A Class of Methods for Evaluating Multiattribute Utilities

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Abstract

A state—for instance a health state—is characterized by a number of attributes to each of which a level is assigned. A specific collection of numerical values, for instance utilities, for all possible states is called a situation. The main purpose of the paper is to develop a class of methods that assign, for a given situation, a numerical value to each possible level of each attribute, intended to measure the contribution of each such level to reaching the perfect state, in which each attribute has maximal level. The paper focuses on methods that share four properties: distribution, zero contribution, homogeneity, and the transfer property. All these methods have the property of marginalism: they measure the effect of lowering, *ceteris paribus*, a certain level by one. Within the class of methods so obtained, special attention is given to the so called egalitarian valuation, which treats lower and higher levels equally.

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1 Introduction

The present work arose from the following practical problem. In an empirical study that was conducted among hearing impaired individuals, each of these individuals described his hearing state and assigned a numerical value to it. The hearing states were composed of levels ('bad', 'reasonable' or 'good') of five different attributes that have to do with hearing abilities. After some numerical operations a collection of numbers between 0 and 1 for all hearing states resulted. The number 0 corresponded to the state where all attributes are at their minimal level, and the number 1 to the combination of all attributes at the maximal

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level. We were interested in the following question: what is the *contribution* of each level of an attribute to reaching the perfect, maximal hearing state?

Such a question arises not only in problems of medical decision making. It may be relevant in any situation where we would like to unravel an overall effect in terms of contributing factors. For instance, in many universities students evaluate courses by assigning levels to different aspects and at the same time providing overall grades. Other examples can be found in economic and social studies, and not only with subjective states and values as in the hearing impairment or student evaluation example, but also with objective data.

The existing literature does provide some methods to answer this question, notably multiattribute utility theory. Our first motivation for nevertheless developing an alternative theory is that standard multiattribute utility theory is based on the assumption of utility independence, see Keeney and Raiffa (1976), in spite of the fact that this assumption is usually violated in practice (e.g., Torrance *et al.*, 1982). The methodology that we propose does not rely on an assumption of utility independence. Instead, we derive a class of methods based on four—in our opinion—quite acceptable conditions. Each member of this class assigns to (e.g.) the hearing situation described above a matrix of 15 numbers, corresponding to the 15 levels. The five low levels (one for each attribute) will always be assigned 0 since they are always reached and do not contribute anything. If the perfect state (every attribute on the highest level) gets assigned 1, then the sum of the remaining 10 numbers in the matrix equals 1. Each number is interpreted as expressing the contribution of reaching the corresponding level to attaining the perfect state. The sum of the numbers per attribute can be seen as representing the contribution of that attribute as a whole. In that sense, it can be used as a check on the appropriateness of the attributes used in a study—obtained, for instance, in an earlier stage by factor analysis.

An important source of inspiration for the proposed methods are some concepts and results in the theory of cooperative games. We mention Dubey (1975) and Einy (1988); and Weber (1988) for the study of the concept of marginalism in Section 5 below. More detailed references will be given at the appropriate places. Our work is also formally related to the work on multichoice games (Hsiao and Raghavan, 1993, and Faigle and Kern, 1992), and our methods could be applied to such games as well.

The present paper is theoretical in nature. For an elaboration of the application of the methodology to the hearing impairment case we refer to Zank *et al.* (2002).

In Section 2 of the paper we introduce the main concepts that play a role in the sequel. The class of methods that we consider is characterized in Section 3. Section 4 deals with the idea of *marginal* contributions. In Section 5 one specific method, the so called egalitarian valuation, is studied in detail. All proofs are collected in Appendix A. Appendix B presents a different way to compute the egalitarian valuation and comments on its relation with the Shapley value (Shapley, 1953).

2 Main concepts

Throughout, the collection of *attributes* $N = \{1, 2, \ldots, n\}$ and the collection of *levels* $M = \{0, 1, 2, \ldots, m\}$ $(m \geqslant 1)$ are fixed. A *state* is a map $\omega : N \to M$. So a state assigns a level to each attribute. Observe that we assume implicitly that each attribute has the same number of levels, namely $m + 1$. This is just for convenience: independent of this assumption, an accurate description of the 'right' levels may be a tricky problem in practical situations. The set of all states is denoted by Ω, hence $\Omega = M^N$.

There is a natural ordering on the set Ω. Identifying $\omega \in \Omega$ with the vector $(\omega(1), \omega(2), \ldots, \omega(n))$ of the levels of the n attributes, we write $\omega \geqslant \omega'$ for $\omega, \omega' \in \Omega$ if this inequality holds for each coordinate, i.e. if $\omega(i) \geqslant \omega'(i)$ for all $i \in N$. This means that the level of every attribute is at least as high in ω as it is in ω'. In practical applications, higher levels are regarded as better.

A *situation* h is a collection of numerical values for all possible $((m+1)^n)$ states satisfying a few additional properties. Formally, it is a map $h : \Omega \to I\!R$ satisfying

(i) $h(0, 0, \ldots, 0) = 0$,

(ii) for all $\omega, \omega' \in \Omega$, $\omega \geqslant \omega'$ implies $h(\omega) \geqslant h(\omega')$.

Property (i) normalizes the value of the worst possible state to zero. Property (ii) states that the assigned values should be monotonic, in acccordance with the implicit assumption that higher levels are better. Let H denote the collection of all situations, hence $H = I\!R^{\Omega}$. Note that (i) and (ii) imply that $h(\omega) \geqslant 0$ for all $h \in H$ and $\omega \in \Omega$.

We wish to say something about the impact or importance of each level of each attribute for any given situation $h \in H$. Formally this means that we will study maps $\varphi : H \to I\!R^{n \times (m+1)}$, i.e., maps that assign to each situation a collection of numbers, one for each level of each attribute. For a situation h, an attribute i and a level j, the number $\varphi_{i,j}(h)$ expresses the importance or contribution of level j of attribute i in situation h. For instance, when health situations are concerned, this number $\varphi_{i,j}(h)$ should give information about what it is worth to reach level j of attribute i on the way to the perfect health state $(m, m, \ldots, m)$.

In order to be justified in attaching this interpretation to such a map φ we need to make some assumptions. The first assumption says that the values assigned by φ should indeed be a distribution of the value of the perfect state.

Distribution For all $h \in H$, $\sum_{i \in N,\ j \in M} \varphi_{i,j}(h) = h(m, m, \ldots, m)$.

The next assumption captures the idea of 'contribution' in a special case. Suppose that, in a certain situation, lowering level j of attribute i to level $j-1$ never makes any difference: the value of h does not change. Then it is natural to assign a value of 0 to the original level j. Formally, this leads to the following condition.

