Proper Rationalizability and Belief Revision in Dynamic Games

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Abstract

In this paper we develop an epistemic model for dynamic games in which players may revise their beliefs about the opponents' preferences (including the opponents' utility functions) as the game proceeds. Within this framework, we propose a rationalizability concept that is based upon the following three principles: (1) at every instance of the game, a player should believe that his opponents are carrying out optimal strategies, (2) a player should only revise his belief about an opponent's relative ranking of two strategies if he is certain that the opponent has decided not to choose one of these strategies, and (3) the players' initial beliefs about the opponents' utility functions should agree on a given profile u of utility functions. Common belief about these events leads to the concept of *persistent rationalizability* for the profile u of utility functions. It is shown that for a given profile u of utility functions, every properly rationalizable strategy for "types with non-increasing type supports" is a persistently rationalizable strategy for u. This result implies that persistently rationalizable strategies always exist for all game trees and all profiles of utility functions.

Keywords: Rationalizability, dynamic games, belief revision.

JEL Classification: C72

1. Introduction

In this paper we are concerned with the problem of how to model rational belief and rational behavior in dynamic games. One of the major challenges in this problem is the issue of belief revision, that is, how players change their belief about the opponents' behavior when they find out that their previous belief has been contradicted by the observed play of the game. In fact, most equilibrium and rationalizability concepts for dynamic games can be classified according to the restrictions they impose upon the players' belief revision policies, as these concepts usually differ as to what players should believe at "zero-probability information sets". In order to illustrate the various restrictions that existing concepts impose upon belief revision, and the impact they bear on the resulting theory of rational behavior, consider the game in Figure 1.

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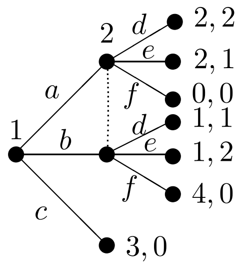

Figure 1

If player 1 believes that player 2 chooses rationally at his information set, then he expects player 2 not to choose f. If player 2, in turn, believes at the beginning that player 1 reasons in this way, and believes at the beginning that player 1 chooses rationally, then player 2 should believe at the beginning that player 1 chooses c. The question remains: What should player 2 do at his information set where he is faced with the fact that player 1 has *not* chosen c? At this information set, player 2 is led to revise his belief about player 1's strategy choice, and player 2's decision at that information set crucially depends upon the way he revises this belief.

There are rationality concepts which at this stage do not impose any restrictions upon player 2's belief revision. For instance, the concept of *common certainty of rationality at the beginning of the game* (Ben-Porath (1997)) requires common belief at the beginning of the game that players choose rationally at each of their information sets, but does not restrict the players' belief revision policies when they find out that their initial belief has been contradicted. In the game of Figure 1, common certainty of rationality at the beginning implies that player 1 should believe that player 2 will not choose f, and that player 2 should believe initially that player 1 chooses c. However, if player 2 is led to revise his belief about player 1 at his information set, he may believe that player 1 has chosen a or b, and as such player 2 may choose both d and e. A similar reasoning holds for the concept of sequential equilibrium (Kreps and Wilson (1982)) in this game: in every sequential equilibrium player 1 is initially believed to choose c, however the concept does not restrict player 2's beliefs at his information set, and hence player 2 is allowed to choose d and e.

The concept of *extensive form rationalizability* (Pearce (1984), Battigalli (1997)), on the other hand, *does* restrict player 2's belief revision procedure, and eventually singles out the choice e for player 2. In words, the concept requires a player, at each of his information sets, to look for the "highest possible degree of interactive belief in rationality"[1] that rationalizes the event of reaching this information set, and the player should then base his current and future beliefs upon this degree until it will be contradicted by some other event in the future. In the game of Figure 1, this means that player 2, upon observing that player 1 has chosen a or b, should attempt to explain this event by a theory in which player 1 is believed to choose rationally. If this is possible, then player 2 should try to find a "more sophisticated" theory explaining this

[1] Battigalli and Siniscalchi (2002) call it "highest possible degree of strategic sophistication".

event in which player 1 is not only believed to choose rationally, but is also believed to believe that player 2 will choose rationally at his information set. If this is not possible, then player 2 should stick to his first theory. If the more sophisticated theory *is* possible, then player 2 should attempt to find a theory with an even higher degree of interactive belief in rationality, and so on. According to this line of reasoning, player 2's "most sophisticated" theory that explains the event of player 1 choosing a or b is the following: player 1 is believed to rationally choose b, and player 1 is believed to believe with high probability that player 2 will irrationally respond with f. As such, player 2 should choose e.

The concepts of *proper equilibrium* (Myerson (1978)) and *proper rationalizability* (Schuhmacher (1999), Asheim (2001)) implicitly impose a different restriction upon player 2's belief revision procedure, and single out the choice d for player 2. The key idea in both concepts is that a player, when choosing his strategy, should not exclude any of the opponents' strategies, yet should deem one opponent strategy "infinitely more likely" than another if he believes the opponent to prefer the former over the latter. Here, the notion of "infinitely more likely" can be made explicit by the use of lexicographic probability distributions, as has been done by Blume, Brandenburger and Dekel (1991a, 1991b) and Asheim (2001) in their characterizations of proper equilibrium and proper rationalizability, respectively. In the game of Figure 1, the reasoning of proper equilibrium and proper rationalizability is as follows. Since player 2 should not exclude that player 1 chooses a or b, he strictly prefers d and e over f. Player 1, knowing this, should thus deem d and e infinitely more likely than f, and hence should strictly prefer c over a and strictly prefer a over b. Player 2, at the beginning of the game, should then deem c infinitely more likely than a, and deem a infinitely more likely than b. This implies that player 2, upon observing that player 1 has chosen a or b, should still deem a infinitely more likely than b, and hence player 2 should choose d at his information set.

