Unfair contests

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Abstract

Real-world contests are often "unfair" in the sense that outperforming all rivals may not be enough to be the winner, because some contestants are favored by the allocation rule, while others are handicapped. Examples of such contests can be inter alia found in the area of litigation and procurement.

This paper analyzes discriminatory contests with a handicap for one of the participants. We first characterize the equilibrium strategies, provide closed form solutions, and illustrate the additional strategic issues arising through the unfairness of contests. We then tackle the issue of the optimal degree of unfairness. From a social point of view, the following trade-off arises: On the one hand, the prize may be awarded to an inferior contestant. But on the other hand, unfair contests lead to a lower overall effort of the contestants and hence reduces inefficient rent-seeking. We characterize situations in which it is optimal for an authority to either stipulate a fair contest, an interior degree of unfairness or even an infinitely unfair contest where the prize is directly awarded to one of the contestants.

Keywords: All-pay auctions, contests, asymmetric allocation rule, rent-seeking games, asymmetric information

JEL-Classification: D44, D88

1 Introduction

Motivation and results One apparently desirable feature of contests, games in which several parties exert costly effort to compete with each other to secure a prize, a rent, or a government contract, is that they should be fair in the sense that the one who performs best should be the winner. In reality, however, contests are often "unfair" because one contestant is *favored* as he need not outperform his competitors to be the winner. Accordingly, other contestants may be *handicapped*, since outperforming their rivals may not be enough to win.

In reality, allocation rules are often asymmetric in the sense described above. For example, in German procurement auctions, although local authorities are in general obliged to choose the firm with the lowest bid, there is a clause according to which it can award the contract to a local firm when this firm's bid is not more than 5 per cent higher than the lowest bid. Under the realistic assumption that preparing a bid itself is costly and that (part of) these costs cannot be recovered independent of which party is awarded the contract, this resembles an unfair contest as described above. As a second example, consider litigation where it is costly for the plaintiff and the defendant to prevail, independent of the actual outcome of the trial (e.g. by searching for favorable evidence and/or by hiring a lawyer). Under the "in dubio pro reo"-rule in criminal law, a defendant will only be convicted if the evidence against him is "abundant", i.e. if his lawyer presents considerably less or worse evidence than the prosecutor. Again, our model can be used to analyze such situations. Furthermore, assume that an enterprise wishes to hire a consulting firm, and suppose that firm A has done some excellent in-house consulting before. Then, we often observe in reality that a potential entrant B is awarded the contract only if the quality of its proposal is considerably above the quality of A's proposal. As long as preparing a proposal is itself costly, this can again be interpreted as an unfair contest in our sense. Finally, when deciding on job promotion, firms often apply some sort of seniority rule which implies that a junior candidate might not get the job although he has performed better than his senior counterpart.

In the first part of the paper, we fully characterize the equilibrium strategies of an unfair two−player discriminatory contest (all-pay auction) using a framework where contestants have private information concerning the value of the prize to them. We show that there exists a unique pure strategy Bayesian Nash Equilibrium and provide a closed form solution of the respective set of first order conditions. With respect to efficiency, it is generally possible that the handicapped contestant exerts more effort than the favored contestant if their valuations are identical, we show that it is not possible for the handicapped contestant to win the contest when his valuation is lower than the favored contestant's valuation. Hence, an inefficient allocation of the prize can only result when the favored contestant is the winner although he has the lower valuation. We then illustrate the additional strategic aspects which arise through the introduction of the asymmetry for uniformly distributed valuations.

In the second part of the paper, we go one step further by asking "What is the optimal degree of unfairness?". Of course, unfair contests may lead to welfare losses whenever a first best requires the prize to be awarded to the "best" contestant. In the procurement example, it will generally be socially optimal to award the contract to the firm with the lowest (marginal) costs. A similar reasoning holds for the other examples mentioned above. The higher the degree of unfairness, the higher the chances of awarding the prize to an inferior contestant and so one obvious drawback from unfair contests is allocative inefficiency.

But on the other hand, there may also be reasons in favor of an unfair contest design. Personal relationships may offer stochastic signals on a contestant's capabilities besides the quality of his proposal, and supporting local firms may be reasonable at least from the local authority's point of view. In our paper, we neglect these aspects by assuming that contestants are ex ante symmetric. But even then, an unfair contest can be superior because of the well-known fact that the private incentive to exert effort in a contest is often far beyond the social value of effort.[1] For instance, the effort exerted for the preparation of proposals in procurement contests may simply be waste from a social point of view.[2]

[1] In fact, it has been argued that the total prize in a rent-seeking contest may be dissipated in the contestants' attempt of securing the prize (see Tullock (1980)). Baye, Kovenock, and de Vries (1999) have shown, that this result can only occur for some realizations of strategies in a mixed strategy equilibrium but not in expectation.

[2] Although this is the assumption we maintain throughout the paper, we are of course aware of the fact that there are also examples of contests where more effort is, at least partly, socially valuable. For example, in the case of R&D races, it is often argued that duplication of effort may be socially desirable as it may lead to innovation and technological progress. Furthermore, in sports contests, more aggregate effort is generally considered a desirable feature as it increases suspense as well as the overall quality of the contest (see e.g. Szymanski (2003)). Accordingly, alternative objective functions of the contest designer have been considered in the literature, including maximizing i) total expected effort and ii) the expected value of the highest effort level (see e.g. Gavious, Moldovanu, and Sela (2002) and Moldovanu and Sela (2002)).

In our paper, total expected effort may decrease in the degree of unfairness, which is the potential advantage of unfair contests so that a trade-off arises between ex-post efficiency and the waste of resources. Given this trade-off, we illustrate that it may be optimal for the contest designer to stipulate a fair contest, an infinitely unfair contest or a contest with an interior degree of unfairness.

Literature It is well-known that discriminatory contests are strategically equivalent to all-pay auctions. In all-pay auctions, each bidder has to pay his bid regardless of whether he wins the auction or not, and in discriminatory contests, each contestant bears his effort costs no matter if he wins or not. Baye, Kovenock, and de Vries (1996) provide a complete analysis of the all-pay auction under complete information. In a framework of asymmetric information, Krishna and Morgan (1997) extend the classic model by Milgrom and Weber (1982) with affiliated signals to first- and second price all-pay auctions. Lizzeri and Persico (2000) analyze under which conditions there exist unique pure strategy equilibria in general auction games, including the all-pay auction.[3] Amann and Leininger (1996) and Maskin and Riley (2000) consider auctions in which contestants are asymmetric in the sense that the valuations of each bidder are drawn from different distributions. This also implies that the bidder with the highest valuation does no longer win the object with certainty. While Maskin and Riley (2000) confine attention to winner-pay auctions, our paper is more related to Amann and Leininger (1996) as they analyze the all-pay auction. Moreover, we adopt and extend their approach for determining the equilibrium bidding strategies from a system of differential equations. As stated above, in all these papers and contrary to our model, the winner of the auction is the high bidder.

In contrast, there are a few papers analyzing contests with handicaps: Konrad (2002) assumes that an incumbent needs to spend less resources than his rival to win a discriminatory contest. However, he restricts attention to complete information, so that only mixed strategy equilibria exist. In the context of bribery games, Lien (1990) and Clark and Riis (2000) consider an all-pay auction where two players compete for a government contract awarded by a corrupt official. In Lien (1990), the players are ex-ante symmetric and the introduction of a handicap unambiguously reduces allocative efficiency. Clark

[3] The issue of existence of pure-strategy equilibria in a more general class of simultaneous games of asymmetric information is also extensively analyzed in Athey (2001).

and Riis (2000) extend the situation to ex-ante asymmetric bidders (with respect to their bidding costs) and show that the official can increase his expected revenue by introducing unfairness in our sense. Bernardo, Talley, and Welch (2000) consider a litigation game in which each litigant's piece of evidence is unequally weighted by the court (in the legal jargon, the court is said to employ a "presumption" in favor of one party). They consider the effect of such a presumption on shirking incentives at an earlier stage of the game. In contrast to our approach, they model this as a non-discriminatory contest (i.e. "Tullock" contest), where the outcome is stochastic even for given efforts levels (see also Kohli and Singh (1999)).

Furthermore, our paper discusses an additional dimension concerning the issue of contest design, and other choice variables so far considered in the literature include i) the number of contestants which are invited to play the contest (Baye, Kovenock, and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2002)), ii) whether there should be only one prize (for the winner), or whether several prizes should be awarded (Moldovanu and Sela (2001), Moldovanu and Sela (2002)), and iii) the desirability of imposing a (symmetric) bid cap (Gavious, Moldovanu, and Sela (2002) and Che and Gale (1998)).