Zero Contribution For all $h \in H, i \in N, j \in M$, if $h(\omega) = h(\omega')$ for all $\omega, \omega' \in \Omega$ with $\omega(i) = j$, $\omega'(i) = j - 1$ and $\omega(k) = \omega'(k)$ for all $k \in N, k \neq i$, then $\varphi_{i,j}(h) = 0$.

Observe that, trivially, this condition implies that $\varphi_{i,0}(h) = 0$ for all $h \in H$ and $i \in N$.

In practical situations the value of the perfect state is often fixed. For instance, in health situations it is usually fixed at 1. If this is not the case, we would want our contribution measure φ to be independent of the value of the perfect state, so that we can always scale it to a fixed number. This is expressed by the following property, where we use the notations $(ah)(\omega) := ah(\omega)$ and $a\varphi(h) := (a\varphi_{i,j}(h))_{i,j}$ for $h \in H$ and $a \in I\!R$.

Homogeneity For all $h \in H$ and all $a \in I\!R$ with $a \geqslant 0$ we have $\varphi(ah) = a\varphi(h)$.

In many practical situations we obtain the numerical values of h by measurement. In health situations for instance, these numbers are often obtained by interviewing individuals. Clearly, individuals or interviewers may make mistakes and, moreover, individual values are subjective. Suppose now that we have obtained two sets of values h and h' of the same population. An obvious way to estimate the 'true' values would be to take the (state-wise) average of h and h'. A corresponding natural condition on φ would be to require that it should not make a difference whether φ is applied to this average or whether the average of the φ-values is calculated. In symbols, we would require

$$\varphi(\frac{h+h'}{2}) = \frac{\varphi(h)+\varphi(h')}{2}.$$

Under homogeneity, this implies

$$\varphi(h+h') = \varphi(h)+\varphi(h').$$

A consequence of this condition would be that it does not make a difference whether we calculate the φ-values either for h and h' and then take the sum, or for the new situations arising from taking (state-wise) minima of h and h' and maxima of h and h', respectively, and then take the sum. It is this weaker condition that will be imposed. The reason for this is partly technical: it allows us to use representations of situations on the basis of so called simple situations, using only positive coefficients. See the Appendix for the details, in particular Lemma A.1. In order to formalize this, define for $h, h' \in H$ the new situations $h \vee h'$ and $h \wedge h'$ by $h \vee h'(\omega) := \max\{h(\omega), h'(\omega)\}$ and $h \wedge h'(\omega) := \min\{h(\omega), h'(\omega)\}$ for all $\omega \in \Omega$.

Transfer Property For all $h, h' \in H$,

$$\varphi(h \vee h') + \varphi(h \wedge h') = \varphi(h) + \varphi(h').$$

The name Transfer Property (cf. Dubey, 1975) comes from the observation that $h \vee h'$ and $h \wedge h'$ arise from h and h' by transferring maximal and minimal values between these two.

A map $\varphi : H \to I\!R^{n\times(m+1)}$ satisfying the Transfer Property is called a *valuation*. See for instance Topkis (1998), and Einy (1988, page 5).

The remainder of the paper is concerned with studying valuations satisfying the properties of Distribution, Homogeneity, and Zero Contribution.

3 A characterization of valuations

In order to characterize valuations we consider their values on a specific subclass of situations. This subclass should be large enough so as to determine a valuation on the whole class H. It should consist of situations that are transparent enough to reveal the implications of a specific valuation for those situations. A subclass of situations satisfying both requirements is the class of so called simple situations. For every $\omega \in \Omega$, $\omega \neq (0, 0, \ldots, 0)$ define the *simple situation* h_ω by

$$h_\omega(\omega') = \begin{cases} 1 & \text{if } \omega' \geqslant \omega \\ 0 & \text{otherwise.} \end{cases}$$

Such a simple situation is characterized by a fixed state that has value equal to 1. All states with at least one level below this given state have zero value, all other states have value equal to 1. The collection of all simple situations is denoted by S. (Simple situations correspond to unanimity games in cooperative game theory.)

An individual reporting a health situation of the form h_ω would probably be a rare event, because such an individual would only be interested in reaching a certain minimal level in any attribute. The point, however, is that these situations are very well suited to fix a valuation.

It turns out that the Transfer Property of a valuation is sufficient to extend it uniquely from simple situations to all situations if we additionally impose Homogeneity. The proof of this result, like all other proofs, is given in Appendix A.

Theorem 3.1 *Let $\varphi' : S \to \mathbb{R}^{n\times(m+1)}$. Then there is a unique homogeneous valuation φ that coincides with φ' on S.*

This theorem is of central importance. It tells us that, in order to define a homogeneous valuation, it is sufficient to choose its values on the class of simple situations. The next theorem tells us how this choice is restricted by the conditions of Distribution and Zero Contribution.

Theorem 3.2 *Let φ be a homogeneous valuation. Then φ satisfies Zero Contribution if, and only if, for every nonzero $\omega \in \Omega$, we have*

(i) $\varphi_{i,j}(h_\omega) = 0$ *for all* $i \in N$ *and* $j \in M$ *with* $\omega(i) \neq j$ *or* $j = 0$,

and φ satisfies Distribution if, and only if, for every nonzero $\omega \in \Omega$, we have

(ii) $\sum_{i\in N, j\in M} \varphi_{i,j}(h_\omega) = 1$.

By combining these two theorems we see that a valuation with the three additional properties of Homogeneity, Distribution, and Zero Contribution is completely determined by distributing, for every nonzero state, the total of 1 over the nonzero levels of that state. There are obviously many ways to do this. A common requirement, however, should be that the values assigned by a valuation are independent of the order in which the attributes are placed. To formalize this we need some notation. For a permutation σ of N, a state $\omega \in \Omega$ and a situation $h \in H$ define $\sigma\omega \in \Omega$ by $\sigma\omega(\sigma(i)) := \omega(i)$ for every $i \in N$, and $\sigma h \in H$ by $\sigma h(\sigma\omega') := h(\omega')$ for every $\omega' \in \Omega$. So σh arises from h by interchanging attributes according to σ.

Attribute Symmetry For every permutation σ of N, every attribute $i \in N$, every level $j \in M$, and every situation $h \in H$, $\varphi_{i,j}(h) = \varphi_{\sigma(i),j}(\sigma h)$.

In order to obtain an attribute symmetric homogeneous valuation it is sufficient to satisfy this property on simple situations, as the following theorem shows.

Theorem 3.3 *Let φ be a homogeneous valuation. Then φ is attribute symmetric if, and only if, for every permutation σ of N, every attribute $i \in N$, every level $j \in M$ and every $\omega \in \Omega$, $\varphi_{i,j}(h_\omega) = \varphi_{\sigma(i),j}(h_{\sigma\omega})$.*

We will now introduce the so called egalitarian valuation. This valuation can be justified as follows. For a nonzero state ω each of the nonzero levels of ω has to be reached in h_ω in order to improve from 0 to 1. Thus, each of these levels is equally important and should be assigned the same value by φ. Formally:

Definition 3.1 The *egalitarian valuation* is the homogeneous valuation, denoted by ε, with for all nonzero $\omega \in \Omega$, all $i \in N$ and all $j \in M$:

$$\varepsilon_{i,j}(h_\omega) = \begin{cases} \frac{1}{n(\omega)} & \text{if } \omega(i) = j \text{ and } j \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $n(\omega)$ denotes the number of nonzero levels of ω.