The crucial belief revision requirement in the argument above to single out the choice d is thus that player 2, when observing that player 1 has chosen a or b, should maintain his initial belief that a is infinitely more likely than b. Within the framework of proper equilibrium and proper rationalizability, this belief revision principle could be stated alternatively as follows: if player 2, at the beginning of the game, believes that player 1 strictly prefers a over b, then player 2 should maintain this belief when observing that player 1 has chosen a or b. This principle thus reflects the idea that a player, upon reaching a new information set, should not change his belief about the opponent's relative ranking of two strategies that both could have led to this information set. We shall refer to this principle as the *proper belief revision principle.*

The concept of extensive form rationalizability, for instance, violates the proper belief revision principle in the game of Figure 1. We have seen that extensive form rationalizability requires player 2 to choose e. Hence, player 2 should believe at the beginning of the game that player 1 believes that player 2 chooses e. Consequently, player 2 believes at the beginning of the game that player 1 strictly prefers c over a, and strictly prefers a over b. However, the key argument in the concept of extensive form rationalizability has been that player 2, upon observing that player 1 has chosen a or b, should believe that player 1 believes that player 2 chooses f with high probability (to be more precise, with probability at least $\frac{2}{3}$). As such, player 2's new belief should be that player 1 prefers b over a, contradicting his initial belief about player 1's relative ranking of a and b.

The main purpose of this paper will be to incorporate the proper belief revision principle in a theory of rational behavior for dynamic games that insists on common belief of rationality throughout the game. More precisely, we shall develop an epistemic framework for dynamic games in which every observed move is to be interpreted as being in accordance with common belief of rationality, while allowing the players to revise their beliefs about the opponents' preferences (including the opponents' utility functions) during the game. In order to achieve common belief of rationality at every information set, it is necessary to allow players to change their beliefs about the opponents' utilities. In fact, Reny (1992, 1993) has shown that for the class of games with perfect information, there are only very few games in which there is no uncertainty about the players' utilities and in which common belief of rationality can be maintained at all decision nodes. In the game of Figure 1, it may be verified easily that common belief of rationality at player 2's information set is not possible if player 2 would not be able to revise his belief about player 1's utilities. Namely, if player 2 observes that player 1 has chosen a or b, and believes that player 1's utilities are as depicted at the terminal nodes, then player 2 should either believe that player 1 has not chosen rationally, or that player 1 believes that player 2 will not choose rationally.

We shall now enter into some details of the epistemic model upon which the concept of rationality shall be built. The basic assumption is that every player, at each of his information sets, faces uncertainty about the opponents' strategy choices, and that his preferences over his own strategies are induced by a subjective probability distribution (or belief) about the opponents' feasible strategy choices and a utility function at the terminal nodes following this information set. That is, every player is assumed to have a preference relation of the expected utility type. However, a player does not only have uncertainty about the opponents' strategies, but also about their preference relations (including their utility functions) at each of their information sets. Consequently, every player, at each of his information sets, should hold a second-order belief about the opponents' feasible strategy choices and the opponents' (first-order) preference relations at each of their information sets. In particular, this second-order belief concerns the possible utility functions held by the opponents at their respective information sets. These second-order beliefs, together with the utility function, induce second-order preference relations for each of the players. By a similar argument as above, one may then argue that a player, at each of his information sets, does not only have uncertainty about the opponents' strategy choices and first-order preference relations, but also about the opponents' second-order preference relations at each of their information sets. This, in turn, will lead to third-order beliefs and third-order preference relations, and so forth. Repeating this argument recursively inevitably leads to a model in which a player, at each of his information sets, holds an infinite hierarchy of preference relations. Within this hierarchy, the k-th order preference relation is induced by (1) a k-th order belief about the opponents' feasible strategy choices and the opponents' first-order, second-order, ..., $(k-1)$-th order preference relations, and (2) a utility function at the terminal nodes.

Our first result states that the model of infinite preference hierarchies described above is "homeomorphic" to a Harsanyi-style model (Harsanyi (1967, 1968)) in which the possible preference hierarchies, reflecting the players' possible assessments of all the relevant uncertain parameters in the game, may be identified with *types.* Every type within this model will then be completely characterized by specifying at each information set a utility function and a belief about the opponents' possible strategy choices *and types*. The justification for using such an implicit type model is not only conceptually relevant, but is also important from a practical viewpoint, as it considerably simplifies the analysis.

Subsequently, we use this epistemic model to develop a theory of rationality for dynamic games. The theory is built upon three conditions that types should satisfy: updating consistency, proper belief revision, and belief in sequential rationality. The first condition simply states that types should update their beliefs according to Bayes' rule, whenever possible. Proper belief revision means that a type should revise his beliefs according to the proper belief revision principle discussed above. By belief in sequential rationality we mean that a type, at each of his information sets, should believe that every opponent is carrying out a strategy that is optimal for him at each of his information sets. A type that, throughout the game, respects common belief about updating consistency, proper belief revision and belief in sequential rationality, is called *persistently rationalizable.*

Within our epistemic framework, the proper belief revision principle may be viewed as an expression of minimal belief change, as it requires a player to adapt his new beliefs to the newly observed behavior by opponents through changing his beliefs about the opponents' preferences in some minimal way. Suppose, for instance, that in the game of Figure 1 player 2 initially believes that player 1 strictly prefers c over a and strictly prefers a over b. (The utilities at the terminal nodes should be ignored at the moment.) Then, upon observing that player 1 has chosen a or b, player 2 should change his beliefs about player 1's preferences if he is to believe that player 1 has acted rationally. However, for player 2's eventual strategy choice it is only relevant how player 2 assesses the relative likelihood of the strategies a and b, and for this assessment it is only relevant what player 2 believes about player 1's relative ranking of a and b. The proper belief revision principle now argues that player 2 should leave his assessment about this relative ranking as it was, since, conditional on player 2's belief that player 1 has rationally chosen a or b, there is no evidence that his initial assessment about this ranking was wrong.