Finally, using a different setup, Bolton and Farrell (1990) analyze the advantages and disadvantages of decentralization, where two firms which have private information about their production costs in each period decide whether or not to sink costs to enter a natural-monopoly market. The basic trade-off is similar to ours: Decentralization tends to induce only a low cost firm end up to enter while a high cost firm prefers to stay out and thus tends to use information efficiently. However, this process only evolves over time, so that it comes with inefficient delay or duplication of effort as there are periods where neither or both firms sink costs in order to enter the market. On the other hand, centralization means that an uninformed agency grants the monopoly right to one of the firms so that there is the chance that it will pick the high cost firm.

The remainder of the paper is organized as follows: In section 2 the basic model is presented. In section 3 we analyze the equilibrium of the contest game and derive our main theoretical results and in section 4 we discuss a numerical example. Section 5 is concerned with the optimal degree of unfairness, and section 6 concludes.

2 The Model

Basic Setup We consider a discriminatory contest (all-pay auction) where 2 risk-neutral contestants indexed $i = 1, 2$ compete for a single prize to be awarded. Each contestant has valuation $v_i \in [0, 1]$ for the prize which drawn from a common distribution function $F(v) \in C^1$ satisfying $F(0) = 0$ where the density function $F'(v)$ is positive valued on $(0, 1)$. The realization of v_i (contestant i's "type") is private information to contestant i. Each contestant can influence his chances of winning the prize by exerting effort which is denoted by b_i. In what follows, we analyze equilibria in which the effort strategy of contestant i is a function of his type, i.e. $b_i : [0, 1] \to \Re_0^+$.

The specific feature of this contest is the allocation rule: Denoting by $W \in \{1, 2\}$ the identity of the winner, we have

$$W = 1 \Leftrightarrow b_1 > t \cdot b_2 \text{ and } W = 2 \Leftrightarrow b_2 > \frac{1}{t} \cdot b_1 \tag{1}$$

where a coin is flipped in case that $b_1 = t \cdot b_2$ holds so that each contestant wins with probability $\frac{1}{2}$. Thus, contestant 1 wins the contest only if he exerts at least t-times as much effort as contestant 2, while contestant 2 wins if he exerts at least $\frac{1}{t}$-times as much effort as contestant 1. Without loss of generality we confine attention to the case $t \geq 1$. Therefore, contestants 1 and 2 will be referred to as the "handicapped" and the "favored" contestant, respectively.[4] The value of t is commonly known in the contest game and we will refer to the case $t = 1$ as a "fair" contest; in this case our model is equivalent to a standard two-player all-pay auction with private values.

Payoffs Following the setup of the model, payoffs for given effort levels b_1 and b_2 are

$$\pi_1(b_1, b_2, v_1; t) = \begin{cases} v_1 - b_1 & \text{if } b_1 > tb_2 \\ \frac{1}{2}v_1 - b_1 & \text{if } b_1 = tb_2 \\ -b_1 & \text{if } b_1 < tb_2 \end{cases} \tag{2}$$

[4] Note that the asymmetry here refers to the allocation rule. This is different to "asymmetric auctions" in the sense of Amann and Leininger (1996) and Maskin and Riley (2000), where the valuations v_1 and v_2 are drawn from different distributions.

and

$$\pi_2(b_1, b_2, v_2; t) = \begin{cases} v_2 - b_2 & \text{if } b_2 > \frac{1}{t} b_1 \\ \frac{1}{2} v_2 - b_2 & \text{if } b_2 = \frac{1}{t} b_1 \\ -b_2 & \text{if } b_2 < \frac{1}{t} b_1 \end{cases} . \tag{3}$$

Finally for t given, *expected* payoffs are[5]

$$\Pi_1(\cdot) = v_1 \cdot \Pr(b_1 > t \cdot b_2(v_2)) - b_1 \tag{4}$$

and

$$\Pi_2(\cdot) = v_2 \cdot \Pr(b_2 > \frac{1}{t} \cdot b_1(v_1)) - b_2. \tag{5}$$

The timing of the game is as follows (see also figure 1 below): At stage 1, the authority chooses t. At stage 2, each contestant's valuation for the contract is determined by a nature's move and privately revealed to that contestant. At stage 3, the contest is played where t is given and commonly known. At stage 4, after observing the effort choices, the prize is awarded by the authority according to the allocation rule.

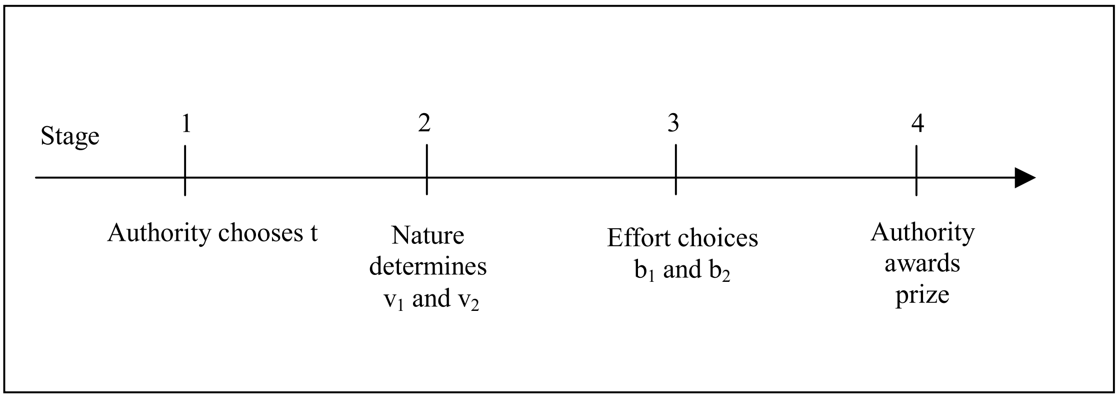


Figure 1: Sequence of Events

[5] Since equilibrium effort strategies will be continuous, the probability of a tie is zero.

3 Equilibrium Analysis

3.1 Uniqueness and existence of equilibrium at the contest stage

Since this is a static game with incomplete information, the equilibrium concept used is Bayesian Nash Equilibrium (BNE). A vector of effort levels $(b_1^*(v_1), b_2^*(v_2))$ is a BNE if the following set of conditions is satisfied:

$$\Pi_i(b_i^*(v_i), b_j^*(v_j); t) \geq \Pi_i(b_i, b_j^*(v_j); t) \text{ for all } v_i \in [0,1], \text{ and } b_i \in \Re_0^+. \tag{6}$$

In equilibrium, no contestant must be able to increase his expected payoff by choosing an effort strategy other than $b_i^*(v_i)$, given that the opponent adheres to his equilibrium strategy. The following definition proves useful for further reference:

Definition 1 *Consider a set $A \subset R$ and a function $z : A \rightarrow \Re$. Then define: $D_z := \{a \in A : z(a) \in \Re^+\}$.*

The restricted domain D_z contains only those elements $a \in A$ whose image $z(a)$ is positive. We can then state the following result concerning the properties of the equilibrium effort strategies:

Lemma 1 (Equilibrium Effort Strategies) $b_i^* : D_{b_i} \rightarrow (0, b_i(1)]$ *where $i = 1, 2$ is an increasing bijection between non-empty subsets of $[0,1]$ and differentiable almost everywhere.*

Proof. See Appendix A. ■

The Lemma simply says that each contestant's equilibrium strategy is a well-behaved and monotonically increasing function in his type.