This definition describes the egalitarian valuation formally for simple situations. Lemma A.4 in Appendix A gives the general form of a valuation based on its values for simple situations.

In Section 5 we will focus on the egalitarian valuation and characterize it by an additional condition called Level Symmetry. In the next section we show that every homogeneous valuation having the Zero Contribution property satisfies a property of marginalism: its values depend only on marginal contributions. This leads to alternative formulations of such valuations, and in particular of the egalitarian valuation.

4 Marginalism

As explained in the Introduction, the idea underlying valuations is that they should represent the contribution of each level to reaching the 'perfect' state. The following property gives this idea a precise meaning. It says that the value assigned to each level of each attribute should only depend on the marginal contributions of that level and not on any other aspects of a situation. (In Young (1985) this idea was exploited to characterize the Shapley value (Shapley, 1953) for cooperative games.)

Marginalism For all $h, h' \in H$, $i \in N$, $j \in M$, if $h(\omega) - h(\omega') = h'(\omega) - h'(\omega')$ for all $\omega, \omega' \in \Omega$ with $\omega(k) = \omega'(k)$ for all $k \neq i$, $\omega(i) = j$, and $\omega'(i) = j - 1$, then $\varphi_{i,j}(h) = \varphi_{i,j}(h')$.

Zero Contribution is in fact a weak form of Marginalism. It turns out that Zero Contribution implies Marginalism for a homogeneous valuation and that, moreover, such a valuation is an additive function of marginal contributions. For $\omega \in \Omega$ denote by $\omega + e_i$ [resp. $\omega - e_i$] the state arising by increasing [resp. decreasing] the level of attribute i by one (if possible).

Theorem 4.1 *Let φ be a homogeneous valuation satisfying Zero Contribution. Then φ satisfies Marginalism and for every $i \in N$ and $j \in M$, $j \neq 0$ there is a collection of real numbers $\{p_\omega : \omega \in \Omega$ with $\omega(i) = j - 1\}$ such that*

$$\varphi_{i,j}(h) = \sum_{\omega \in \Omega:\ \omega(i) = j-1} p_\omega [h(\omega + e_i) - h(\omega)]$$

for all $h \in H$. If, additionally, φ satisfies Distribution then these numbers p_ω sum to one.

This theorem implies that for instance the egalitarian valuation ε has an additive representation on the basis of marginal contributions. The result can be seen as the analogon of similar results for cooperative games in Weber (1988).

For attribute symmetric valuations we have the following result. To avoid confusion we will sometimes use a notation like p^i_ω instead of p_ω as in Theorem 4.1.

Theorem 4.2 *Let φ be a homogeneous valuation satisfying Zero Contribution and Attribute Symmetry, and let for $i \in N$ and $0 \neq j \in M$ the weights p^i_ω be as determined in Theorem 4.1. Let σ be a permutation of N. Then for every ω with $\omega(i) = j - 1$ we have*

$$p^i_\omega = p^{\sigma(i)}_{\sigma\omega}.$$

Theorem 4.2 implies that under Attribute Symmetry the weights p_ω depend only on the level and not on the particular attribute to which that level is assigned. Obviously, this theorem applies to the egalitarian valuation in particular.

5 The egalitarian valuation

In this section we focus on one particular method, the egalitarian valuation, as described in Definition 3.1. The main result is an axiomatic characterization of this valuation.

Let $i \in N$ be an attribute, $h \in H$ a situation, and let $l, l' \in M \backslash \{0\}$ be distinct levels. We call l and l' *symmetric* in attribute i for the situation h if for all $\omega, \omega' \in \Omega$ with $\omega(i) = l, \omega'(i) = l'$ and $\omega(k) = \omega'(k)$ for $k \neq i$ the following holds:

$$h(\omega) - h(\omega - e_i) = h(\omega') - h(\omega' - e_i).$$

In words, two levels of an attribute are symmetric if decreasing them by 1 has the same effect on the value of a state. An intuitive requirement is that to such levels a method that is intended to measure contributions should assign the same value. Formally, this leads to the following condition for a valuation φ on H.

Level Symmetry For any $h \in H$, $i \in N$ we have $\varphi_{i,l}(h) = \varphi_{i,l'}(h)$ for all $l, l' \in M \backslash \{0\}$ that are symmetric in attribute i for the situation h.

We now have the following characterization of the egalitarian valuation.

Theorem 5.1 $\varphi : H \rightarrow I\!R^{n\times(m+1)}$ *is a homogeneous valuation satisfying Zero Contribution and Distribution, Attribute Symmetry and Level Symmetry if, and only if, φ is the egalitarian valuation.*

If we take $m = 1$ and drop Level Symmetry, which has no bite anyway in that case, then basically this theorem characterizes the Shapley value (Shapley, 1953). Therefore the class of methods satisfying all the conditions in this theorem but not necessarily Level Symmetry, extends the Shapley value. Appendix B elaborates on how the Shapley value is extended by the egalitarian valuation.

Some remarks on the computation of valuations and in particular the egalitarian valuation are in order. One way to compute a valuation for a given situation $h \in H$ is to first compute the representation of h on simple situations h_ω (see Lemma A.1 and formulas (1) and (2) in Appendix A) and then compute the valuation according to Lemma A.4. Alternatively, one may determine the weights p_ω as in Theorem 4.1 and then use the expression in Theorem 4.1 to calculate a valuation on h. These weights p_ω are derived in Lemmas A.5 and A.6: they are related to the values assigned to specific situations by the linear extension of a valuation to the class of nonmonotonic situations. For the egalitarian valuation these values are given in Theorem B.1 in Appendix B.

A Proofs

For $h \in H$ and $\alpha \in I\!R_+$ (the set of nonnegative real numbers) denote by $\Omega(\alpha, h)$ the set of minimal states with value α under h, i.e.

$$\Omega(\alpha, h) := \{\omega \in \Omega : h(\omega) = \alpha \text{ and } h(\omega') < \alpha \text{ for all } \omega' \in \Omega \text{ with } \omega \geqslant \omega',\ \omega \neq \omega'\}.$$

As before, let $\vee$ denote the maximum operator.