An important ingredient in the concept of persistent rationalizability is the possibility for types to revise their beliefs about the opponents' utility functions during the game. As such, we explicitly allow for uncertainty about the utility functions in the game. With respect to the latter aspect, the literature on noncooperative games can roughly be divided into three categories. The first, and largest, category contains concepts in which the players' beliefs about the opponents' utility functions should at all times agree on an exogenously given profile of utility functions. In these concepts there is thus no room for uncertainty about utilities. The second category contains concepts where players may have uncertainty about the opponents' utilities *during* the game, but where the players' beliefs *at the beginning* of the game should agree on an exogenously given profile of utility functions. The model of *games with randomly disturbed payoffs*, as proposed by Harsanyi (1973), represents such a situation: the uncertainty about the opponents' utilities is modeled by a sequence of utility perturbations around a fixed profile of utilities, whereas the assumption that the perturbation vanishes in the limit guarantees that the players' beliefs at the beginning of the game should (approximately) agree on this particular profile of utilities. Harsanyi's model has subsequently been applied by Fudenberg, Kreps and Levine (1988) and Dekel and Fudenberg (1990) for their analysis of the robustness of rationality concepts against infinitesemal uncertainty about the opponents' utilities. Other applications of Harsanyi's model are, among others, Zauner (2002) and Stinchcombe and Zauner (2002). The concept of *preference conjecture equilibrium* put forward in Perea (2002) also belongs to this second category. The third category, finally, consists of concepts in which players have uncertainty about the opponents' utilities, and in which the initial beliefs do not have to agree on one particular profile of utilities. A prominent contribution in this category is Battigalli (2003), who refers to such situations as "games with genuine incomplete information" and studies the concepts of weak and strong rationalizability within such situations.

This paper should be placed in the second category since we impose the additional condition that the players' initial beliefs about the opponents' utilities agree on an exogenously given profile u of utility functions. We say that a type is *persistently rationalizable for* u if (1) it is persistently rationalizable, (2) has the utility function as prescribed by u, and (3) respects common belief about the event that types *initially* believe that the opponents have utility functions as specified by u. Accordingly, a *strategy* is called persistently rationalizable for u if there is a persistently rationalizable *type* for u such that the strategy is optimal for this type at each of his information sets.

As to illustrate this concept, consider again the game in Figure 1. Let $u = (u_1, u_2)$ be the pair of utility functions depicted at the terminal nodes. We verify that the only strategy for player 2 which is persistently rationalizable for u coincides with the unique properly rationalizable strategy in this game, d. Suppose, namely, that type t_2 for player 2 would be persistently rationalizable for u. Since t_2 initially believes that player 1 initially believes that player 2 has utility function u_2 and chooses rationally, t_2 initially believes that player 1 initially believes that player 2 will not choose f. Combining this insight with the requirement that t_2 initially believes that player 1 has utility function u_1, it follows that t_2 initially believes that player 1 prefers strategy a over strategy b. By proper belief revision, t_2 must still believe that player 1 prefers a over b if he observes that player 1 has chosen a or b. Since t_2 should believe at his information set that player 1 has chosen rationally, t_2 should believe at his information set that player 1 has chosen a. But then, t_2 strictly prefers d at his information set, and hence player 2's unique persistently rationalizable strategy for u is d.

The main result in this paper shows that the relationship in the example between properly rationalizable strategies for u on the one hand and persistently rationalizable strategies for u on the other hand, is not a coincidence. We shall prove namely that some refinement of proper rationalizability always implies persistent rationalizability. The refinement we adopt uses Asheim's (2001) characterization of proper rationalizability, and imposes an additional restriction upon the lexicographic beliefs of properly rationalizable types. The additional requirement states that, whenever a type t in the k-th layer of his lexicographic belief assigns positive probability to some opponent's type, he should also have assigned positive probability to this type in his $(k-1)$-th layer. That is, the "type support" of type t should be non-increasing if we step down one layer in his lexicographic belief. Types with this additional property are called "properly rationalizable with non-increasing type supports", and strategies that may be chosen optimally by such types are referred to as "properly rationalizable strategies for types with non-increasing type supports". From Asheim (2001) it easily follows that such types and strategies always exist. Our main result then states that for every game tree and every possible profile u of utility functions, every properly rationalizable strategy for u for types with non-increasing type supports is persistently rationalizable for u.

This result has at least three important implications. First of all, it guarantees existence of persistently rationalizable types and strategies for every profile of utility functions. In other words, for every game tree and every profile u of utility functions we may formulate a non-contradictory theory of rational behavior in which players hold utility functions as specified by u, initially believe that the opponents have utility functions as specified by u, believe at every instance of the game that the opponents are carrying out optimal strategies, and in which players may revise their beliefs about the opponents' preferences within the limits of proper belief revision. Secondly, it shows that properly rationalizable strategies for types with non-increasing type supports may always be motivated by persistently rationalizable types. As such, the result offers an alternative interpretation of the concept of properly rationalizable strategies within a framework in which common belief of rationality is required at all information sets and in which players may revise their beliefs about the opponents' utility functions during the course of the game. Finally, note that the concept of properly rationalizable strategies for types with non-increasing type supports depends solely upon the pure reduced normal form of a dynamic game. By Thompson (1952) and Elmes and Reny (1994), this concept is thus invariant under the application of certain "irrelevant" transformations of the game tree. Consequently, the result implies that it is possible to find a refinement of persistent rationalizability that is invariant under all Thompson-Elmes-Reny transformations. This property is similar in spirit to the results by van Damme (1984) and Kohlberg and Mertens (1986) who showed that the extensive form rationality concepts of quasi-perfect and sequential equilibrium have a refinement that is invariant under all Thompson-Elmes-Reny transformations, namely proper equilibrium. For a study of the relationships between persistent rationalizability and rationality concepts other than proper rationalizability, the reader is referred to Perea (2003).

The outline of this paper is as follows. In Section 2 we first present some preliminary definitions and notation in extensive form games. Section 3 lays out the epistemic model we use, and contains the above mentioned representation result for preference hierarchies. The concept of persistent rationalizability is introduced in Section 4. Finally, in Section 5, we prove our main result concerning the relationship between proper and persistent rationalizability. The more technical proofs are contained in the appendix.

2. Extensive Form Structures

In this section we present the notation and some basic definitions in extensive form games that will be employed throughout this paper. The rules of the game are represented by an *extensive form structure* $\mathcal{S}$ consisting of a finite game tree, a finite set of players I, a finite collection H_i of information sets for each player i and at each information set $h_i \in H_i$ a finite collection $A(h_i)$ of actions for the player. The set of terminal nodes in $\mathcal{S}$ is denoted by Z, whereas $H = \cup_{i \in I} H_i$ denotes the collection of all information sets. We assume throughout that the extensive form structure satisfies perfect recall and that no chance moves occur. The latter assumption is not crucial for our analysis, but simplifies the presentation.