Uniqueness of Equilibrium We first show that an equilibrium is unique whenever it exists. The issue of existence is addressed below. Note that Lemma 1 also ensures existence of the inverse mapping $\rho_i : (0, b_i^*(1)] \rightarrow D_{b_i}$, i.e. $\rho_i(b) \equiv b_i^{-1}(b)$ is the valuation contestant i must have in order to choose effort level b. Equipped with this result we can now characterize the equilibrium effort strategies in more detail. The maximization problem for contestant 1 when contestant 2 is playing some strategy $b_2(v_2)$ is given by

$$\max_{b_1} \; v_1 \cdot \Pr(b_1 > t \cdot b_2(v_2)) - b_1 = \max_{b_1} \; v_1 \cdot F(\rho_2(\frac{b_1}{t})) - b_1, \tag{7}$$

while for contestant 2, when contestant 1 is playing strategy $b_1(v_1)$ we have

$$\max_{b_2} \ v_2 \cdot \Pr(b_2 > \frac{1}{t} \cdot b_1(v_1)) - b_2 = \max_{b_2} \ v_2 \cdot F(\rho_1(t \cdot b_2)) - b_2. \tag{8}$$

The first order conditions of these maximization problems are given by the following system of ordinary first order differential equations:

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{t})) \cdot \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{1}{t} = 1. \tag{9}$$

$$v_2 \cdot F'(\rho_1(t \cdot b_2(v_2))) \cdot \rho_1'(t \cdot b_2(v_2)) \cdot t = 1. \tag{10}$$

For a given set of initial conditions, this system determines a unique trajectory of effort strategies. That there is only a single pair of initial conditions (such that a solution to Eqns. (9) and (10) is indeed unique) follows from the subsequent results concerning the properties of the *equilibrium effort distributions* $G_{i=1,2} := F(\rho_i(b_i^*)) : D_{G_i} \to (0,1]$:[6]

Lemma 2 (Equilibrium Effort Distributions) *In any BNE, the effort distributions G_1 and G_2 have the following properties:*
(i) $D_{G_1} = (0, b_1^*(1)]$ and $D_{G_2} = (0, b_2^*(1)]$ where $b_1^*(1) = t \cdot b_2^*(1)$.
(ii) G_i is continuous and strictly monotone increasing $\forall i = 1, 2$.
(iii) If $G_i(0) > 0$, then $G_{j \neq i}(0) = 0$.
(iv) There is a single set of admissible initial conditions.

Proof. See Appendix B. ■

Part (i) of the Lemma characterizes one main difference of our model compared to the standard model with $t = 1$. Clearly, it can never be optimal for (the favored) contestant 2 to exert more than $\frac{1}{t}$-times the maximum effort of (the handicapped) contestant 1 since he already wins with probability one when choosing $b_2 = \frac{1}{t} \cdot b_1(1)$. Part ii) follows from the fact that, in equilibrium, effort distributions must ensure that no contestant can increase his expected profit by choosing a lower effort level while leaving the probability of winning the contest unchanged. Part iii) says that only one contestant's effort function can have an atom at zero. Intuitively, this follows from the fact that, given that one contestant's effort function has an atom at zero, the other contestant would always be better of by choosing a strictly positive effort level whenever his own valuation is positive. As one

[6] Similar statements for the case $t = 1$ have for example been derived by Amann and Leininger (1996).

consequence, the coexistence of different sets of admissible initial conditions is ruled out as stated in part iv).[7]

Existence of Equilibrium Rather than modifying equations (9) and (10) directly, we extend the method adopted by Amann and Leininger (1996) who have analyzed the case $t = 1$ for valuations v_1 and v_2 drawn from different distributions. The advantage of this method is that it simplifies the problem of simultaneously solving a system of differential equations into a sequential procedure.

Consider a bijection $k(v_1; t)$ which maps every type of contestant 1 onto that type of contestant 2 whose equilibrium effort level is *1/t- times as much as* contestant $1'$s so that

$$k(v_1; t) = \rho_2(\frac{b_1^*(v_1)}{t}). \tag{11}$$

Analogously, $k^{-1}(v_2; t) = \rho_1(t \cdot b_2^*(v_2))$ gives that type of contestant 1 who will choose t-times as much effort as contestant 2 when his type is v_2. Note that due to our previous results, Eqn. (11) defines indeed a bijection between the domains D_{b_1} and D_{b_2} of the two equilibrium strategies which is differentiable almost everywhere. It turns out that rewriting the first order conditions with $k(v_1; t)$ and a separation of dependent and independent variables provides a closed form solution for $k(\cdot)$:

Lemma 3 *Define* $H(x) := \int_x^1 \frac{F'(y)}{y} dy$ *so that* $\frac{d}{dx}H(\cdot) = -\frac{F'(x)}{x} < 0$*. Then we have:*
i) $k(v_1; t) = H^{-1}(tH(v_1))$ *satisfying* $\frac{d}{dt}k(v_1; \cdot) < 0$ *and* $k(v_1; 1) = v_1$*.*
ii) $k^{-1}(v_2; t) = H^{-1}(\frac{1}{t}H(v_2))$ *satisfying* $\frac{d}{dt}k(v_2; \cdot) > 0$ *and* $k^{-1}(v_2; 1) = v_2$*.*
iii) $\lim_{t\to\infty} k^{-1}(v_2; t) = 1$ *and* $\lim_{t\to\infty} k(v_1; t) = 0$ *which implies that contestant* 2*'s (contestant* 1*'s) probability of winning tends to* 1 *(*0*) as* $t \to \infty$*.*

Proof. See Appendix C. ■

Equipped with a closed form solution for $k(\cdot)$ and its inverse, a simple quadrature provides us with the equilibrium strategies:

Theorem 1 *There exists a unique pure-strategy Bayesian Nash-Equilibrium in which*

[7] This follows from the no-crossing property of equilibrium bid functions as established by Lizzeri and Persico (2000).

contestant 1 *(the handicapped contestant) chooses*

$$b_1^*(v_1) = \int_{\max\{0,k^{-1}(0;t)\}}^{v_1} t \cdot k(V;t)F'(V)dV \tag{12}$$

and in which contestant 2 *(the favored contestant) chooses*

$$b_2^*(v_2) = \frac{b_1^*(k^{-1}(v_2;t))}{t}. \tag{13}$$

Proof. See Appendix D. ■

3.2 The impact of $t > 1$ on ex-post efficiency

In our framework, the allocation of the prize does not only depend on who exerts more effort but also on the identity of a contestant. Hence, we can not exclude that the prize is awarded to a contestant whose valuation is lower than that of his competitor. Furthermore, without further information on the distribution $F(\cdot)$, one can not say which contestant will exert more effort when valuations are identical. However, we can show that the handicapped contestant 1 will generically not win the contest if his valuation is lower. This means that, even if being handicapped induces him to choose higher effort levels for some realizations of v_1, this can never outweigh his handicap. It follows that an inefficient allocation of the prize can only result when (the favored) contestant 2 wins the auction although he has a lower valuation. This is expressed in the following result, where $W^* \in \{1, 2\}$ denotes the identity of the winner in equilibrium:

Theorem 2 *i) In equilibrium, there can only exist the case where* $v_1 > v_2$ *but* $W^* = 2$*, while the case where* $v_2 > v_1$ *but* $W^* = 1$ *does not occur with positive probability.*
ii) The probability of an inefficient allocation is therefore given by

$$p^*(t) := \int_0^1 \left(F(v_1) - F(k(v_1;t))\right) F'(v_1)dv_1 \tag{14}$$

satisfying $p^*(1) = 0$*,* $\frac{d}{dt}p^*(\cdot) > 0$ *and* $\lim_{t\to\infty} p^*(t) = \frac{1}{2}$*.*

Proof. See Appendix E. ■

For each v_1, contestant 1 has the higher valuation with probability $F(v_1)$ but will be the winner only when $v_2 < k(v_1;t)$ which occurs with probability $F(k(v_1;t))$. It is well known that the case $t = 1$ (the standard all-pay auction) allocates the prize efficiently since equilibrium effort strategies are strictly increasing in the valuations so that the contestant with the highest valuation will choose the highest effort level. When t increases, exerting the highest effort level does not guarantee victory so that the allocation will be distorted. As the handicap goes to infinity, contestant 2 becomes the winner with probability 1, while he has the higher valuation only with probability $\frac{1}{2}$ since $F(\cdot)$ is the same for both contestants.

4 An example: Uniform distribution

Equilibrium Strategies To illustrate our main results, we consider the case where the v_i are uniformly distributed, i.e. $F(v) = v$. Applying Lemma 3, we get $H(x) = \int_x^1 \frac{1}{y}dy = -\ln x$, so that $k(\cdot)$ is then implicitly given by $-\ln k = -\ln v_1^t$ which leads to $k(v_1;t) = v_1^t$. Substituting in Eqn. (45) yields

$$b_1^*(v_1;t) = \int_0^{v_1} t \cdot V^t dV = \frac{t}{t+1} v_1^{t+1} \tag{15}$$

and, by definition of $k(v_1;t)$,

$$b_2^*(v_2;t) = \frac{b_1^*(k^{-1}(v_2))}{t} = \frac{1}{t+1} v_2^{(t+1)/t}. \tag{16}$$

The equilibrium effort distributions are $G_1(b_1^*;t) = \rho_1(b_1^*;t) = ((\frac{1+t}{t})b_1^*)^{\frac{1}{t+1}}$ and $G_2(b_2^*;t) = \rho_2(b_2^*;t) = ((1+t)b_2^*)^{\frac{t}{t+1}}$ which both satisfy $G_i(0) = 0$ (and hence are atomless) and $G_i(b^*(1)) = 1$. Clearly, $b_i^*(v_i;t)$ is increasing in v_i, also satisfying $b_i^*(0) = 0$. Moreover, the equilibrium effort strategies satisfy the support constraint $b_1^*(1) = \frac{t}{t+1} = t \cdot b_2^*(1) = \frac{1}{1+t}$ as required by Lemma 2.

