin{aligned} W^y \ &: = \{S \subseteq N \mid \text{for all } p \in P^N : f(p) \neq x \text{ and } S^{(y,z)}(p) = S \\ &\qquad \text{implies } f(p) = y\}, \\ W^z \ &: = \{S \subseteq N \mid \text{for all } p \in P^N : f(p) \neq x \text{ and } S^{(z,y)}(p) = S \\ &\qquad \text{implies } f(p) = z\}. \end{aligned}$$

Then $N \in W^y$ by surjectivity and claim 5, as $S^{(y,z)}(p) \subseteq N = S^{(y,z)}(q)$ for all p such that $f(p) = y$ and $q \in top(y) \subset U$. Likewise, $N \in W^z$.

Claim 6: $f(xyz^S, xzy^{N-S}) = x$ implies $S = N$ or $S = \emptyset$.

To the contrary let $i \in S$ and $j \in N - S$. Consider $r = (yxz^{\{i\}}, xyz^{S-\{i\}}, xzy^{N-S})$. As $r \in U$, and therefore $r \notin M_f$, we have $f(r) \neq z$. But similarly $f(\overline{r}) \neq y$, where $\overline{r} = (zxy^{\{j\}}, xyz^S, xzy^{N-S-\{j\}})$. By claim 4 $f(r) \neq x$. Thus $f(r) = y$ and $f(\overline{r}) \neq y, \overline{r} \in U$. As $S^{(y,z)}(r) = S^{(y,z)}(\overline{r}) \neq \emptyset$ we have a contradiction with claim 5. So $S = N$ or $S = \emptyset$, proving claim 6.

But then $W^y = 2^N \setminus \{\emptyset\}$ and $W^z = \{N\}$ or $W^y = \{N\}$ and $W^z = 2^N \setminus \{\emptyset\}$. Moreover, we have either $f^{-1}(\{x\}) = \{xyz^N\}$ if $W^z = 2^N \setminus \{\emptyset\}$, or $f^{-1}(\{x\}) = \{xzy^N\}$ if $W^y = 2^N \setminus \{\emptyset\}$. So, $f = m^t$ for some $t \in \{xyz, xzy\}$. ■

7 Conclusion

In this paper we have found the minimally manipulable surjective and non-dictatorial social choice functions with three alternatives and more than two agents. They turn out to be anonymous. This is in contrast to the social choice functions attaining the global minimum 0 in the class of all social choice functions, which are the dictatorial ones. The second smallest value that the function $K(f) := |M_f|$, f a social choice function for three alternatives, takes is n. This value is not attained, as one might expect, by the almost dictatorial social choice functions that give the minimum in the case of two agents and four or more alternatives, see Fristrup and Keiding (1998), but it is attained for all numbers of agents larger than two by six anonymous social choice functions similar in structure. For the case of two agents these six social choice functions are only part of the set of all minimally manipulable social choice functions, see Kelly (1988) for other examples of social choice functions attaining the minimum in this case. For more than three alternatives it is not clear whether the almost dictatorial social choice functions are minimally manipulable for more than two agents. We show this however if surjectivity is replaced by unanimity. In that case almost dictatorial social choice functions are the minimally manipulable social choice functions if in addition $n \geq 3$. The case of four alternatives and three agents is after this paper the smallest unsolved case with surjectivity, and for two agents the minimally manipulable rules are not characterized. It would be of interest to see whether only almost dictatorial social choice functions are best in the case of two agents and more than three alternatives, as this might help with an induction. The following table summarizes these results about minimal manipulability. FK (1998) stands for Fristup and Keiding (1998), and K (1988), K (1989) stand for Kelly (1988) and Kelly (1989) respectively.