Lemma A.1 *Let $h \in H$, with range $h(\Omega) = \{0, \alpha_1, \ldots, \alpha_r\}$. Then*

$$h = \bigvee_{i=1}^{r} \bigvee_{\omega \in \Omega(\alpha_i, h)} \alpha_i h_\omega. \tag{1}$$

Proof. Take $0 \neq \omega' \in \Omega$ arbitrary, and suppose $h(\omega') = \alpha_i$ for some $i \in \{1, \ldots, r\}$. Note that, by the monotonicity of h (property (ii) in the definition of a situation) and the assumption that the zero state always has value zero (property (i)), the range of h can only contain nonnegative numbers. For the right hand side of (1) we obtain

$$\bigvee_{i=1}^{r} \bigvee_{\omega \in \Omega(\alpha_i, h)} \alpha_i h_\omega(\omega') \geqslant \alpha_i.$$

Suppose this inequality were strict. Then there would be a $k \in \{1, \ldots, r\}$ and an $\omega'' \in \Omega(\alpha_k, h)$ with $\omega'' \leqslant \omega'$ and $\alpha_k h_{\omega''}(\omega') > \alpha_i$, in particular $h(\omega'') = \alpha_k > \alpha_i = h(\omega')$. Because $\omega'' \leqslant \omega'$ this contradicts the monotonicity of h. □

It can be shown that the representation in (1) of $h \in H$ is in fact unique among representations of this form. This fact, however, will not be used in the sequel. For our purposes, it is sufficient to note that every $h \in H$ has a representation of the form

$$h = \bigvee_{i \in I} \alpha_i h_{\omega^i} \tag{2}$$

on the collection of simple situations, where I is some finite index set.

As before, let $\wedge$ denote the minimum operator. The following lemma expresses the valuation of a maximum of a collection of situations in terms of their minima.

Lemma A.2 *Let φ be a valuation and let $h_1, h_2, \ldots, h_r \in H$. Then $\bigvee_{i=1}^r h_i \in H$ and*

$$\begin{aligned}\varphi(\textstyle\bigvee_{i=1}^r h_i) \;=\; & \textstyle\sum_{i=1}^r \varphi(h_i) - \sum_{i<j}^r \varphi(h_i \wedge h_j) \\ & + \textstyle\sum_{i<j<\ell}^r \varphi(h_i \wedge h_j \wedge h_\ell) \\ & \vdots \\ & +(-1)^{r+1} \varphi\left(\textstyle\bigwedge_{i=1}^r h_i\right).\end{aligned}$$

Proof. Straightforward, using the Transfer Property, by induction on r. □

Also the proof of the following lemma is straightforward and left to the reader (cf. Einy, 1988, Lemma 11).

Lemma A.3 *Let $\alpha, \beta \in I\!R_+$ and $\omega, \omega' \in \Omega$. Then*

$$\alpha h_\omega \wedge \beta h_{\omega'} = (\alpha \wedge \beta) h_{\omega \vee \omega'},$$

in particular, $\alpha h_\omega \wedge \beta h_{\omega'}$ is of the form $\gamma h_{\omega''}$ for some $\gamma \in I\!R_+$ and $\omega'' \in \Omega$.

The next lemma gives an expression for homogeneous valuations on H with the aid of simple situations in $S \subset H$. It follows from Lemmas A.2 and A.3.

Lemma A.4 *Let φ be a homogeneous valuation, and let $h \in H$ have the representation (2) with $I = \{1, \ldots, r\}$. Then*

$$\begin{aligned} \varphi(h) \quad = \quad & \textstyle\sum_{i=1}^r \alpha_i \varphi(h_{\omega^i}) - \sum_{i,j \in I, i<j} (\alpha_i \wedge \alpha_j) \varphi(h_{\omega^i \vee \omega^j}) \\ & + \textstyle\sum_{i,j,\ell \in I, i<j<\ell} (\alpha_i \wedge \alpha_j \wedge \alpha_\ell) \varphi(h_{\omega^i \vee \omega^j \vee \omega^\ell}) \\ & \vdots \\ & +(-1)^{r+1} \wedge_{i=1}^r \alpha_i \varphi\left(h_{\vee_{i \in I} \omega^i}\right). \end{aligned}$$

Proof of Theorem 3.1. Follows from Lemmas A.4 and A.3. □

Proof of Theorem 3.2. The only-if parts are left to the reader.

Let φ be a homogeneous valuation satisfying (i) and (ii) in the Theorem for every nonzero $\omega \in \Omega$.

We first show Zero Contribution of φ. Let i, j, h be as in the formulation of Zero Contribution. Let $h = \vee_{k \in I} \alpha_k h_{\omega^k}$ as in (2). Because every α_k is positive, it follows that the condition in Zero Contribution holds for every h_{ω^k}, and hence $\varphi_{i,j}(h_{\omega^k}) = 0$ for every $k \in I$ by (i). Using Lemmas A.3, A.4, (i) and induction, it is sufficient to prove that, for all $k, \ell \in I$ and ω, ω' as in the definition of Zero Contribution we have: $h_{\omega^k \vee \omega^\ell}(\omega) - h_{\omega^k \vee \omega^\ell}(\omega') = 0$. If $h_{\omega^k \vee \omega^\ell}(\omega) = 0$ then also $h_{\omega^k \vee \omega^\ell}(\omega') = 0$ because $\omega' \leqslant \omega$. If $h_{\omega^k \vee \omega^\ell}(\omega) = 1$ then $\omega \geqslant \omega^k$ and $\omega \geqslant \omega^\ell$, hence $h_{\omega^k}(\omega) = 1$ and $h_{\omega^\ell}(\omega) = 1$. This implies that also $h_{\omega^k}(\omega') = 1$ and $h_{\omega^\ell}(\omega') = 1$ because both situations satisfy the condition in Zero Contribution. Therefore, $h_{\omega^k \vee \omega^\ell}(\omega') = 1$, which completes the proof.

In order to prove Distribution for φ, we use induction on r, with r as in Lemma A.4. For $r = 1$, Distribution follows from homogeneity of φ and (ii). For arbitrary r and repeatedly applying (ii), Lemma A.4 implies (with notation as there)

$$\begin{aligned} \textstyle\sum_{i \in N, j \in M} \varphi_{i,j}(h) \quad = \quad & \textstyle\sum_{k=1}^r \alpha_k - \sum_{k<\ell}^r \alpha_k \wedge \alpha_\ell \\ & + \textstyle\sum_{k<\ell<p}^r \alpha_k \wedge \alpha_\ell \wedge \alpha_p \\ & \vdots \\ & +(-1)^{r+1} \wedge_{k=1}^r \alpha_k. \end{aligned}$$

To prove Distribution, we have to prove that the right hand side of this equality is equal to $h(m, m, \ldots, m)$, hence to $\vee_{k=1}^r \alpha_k$. This can be proved by induction on r, and is left to the reader. □