For the comparative statics with respect to t, let us first consider two polar cases: For $t = 1$, we have $b_i^*(v_i;1) = \frac{1}{2}v_i^2$ for $i = 1, 2$ which is simply the standard symmetric equilibrium of the two player all-pay auction with private values. If t becomes large, equilibrium bids converge to zero, i.e. $\lim_{t\to\infty} \frac{t}{t+1} v_1^{t+1} = \lim_{t\to\infty} \frac{1}{t+1} v_2^{(t+1)/t} = 0$. Intuitively, when the handicap becomes infinitely strong, then there is no point for the handicapped contestant to exert effort at all as he will never be the winner of the contest. Analogously for the favored contestant, an arbitrarily small amount of effort ensures winning the object with certainty.

Interestingly, the results for intermediate values of t are not as clear-cut. As an illustration, figure 1 shows contestant 2's equilibrium strategy $b_2^*(v_2;t)$ as a function of t (where $t \geq 1$) for $v_2 = \frac{1}{3}$:

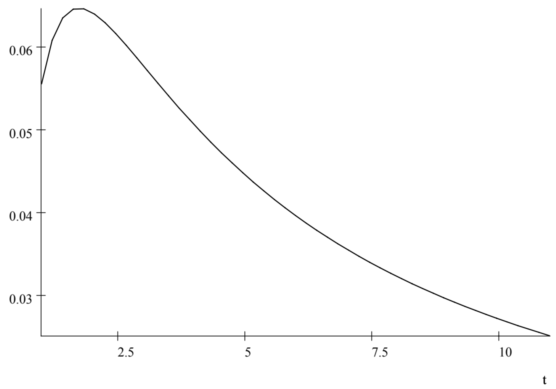


Figure 2: $b_2^*(v_2 = \frac{1}{3})$ as a function of t.

As the marginal cost from increasing b_i is always equal to 1, the intuition behind this non-monotonicity result can best be explained by looking at the marginal benefit, which is denoted by MB_i. We confine attention to an illustration for contestant 2; the case is analogous for contestant 1. Generally, when t increases by Δt, there are two effects which will be analyzed subsequently:

1. contestant 2 wins the contest not only when $b_2 > \frac{1}{t}b_1$ but already when $b_2 > \frac{1}{t+\Delta t}b_1$ (the "direct effect").

2. as t changes, also $b_1^*(v_1;t)$ changes by $\frac{d}{dt}b_1^*(v_1;\cdot)\cdot\Delta t$ and this changes the distribution of effort $G_1(\cdot)$ which contestant 2 faces (the "indirect effect").

Direct effect By setting $t = \widetilde{t}$, we fix $b_1^*(v_1;\widetilde{t})$ and look how a change in t affects the marginal benefit from increasing b_2. In this case, we can write expected benefit as $v_2 \cdot \Pr(b_2 > \frac{1}{t}b_1^*(v_1;\widetilde{t})) = v_2 \cdot \rho_1(tb_2;\widetilde{t})$ so that $MB_2(tb_2;\widetilde{t})$ at the equilibrium level b_2^* is $MB_2(tb_2^*;\widetilde{t}) = \frac{d}{db_2}(v_2 \cdot \rho_1(tb_2^*;\widetilde{t}))$. Taking the derivative w.r.t. t and then substituting back $\widetilde{t} = t$ we get

$$\frac{d}{dt}MB_2(\cdot) = \frac{1}{(t+1)\,t} > 0 \text{ for all } v_2 \in [0,1] \text{ and } t \geq 1. \tag{17}$$

The direct effect unambiguously increases contestant 2's marginal benefit and thus ceteris paribus also increase equilibrium effort, because he wins for more realizations of v_1 due to the more advantageous allocation rule.

Indirect effect For the indirect effect, we fix the allocation rule at $t = \widetilde{t}$ and analyze the effect of a change of t on $b_1^*(\cdot)$ and thus on the distribution of equilibrium effort levels $G_1(\cdot)$ which contestant 2 faces. For this case we can write expected benefit as $v_2 \cdot \Pr(b_2 > \frac{1}{t} b_1^*(v_1; t)) = v_2 \cdot \rho_1(\widetilde{t} b_2; t)$ so that $MB_2(\widetilde{t} b_2; t)$ at b_2^* is $MB_2(\widetilde{t} b_2^*; t) = \frac{d}{db_2}(v_2 \cdot \rho_1(\widetilde{t} b_2^*; t))$. Taking the derivative w.r.t. t and then substituting back $\widetilde{t} = t$, we get

$$\frac{d}{dt} MB_2(\cdot) = -\frac{(\ln v_2)\, t + \ln v_2 + 1 + t^2 + t}{t\,(t+1)^2} \lesseqgtr 0 \tag{18}$$

which is positive (negative) for $v_2 < (>) e^{-\frac{t+1+t^2}{t+1}}$ where $0 < e^{-\frac{t+1+t^2}{t+1}} < 1$ all $t > 1$.

In order to understand the intuition behind this effect, we have to check how different types of contestant 1 react to a change in t: Define $t_i^{\max}(v_i) \in \arg\max_t b_i^*(v_i, t)$ which leads to

$$t_1^{\max}(v_1) = \frac{1}{2 \ln v_1} \left(-\ln v_1 - \sqrt{(\ln^2 v_1 - 4 \ln v_1)} \right) \tag{19}$$

which is increasing in v_1 as

$$\frac{d}{dv_1} t_1^{\max}(v_1) = -\frac{1}{(\ln v_1)\, v_1 \sqrt{((\ln v_1)(\ln v_1 - 4))}} > 0. \tag{20}$$

Moreover, $t_1^{\max}(v_1) = 1 \Leftrightarrow v_1 = e^{-\frac{1}{2}}$. Finally, $\lim_{v_1 \to 1} t_1^{\max}(v_1) = \infty$, so that $b_1^*(v_1; t)$ is monotone decreasing in t for $0 < v_1 \leq e^{-\frac{1}{2}}$, and concave in t with an interior maximum at $t_1^{\max}(v_1)$ for $e^{-\frac{1}{2}} < v_1 < 1$.

Thus, for all v_1, at some point the handicap becomes too strong, so that it is optimal to "give up" and exert less effort. But those types with $v_1 > e^{-\frac{1}{2}}$ are at least willing to "fight" against the stronger handicap by increasing b_1 as long as t is not yet too large. How does this impact on the indirect effect on MB_2 in Eqn. (18)? From the definition of $k(\cdot)$ it follows that in equilibrium, contestant 2 wins whenever $v_1 < k^{-1}(v_2) = v_2^{\frac{1}{t}}$. It follows that when v_2 is high, contestant 2 will be the winner for most realizations of v_1 and so when t increases, his incentive to increase b_2 to win for even more realizations of v_1 is relatively low. Therefore, the expression in (18) is negative. On the other hand when v_2 is low, contestant 2 has an incentive to exert more effort as this allows to win against some of those types of contestant 1 for which $\frac{d}{dt}b_1^*(\cdot) < 0$ holds.

Clearly, the total effect is just the sum of the direct and the indirect effect the following figure illustrates them for $v_2 = \frac{1}{3}$(direct effect = dotted line, the indirect effect = dashed line, total effect = solid line):

Allocation Ex Post Concerning the allocation of the object, recall that for the uniform distribution case we have $k(v_1;t) = v_1^t$ so that from Theorem 2, the probability of misallocation is given by

$$p^*(t) \equiv \int_0^1 \left(v_1 - v_1^t\right) dv_1 = \frac{t-1}{2(t+1)} \tag{21}$$

yielding $p^*(1) = 0$, $\frac{d}{dt}p^*(\cdot) = \frac{1}{(t+1)^2} > 0$ and $\lim_{t\to\infty} p^*(t) = \frac{1}{2}$. For each v_1, contestant 1 has the higher valuation with $\Pr(v_2 < v_1) = v_1$, but is the winner only with $\Pr(v_2 < k(v_1;t) = v_1^t$.