	$m = 3$		
n	m_F	$f \in F^*$	in:
$= 2$	2	if $f = m^t, t \in P$	K (1988) m_F
≥ 3	n	iff $f = m^t, t \in P$	Section 6
	m_G	$g \in G^*$	
$= 2$	2	examples	K (1989)
≥ 3	$2n - 1$	iff almost dictatorial	Section 5
	$m \geq 4$		
n	m_F	$f \in F^*$	
$= 2$	$\frac{m!}{2}$	if almost dictatorial	FK (1998)
≥ 3	$\leq (n-1)(\frac{m!}{2} - 1) + 1$	?	FK (1998)
	m_G	$g \in G^*$	
$= 2$	$\frac{m!}{2}$	if almost dictatorial	FK (1998)
≥ 3	$(n-1)(\frac{m!}{2} - 1) + 1$	iff almost dictatorial	Section 5

We relate the result in Section 6 also to a conjecture made in Kelly (1988). The Kemeny distance between two preferences t and t' is defined to be the minimal number of transpositions of adjacent positions in the preference t necessary to obtain the preference t'. Local strategy-proofness is said to hold if there is a $\delta > 1$ such that to manipulate, an agent always has to change to a preference t' that has a larger Kemeny distance from his true preference t than δ. In the minimally manipulable social choice functions characterized here, the manipulating agent changes from xyz to xzy. So the Kemeny distance is 1. Now, in Kelly (1988) the conjecture was that local strategy-proof social choice functions and minimally manipulable social choice functions have a nonempty intersection. So instead of looking for minimally manipulable social choice functions one can look for local strategy-proof social choice functions, which might be easier. The result from Section 6 says that this is not true for three alternatives and more than two agents. This relation is also not true for almost dictatorial social choice functions and two agents. But since Fristrup and Keiding (1998) do not prove that these are all minimally manipulable social choice functions for two agents, this does not say that the relation is not true in this case. So, it is of interest not only to find the minimum in the other cases than the ones considered here, but to characterize also all social choice functions that give this minimum, as was achieved here. In the case of three alternatives and two agents the conjecture will remain true of course, since this was the example given in Kelly (1988).

In another interesting paper by Kelly (1993), a computer draws social choice functions uniformly from all social choice functions satisfying axioms like anonymity, neutrality and Pareto optimality, or combinations thereof. He then investigates the sample distributions with respect to the number of manipulable profiles of these social choice functions. One sees that imposing anonymity in the case of three alternatives causes a shift towards social choice functions with less manipulable profiles, compared to the sample obtained without any constraints. We saw here that the minimally manipulable social choice functions for three alternatives are anonymous. So in this sense anonimity is for three alternatives a good property when looking for minimally manipulable social choice functions. It would be interesting to see what the distributions look like for more than three alternatives, to see what the effect is there.

8 Appendix

In this appendix we study the case where $n = 3$ and so Lemma 12 cannot be used to show that $M_S^3 \cap M_T^3 = \varnothing$ if $S, T \in \mathcal{S}$, $S \neq T$. Instead we use even more elementary techniques.

Proposition 16 *Let $f : P^{\{1,2,3\}} \to \{x, y, z\}$ be a unanimous and nondictatorial social choice function. Suppose that all $f_S, S \in \mathcal{S}$, are nondictatorial. Then*

$$|M_f| \geq 6.$$

Proof. By Theorem 6 there are $p_1, p_2 \in M_{f_{\{1,2\}}}, p_3, p_4 \in M_{f_{\{1,3\}}}$ and $p_5, p_6 \in M_{f_{\{2,3\}}}$ such that

$$\begin{aligned} p_1 &= (t_1, t_1, \widetilde{t}_1), p_3 = (t_3, \widetilde{t}_3, t_3), p_5 = (\widetilde{t}_5, t_5, t_5), \\ p_2 &= (t_2, t_2, \widetilde{t}_2), p_4 = (t_4, \widetilde{t}_4, t_4), p_6 = (\widetilde{t}_6, t_6, t_6), \end{aligned}$$