The concept of strategy we use in this paper is different from the usual one since it does not require a player to specify actions at information sets that are avoided by the same strategy. It thus coincides with the concept of *plan of action* in Rubinstein (1991). The use of this alternative definition is not really relevant for the analysis, but rather avoids the inclusion of redundant information in the definition of a strategy. Formally, let $\tilde{H}_i \subseteq H_i$ be some collection of information sets for player i, not necessarily containing *all* player i information sets, and let s_i be a mapping that assigns to every $h_i \in \tilde{H}_i$ some available action $s_i(h_i) \in A(h_i)$. We say that some information set $h^* \in H$ is *avoided* by the mapping s_i if for every profile of actions $(a(h))_{h \in H}$ with $a(h) \in A(h)$ for all h and $a(h_i) = s_i(h_i)$ for all $h_i \in \tilde{H}_i$, it holds that $(a(h))_{h \in H}$ avoids the information set h^*. We say that s_i is a *strategy* if its domain $\tilde{H}_i$ is equal to the collection of player i information sets that are not avoided by s_i. Obviously, every strategy s_i can be obtained by first prescribing some action at all player i information sets (that is, defining a strategy in the usual sense) and then deleting those player i information sets that are avoided by it. Let S_i denote the set of player i strategies, and let $S = \times_{i \in I} S_i$ be the set of all strategy profiles.

Throughout the paper, we shall make the assumption that the extensive form structure is with *observable deviators* (see Battigalli (1996), among others). In order to formalize this condition, we need the following definitions. For a given information set h, let $S(h)$ be the set of strategy profiles that reach h. For a given player i, not necessarily the player who moves at h, let $S_i(h)$ be the set of strategies s_i that do not avoid h. We say that $\mathcal{S}$ is with observable deviators if $S(h) = \times_{i \in I} S_i(h)$ for every information set h. That is, an information set h can only be avoided if there is at least one player who chooses a strategy that already avoids h by itself.

3. Epistemic Framework

In this section we formally model the players in an extensive form structure as decision makers under uncertainty. In order to do so, we first introduce some preliminary decision theoretic and epistemic concepts upon which this model shall be built.

3.1. Preference Hierarchies

The decision theoretic framework to be presented here is based on the models by Savage (1954) and Anscombe and Aumann (1963) for decision making under uncertainty. Let X be a compact metric space provided with some topology, and Y some finite set. Let $\Delta(Y)$ denote the space of probability distributions on Y, endowed with the natural topology. By $\mathcal{F}(X, Y)$ we denote the set of all measurable functions $f : X \to \Delta(Y)$ to which we shall refer as *acts*[2]. The set X is to be interpreted as the space of relevant variables about which the decision maker has uncertainty, whereas Y represents the set of possible consequences. As such, $\Delta(Y)$ contains all objective lotteries on the possible consequences. For a given act f in $\mathcal{F}(X, Y)$ and $x \in X$, let $f(x) \in \Delta(Y)$ be the objective lottery induced by x on Y, and let $f(x)(y)$ be the objective probability that $f(x)$ assigns to consequence y. By $\mathcal{P}^{eu}(X, Y)$ we denote the set of all nontrivial preference relations on $\mathcal{F}(X, Y)$ that are of the expected utility type, that is, for which is there is

[2]The definition of an act as we use it coincides with the notion of *compound horse lottery* in Anscombe and Aumann (1963).

some probability distribution μ on X and some nonconstant von Neumann-Morgenstern utility function $u : Y \to \mathbb{R}$ such that act f is weakly preferred over act g if and only if

$$\int_X u(f(x))\ d\mu \geq \int_X u(g(x))\ d\mu.$$

Here,

$$u(f(x)) = \sum_{y \in Y} f(x)(y)\ u(y)$$

denotes the expected utility induced by the objective lottery $f(x) \in \Delta(Y)$ and the utility function u.

Since for a given preference relation $p \in \mathcal{P}^{eu}(X, Y)$, the probability distribution μ is unique and the utility function u is unique up to some positive affine transformation, we may uniquely identify every $p \in \mathcal{P}^{eu}(X, Y)$ with a pair (μ, u) where μ is a subjective probability distribution on X and $u : Y \to \mathbb{R}$ with $\min_{y \in Y} u(y) = 0$ and $\max_{y \in Y} u(y) = 1$. Let $U(Y)$ be the set of all utility functions $u : Y \to \mathbb{R}$ with the latter property, and let $\Delta(X)$ be the set of probability distributions on X. Hence, we may identify $\mathcal{P}^{eu}(X, Y)$ with the set $\Delta(X) \times U(Y)$. Let τ_1 be the weak topology on $\Delta(X)$, let τ_2 be the natural topology on $U(Y)$ and τ the product topology on $\mathcal{P}^{eu}(X, Y)$ induced by τ_1 and τ_2. Then, the topological space $(\mathcal{P}^{eu}(X, Y), \tau)$ is a compact metric space.

Having established the model for individual decision making under uncertainty, we may now formalize an epistemic model for extensive form games in which players, at each of their information sets, have uncertainty about the opponents' strategy choices, uncertainty about the opponents' first-order preference relations (including their utility functions), uncertainty about the opponents' second-order preference relations, and so forth. This will eventually lead to the concept of *preference hierarchies* for players. The epistemic model combines elements from Epstein and Wang (1996) and Battigalli and Siniscalchi (1999). Epstein and Wang (1996) propose a model for *static* games in which players have uncertainty about the opponents' preference relations (possibly including the opponents' utility functions) and players may hold preference relations that do not conform to expected utility. Battigalli and Siniscalchi (1999), in turn, propose a model for *dynamic* games in which players hold expected utility preferences, players have no doubts about the opponents' utility functions but have uncertainty about the opponents' subjective probability distributions.