5 The optimal degree of unfairness

After having analyzed the equilibrium of the continuation game in which contestants choose their effort levels for a given level of unfairness, t, we now determine the level of t which an authority should set in trying to minimize social costs. To make things concrete, let's assume that the prize is a contract for a public service which grants some monopoly power to the winner and firms can spend resources to be awarded this contract. From a social point of view, these resources spent are a pure waste as their use is not productive. The overall value of this monopoly right to each contestant will generally depend on (privately known) marginal costs of production. In such a setting, it seems reasonable to assume that the value of the contract to a contestant is the higher, the lower his marginal costs.

5.1 The benefits and costs of unfair contests

Given the properties of the stage game equilibrium at the continuation stage, the authority's goal is to minimize expected social costs associated with the allocation of the prize.

As explained in the introduction, there are two types of social costs - the contestants' (socially useless) efforts spent in competing for the monopoly right, and potential ex-post inefficiencies as this monopoly right might be awarded to the contestant with the higher marginal cost. The introduction of a handicap might well reduce wasteful effort spending (we will refer to this as the benefits of unfairness) while we already saw in Theorem 2 that any handicap induces allocative inefficiencies (we will refer to this as the costs of unfairness).

Benefits The total effort spending in the contest is given by

$$\Sigma(t) \equiv E[b_1^*(v_1;t)] + E[b_2^*(v_2;t)] \tag{22}$$

Although it is clearly plausible that there is a positive relationship between the contestants' effort costs and the social costs of the rent seeking activity, they do not necessarily have to be identical. For our specific example, private effort costs may simply be the disutility of effort, while the social costs might also include forgone benefits from other potential activities which were not carried out as each contestant spends his resources in competing for the contract. Therefore, we will assume that the total effort enters the objective function of the authority via some function $\Psi(\Sigma(t))$ satisfying $\Psi'(\cdot) > 0$ and $\Psi(0) = 0$.

Costs As we have seen above in Theorem 2, the disadvantage of choosing an unfair contest design is that the monopoly right is not necessarily awarded efficiently. Whenever the favored contestant is awarded the contract although he has the lower valuation, this reduces the *private* surplus by $v \equiv |v_2 - v_1|$. There is also a social loss as it is in the authority's interest to award the contract to the contestant with the lower marginal cost (for example, in the simplest monopoly model with linear demand, the monopoly price is increasing in his marginal costs of production, see e.g. Tirole (1988, p. 66)). However, following the reasoning above, private and social loss need not necessarily to be identical as also consumer surplus may have to be taken into account.

We therefore assume that v enters the objective function of the social planner via a function $\phi(v)$ satisfying $\phi'(\cdot) > 0$ and $\phi(0) = 0$. It then follows that the expected welfare loss resulting from an inefficient allocation of the prize is given by

$$\Pi(t) \equiv \int_0^1 \int_{k(v_1;t)}^{v_1} \phi(v) F'(v_2) dv_2 F'(v_1) dv_1. \tag{23}$$

The lower bound of the inner integral is $k(v_1;t)$ which, recall, gives that type of (the favored) contestant 2 who chooses $\frac{1}{t}$-times of the effort level which contestant 1 would choose if his type were v_1. Thus, contestant 1 loses the contest whenever contestants 2's type is larger than $k(v_1;t)$. However, this is only socially undesirable as long as $v_2 < v_1$ which explains the upper bound of the inner integral. As for the outer integral, the authority has to take expectations over v_1.

Social costs Given our previous discussion, the authority's objective is to minimize the following social cost function

$$SC(t) = \Psi(t) + \Pi(t). \tag{24}$$

by choice of t. In general, the optimal level of unfairness denoted by t^* will depend on the nature of $\Psi(\cdot)$ and $\phi(\cdot)$. The properties of the social cost function (and its constituents) can be summarized as follows:

Lemma 4 *(i)* $\frac{d\Pi(t)}{dt} > 0 \; \forall t \in (1, \infty)$. *(ii)* $\lim_{t \to 1} \frac{dSC}{dt} = 0$. *(iii)* $\lim_{t \to \infty} \frac{dSC}{dt} = 0$.

Proof. See Appendix F. ■

Part (i) simply says that the social loss due to an inefficient allocation of the contest strictly increases in t for any t bounded away from 1 or infinity. But as both, marginal costs ($\frac{d\Pi}{dt}$) and marginal benefits ($\frac{d\Psi}{dt}$) vanish at the boundary (see part (ii)), it follows that corner solutions with either $t^* = 1$ or $t^* = \infty$ may well emerge. In the first case, social costs from inefficient allocation are so high that the authority prefers a fair contest. In the second case, the rent-seeking effect of wasting socially valuable effort dominates, and the authority actually awards the project to one contestant without procuring a contest at all. In all other cases, an interior solution with $1 < t^* < \infty$ arises. In the following we return to the example of uniformly distributed types ($F(v) = v$) to provide examples for Ψ and Φ that exhibit both corner solutions interior optimal degrees of unfairness.

5.2 Example revisited: Uniform distribution

Given the equilibrium efforts as in (15) and (16), *expected* equilibrium efforts are now given by

$$E[b_1^*(v_1;t)] = \int_0^1 \frac{t}{t+1} v_1^{t+1} dv_1 = \frac{t}{(t+1)(t+2)} \tag{25}$$

$$E[b_1^*(v_1;t)] = \int_0^1 \frac{1}{t+1} v_2^{(t+1)/t} dv_2 = \frac{t}{(t+1)(2t+1)} \tag{26}$$

Linear Case Assume first that $\Psi(\Sigma(t)) = x \cdot \Sigma(t)$ where $x > 0$ and $\phi(v) = v$. Thus, social costs of excessive effort and inefficient allocation are linear in the private costs, where x can be interpreted as the relative weight put on the loss from excessive effort. In this case, we have

$$SC(t) = x \cdot \frac{3t}{(2t+1)(t+2)} + \frac{t^2 - 2t + 1}{3(2t+1)(t+2)} = \frac{9xt - 2t + t^2 + 1}{3(2t+1)(t+2)}. \tag{27}$$

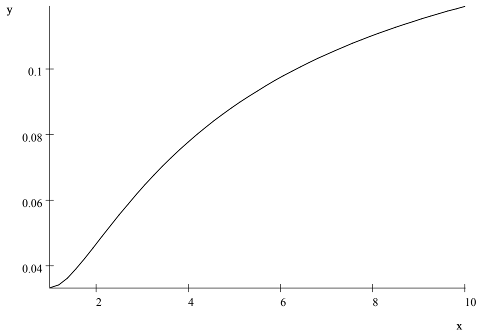
as

$$\begin{aligned} \Pi(t) &= \left| \int_0^1 \int_{v_1^t}^{v_1} (v_2 - v_1) dv_2 dv_1 \right| = \left| \int_0^1 (-\frac{1}{2}v_1^2 - \frac{1}{2}\left(v_1^2\right)^t + v_1^{t+1}) dv_1 \right| \\ &= \frac{t^2 - 2t + 1}{3(2t+1)(t+2)}. \end{aligned} \tag{28}$$

Taking the derivative with respect to t and simplifying yields

$$\frac{d}{dt} SC(t) = -\frac{3(t-1)(t+1)(2x-1)}{(2t+1)^2 (t+2)^2} \tag{29}$$

which is weakly positive (negative) for $x < (>) \frac{1}{2}$. Thus in the linear case, the objective function is monotone which leads to corner solutions. If the social loss from wasting effort is small compared to the loss from inefficiently awarding the prize, then the authority optimally stipulates a fair contest with $t = 1$ (see figure 3a). Hence, there is no allocation inefficiency at all. On the other hand, when the social loss from effort is high, then the authority will make the contest arbitrarily unfair which is equivalent to either randomly awarding the prize to one of the contestants, or to completely forestalling entry for the handicapped contestant (see figure 3b). As a consequence, the authority will choose the "false" contestant with probability $\frac{1}{2}$.



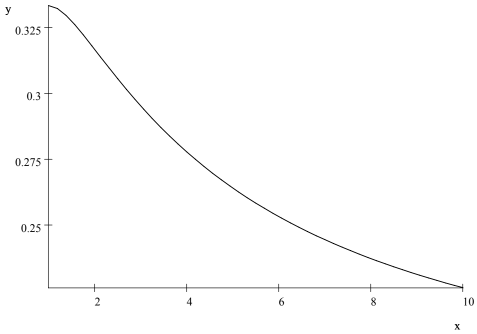


Fig. 4a: Exp. Soc. Loss, $x = 0.1$. Fig. 4b: Exp. Soc. Loss, $x = 1$.