and $p_i \neq p_{i+1}, i \in \{1,3,5\}$, $t_i \neq \widetilde{t}_i, i \in \{1,2,3,4,5,6\}$, as otherwise these profiles cannot be manipulable because of unanimity. This also implies that $|\{p_1, p_2, p_3, p_4, p_5, p_6\}| = 6$, since otherwise $t_i = \widetilde{t}_i$ for some $i \in \{1,2,3,4,5,6\}$. For each p_i there is a $\overline{t}_i \in P$ and an $S \in \mathcal{S}$, such that $p_i \notin M_f$ and p_i is manipulable towards $q_i := (\overline{t}_i^S, \widetilde{t}_i)$ by S or $p_i \in M_f$ and p_i is manipulable towards $q_i := (t_i^S, \overline{t}_i)$ by $N - S$. If $p_i \notin M_f$ then $\overline{t}_i \neq t_i$ and if $p_i \in M_f$ then $\overline{t}_i \neq \widetilde{t}_i$.

Claim: Let $p_i \notin M_f$. Then $|M_f| \geq 6$ or p_i is manipulable towards some $q_i = (\overline{t}_i^S, \widetilde{t}_i), S \in \mathcal{S}$, such that $\overline{t}_i \notin \{t_i, \widetilde{t}_i\}$.

As $p_i \notin M_f$ a $q_i = (\overline{t}_i^S, \widetilde{t}_i)$ towards which p_i is manipulable must exist and as $p_i \neq q_i$ $t_i \neq \overline{t}_i$. Suppose to the contrary that $|M_f| \leq 5$ and $\overline{t}_i \in \{t_i, \widetilde{t}_i\}$, i.e. $\overline{t}_i = \widetilde{t}_i$. Without loss of generality let $S = \{1,2\}$, i.e. $i \in \{1,2\}$. We omit the subscript i. Let $u := f(p), v := f(q)$. Then $v \neq u$ and $(v,u) \in t$. As $\overline{t} = \widetilde{t}$ and $f(q) = f(\overline{t}, \overline{t}, \widetilde{t}) = v$ unanimity implies that $\widetilde{t} = \overline{t} \in \{vwu, vuw\}$. Then, $t = wvu$ as by unanimity and $f(p) \neq v$ $t \in P - \{vwu, vuw\}$, and $(v,u) \in t$. This in turn implies that $\widetilde{t} = \overline{t} = vuw$ as otherwise $p = (wvu, wvu, vwu)$ is manipulable under f by $\{3\}$ towards (wvu, wvu, wvu) by unanimity. Hence,

$$p = (wvu, wvu, vuw), q = (vuw, vuw, vuw).$$

Now, unanimity implies that p is also manipulable towards (vwu, vwu, vuw) by the coalition $\{1,2\}$ and so, as $p \notin M_f$,

$$M_f \supseteq \{(t, \overline{t}, \overline{t}), (\overline{t}, t, \overline{t}), (t, vwu, \overline{t}), (vwu, t, \overline{t})\}.$$

Consider p_3, p_4, p_5 and p_6. Then $|M_f| \leq 5$ implies that $|M_f \cap \{p_3, p_4, p_5, p_6\}| \leq 1$. So, without loss of generality $p_3, p_4 \notin M_f$. But then

$$M_f \supseteq \{(t_3, \widetilde{t}_3, \overline{t}_3), (\overline{t}_3, \widetilde{t}_3, t_3), (t_4, \widetilde{t}_4, \overline{t}_4), (\overline{t}_4, \widetilde{t}_4, t_4)\},$$

and $|M_f| \leq 5$ implies that without loss of generality $\{(t_3, \widetilde{t}_3, \overline{t}_3), (\overline{t}_3, \widetilde{t}_3, t_3)\} = \{(\overline{t}, t, \overline{t}), (vwu, t, \overline{t})\}$, since $(\overline{t}, t, \overline{t}), (vwu, t, \overline{t})$ are the only profiles in

$$\{(t, \overline{t}, \overline{t}), (\overline{t}, t, \overline{t}), (t, vwu, \overline{t}), (vwu, t, \overline{t})\}$$

where the second agent has an identical preference. But this implies that $t_3 = \overline{t} = \overline{t}_3$, a contradiction. This finishes the proof of the claim.