Consider some player i in an extensive form structure. Let h_0 be the information set that coincides with the beginning of the game, and let $H_i^* = H_i \cup \{h_0\}$. The primary source of uncertainty faced by player i at information set $h_i \in H_i^*$ concerns the strategy choices by his opponents. We may thus define the first-order space of uncertainty $X_i^1(h_i)$ by

$$X_i^1(h_i) = S_{-i}(h_i),$$

where $S_{-i}(h_i) = \times_{j \neq i} S_j(h_i)$. If $h_i = h_0$, we set $S_j(h_i) = S_j$ for all players j. Let $Z(h_i)$ be the set of terminal nodes that follow h_i. Every player i strategy $s_i \in S_i(h_i)$ may now be identified with an act $f_{s_i} : X_i^1(h_i) \to \Delta(Z(h_i))$ assigning to every $s_{-i} \in X_i^1(h_i)$ the lottery that assigns probability one to the terminal node $z \in Z(h_i)$ reached by the strategy profile (s_i, s_{-i}). Hence, every strategy $s_i \in S_i(h_i)$ corresponds to an act in $\mathcal{F}(X_i^1(h_i), Z(h_i))$. We assume that player i holds a nontrivial preference relation of the expected utility type $p_i^1(h_i) \in \mathcal{P}^{eu}(X_i^1(h_i), Z(h_i))$. We refer to $\mathcal{P}^{eu}(X_i^1(h_i), Z(h_i))$ as the set of *first-order* preference relations for player i at h_i.

At information set h_i, player i does not only have uncertainty about the strategies chosen by the opponents, but also about the first-order preference relations held by his opponents at each of their information sets. The second-order space of uncertainty for player i at h_i is therefore given by

$$\begin{aligned} X_i^2(h_i) &= S_{-i}(h_i) \times (\times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^1(h_j), Z(h_j))) \\ &= X_i^1(h_i) \times (\times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^1(h_j), Z(h_j))), \end{aligned}$$

which, together with the product topology induced by the topologies on $X_i^1(h_i)$ and $\mathcal{P}^{eu}(X_j^1(h_j), Z(h_j))$, is a compact metric space.

By the same argument as above, player i at h_i is assumed to hold a *second-order* preference relation $p_i^2(h_i) \in \mathcal{P}^{eu}(X_i^2(h_i), Z(h_i))$. Since player i has uncertainty about the second-order preference relations held by the other players at each of their information sets, the third-order space of uncertainty at h_i becomes

$$X_i^3(h_i) = X_i^2(h_i) \times (\times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^2(h_j), Z(h_j))),$$

which, together with the induced product topology, is again a compact metric space. By repeating this construction, we obtain an infinite sequence of "succesively richer" spaces of uncertainty, defined by

$$X_i^k(h_i) = X_i^{k-1}(h_i) \times (\times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), Z(h_j)))$$

for $k \geq 2$, which are all compact metric spaces.

A *preference hierachy* for player i at h_i is a sequence $p_i(h_i) = (p_i^k(h_i))_{k \in \mathbb{N}}$ where $p_i^k(h_i) \in \mathcal{P}^{eu}(X_i^k(h_i), Z(h_i))$ for all k. Hence, it specifies an infinite hierarchy of expected utility preference relations over succesively richer spaces of uncertainty. A vector $p_i = (p_i(h_i))_{h_i \in H_i^*}$, specifying a preference hierarchy at each of player i information sets, is simply called a *preference hierarchy* for player i. Let P_i be the set of all preference hierarchies for player i.

3.2. Coherence

Let $\mathcal{F}(X_i^k(h_i), Z(h_i)|\ X_i^{k-1}(h_i))$ be the set of acts from $X_i^k(h_i)$ to $\Delta(Z(h_i))$ which only depend upon the argument in $X_i^{k-1}(h_i)$. For a given preference relation $p_i^k(h_i) \in \mathcal{P}^{eu}(X_i^k(h_i), Z(h_i))$, let $mrg(p_i^k(h_i)|\ X_i^{k-1}(h_i))$ be the marginal induced by $p_i^k(h_i)$ on the set of acts $\mathcal{F}(X_i^k(h_i), Z(h_i)|\ X_i^{k-1}(h_i))$. In the obvious way, $mrg(p_i^k(h_i)|\ X_i^{k-1}(h_i))$ may be identified with a preference relation on the set of acts $\mathcal{F}(X_i^{k-1}(h_i), Z(h_i))$. We say that a preference hierarchy p_i is *coherent* if for every information set $h_i \in H_i^*$ and every $k \geq 2$ it holds that

$$mrg(p_i^k(h_i)|\ X_i^{k-1}(h_i)) = p_i^{k-1}(h_i).$$

In other words, a coherent preference hierarchy always exhibits a sequence of preference relations that do not contradict one another at overlapping layers. Let P_i^c be the set of coherent preference hierarchies for player i, and let $P_{-i} = \times_{j \neq i} P_j$ be the set of all opponents' preference hierarchies. Below, we shall prove that the set P_i^c is homeomorphic to $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$. In order to do so, we use the following version of Kolmogorov's Existence Theorem, which can be found in Dellacherie and Meyer (1978).

Lemma 3.1. *Let $(Z^k)_{k \in \mathbb{N}}$ be a sequence of compact metric spaces, and $(\mu^k)_{k \in \mathbb{N}}$ a sequence of probability measures $\mu^k \in \Delta(Z^1 \times ... \times Z^k)$ with $mrg(\mu^k | \, Z^1 \times ... \times Z^{k-1}) = p^{k-1}$ for all $k \geq 2$. Then, there is a unique probability measure $\mu \in \Delta(\times_{k \in \mathbb{N}} Z^k)$ such that $mrg(\mu | \, Z^1 \times ... \times Z^k) = \mu^k$ for every k.*

Here, by $mrg(\mu^k | \, Z^1 \times ... \times Z^{k-1})$ we denote the marginal induced by probability distribution μ^k on $Z^1 \times ... \times Z^{k-1}$. We are now able to derive the following result.

Lemma 3.2. *For every player i, the space P_i^c of coherent preference hierarchies is homeomorphic to the space $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$.*