Quadratic Case In this case, we continue to assume that the social cost of inefficient allocation is equal to the private costs, but that social costs of excessive effort is quadratic in the private costs, i.e. $\phi(v) = v$ and $\Psi(\Sigma(t)) = (\Sigma(t))^2$. Social costs are then

$$SC(t) = \left(\frac{3t}{(2t+1)(t+2)}\right)^2 + \frac{t^2 - 2t + 1}{3(2t+1)(t+2)} = \frac{21t^2 + t^3 + 2t^4 + t + 2}{4(2t+1)^2(t+2)^2}. \qquad (30)$$

As is illustrated in figure 4 below, this gives rise for an interior solution at $t^* \approx 3.1861$, so that a finite degree of unfairness is optimally chosen by the authority:

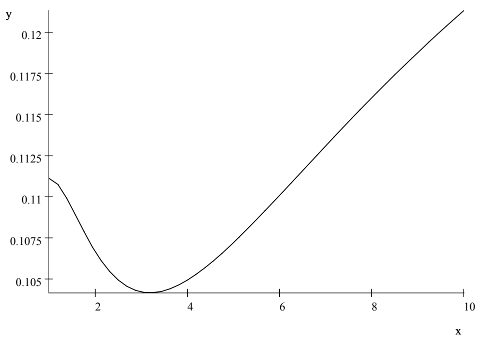


Figure 5: An interior solution for t^*.

We briefly summarize our results for these two examples as follows:

Result 3 *i) In the linear case where $\Psi(\Sigma(t)) = x \cdot \Sigma(t)$ and $\phi(v) = v$, the optimal level t^* chosen by the authority is given by*

$$t^* = \begin{cases} 1 & \textit{for } x \leq \frac{1}{2} \\ \infty & \textit{for } x \geq \frac{1}{2} \end{cases} .$$

ii) In the quadratic case where $\phi(v) = v$ and $\Psi(\Sigma(t)) = (\Sigma(t))^2$, the authorities optimal policy is an interior degree of unfairness ($t^* \approx 3.1861$).

6 Conclusion

We have analyzed a two-player discriminatory contest which is potentially unfair, as an authority has the option of setting an asymmetric allocation rule which is favoring one contestant while handicapping the other. We show that there exists a unique pure strategy equilibrium and that, for a given handicap, it is never possible that the handicapped contestant is awarded the prize when he has the lower valuation. As a result, inefficiencies based on inefficient allocation arise only from the possibility that the favored player is awarded the contract although his valuation is lower. This inefficiency is increasing in the degree of unfairness (t). On the other hand, total expected effort may decrease in t, so that there is a potential trade-off between these two types of social costs.

It may turn out that either a fair contest or no contest at all is the optimal choice for the authority. Intuitively, the first case is likely whenever social costs are very sensitive to allocative efficiency (e.g. if it is important to award a procurement to the low cost firm, or to avoid errors in court). By contrast, directly awarding the contract to one of the contestants (which may be chosen at random) makes sense whenever social costs focus on the wasteful effort spending (e.g. because opportunity costs are high from a social point of view as, for instance, in the application procedures for research grants). In less extreme settings, interior solution may arise, and this may justify unfair contests as frequently observed in reality. Coming back to the possibility that effort may also be desirable from a social point of view, a fair contest would always be optimal because it would lead to ex-post efficiency and to maximum effort incentives.

Of course, there may exist additional arguments in the contest designer's objective function. For instance, he explicitly wants to support local suppliers, or he believes that penalizing an innocent defendant is worse than acquitting a defendant who is guilty. Since there is one degree of freedom when deciding which party to favor, the handicap can be set to take such issues into account.

A Proof of Lemma 1

To prove the several characteristics of the equilibrium effort strategies, we proceed in three steps. First, we show that the structure of the payoff function induces non-decreasing strategies. Together with continuity, this in turn implies strict monotonicity and therefore differentiability and bijectivity on the restricted domain D_{b_i}.

As a first step consider monotonicity. Under slight abuse of notation, for any $v_i', v_i \in [0,1]$ with $v_i' > v_i$ incentive compatibility requires

$$\begin{aligned} \Pi_i(b_i(v_i), v_i; t) &\geq \Pi_i(b_i(v_i'), v_i; t) \\ \Pi_i(b_i(v_i'), v_i'; t) &\geq \Pi_i(b_i(v_i), v_i'; t) \end{aligned}$$

Taking the sum of both conditions and reordering yields:

$$\Pi_i(b_i(v_i'), v_i'; t) - \Pi_i(b_i(v_i'), v_i; t) \geq \Pi_i(b_i(v_i), v_i'; t) - \Pi_i(b_i(v_i), v_i; t).$$

Using the explicit structure of the pay-off function, this leads to

$$\begin{aligned} (v_1' - v_1)\Pr(b_1(v_1') > t \cdot b_2) &\geq (v_1' - v_1)\Pr(b_1(v_1) > t \cdot b_2) \\ (v_2' - v_2)\Pr(b_2(v_2') > \frac{1}{t} \cdot b_1) &\geq (v_2' - v_2)\Pr(b_2(v_2) > \frac{1}{t} \cdot b_1) \end{aligned}$$

But this only holds if $b_i(v_i') \geq b_i(v_i)$ which proves monotonicity.

We will prove continuity by contradiction. Assume that b_1 is not continuous at $x \in (0, b_1(1))$. Stated differently $b_1(x) > \lim_{\epsilon \to 0} b_1(x - \epsilon) \equiv \underline{b}_1(x)$. This implies, that contestant 2 will not choose some effort level $b_2 \in (\underline{b}_1(x)/t, b_1(x)/t)$ as he can always reduce costs while the probability of winning the contest remains unchanged. Anticipating this, there is no reason for contestant 1 to increase effort from $\underline{b}_1(x)$ to $b_1(x)$. Hence, we end up with a contradiction. Note, that the same result can be derived for the continuity of strategies of the favored player by a permutation of indices and the appropriate modification of probabilities of winning the contest. Furthermore, as $F'(v) \neq 0 \ \forall v \neq 0$, this result holds for all $v_1 \in (0, 1]$.

Now assume that $b_i(v_i)$ is not strictly increasing on the restricted domain D_{b_i}. That means, there is an interval $I \subseteq (0,1]$ of finite length with $b_i(v_i) \equiv \underline{b} > 0 \forall v_i \in I$. Given such a strategy profile of contestant i, contestant j maximizes his expected payoff as given by Eqn. (4) or (5). To be specific, let $i = 1$ and $j = 2$. Now assume contestant 2 chooses $(\underline{b} - \epsilon)/t$ for some valuation v_2. Then his pay-off is

$$v_2 \Pr(\underline{b} - \epsilon > b_1) - (\underline{b} - \epsilon)/t.$$

Now assume contestant 2 chooses $(\underline{b} + \epsilon)/t$ instead. His expected pay-off function is then

$$v_2 \Pr(\underline{b} + \epsilon > b_1) - (\underline{b} + \epsilon)/t$$

contestant 2 profits from such a deviation as can be seen when $\epsilon \rightarrow 0$

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} (v_2 \Pr(b_1 &> \underline{b} + \epsilon) - (\underline{b} + \epsilon)/t - (v_2 \Pr(b_1 > \underline{b} - \epsilon) - (\underline{b} - \epsilon)/t)) \\
&= \lim_{\epsilon \rightarrow 0} (v_2 (\Pr(b_1 \in [\underline{b} - \epsilon, \underline{b} + \epsilon])) - 2\frac{\epsilon}{t}) \\
&= v_2 \Pr(b_1 = \underline{b}) > 0
\end{aligned}$$

Therefore contestant 2 will always exert effort slightly above $\underline{b}/t$ instead of slightly below, but that contradicts continuity. Analogously, a gap in effort strategies of contestant 1 can be deduced from a plateau in contestant 2's equilibrium strategy. This proves strict monotonicity on the restricted domain. Therefore effort strategies are differentiable almost everywhere and a bijection from the restricted domain D_{b_i} onto $(0, b_i(1)]$. Finally, D_{b_i} has to be non-empty, as it can never be part of an equilibrium that both contestants or only one contestant choose an effort level of zero for the entire valuation space.

B Proof of Lemma 2

Part i) Clearly, $b_i(0) = 0$ determines the lower bound of D_{G_i}. Moreover, denoting by $b_i^{\max}$ the maximum effort level of contestant i, it follows from Lemma 1 that $\rho(b_i^{\max}) = \max\{v_i\} = 1$. This implies that contestant 1 can never be better off by exerting effort excessively, i.e. $b_1 \leq t \cdot b_2^{max}$ has to hold. Analogously, neither will contestant 2 exert more effort more than necessary to win the contest with probability 1, i.e. $b_2 \leq \frac{1}{t} \cdot b_1^{max}$ has to hold. Of course, this must also be true for $b_1^{\max}$ and $b_2^{\max}$, respectively, i.e. $b_1^{\max} \leq t \cdot b_2^{max}$ and $b_2^{\max} \leq \frac{1}{t} \cdot b_1^{max}$ must hold. Rearranging yields

$$b_2^{max} \leq \frac{1}{t} \cdot b_1^{max} \leq b_2^{\max}$$

from which it follows that $b_2^{max} = \frac{1}{t} \cdot b_1^{max}$ or equivalently, $b_1^{max} = t \cdot b_2^{max}$ must hold. We refer to this as the *final condition.*

Part ii) Follows immediately from our assumptions on $F(v)$ and Lemma 1.