Now, we consider three cases.

Case 1: There is an $i \in \{1,3,5\}$ such that $p_i, p_{i+1} \notin M_f$.

Without loss of generality $i = 1$. By the claim there are $\overline{t}_1, \overline{t}_2 \in P$ such that $|\{t_1, \widetilde{t}_1, \overline{t}_1\}| = |\{t_2, \widetilde{t}_2, \overline{t}_2\}| = 3$ and

$$M_f \supseteq \{(t_1, \overline{t}_1, \widetilde{t}_1), (\overline{t}_1, t_1, \widetilde{t}_1), (t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2)\}.$$

We show that $|\{(t_1, \overline{t}_1, \widetilde{t}_1), (\overline{t}_1, t_1, \widetilde{t}_1), (t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2)\}| = 4$. So, suppose contrapositive that $|\{(t_1, \overline{t}_1, \widetilde{t}_1), (\overline{t}_1, t_1, \widetilde{t}_1), (t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2)\}| \leq 3$. As

$$|\{t_1, \widetilde{t}_1, \overline{t}_1\}| = |\{t_2, \widetilde{t}_2, \overline{t}_2\}| = 3$$

and $p_1 \neq p_2$ this implies that $t_1 = \overline{t}_2, \overline{t}_1 = t_2$ and $\widetilde{t}_1 = \widetilde{t}_2$. Hence, if we let $t = t_1, \overline{t}_1 = \overline{t}$ and $\widetilde{t}_1 = \widetilde{t}$ then $p_1 = (t, t, \widetilde{t}) = q_2$ and $p_2 = (\overline{t}, \overline{t}, \widetilde{t}) = q_1$. An application of Lemma 4 yields that $t = uvw$ and $\overline{t} = vuw$ where $v = f(p_1)$ and $u = f(p_2)$. By unanimity then $\widetilde{t} \in \{wuv, wvu\}$. But then, if $\widetilde{t} = wuv$ $p_1 \in M_f$ by unanimity and if $\widetilde{t} = wvu$ $p_2 \in M_f$ by unanimity, a contradiction. Hence,

$$|\{(t_1, \overline{t}_1, \widetilde{t}_1), (\overline{t}_1, t_1, \widetilde{t}_1), (t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2)\}| = 4.$$

Note that we are now in a similar situation as in the end of the proof of the claim, except that we have additionally that $|\{t_1, \widetilde{t}_1, \overline{t}_1\}| = 3$ and $|\{t_2, \widetilde{t}_2, \overline{t}_2\}| = 3$. So, by considering p_3, p_4, p_5 and p_6, we can obtain the same contradiction. This finishes case 1.

Case 2: There is an $i \in \{1,3,5\}$ such that $p_i, p_{i+1} \in M_f$.

Then $|M_f| \leq 5$ and case 1 implies that without loss of generality either $p_1, p_2, p_3, p_5 \in M_f$ and $p_4, p_6 \notin M_f$ or $p_1, p_2, p_3, p_4, p_5 \in M_f$ and $p_6 \notin M_f$. In both cases we can choose $t_6, \overline{t}_6, \widetilde{t}_6 \in P$ by the claim such that $|\{t_6, \overline{t}_6, \widetilde{t}_6\}| = 3$ and $(\widetilde{t}_6, t_6, \overline{t}_6), (\widetilde{t}_6, \overline{t}_6, t_6) \in M_f$. Then

$$\{p_1, p_2, p_3, p_5\} \cap \{(\widetilde{t}_6, t_6, \overline{t}_6), (\widetilde{t}_6, \overline{t}_6, t_6)\} = \varnothing,$$

contradicting $|M_f| \leq 5$. This finishes case 2.

Case 3: For all $i \in \{1,3,5\}$ either $p_i \in M_f$ and $p_{i+1} \notin M_f$ or $p_i \notin M_f$ and $p_{i+1} \in M_f$.