Proof of Theorem 3.3. For the proof of the only-if part it is sufficient to check (left to the reader) that for every $\omega \in \Omega$ and every permutation σ of N it holds that $\sigma h_\omega = h_{\sigma\omega}$. For the if-part, it is sufficient to check that equation (2) implies

$$\sigma h = \bigvee_{i \in I} \alpha_i h_{\sigma \omega^i}$$

for every permutation σ of N. $\square$

In order to prepare for the proof of Theorem 4.1 we first extend the set of situations H to the larger set $\tilde{H}$ by dropping the monotonicity requirement (ii) in the definition of a situation. With the operations $(h+h')(\omega) = h(\omega)+h(\omega')$ and $(ah)(\omega) = ah(\omega)$ for all $h, h' \in \tilde{H}$, $\omega \in \Omega$, and $a \in I\!R$, the set $\tilde{H}$ is a linear space with the collection of simple situations as a basis. It is easily seen that the restriction of any map $\tilde{\varphi} : \tilde{H} \to I\!R^{n\times(m+1)}$ that is linear, i.e.

$$\tilde{\varphi}(ah + bh') = a\tilde{\varphi}(h) + b\tilde{\varphi}(h') \text{ for all } a, b \in I\!R \text{ and } h, h' \in \tilde{H},$$

to H is a homogeneous valuation (i.e., satisfying the transfer property) φ on H, uniquely determined by its values on simple situations. Hence there is a one-to-one correspondence between homogeneous valuations on H and their linear extensions on $\tilde{H}$. So we can derive properties of a homogeneous valuation φ on H from properties of its linear extension $\tilde{\varphi}$ on $\tilde{H}$.

As a first step we prove the following lemma.

Lemma A.5 *Let $\varphi : \tilde{H} \to I\!R^{n\times(m+1)}$ be a linear map. Let $i \in N$ and $j \in M$. Then there exists a collection of constants $\{a_\omega : \omega \in \Omega\}$, such that for any $h \in \tilde{H}$*

$$\varphi_{i,j}(h) = \sum_{\omega\in\Omega} a_\omega h(\omega).$$

Proof. For $\omega \in \Omega$, $\omega \neq (0, \ldots, 0)$ we define the situation $\kappa_\omega \in \tilde{H}$ as follows:

$$\kappa_\omega(\widetilde{\omega}) := \begin{cases} 1 & \text{if } \widetilde{\omega} = \omega \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the set $\{\kappa_\omega : \omega \in \Omega, \omega \neq (0, \ldots, 0)\}$ is also a basis for $\tilde{H}$. Every $h \in \tilde{H}$ can be uniquely written as

$$h = \sum_{\omega\in\Omega\setminus\{(0,\ldots,0)\}} h(\omega)\kappa_\omega,$$

and linearity of φ implies

$$\varphi(h) = \sum_{\omega\in\Omega\setminus\{(0,\ldots,0)\}} h(\omega)\varphi(\kappa_\omega).$$

Therefore, with $a_\omega := \varphi_{i,j}(\kappa_\omega)$ for $\omega \in \Omega\setminus\{(0, \ldots, 0)\}$ and $a_{(0,\ldots,0)}$ arbitrary, we obtain

$$\varphi_{i,j}(h) = \sum_{\omega\in\Omega} a_\omega h(\omega).$$

This completes the proof of the lemma. $\square$

In the following two lemmas we include the properties of Zero Contribution and Distribution, the definitions of which are extended in the obvious way to maps $\varphi : \tilde{H} \to I\!R^{n\times(m+1)}$.

Lemma A.6 *Let $\varphi : \tilde{H} \to I\!R^{n\times(m+1)}$ be linear and satisfy Zero Contribution. Let $i \in N$ and $j \in M\setminus\{(0, \ldots, 0)\}$. Then there exists a collection of constants $\{p_\omega : \omega \in \Omega,\ \omega(i) = j - 1\}$, such that for any $h \in \tilde{H}$*

$$\varphi_{i,j}(h) = \sum_{\omega\in\Omega:\ \omega(i)=j-1} p_\omega[h(\omega + e_i) - h(\omega)].$$

Proof. Suppose $\omega \in \Omega$ is a state with $\omega(i) \notin \{j, j-1\}$. As in the proof of Lemma A.5 define $a_\omega := \varphi_{i,j}(\kappa_\omega)$. Observe that $\kappa_\omega(\tilde{\omega}) = \kappa_\omega(\tilde{\omega} + e_i) = 0$ for all $\tilde{\omega} \in \Omega$ with $\tilde{\omega}(i) = j-1$. Therefore, by Zero Contribution, we have $\varphi_{i,j}(\kappa_\omega) = 0$, hence $a_\omega = 0$.

Let $h \in \tilde{H}$. Then, by Lemma A.5 and the preceding argument,

$$\begin{aligned}
\varphi_{i,j}(h) &= \sum_{\omega\in\Omega} a_\omega h(\omega) \\
&= \sum_{\omega\in\Omega:\ \omega(i)\in\{j,j-1\}} a_\omega h(\omega) \\
&= \sum_{\omega\in\Omega:\ \omega(i)=j-1} [a_\omega h(\omega) + a_{\omega+e_i} h(\omega+e_i)].
\end{aligned}$$

We will show by induction that $a_{\omega+e_i} = -a_\omega$ for all $\omega \in \Omega$ with $\omega(i) = j-1$.

Suppose $\tilde{\omega} \in \Omega$ with $\tilde{\omega}(i) = j-1$, then $h_{\tilde{\omega}}(\omega + e_i) = h_{\tilde{\omega}}(\omega)$ for every $\omega \in \Omega$ with $\omega(i) = j-1$. Therefore Zero Contribution implies

$$\varphi_{i,j}(h_{\tilde{\omega}}) = 0. \tag{3}$$

Let now $\tilde{\omega}$ be such that $\tilde{\omega}(i) = j-1$ and $\tilde{\omega}(k) = m$ for all $k \in N$, $k \neq i$. Then

$$\begin{aligned}
\varphi_{i,j}(h_{\tilde{\omega}}) &= \sum_{\omega\in\Omega:\ \omega(i)=j-1} [a_\omega h_{\tilde{\omega}}(\omega) + a_{\omega+e_i} h_{\tilde{\omega}}(\omega+e_i)] \\
&= a_{\tilde{\omega}} + a_{\tilde{\omega}+e_i}.
\end{aligned}$$