Proof. Let $p_i \in P_i^c$. Then, $p_i = (p_i(h_i))_{h_i \in H_i^*}$ where $p_i(h_i) = (p_i^k(h_i))_{k \in \mathbb{N}}$ and $p_i^k(h_i) \in \mathcal{P}^{eu}(X_i^k(h_i), Z(h_i))$ for all k. Hence, every $p_i^k(h_i)$ can be uniquely identified with a pair $(\mu_i^k(h_i), u_i^k(h_i))$ where $\mu_i^k(h_i) \in \Delta(X_i^k(h_i))$ and $u_i^k(h_i) \in U(Z(h_i))$. Since $mrg(p_i^k(h_i) | \, X_i^{k-1}(h_i)) = p_i^{k-1}(h_i)$, it follows that $u_i^k(h_i) = u_i^{k-1}(h_i)$ and $mrg(\mu_i^k(h_i) | \, X_i^{k-1}(h_i)) = \mu_i^{k-1}(h_i)$ for all $k \geq 2$. Let $M_i(h_i)$ be the set of infinite hierarchies of probability measures $(\mu_i^k(h_i))_{k \in \mathbb{N}}$ with $\mu_i^k(h_i) \in \Delta(X_i^k(h_i))$ for all k and $mrg(\mu_i^k(h_i) | \, X_i^{k-1}(h_i)) = \mu_i^{k-1}(h_i)$ for all $k \geq 2$. From the above, it follows that P_i^c is homeomorphic to the space $\times_{h_i \in H_i^*}(M_i(h_i) \times U(Z(h_i)))$. Since $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$ is homeomorphic to $\Delta(S_{-i}(h_i) \times P_{-i}) \times U(Z(h_i))$ for every $h_i \in H_i^*$, it suffices to show that $M_i(h_i)$ is homeomorphic to $\Delta(S_{-i}(h_i) \times P_{-i})$ for every $h_i \in H_i^*$.

Let $h_i \in H_i^*$ be fixed. Define $Z^1 = S_{-i}(h_i)$ and $Z^k = \times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), Z(h_j))$ for $k \geq 2$. Then, by construction, $X_i^k(h_i) = Z^1 \times ... \times Z^k$ for all k. Hence $M_i(h_i)$ is the set of all hierarchies $\mu_i(h_i) = (\mu_i^k(h_i))_{k \in \mathbb{N}}$ with $\mu_i^k(h_i) \in \Delta(Z^1 \times ... \times Z^k)$ for all k and $mrg(\mu_i^k(h_i) | \, Z^1 \times ... \times Z^{k-1}) = \mu_i^{k-1}(h_i)$ for all $k \geq 2$. Since every Z^k is a compact metric space, we know by Lemma 3.1 that for each $(\mu_i^k(h_i))_{k \in \mathbb{N}} \in M_i(h_i)$ there is a unique probability measure $\mu_i(h_i) \in \Delta(\times_{k \in \mathbb{N}} Z^k)$ such that $mrg(\mu_i(h_i) | \, Z^1 \times ... \times Z^k) = \mu_i^k(h_i)$ for all k. Let f be the function which assigns to every $(\mu_i^k(h_i))_{k \in \mathbb{N}} \in M_i(h_i)$ this particular $\mu_i(h_i) \in \Delta(\times_{k \in \mathbb{N}} Z^k)$. Then, it may be verified that f is a homeomorphism with $f(M_i(h_i)) = \Delta(\times_{k \in \mathbb{N}} Z^k)$, and hence $M_i(h_i)$ is homeomorphic to $\Delta(\times_{k \in \mathbb{N}} Z^k)$. By definition,

$$\times_{k \in \mathbb{N}} Z^k = S_{-i}(h_i) \times (\times_{k \geq 2}(\times_{j \neq i} \times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), Z(h_j)))$$

which is homeomorphic to the space

$$\begin{aligned} & S_{-i}(h_i) \times (\times_{j \neq i} \times_{h_j \in H_j^*} \times_{k \geq 2} \mathcal{P}^{eu}(X_j^{k-1}(h_j), Z(h_j))) \\ = \; & S_{-i}(h_i) \times (\times_{j \neq i} P_j) = S_{-i}(h_i) \times P_{-i}. \end{aligned}$$

This implies that $\Delta(\times_{k \in \mathbb{N}} Z^k)$ is homeomorphic to $\Delta(S_{-i}(h_i) \times P_{-i})$, and hence $M_i(h_i)$ is homeomorphic to $\Delta(S_{-i}(h_i) \times P_{-i})$, which completes the proof. ■

Hence, there is a homeomorphism f_i from P_i^c to $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$ for every player i. Hence, every preference hierarchy $p_i \in P_i^c$ can be identified with the vector

$$f_i(p_i) = (\mu_i(p_i, h_i), u_i(p_i, h_i))_{h_i \in H_i^*}$$

where $\mu_i(p_i, h_i) \in \Delta(S_{-i}(h_i) \times P_{-i})$ and $u_i(p_i, h_i) \in U(Z(h_i))$. A subset $E \subseteq S_{-i}(h_i) \times P_{-i}$ is called an *event* at information set h_i. We say that preference hierarchy $p_i \in P_i^c$ *believes* the event E at information set h_i if

$$\text{supp } \mu_i(p_i, h_i) \subseteq E.$$

We do not only require that every preference hierarchy is coherent, but also that there be *common belief* among the players that all preference hierarchies are coherent. This may be formalized as follows. Let $P_{-i}^c = \times_{j \neq i} P_j^c$. Define the sets $P_i^{c,1}, P_i^{c,2}, \ldots$ by

$$\begin{aligned} P_i^{c,1} &= \{p_i \in P_i^c \mid p_i \text{ believes } S_{-i}(h_i) \times P_{-i}^c \text{ at every } h_i \in H_i^*\}, \\ P_i^{c,k} &= \{p_i \in P_i^{c,k-1} \mid p_i \text{ believes } S_{-i}(h_i) \times P_{-i}^{c,k-1} \text{ at every } h_i \in H_i^*\} \end{aligned}$$

for $k \geq 2$. Define $P_i^{c,\infty} = \cap_{k \in \mathbb{N}} P_i^{c,k}$ for all players i. We say that $P_i^{c,\infty}$ is the set of preference hierarchies for player i that *respect common belief of coherence.* We now obtain the following representation result for infinite preference hierarchies respecting common belief of coherence. The result is similar in spirit to results in Armbruster and Böge (1979), Böge and Eisele (1979), Mertens and Zamir (1985) and Epstein and Wang (1996). The proof for this result can be found in the appendix.