Without loss of generality $p_1, p_3, p_5 \in M_f$. By the claim there are $t_i, \overline{t}_i, \widetilde{t}_i \in P, i \in \{2,4\}$, such that

$$M_f \supseteq \{(t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2), (t_4, \widetilde{t}_4, \overline{t}_4), (\overline{t}_4, \widetilde{t}_4, t_4)\}$$

and $|\{t_i, \overline{t}_i, \widetilde{t}_i\}| = 3$ for all $i \in \{2,4\}$. Then $|M_f| \leq 5$ implies that

$$\{(t_2, \overline{t}_2, \widetilde{t}_2), (\overline{t}_2, t_2, \widetilde{t}_2)\} = \{(t_4, \widetilde{t}_4, \overline{t}_4), (\overline{t}_4, \widetilde{t}_4, t_4)\},$$

and so $\overline{t}_2 = \widetilde{t}_4 = t_2$, a contradiction. This finishes case 3.

All three cases together show that under the assumptions of the proposition $|M_f| \leq 5$ leads to a contradiction. Hence, $|M_f| \geq 6$, proving the proposition. ■

Proposition 17 *Let $f : P^{\{1,2,3\}} \to A$ be a unanimous and nondictatorial social choice function. Suppose that all $f_S, S \in \mathcal{S} - \{\{1\}\}$, are nondictatorial. Then*

$$|M_f| \geq m! > 2\left(\frac{m!}{2} - 1\right) + 1.$$

Proof. As $N = \{1,2,3\}$ we have $\mathcal{S} - \{\{1\}\} = \{\{1,2\},\{1,3\}\}$. By Lemma 6 $|\Pi_S(M_{f_S})| \geq \frac{m!}{2}$ for all $S \in \mathcal{S} - \{\{1\}\}$. Without loss of generality $k := |\Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) \cap M_f| \leq |\Pi_{\{1,3\}}(M_{f_{\{1,3\}}}) \cap M_f|$. If the following claim is proven the proposition follows immediately, as then

$$\begin{aligned} |M_f| \;&\geq\; 2 * |\Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f| + |\Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) \cap M_f| \\ &\quad + |\Pi_{\{1,3\}}(M_{f_{\{1,3\}}}) \cap M_f| \\ &\geq\; 2 * \left(\frac{m!}{2} - k\right) + k + k = m! > 2\left(\frac{m!}{2} - 1\right) + 1. \end{aligned}$$

<u>Claim:</u> To each $p \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ we can associate two profiles $r_p^1, r_p^2 \in M_f$ such that $|r_p^1(N)| = |r_p^2(N)| = 3$ and $\{r_p^1, r_p^2\} \cap \{r_{\widetilde{p}}^1, r_{\widetilde{p}}^2\} = \emptyset$ for all $\widetilde{p} \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$, $\widetilde{p} \neq p$.

Let $p = (t_p, t_p, \widetilde{t}_p) \notin M_f$ be manipulable towards $q = (\overline{t}_p, \overline{t}_p, \widetilde{t}_p)$. By Lemma 5 we can assume that $\overline{t}_p \notin \{t_p, \widetilde{t}_p\}$. For all such p by Lemma 2 $r_p^1 := (t_p, \overline{t}_p, \widetilde{t}_p) \in M_f$ and $r_p^2 := (\overline{t}_p, t_p, \widetilde{t}_p) \in M_f$. If $\{r_p^1, r_p^2\} \cap \{r_{\widetilde{p}}^1, r_{\widetilde{p}}^2\} = \emptyset$ for all $p, \widetilde{p} \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$, $\widetilde{p} \neq p$, we are done.