Also, $\varphi_{i,j}(h_{\tilde{\omega}}) = 0$ by (3), so that

$$a_{\tilde{\omega}+e_i} = -a_{\tilde{\omega}}.$$

Next let $\tilde{\omega}$ be arbitrary with $\tilde{\omega}(i) = j-1$, and suppose as induction hypothesis that for all $\omega' \geqslant \tilde{\omega}$, with $\omega' \neq \tilde{\omega}$ and $\omega' = j-1$ we have

$$a_{\omega'+e_i} = -a_{\omega'}.$$

Then

$$\begin{aligned}
\varphi_{i,j}(h_{\tilde{\omega}}) &= \sum_{\omega\in\Omega:\ \omega(i)=j-1} [a_\omega h_{\tilde{\omega}}(\omega) + a_{\omega+e_i} h_{\tilde{\omega}}(\omega+e_i)] \\
&= \sum_{\omega\in\Omega:\ \omega(i)=j-1,\ \omega\geqslant\tilde{\omega}} [a_\omega + a_{\omega+e_i}] \\
&= \sum_{\omega\in\Omega:\ \omega(i)=j-1,\ \omega\geqslant\tilde{\omega},\ \omega\neq\tilde{\omega}} [a_\omega + a_{\omega+e_i}] + [a_{\tilde{\omega}} + a_{\tilde{\omega}+e_i}] \\
&= a_{\tilde{\omega}} + a_{\tilde{\omega}+e_i},
\end{aligned}$$

where the last equality follows by the induction hypothesis. Hence, with $\varphi_{i,j}(h_{\tilde{\omega}}) = 0$ by (3), we have

$$a_{\tilde{\omega}+e_i} = -a_{\tilde{\omega}}.$$

Therefore, by induction,

$$a_{\tilde{\omega}+e_i} = -a_{\tilde{\omega}}$$

for all $\tilde{\omega} \in \Omega$ with $\tilde{\omega}(i) = j - 1$. Hence we can write

$$\varphi_{i,j}(h) = \sum_{\omega \in \Omega:\ \omega(i)=j-1} [a_\omega h(\omega) - a_\omega h(\omega + e_i)].$$

The proof of the lemma is complete by taking $p_\omega := -a_\omega$ and observing that these weights are independent of the situation $h \in \tilde{H}$. $\square$

Lemma A.7 *Let φ as in Lemma A.6 additionally satisfy Distribution. Then the weights p_ω as in Lemma A.6 sum to one.*

Proof. Let $\tilde{\omega} = (0, 0, \ldots, 0) + je_i$, then by Lemma A.6

$$\begin{aligned}
\varphi_{i,j}(h_{\tilde{\omega}}) &= \sum_{\omega \in \Omega:\ \omega(i)=j-1} p_\omega [h_{\tilde{\omega}}(\omega + e_i) - h_{\tilde{\omega}}(\omega)] \\
&= \sum_{\omega \in \Omega:\ \omega(i)=j-1} p_\omega.
\end{aligned}$$

Distribution implies

$$\sum_{k=1}^{n} \sum_{l=1}^{m} \varphi_{k,l}(h_{\tilde{\omega}}) = 1,$$

and Zero Contribution implies $\varphi_{k,l}(h_{\tilde{\omega}}) = 0$ if $k \neq i$ and $\varphi_{i,l}(h_{\tilde{\omega}}) = 0$ if $l \neq \tilde{\omega}(i) = j$. Hence

$$1 = \sum_{k=1}^{n} \sum_{l=1}^{m} \varphi_{k,l}(h_{\tilde{\omega}}) = \varphi_{i,j}(h_{\tilde{\omega}}) = \sum_{\substack{\omega \in \Omega \\ \omega(i)=j-1}} p_\omega.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 4.1. The Theorem follows from Lemmas A.6 and A.7 and the remarks about the extension of homogeneous valuations to $\tilde{H}$ preceding Lemma A.5. $\square$

We proceed with proving Theorem 4.2. Like Theorem 4.1 we will prove Theorem 4.2 for the more general case of a linear map $\varphi : \tilde{H} \to I\!R^{n \times (m+1)}$. More specifically, the theorem is implied by the following lemma, in which the condition of Attribute Symmetry is the obvious extension to this more general case.

In the proof of Lemma A.8 we use the following definition. Call two attributes $i, k \in N$ *symmetric in* $h \in \tilde{H}$ if for the permutation $\sigma : N \to N$ with $\sigma(i) = k$, $\sigma(k) = i$, and $\sigma(l) = l$ for all $l \neq i, k$, we have $h(\omega) = \sigma h(\omega)$ for all $\omega \in \Omega$.

Lemma A.8 *Let $\varphi : \tilde{H} \to I\!R^{n \times (m+1)}$ be linear and satisfy Attribute Symmetry and Zero Contribution. Let further p_ω be the weights determined in Lemma A.6. Then, for every permutation σ of N,*

$$p_\omega^i = p_{\sigma\omega}^{\sigma(i)}.$$

Proof. Lemma A.6 implies that for every $i \in N$ and $j \in M,\ j \neq 0$ there is a collection of real numbers $\{p_\omega : \omega \in \Omega \text{ with } \omega(i) = j-1\}$ with

$$\varphi_{i,j}(h) = \sum_{\omega\in\Omega:\ \omega(i)=j-1} p_\omega[h(\omega + e_i) - h(\omega)]$$

for all $h \in \tilde{H}$.

Suppose that attributes $1, 2 \in N$ are symmetric in $h \in \tilde{H}$. Hence, for the permutation $\sigma : N \to N$, with $\sigma(1) = 2, \sigma(2) = 1$ and $\sigma(i) = i$ if $i \neq 1, 2$, we have $h(\omega) = \sigma h(\omega)$ for all $\omega \in \Omega$. Then Attribute Symmetry implies

$$\varphi_{1,j}(h) = \varphi_{2,j}(h),$$

for all $j = 1, \ldots, m$. Substituting the expression above we obtain

$$\sum_{\omega\in\Omega:\ \omega(1)=j-1} p_\omega[h(\omega + e_1) - h(\omega)] = \sum_{\tilde{\omega}\in\Omega:\ \tilde{\omega}(2)=j-1} \tilde{p}_{\tilde{\omega}}[h(\tilde{\omega} + e_2) - h(\tilde{\omega})],$$

where $p_\omega = p^1_\omega$ and $\tilde{p}_{\tilde{\omega}} = p^2_{\tilde{\omega}}$. Because $\sigma h = h$ this implies

$$0 = \sum_{\omega\in\Omega:\ \omega(1)=j-1} p_\omega[h(\omega + e_1) - h(\omega)] - \sum_{\tilde{\omega}\in\Omega:\ \tilde{\omega}(2)=j-1} \tilde{p}_{\tilde{\omega}}[\sigma h(\tilde{\omega} + e_2) - \sigma h(\tilde{\omega})],$$

and using $\sigma h(\sigma\omega') = h(\omega')$ we obtain

$$\begin{aligned}
0 &= \sum_{\omega\in\Omega:\ \omega(1)=j-1} p_\omega[h(\omega + e_1) - h(\omega)] - \sum_{\tilde{\omega}\in\Omega:\ \tilde{\omega}(2)=j-1} \tilde{p}_{\tilde{\omega}}[h(\sigma^{-1}(\tilde{\omega} + e_2)) - h(\sigma^{-1}\tilde{\omega})] \\
&= \sum_{\omega\in\Omega:\ \omega(1)=j-1} p_\omega[h(\omega + e_1) - h(\omega)] - \sum_{\tilde{\omega}\in\Omega:\ \tilde{\omega}(2)=j-1} \tilde{p}_{\tilde{\omega}}[h(\sigma^{-1}\tilde{\omega} + e_1) - h(\sigma^{-1}\tilde{\omega})] \\
&= \sum_{\omega\in\Omega:\ \omega(1)=j-1} p_\omega[h(\omega + e_1) - h(\omega)] - \sum_{\omega\in\Omega:\ \omega(1)=j-1} \tilde{p}_{\sigma\omega}[h(\omega + e_1) - h(\omega)] \\
&= \sum_{\omega\in\Omega:\ \omega(1)=j-1} [p_\omega - \tilde{p}_{\sigma\omega}][h(\omega + e_1) - h(\omega)].
\end{aligned}$$