Part iii) Suppose, $G_j(0) = g > 0$. We show that, for all $v_i \in [0, 1]$, there is some positive effort level $x > 0$ for contestant i such that he is strictly better off than with choosing $b_i = 0$: With $b_i = 0$, contestant i's loses whenever $b_j > 0$ (which happens with probability $(1 - g)$) wins with probability $\frac{1}{2}$ whenever $b_j = 0$ (which happens with probability g) so that his expected payoff is simply $v_i \cdot \frac{g}{2}$. When choosing a positive effort level $x > 0$, he wins with certainty when $b_j = 0$ and, depending on x (and t), may even win when $b_j > 0$. Thus we have:

$$\Pi_i(x, \cdot) = v_i \cdot G_j(x) - x \geq v_i \cdot g - x > v_i \cdot \frac{g}{2} = \Pi_i(0, \cdot)$$

where the last inequality holds whenever $x < \frac{g}{2} \cdot v_i$, so that for all $v_i > 0$, there exist $x > 0$ which satisfies this condition.

Part iv) As the first order conditions consist of two ordinary first order differential equations which are Lipschitz continuous for $v_i > 0$, any set of initial conditions ($b_i(v_i) = c_i, i = 1, 2$) determines unique trajectories $b_i(v_i)$. In the following, we show that part (i) and part (iii) together with the so-called no-crossing property of equilibrium effort levels (see Lizzeri and Persico (2000)) implies, that there is only *one* admissible set of initial conditions.

First note, that the final condition in part (i) reduces the freedom to choose initial conditions by one, as for a given $b_i(1)$, $b_{j \neq i}(1)$ is fixed. On the other hand part (iii) requires that at least one contestant i chooses a finite level of effort for every positive valuation $b_i(v_i) > 0 \forall v_i > 0$.

Consequently, for two sets of initial conditions to coexist, in at least one set one of the contestant's effort-distributions has to have an atom at zero. Furthermore, one of the two following properties of the corresponding equilibrium effort functions would have to hold.[8] (a) The atom of one contestant's effort distribution is smaller against a tougher strategy of his opponent. (b) At least one contestant chooses the same effort for a given valuation against two distinct opponent's strategies. In the following we show that none of the two requirements can be fulfilled in equilibrium.

As to (a), consider the first order conditions for $\underline{v}_i$ given by $F(\underline{v}_i) \equiv G_i(0)$ (contestant i's type who only just exerts zero effort) and denoting contestant i's type who only just chooses zero effort and a second equilibrium denoted by $\widetilde{(\cdot)}$

$$\begin{aligned} \underline{v}_i \frac{d}{db} G_j(0) &= 1 \\ \widetilde{\underline{v}}_i \frac{d}{db} \widetilde{G}_j(0) &= 1 \end{aligned}$$

But (a) requires that $\frac{d}{db} G_j(0) > \frac{d}{db} \widetilde{G}_j(0)$ and $\underline{v}_i > \widetilde{\underline{v}}_i$ are satisfied simultaneously which is a contradiction to the structure of the first order conditions.

A similar argument contradicts (b). The first order conditions [9] for player 1 with valuation v_1 against two distinct strategies of player 2 (once more distinguished by $\widetilde{(\cdot)}$)

$$\begin{aligned} v_1 \frac{d}{db_1} G_2(b_1/t) &= 1 \\ v_1 \frac{d}{db_1} \widetilde{G}_2(b_1/t) &= 1 \end{aligned} \tag{31}$$

can not be fulfilled simultaneously. Therefore coexisting sets of initial conditions are not feasible.

C Proof of Lemma 3

Using $k(v_1; t)$, the first order conditions (9) and (10) can be transformed into a set of differential equations expressed in a single variable v_1. Substituting $k(v_1)$ for v_2 in Eqn.

[8] To see this it suffices to plot ρ_i against b_i for $i = 1, 2$ as detailed in Lizzeri and Persico (2000).

[9] Once again we restrict ourselves to the favored bidder without loss of generality as the argument is independent of t.

(10) yields

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{t})) \cdot \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{1}{t} \;=\; 1 \tag{32}$$

$$k(v_1) \cdot F'(\rho_1(t \cdot b_2(k(v_1)))) \cdot \rho_1'(t \cdot b_2(k(v_1))) \cdot t \;=\; 1. \tag{33}$$

We can also make use of the identity of the two equations to yield

$$v_1 \cdot F'(\rho_2(\frac{b_1}{t})) \cdot \rho_2'(\frac{b_1}{t}) \cdot \frac{1}{t} = k(v_1) \cdot F'(\rho_1(t \cdot b_2(k(v_1))) \cdot \rho_1'(t \cdot b_2(k(v_1)) \cdot t. \tag{34}$$

Moreover, it follows from the definition of $k(\cdot)$ that

$$\frac{dk(v_1;t)}{dv_1} = \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{db_1(v_1)}{dv_1} \cdot \frac{1}{t}. \tag{35}$$

Thus, we can re-write Eqn. (34) as

$$\begin{aligned}
&v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} = \\
&\qquad k(v_1) \cdot F'(\rho_1(tb_2(\rho_2(\frac{b_1(v_1)}{t})))) \cdot \rho_1'(tb_2(\rho_2(\frac{b_1(v_1)}{t})) \cdot t \\
&\Leftrightarrow v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} = k(v_1) \cdot F'(\rho_1(b_1)) \cdot \rho_1'(b_1) \cdot t \\
&\Leftrightarrow v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} = k(v_1) \cdot F'(v_1) \cdot \rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} \cdot t.
\end{aligned} \tag{36}$$

Finally, as $\rho_1(b_1(v_1)) = v_1$, it follows that $\rho_1'(b_1) = \frac{dv_1}{db_1}$ which implies that $\rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} = 1$. Hence, we end up with a single ordinary differential equation

$$\frac{dk(\cdot;t)}{dv_1} = \frac{t \cdot k(v_1;t) \cdot F'(v_1)}{v_1 \cdot F'(k(v_1;t))}, \tag{37}$$

where the boundary condition $k(1;t) \equiv 1$ and our assumptions on $F(v)$ guarantee a unique solution for $k(\cdot)$. Analogously, we get

$$\frac{dk^{-1}(\cdot;t)}{dv_2} = \frac{k^{-1}(v_2;t) \cdot F'(v_2)}{t \cdot v_2 \cdot F'(k^{-1}(v_2;t))}. \tag{38}$$

To derive a solution in closed form, we separate dependent and independent variables of differential equations (37) and (38) to yield

$$\frac{dk}{k}F'(k) = t\frac{dv_1}{v_1}F'(v_1) \tag{39}$$

$$\frac{dk^{-1}}{k^{-1}}F'(k^{-1}) = \frac{dv_2}{v_2}F'(v_2). \tag{40}$$

With $H(x) = \int_x^1 \frac{F'(y)}{y}dy$, integration yields

$$H(k) = tH(v_1) \tag{41}$$

$$H(k^{-1}) = \frac{1}{t}H(v_2) \tag{42}$$

which is equivalent to $k(v_1;t) = H^{-1}(tH(v_1))$ and $k^{-1}(v_2;t) = H^{-1}(\frac{1}{t}H(v_2))$ as stated in the Lemma. Finally note that, by definition, $H(1) = 0$ so that $H^{-1}(0) = 1$. As $t \to \infty$, $k^{-1}(v_2;\cdot) \to 1$. This also implies that contestant 2 will be the winner with probability 1 when $t \to \infty$: To see this, note that contestant 2 wins whenever $v_1 \leq k^{-1}(v_2;t)$ which occurs with probability $F(k^{-1}(v_2;t))$ and which tends to 1 as $t \to \infty$. Analogously, contestant 1 wins whenever $v_2 \leq k(v_1;t)$ which occurs with probability $F(k(v_1;t))$ and which must tend to zero as $t \to \infty$ and thus $\lim_{t\to\infty} k(v_1;t) = 0$ must hold.