Lemma 3.3. *For every player i, the space of preference hierarchies $P_i^{c,\infty}$ respecting common belief of coherence is homeomorphic to the space $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}^{c,\infty}, Z(h_i))$.*

3.3. Types and Common Belief

In view of Lemma 3.3, we may identify each preference hierarchy $p_i \in P_i^{c,\infty}$ with a vector specifying at each information set $h_i \in H_i^*$ an expected utility preference relation $(\mu_i(p_i, h_i), u_i(p_i, h_i))$ where $\mu_i(p_i, h_i)$ is a probability measure on $S_{-i}(h_i) \times P_{-i}^{c,\infty}$ and $u_i(p_i, h_i)$ is a utility function from $Z(h_i)$ to the real numbers. A preference hierarchy $p_i \in P_i^{c,\infty}$ is called a *type* for player i, and by $T_i = P_i^{c,\infty}$ we denote the set of all player i types. Hence, every type $t_i \in T_i$ corresponds to a vector $(\mu_i(t_i, h_i), u_i(t_i, h_i))_{h_i \in H_i^*}$ where $\mu_i(t_i, h_i)$ is a probability distribution on $S_{-i}(h_i) \times T_{-i}$ and $u_i(t_i, h_i)$ is a utility function on $Z(h_i)$ for every information set $h_i \in H_i^*$. Using Lemma 3.3, we thus obtain the following representation result for types.

Corollary 3.4. *For every player i, the space T_i of player i types is homeomorphic to the space $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times T_{-i}, Z(h_i))$.*

We now formalize what it means that a type respects *common belief* about the event that types have certain properties. In order to do so, we use the following definitions. For a given type t_i, information set $h_i \in H_i^*$, and opponent j, let $\mu_i(t_i, h_i \mid T_j)$ be the marginal of the probability distribution $\mu_i(t_i, h_i)$ on the set of player j types. By

$$T_j^1(t_i, h_i) = \text{supp } \mu_i(t_i, h_i \mid T_j)$$

we denote the set of player j types that t_i attaches positive probability to at h_i, whereas

$$T_j^1(t_i) = \cup_{h_i \in H_i^*} T_j^1(t_i, h_i)$$

is the set of player j types that t_i attaches positive probability to somewhere in the game. For $j = i$, we define $T_i^1(t_i) = \{t_i\}$. Let

$$T^1(t_i) = \cup_{j \in I} T_j^1(t_i).$$

By

$$T^2(t_i) = \bigcup_{t \in T^1(t_i)} T^1(t)$$

we denote the set of types that (1) are attached positive probability by t_i, or (2) are attached positive probability by some type that is attached positive probability by t_i. In other words, $T^2(t_i)$ contains all those types to which t_i, directly or first-order-indirectly, assigns positive probability. By repeating this argument recursively, we obtain that

$$T^k(t_i) = \bigcup_{t \in T^{k-1}(t_i)} T^1(t)$$

for $k \geq 2$ represents the set of types to which t_i, directly or k-th-order-indirectly, assigns positive probability. By $T^\infty(t_i) = \cup_{k \in \mathbb{N}} T^k(t_i)$ we denote the set of all types to which t_i, directly or indirectly, assigns positive probability.

Now, let $\tilde{T} \subseteq \times_{j \in I} T_j$ be some set of profiles of types, or, simply, and event. We say that type t_i *respects common belief about* $\tilde{T}$ if $T^\infty(t_i) \subseteq \tilde{T}$. That is, t_i believes that all opponents' types belong to $\tilde{T}$, believes that all opponents' types believe that all the other players' types belong to $\tilde{T}$, and so forth.

4. Persistent Rationalizability

In the concept of persistent rationalizability we impose three conditions on types, to which we refer as common belief about *updating consistency, proper belief revision* and *belief in sequential rationality.* Types that satisfy these requirements are called *persistently rationalizable,* and strategies that are sequentially optimal for a persistently rationalizable type are called *persistently rationalizable strategies.*

In the previous section, we have seen that every type $t_i \in T_i$ corresponds to a vector $(\mu_i(t_i, h_i), u_i(t_i, h_i))_{h_i \in H_i^*}$, where $\mu_i(t_i, h_i)$ is a probability measure on $S_{-i}(h_i) \times T_{-i}$ and $u_i(t_i, h_i)$ is a utility function on $Z(h_i)$ for every information set $h_i \in H_i^*$. Updating consistency states that, whenever the game moves from a player i information set h_i^1 to another player i information set h_i^2, player i should derive his new belief $\mu_i(t_i, h_i^2)$ from his old belief $\mu_i(t_i, h_i^1)$ by Bayesian updating, if possible.

Definition 4.1. *A type t_i is said to be updating consistent if for all information sets $h_i^1, h_i^2 \in H_i^*$, where h_i^2 follows h_i^1, it holds that*

$$\mu_i(t_i, h_i^2)(E) = \frac{\mu_i(t_i, h_i^1)(E)}{\mu_i(t_i, h_i^1)(S_{-i}(h_i^2) \times T_{-i})}$$

for all events $E \subseteq S_{-i}(h_i^2) \times T_{-i}$, whenever $\mu_i(t_i, h_i^1)(S_{-i}(h_i^2) \times T_{-i}) > 0$.

While updating consistency states how to change the belief when the observed behavior is still in accordance with the previously held beliefs, *proper belief revision* imposes a condition upon the players' belief revision policies when the observed behavior contradicts the previous beliefs. In words, the condition states that, whenever player i at some information set h_i is led to revise his beliefs about opponent j's preference relation, then he should not change his belief about j's relative ranking of two strategies s_j and s_j' unless i is absolutely certain that j has decided not to choose one of these strategies. More precisely, if player i finds himself at information set h_i, then he is certain that player j has chosen some strategy in $S_j(h_i)$, without knowing for sure which strategy in $S_j(h_i)$ has been chosen. As such, proper belief revision states that player i should not revise his belief concerning player j's preferences over strategies in $S_j(h_i)$.

Definition 4.2. *A type t_i is said to satisfy proper belief revision if for every two information sets h_i^1, h_i^2 in H_i^* such that h_i^2 follows h_i^1 the following holds: if $t_j^2 \in$ supp $\mu_i(t_i, h_i^2| \, T_j)$ then there exists some $t_j^1 \in$ supp $\mu_i(t_i, h_i^1| \, T_j)$ such that t_j^1 and t_j^2 hold the same preference relation over strategies in $S_j(h_j) \cap S_j(h_i^2)$ at every $h_j \in H_j^*$.*

Here, $\mu_i(t_i, h_i^2| \, T_j)$ denotes the marginal of the probability distribution $\mu_i(t_i, h_i^2) \in \Delta(S_{-i}(h_i^2) \times T_{-i})$ on T_j.