Let $\omega' \in \Omega$ such that $\omega' = (j-1, l, \omega'(3), \ldots, \omega'(n))$ for some $l \in M\backslash\{0\}$. We define the situation h by

$$h := \kappa_{\omega'} + \kappa_{\sigma\omega'}.$$

Then the attributes 1 and 2 are symmetric in h. For $\omega \in \Omega$ with $\omega(1) = j-1$ we have

$$h(\omega) = \begin{cases} 1 & \text{if } \omega = \omega' \text{ and } l \neq j-1 \\ 2 & \text{if } \omega = \omega' \text{ and } l = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
0 &= \sum_{\omega\in\Omega:\ \omega(1)=j-1} [p_\omega - \tilde{p}_{\sigma\omega}][h(\omega + e_1) - h(\omega)] \\
&= \begin{cases} 2[p_{\omega'} - \tilde{p}_{\sigma\omega'}] & \text{if } l = j-1 \\ p_{\omega'} - \tilde{p}_{\sigma\omega'} & \text{otherwise} \end{cases}
\end{aligned}$$

or equivalently $p_{\omega'} = \tilde{p}_{\sigma\omega'}$. Because the choice of $l, j \in M\backslash\{0\}$ was arbitrary we conclude that $p_\omega = \tilde{p}_{\sigma\omega}$ for any $\omega \in \Omega$.

The proof for any pair of symmetric attributes $i, k \in N$ is similar, so that the theorem follows from the fact that every permutation can be written as a sequence of 2-attribute exchanges. □

We proceed with the proof of Theorem 5.1, the characterization of the egalitarian evaluation.

Proof of Theorem 5.1. The proof of the if-part of the theorem is left to the reader.

For the only-if part, let φ be a homogeneous valuation satisfying Zero Contribution, Distribution, Attribute Symmetry and Level Symmetry. By Theorem 3.1 it is sufficient to prove that φ coincides with the egalitarian valuation on all simple situations.

Let $k \in \{0, \ldots, n-1\}$ be arbitrary, and let $O \subset N$ be any collection of k distinct attributes. Consider $\omega \in \Omega$ with $\omega(i) = 0$ for all $i \in O$ and $\omega(i) > 0$ for all $i \in N\backslash O$. Distribution implies

$$1 = \sum_{i=1}^{n} \sum_{j=0}^{m} \varphi_{i,j}(h_\omega),$$

and Zero Contribution implies $\varphi_{i,j}(h_\omega) = 0$ for all $i \in O$ and $\varphi_{i,j}(h_\omega) = 0$ for all $i \in N\backslash O$ if $j \neq \omega(i)$. Substitution into the above equation gives

$$1 = \sum_{i \in N\backslash O} \varphi_{i,\omega(i)}(h_\omega).$$

We show by induction, that $\varphi_{i,\omega(i)}(h_\omega) = 1/(n-k)$ for all $i \in N\backslash O$. Let $l := \max_{i \in N\backslash O} \{\omega(i)\}$. If for all $i \in N\backslash O$ we have $\omega(i) = l$, then we conclude by Attribute Symmetry that $\varphi_{i,\omega(i)}(h_\omega) = 1/(n-k)$ for all these i.

Let there exist an $s \in N\backslash O$ such that $\omega(s) < l$. Assume that for all $\tilde{\omega} \in \Omega$ with $\tilde{\omega}(i) = 0$ for all $i \in O$ and $\omega(i) > 0$ for all $i \in N\backslash O$, such that

$$|\{i \in N\backslash O : \tilde{\omega}(i) = l\}| > |\{i \in N\backslash O : \omega(i) = l\}|$$

we have $\varphi_{i,\tilde{\omega}(i)}(h_{\tilde{\omega}}) = 1/(n-k)$.

Take any $s \in N\backslash O$ such that $\omega(s) < l$. Define ω' as follows

$$\omega'(i) = \begin{cases} l & \text{if } i = s \\ \omega(i) & \text{otherwise,} \end{cases}$$

and define the situation $h := h_\omega + h_{\omega'}$. Then, it is easy to show that the levels $\omega(s)$ and l are symmetric for s in h. Hence, by Level Symmetry we have $\varphi_{s,\omega(s)}(h) = \varphi_{s,l}(h)$. With the Transfer Property this implies that

$$\varphi_{s,\omega(s)}(h_\omega) + \varphi_{s,\omega(s)}(h_{\omega'}) = \varphi_{s,l}(h_\omega) + \varphi_{s,l}(h_{\omega'}).$$

Note also that by Zero Contribution (and the fact that $\omega(s) \neq l = \omega'(s)$) we have $\varphi_{s,\omega(s)}(h_{\omega'}) = \varphi_{s,l}(h_\omega) = 0$ so that

$$\varphi_{s,\omega(s)}(h_\omega) = \varphi_{s,l}(h_{\omega'}).$$

By the induction assumption $\varphi_{s,l}(h_{\omega'}) = 1/(n-k)$ for all $i \in N\backslash O$. Therefore $\varphi_{s,\omega(s)}(h_\omega) = 1/(n-k)$. By induction it follows that $\varphi_{i,\omega(i)}(h_\omega) = 1/(n-k)$ for all $i \in N\backslash O$ such that $\omega(i) < l$.

Recall now that by Distribution and Zero Contribution

$$1 = \sum_{i \in N \setminus O} \varphi_{i,\omega(i)}(h_\omega),$$

which by the above findings is equivalent to

$$\begin{aligned} 1 &= \sum_{i:\ 0<\omega(i)<l} \varphi_{i,\omega(i)}(h_\omega) + \sum_{i:\ \omega(i)=l} \varphi_{i,\omega(i)}(h_\omega) \\ &= \sum_{i:\ 0<\omega(i)<l} \frac{1}{n-k} + \sum_{i:\ \omega(i)=l} \varphi_{i,\omega(i)}(h_\omega). \end{aligned}$$

Using Attribute Symmetry in the second sum we find that $\varphi_{i,\omega(i)}(h_\omega) = 1/(n-k)$ for all $i \in N \setminus O$. Hence, $\varphi = \varepsilon$ for this case.