D Proof of Theorem 1

Recall that the derivative of the equilibrium effort strategy with respect to v_1 must satisfy $\frac{db_1(v_1)}{dv_1} = \frac{1}{\rho_1'(b_1)}$. Moreover, using the definition of $k(\cdot)$, we have $\rho_1'(b_1) = \rho_1'(t \cdot b_2(k(v_1;t)))$ such that $\frac{db_1(v_1)}{dv_1} = \frac{1}{\rho_1'(t \cdot b_2(k(v_1;t)))}$ holds. From Eqn. (33) it also follows that

$$\frac{1}{\rho_1'(t \cdot b_2(k(v_1;t))} = t \cdot k(v_1;t) \cdot F'(v_1) \tag{43}$$

must hold in equilibrium so that we have

$$\frac{db_1(v_1)}{dv_1} = t \cdot k(v_1;t) \cdot F'(v_1). \tag{44}$$

Together with $b_1(\max\{0, k^{-1}(0;t)\}) = 0$ and the definition of $k(v_1;t)$, closed form solutions for the equilibrium effort strategies are given by

$$b_1^*(v_1) = \int_{\max\{0,k^{-1}(0)\}}^{v_1} t \cdot k(V;t) \cdot dF(V) \tag{45}$$

$$b_2^*(v_2) = \frac{b_1^*(k^{-1}(v_2))}{t} \tag{46}$$

as stated in the Theorem.

E Proof of Theorem 2

Part i) As for the first case, in any BNE, contestant 1 loses the contest whenever $b_1^*(v_1) < t \cdot b_2^*(v_2) \Leftrightarrow v_2 > k(v_1;t)$ which simply follows from the definition of $k(v_1;t)$: Since $k(v_1;t)$ gives that type of contestant 2 who bids $\frac{1}{t}-$times as much as contestant 1 (which would result in a tie), contestant 1 loses the contest whenever $v_2 > k(v_1;t)$. To have an inefficient allocation, also $v_1 > v_2$ must hold. As we have seen for the symmetric case with $t = 1$, $H^{-1}(H(v_1)) = v_1$ leads to $k(v_1;1) = v_1$. Since it has been shown in Lemma 3 that $k(v_1;t)$ is decreasing in t, it follows that for all $t > 1$ there exist v_1, v_2 such that $v_2 > k(v_1;t)$ even when $v_1 > v_2$. Therefore the joint event $\{v_2 > k(v_1;t)\} \wedge \{v_1 > v_2\}$ has positive probability.

Contrary to that consider the second case: contestant 2 loses whenever $b_2^*(v_2) < \frac{1}{t} \cdot b_1^*(v_1) \Leftrightarrow k^{-1}(v_2;t) < v_1$. Again, for this outcome to be inefficient, we must also have $v_2 > v_1$. For $t = 1$ we get $k^{-1}(v_2;1) = v_2$. However, contrary to the first case, since $k^{-1}(v_2;t)$ is increasing in t (see Lemma 3 again). Therefore, for all $t > 1$ the joint event $\{k^{-1}(v_2;t) < v_1\} \wedge \{v_2 > v_1\}$ has probability measure zero.

Part ii) Note that for all v_1, $k(v_1;t) \leq v_1$ holds as for $t = 1$, we have $k(v_1;1) = v_1$, and $k(v_1;t)$ was shown to be decreasing in t (see Lemma 3). In Eqn. (14) for each v_1, $F(v_1)$ is the probability that contestant 1 has the higher valuation, while the probability that contestant 1 is the winner is only $F(k(v_1;t)) \leq F(v_1)$ so that by integrating over all v_1, the result follows. Moreover, for $t = 1$, we have $k(v_1;t) = v_1$ so that the integrand in Eqn. (14) is zero. For the derivative w.r.t. t, we have $\frac{d}{dt}p^*(\cdot) = \int_0^1 -F'(k(v_1;t)\frac{dk(\cdot)}{dt}dv_1 > 0$ as we have shown in Lemma (3) that $k(\cdot)$ is decreasing in t. Finally, we know from Lemma

3, part iii) that contestant 2's probability of winning tends to 1 as $t \to \infty$. It remains to show that contestant 2 has the lower valuation with probability $\frac{1}{2}$: Since for each v_1, $\Pr(v_2 \leq v_1)$ is $F(v_1)$, taking expectations over v_1 yields $\int_0^1 F(v_1)F'(v_1)dv_1$. Integration by parts then gives

$$\begin{aligned}
\int_0^1 F(v_1)F'(v_1)dv_1 &= F(v_1)^2\Big|_0^1 - \int_0^1 F(v_1)F'(v_1)dv_1 \\
&= 1 - \int_0^1 F(v_1)F'(v_1)dv_1 \Leftrightarrow \\
2\int_0^1 F(v_1)F'(v_1)dv_1 &= 1 \Leftrightarrow \int_0^1 F(v_1)F'(v_1)dv_1 = \frac{1}{2}.
\end{aligned}$$

F Proof of Lemma 4

Part (i) As we already saw in Lemma 3 that $\frac{dk(v_1;t)}{dt} < 0$, Part (i) follows directly from

$$\begin{aligned}
\frac{d\Pi}{dt} &= \frac{d}{dt}\int_0^1 \int_{k(v_1;t)}^{v_1} \phi(v)F'(v_2)dv_2F'(v_1)dv_1 \\
&= -\int_0^1 \left(\phi(|k(v_1;t) - v_1|)F'(k(v_1;t))\frac{dk(v_1;t)}{dt}\right) F'(v_1)dv_1 \qquad (47)
\end{aligned}$$

which is strictly positive as long as $t \in (0,\infty)$ (as $k(v_1;t) < v_1$ in this case and, furthermore $F'(.) > 0$ for $v \in (0,1)$ by assumption). $\frac{d\Pi}{dt} = 0$ if $t = 1$ as $v = 0$ in this case. Furthermore $\lim_{t\to\infty} \frac{d\Pi}{dt} = 0$ as we already saw in Lemma 3 that $\lim_{t\to\infty} k(v_1;t) = 0$ which implies that $\lim_{t\to\infty} \frac{dk(v_1;t)}{dt} = 0$.

Part(ii) and (iii) This given, we are left with the proof that $\frac{d\Psi}{dt}$ vanishes if $t = 1$ or $t \to \infty$. Note that we can restrict ourselves to an investigation of $\frac{d\Sigma}{dt}$ as $\Psi'(.) > 0$ by assumption. It will prove useful to rewrite $b_2(v_2)$ as follows. Analogously to the proof of Theorem 1, we can extract the slope of $b_2(v_2)$ through

$$\begin{aligned}
\frac{db_2(v_2)}{dv_2} &= \frac{1}{\rho_2'(b_2)} = \frac{1}{\rho_2'(b_1(k^{-1}(v_2;t))/t)} \\
&= \frac{1}{t}k^{-1}(v_2;t)F'(\rho_2(b_1(k^{-1}(v_2;t))/t)) \\
&= \frac{1}{t}k^{-1}(v_2;t)F'(v_2) \qquad (48)
\end{aligned}$$

where the first row follows from the definition of $k^{-1}(.)$ and the consecutive step uses Eqn. (9). The last row is an application of the definition of $k^{-1}(v_2;t)$. Then, $b_2(v_2)$ is given by

$$b_2(v_2) = \int_{\max\{k(0;t),0\}}^{v_2} \frac{1}{t} k^{-1}(V;t) F'(V) dV \tag{49}$$

With an index permutation $\frac{d\Sigma}{dt}$ can be rewritten as[10]

$$\frac{d\Sigma}{dt} = \int_0^1 \int_0^{v_1} \left(k(v_2;t) - \frac{1}{t^2} k^{-1}(v_2;t) + t\frac{dk(v_2;t)}{dt} + \frac{1}{t}\frac{dk^{-1}(v_2;t)}{dt} \right) F'(v_2) dv_2 F'(v_1) dv_1. \tag{50}$$

This expression, however, vanishes at $t = 1$ and for $t \to \infty$. To see this recall that $k(v;1) = k^{-1}(v;1) = v$, $\frac{dk(v;t)}{dt}|_{t=1} = -\frac{dk^{-1}(v;t)}{dt}|_{t=1}$, $\lim_{t\to\infty} k(v;t) = 0$, and $\lim_{t\to\infty} k^{-1}(v;t) = 1$ (see Lemma 3).

[10] For the ease of exposition, we neglect the $\max\{k^{-1}(0;t), 0\}$ term and its analogue for $b_2(v_2)$ in the equilibrium bid functions, as it is easy to show that this is without impact on the t-dependence.

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