So, suppose that there is an $r \in \{r_p^1, r_p^2\} \cap \{r_{\widetilde{p}}^1, r_{\widetilde{p}}^2\}$ for some $p, \widetilde{p} \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$, $\widetilde{p} \neq p$. By Lemma 12 then $p = (t, t, \widetilde{t}) = \widetilde{q}$ and $q = (\overline{t}, \overline{t}, \widetilde{t}) = \widetilde{p}$ for some $t, \widetilde{t}, \overline{t} \in P$ such that $|\{t, \widetilde{t}, \overline{t}\}| = 3$. Let $x := f(p)$ and $f(\widetilde{p}) = y$. Then $x \neq y$, $(y,x) \in t$ and $(x,y) \in \overline{t}$ because f is manipulable from p to $\widetilde{p}$ and from $\widetilde{p}$ to p. Without loss of generality we may assume that $(x,y) \in \widetilde{t}$. As f is unanimous and $f(\widetilde{p}) = y$ either $\widetilde{t} \neq x \ldots$ or $\overline{t} \neq x \ldots$. As, $m \geq 4$ there are $(m-2)! \geq 2$ preferences $\widehat{t}$ such that $\widehat{t} = xy\ldots$, hence we can choose such a $\widehat{t} \notin \{\widetilde{t}, \overline{t}\}$. Let $\widehat{p} = (\widehat{t}, \widehat{t}, \widetilde{t})$ and

$$\begin{aligned} \widehat{r}^1 &= (\widehat{t}, \overline{t}, \widetilde{t}), \widehat{r}^2 = (\overline{t}, \widehat{t}, \widetilde{t}), \\ \overline{r}^1 &= (\widehat{t}, t, \widetilde{t}), \overline{r}^2 = (t, \widehat{t}, \widetilde{t}). \end{aligned}$$

We need the following subclaim concerning these profiles.

Subclaim: Let $\widehat{r}^i \notin M_f$ for some $i \in \{1,2\}$. Then $\overline{r}^i \in M_f$ and $\widehat{p} \notin \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$. Furthermore then for all $p' \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ such that p' is manipulable by $\{1,2\}$ towards q' with $|p'(N) \cup q'(N)| = 3$ and $\widehat{r}^i$ or $\overline{r}^i, i \in \{1,2\}$, is on a shortest path from p' to q', $p' = \widetilde{p}$ or $p' = p$ respectively.

Without loss of generality $i = 1$. If $f(\widehat{r}^1) \in A - \{x,y\}$ then agent 1 can manipulate at $\widehat{r}^1$ towards $\widetilde{p}$, contradicting $\widehat{r}^1 \notin M_f$. If $f(\widehat{r}^1) = x$ then $\widetilde{p}$ is manipulable by agent 1 towards $\widehat{r}^1$, contradicting $\widetilde{p} \notin M_f$. Thus

$$f(\widehat{r}^1) = y.$$

If $f(\widehat{p}) = y$, then $\widehat{p} \in M_f$ as agent 3 can deviate towards $(\widehat{t}, \widehat{t}, \widehat{t})$ and $(x,y) \in \widetilde{t}$. If $f(\widehat{p}) \in A - \{x,y\}$ then $\widehat{p} \in M_f$ as agent 2 can deviate towards $\widehat{r}^1$ and $\widehat{p}(2) = \widehat{t} = xy\ldots$. If $f(\widehat{p}) = x$ then $\widehat{p} \notin \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ as $\widehat{p}(\{1,2\}) = \widehat{t} = x\ldots$, so coalition $\{1,2\}$ has no incentive to manipulate. Hence, for all values of $f(\widehat{p})$ either $\widehat{p} \in M_f$ or $\widehat{p} \notin \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ implying in particular that

$$\widehat{p} \notin \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f.$$

If $f(\overline{r}^1) = x$ then agent 2 can manipulate at $\widehat{r}^1$ towards $\overline{r}^1$, contradicting $\widehat{r}^1 \notin M_f$. Hence, $f(\overline{r}^1) \neq x$. But then agent 1 can manipulate at $\overline{r}^1$ towards p, so

$$\overline{r}^1 \in M_f.$$

Now, to end the proof of the subclaim let $p' \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ be such that p' is manipulable by $\{1,2\}$ towards q' with $|p'(N) \cup q'(N)| = 3$ and $\widehat{r}^i$ or $\overline{r}^i, i \in \{1,2\}$, is on a shortest path from p' to q'. Then $\widehat{r}^i$ ($\overline{r}^i$) is also on a shortest path from $\widehat{p}$ to $\widetilde{p}$ ($\widehat{p}$ to p) and $\widehat{p} \neq p'$ as $\widehat{p} \notin \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ and $p' \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$. Hence, by part (2) of Lemma 12 $p' = \widetilde{p}$ ($p' = p$). This proves the subclaim.