Because O was an arbitrary selection of k distinct attributes, and $k \in \{0, \ldots, n-1\}$ was arbitrary too, we have $\varphi = \varepsilon$ on the set of simple situations S. This concludes the proof of the theorem. □

B Additional coefficients for the egalitarian valuation

In this Appendix we provide a theorem describing the values assigned by the egalitarian valuation to the situations $\kappa_\omega \in \widetilde{H}$. These values are of interest because they give the weights p_ω in Theorem 4.1 through the relation $p_\omega = -\varphi_{i,j}(\kappa_\omega)$ (cf. Lemmas A.5 and A.6 and their proofs).

Henceforth, assume $m \geqslant 2, n \geqslant 2$ to avoid trivial or special cases. Let $k \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, m\}$. Define the state $\omega_{k,j} \in \Omega$ as follows:

$$\omega_{k,j} := (\underbrace{0, \ldots, 0}_{k}, j, \underbrace{m, \ldots, m}_{n-k-1}).$$

The following theorem is formulated for states $\omega_{k,j}$. Similarly, the result holds for states $\sigma(\omega_{k,j})$, for any permutation σ of N, because the egalitarian valuation satisfies Attribute Symmetry. However, to avoid tedious notation, in what follows we suppress σ.

Theorem B.1 *Let $\varepsilon : \widetilde{H} \to I\!R^{n \times (m+1)}$ be the egalitarian valuation. Then, the following hold for $\kappa_{\omega_{k,j}}$:*

(i) If $k = 0$ and $j = m$, then

$$\varepsilon_{s,t}(\kappa_{\omega_{k,j}}) = \frac{1}{n} \cdot \begin{cases} 1 & \text{for } s = 1, \ldots, n \text{ and } t = m \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $k > 0$ and $j = m$, then

$$\varepsilon_{s,t}(\kappa_{\omega_{k,j}}) = \frac{(k-1)!}{\prod_{r=0}^{k} (n-r)} \cdot \begin{cases} k-n & \text{for } s = 1, \ldots, k \text{ and } t = 1 \\ k & \text{for } s = k+1, \ldots, n \text{ and } t = m \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $j \in \{1, \ldots, m-1\}$, then

$$\varepsilon_{s,t}(\kappa_{\omega_{k,j}}) = \frac{k!}{\prod_{r=0}^{k}(n-r)} \cdot \begin{cases} 1 & \text{for } s = k+1 \text{ and } t = j \\ -1 & \text{for } s = k+1 \text{ and } t = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

(iv) Finally, if a state ω has at least two levels in $\{1, \ldots, m-1\}$, then $\varepsilon_{s,t}(\kappa_\omega) = 0$ for all $s = 1, \ldots, n$ and $t = 1, \ldots, m$.

The proof of this theorem is rather technical and available from the authors upon request.

As observed before every homogeneous valuation satisfying Zero Contribution, Distribution, and Attribute Symmetry extends the Shapley value of cooperative game theory (Shapley, 1953). The egalitarian valuation does this in a even more special way, as we will demonstrate now. Let $h \in H$ be any situation. Theorem 4.1 implies:

$$\varepsilon_{i,j}(h) = \sum_{\omega \in \Omega:\ \omega(i)=j-1} p_\omega[h(\omega + e_i) - h(\omega)].$$

According to the last two statements in Theorem B.1, the weight p_ω $(= -\varepsilon_{i,j}(\kappa_\omega))$ is equal to 0 if there exists an attribute $s \neq i$ for which $\omega(s) \notin \{0, m\}$. This follows from statement (iii) if $j = 1$, and from statement (iv) if $j > 1$. Consequently, the egalitarian valuation takes marginal contributions in attribute i into account only when the levels of all other attributes are either 0 or m, ignoring contributions elsewhere. Hence, in the formula above summation over the set $\{\omega \in \Omega :\ \omega(i) = j-1\}$ can be reduced to summation over the set $C := \{\omega \in \Omega : \omega(i) = j-1, \omega(s) \in \{0, m\}$ for all $s \neq i\}$.

Let us take a closer look at the states in this set. Obviously, each $\omega \in C$ can be written as $\sigma(\omega_{k,j-1})$ for a permutation σ of N , and a state $\omega_{k,j-1}$ as defined before Theorem B.1. By Attribute Symmetry (Theorem 4.2) we know that $p^i_{\sigma(\omega_{k,j-1})} = p^{k+1}_{\omega_{k,j-1}}$, and by the definition of the weights p_ω (see the proofs of Lemmas A.5 and A.6) we have

$$p_{\omega_{k,j-1}} = -\varepsilon_{k+1,j}(\kappa_{\omega_{k,j-1}}).$$

If $j \in \{2, \ldots, m\}$ statement (iii) of Theorem B.1 implies

$$p_{\omega_{k,j-1}} = -\varepsilon_{k+1,j}(\kappa_{\omega_{k,j-1}}) = \frac{k!}{\prod_{r=0}^{k}(n-r)},$$

which can be rewritten as

$$p_{\omega_{k,j-1}} = \frac{k!(n-1-k)!}{n!}.$$

If $j = 1$, statement (ii) of Theorem B.1 implies

$$p_{\omega_{k,0}} = -\varepsilon_{k+1,1}(\kappa_{\omega_{k,0}}) = -\varepsilon_{k+1,1}(\kappa_{\omega_{k+1,m}}) = -\frac{k!(k+1-n)}{\prod_{r=0}^{k+1}(n-r)},$$

which after rearranging yields

$$p_{\omega_{k,0}} = \frac{k!(n-1-k)!}{n!}.$$

Hence, for all $j = 1, \ldots, m$ the weights $p_{\omega_{k,j-1}} = \frac{k!(n-1-k)!}{n!}$ depend only on the number of levels that are equal to 0.

To compare this with the Shapley value in cooperative game theory note that attributes can be viewed as players, and consider the marginal contribution $h(\omega_{k,j}) - h(\omega_{k,j-1})$. This contribution can be seen as the result of player $k+1$ increasing his level from $j-1$ to j in a coalition consisting of $n-(k+1)$ players already there at full strength (that is, players $k+2, \ldots, n$ at their maximum level m) and the remaining players not (yet) there (that is, players $1, \ldots, k$ at level 0). In the egalitarian valuation this contribution is weighted by $p_{\omega_{k,j-1}} = \frac{k!(n-1-k)!}{n!}$, which is exactly the coefficient in the Shapley value that weighs the contribution of a player joining a coalition of size k or of size $n-k-1$ in random order (see for instance Weber, 1988, page 118).

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