Now we consider three cases.

Case 1: $\{\widehat{r}^1, \widehat{r}^2\} \cap M_f = \emptyset$.

Then by the subclaim $\overline{r}^1, \overline{r}^2 \in M_f$. By the furthermore part of the subclaim $\overline{r}^1, \overline{r}^2$ can be uniquely associated to p. Then $r^1_{\widetilde{p}}, r^2_{\widetilde{p}}$ can be uniquely associated to $\widetilde{p}$.

Case 2: $\{\widehat{r}^1, \widehat{r}^2\} \cap M_f = \{\widehat{r}^i\}, i \in \{1,2\}$.

Without loss of generality $i = 1$. By the furthermore part of the claim we can uniquely associate $\overline{r}^2, r^1_p$ to p and $\widehat{r}^1, r^2_p$ to $\widetilde{p}$.

Case 3: $\{\widehat{r}^1, \widehat{r}^2\} \subset M_f$.

Now suppose, otherwise we are obviously done, that there is an $\widehat{r}^i, i \in \{1,2\}$, and a $p' \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$, manipulable by $\{1,2\}$ towards q' with

$|p'(N) \cup q'(N)| = 3$, such that $\widehat{r}^i$ is on a shortest path from p' to q'. Then, by part (2) of Lemma 12 $\{p', q'\} = \{\widetilde{p}, \widehat{p}\}$. If $p' = \widetilde{p}$ for all such p' we are done, since we can uniquely associate $\widehat{r}^1, \widehat{r}^2$ to $\widetilde{p}$ and r_p^1, r_p^2 to p. So, suppose that $p' = \widehat{p}$ and $q' = \widetilde{p}$, i.e. $\widehat{p} = (\widehat{t}, \widehat{t}, \widetilde{t})$ is manipulable towards $\widetilde{p} = (\overline{t}, \overline{t}, \widetilde{t})$ by coalition $\{1, 2\}$. As $f(\widetilde{p}) = y$ this implies that $(y, f(\widehat{p})) \in \widehat{t}$ and $f(\widehat{p}) \neq y$. Since $\widehat{t} = xy \ldots$ and $f(p) = x$, $p = (t, t, \widetilde{t})$, it follows that $\widehat{p}$ is also manipulable by $\{1, 2\}$ towards p. As $\widehat{p} = p' \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$ this implies that $\overline{r}^1, \overline{r}^2 \in M_f$. So, we have found six manipulable profiles $\{r_p^1, r_p^2, \overline{r}^1, \overline{r}^2, \widehat{r}^1, \widehat{r}^2\}$ for the three distinct profiles $p, \widetilde{p}, \widehat{p} \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$. We are done if we can show that these six manipulable profiles are uniquely associated to $p, \widetilde{p}, \widehat{p}$. The profiles $r_p^1, r_p^2, \widehat{r}^1, \widehat{r}^2$ are by construction not associated to any $p" \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f, p'' \notin \{p, \widetilde{p}, \widehat{p}\}$. Suppose that $\overline{r}^i, i \in \{1, 2\}$, is associated to $p" \in \Pi_{\{1,2\}}(M_{f_{\{1,2\}}}) - M_f$, i.e. on a path from $p"$ to $q"$. Again then, by part (2) of Lemma 12 $\{p'', q''\} = \{p, \widehat{p}\}$, and this finishes case 3.

Cases 1 to 3 show that the claim is true, and this proves the proposition by the remarks before the claim. ■

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