We finally define belief in sequential rationality. For a given strategy s_i, let $H_i^*(s_i)$ be the set of information sets in H_i^* that are not avoided by s_i. A strategy-type pair $(s_i, t_i) \in S_i \times T_i$ is called *sequentially rational* if at every information set $h_i \in H_i^*(s_i)$, we have that

$$u_i(s_i, t_i| \, h_i) = \max_{s_i' \in S_i(h_i)} u_i(s_i', t_i| \, h_i).$$

Here, $u_i(s_i, t_i| \, h_i)$ denotes the expected utility induced by strategy s_i with respect to the probability distribution $mrg(\mu_i(t_i, h_i)| \, S_{-i}(h_i)) \in \Delta(S_{-i}(h_i))$ and the utility function $u_i(t_i, h_i)$. Let $(S_i \times T_i)^{sr}$ be the set of sequentially rational strategy-type pairs for player i, and let $(S_{-i} \times T_{-i})^{sr} = \times_{j \neq i}(S_j \times T_j)^{sr}$.

Definition 4.3. *A type t_i is said to believe in sequential rationality if supp $\mu_i(t_i, h_i) \subseteq (S_{-i} \times T_{-i})^{sr}$ for every $h_i \in H_i^*$.*

Definition 4.4. *A type t_i is called persistently rationalizable if it respects common belief about the events that (1) types are updating consistent, (2) types satisfy proper belief revision, and (3) types believe in sequential rationality. A strategy $s_i \in S_i$ is called persistently rationalizable if there is some persistently rationalizable type t_i such that (s_i, t_i) is sequentially rational.*

We finally impose an exogenous restriction upon the players' utility functions, and the players' initial beliefs about the opponents' utility functions. Let $\mathcal{S}$ be an extensive form structure and $u = (u_i)_{i \in I}$ an exogenously given profile of utility functions. We say that a type t_i *initially believes* u if $\mu_i(t_i, h_0)$ assigns probability one to the event that every opponent j has some type t_j with $u_j(t_j, h_j) = u_j|_{Z(h_j)}$ for all $h_j \in H_j^*$. Here, $u_j|_{Z(h_j)}$ denotes the restriction of the utility function u_j to the terminal nodes following h_j.

Definition 4.5. *We say that a type t_i is persistently rationalizable for $(\mathcal{S}, u)$ if (1) t_i is persistently rationalizable, (2) $u_i(t_i, h_i) = u_i|_{Z(h_i)}$ for all $h_i \in H_i^*$, and (3) t_i respects common belief about the event that types initially believe u.*

5. Relation to Proper Rationalizability

Schuhmacher (1999) introduced the concept of *proper rationalizabily* as a rationalizability-type analogue to proper equilibrium, and showed that it uniquely selects the backward induction strategies in generic games with perfect information. Subsequently, Asheim (2001) provided a characterization of proper rationalizability in terms of lexicographic beliefs for the case of two players. In this section, we introduce a refinement of properly rationalizable strategies to which we shall refer as "properly rationalizable strategies for types with non-increasing type supports". We show that for a given extensive form structure $\mathcal{S}$ and profile u of utility functions, every properly rationalizable strategy for $(\mathcal{S}, u)$ for types with non-increasing type supports is persistently rationalizable for $(\mathcal{S}, u)$. Since properly rationalizable strategies for types with non-increasing type supports always exist for every $(\mathcal{S}, u)$, this result implies the existence of persistently rationalizable strategies for every $(\mathcal{S}, u)$. For the definition of properly rationalizable strategies for types with non-increasing type supports, we use Asheim's characterization of proper rationalizability and extend it to games with more than two players.

5.1. Proper Rationalizability

Instead of iteratively constructing the type space of each player, as we have done in Section 3, we now assume the existence of some type space R_i for every player i with the property that every type r_i can be identified with some preference relation $q_i(r_i)$ on the set of acts $\mathcal{F}(S_{-i} \times R_{-i}, Z)$.[3] Hence, the exhaustive set of relevant variables about which player i may be uncertain is given by the opponents' strategy choices and types. From these preference relations we may actually derive a first-order preference relation, second-order preference relation, and so on, similar to Section 3. In Asheim's model, it is assumed that every preference relation $q_i(r_i)$ on $\mathcal{F}(S_{-i} \times R_{-i}, Z)$ is of the *lexicographic expected utility* type, that is, there is some lexicographic probability distribution $\lambda_i(r_i)$ on $S_{-i} \times R_{-i}$, and some utility function $u_i(r_i)$ from Z to the real numbers which together represent $q_i(r_i)$. By a *lexicographic probability distribution* we mean a vector $\lambda_i = (\lambda_i^1, \lambda_i^2, ..., \lambda_i^K)$ of probability distributions on $S_{-i} \times R_{-i}$, and we call λ_i^k the k-th order belief in λ_i. The interpretation is that player i assigns "infinitely more importance" to his k-th order beliefs than to his $(k+1)$-th order beliefs, without completely discarding the latter beliefs when reaching a decision. For every $(s_{-i}, r_{-i}) \in S_{-i} \times R_{-i}$, let $k(s_{-i}, r_{-i})$ be the first k for which (s_{-i}, r_{-i}) lies in the support of λ_i^k. We say that (s_{-i}, r_{-i}) is *infinitely more likely* than (s'_{-i}, r'_{-i}) in λ_i if $k(s_{-i}, r_{-i}) < k(s'_{-i}, r'_{-i})$.

Such a lexicographic probability distribution λ_i and a utility function u_i induce the preference relation q_i on $\mathcal{F}(S_{-i} \times R_{-i}, Z)$ defined as follows: act f is weakly preferred to act g if and only if for every $k \in \{1, ..., K\}$ with $u_i(f, \lambda_i^k) < u_i(g, \lambda_i^k)$ there is some $l < k$ with $u_i(f, \lambda_i^l) > u_i(g, \lambda_i^l)$.

[3] Here, we use different symbols for types and preferences, as to distinguish them from the types and preferences introduced in Section 3.

Here, $u_i(f, \lambda_i^k)$ denotes the expected utility of choosing act f while holding utility function u_i and subjective probability distribution λ_i^k.

Since the preference relation $q_i(r_i)$ can be represented by a pair $(\lambda_i(r_i), u_i(r_i))$, we may as well identify each type $r_i \in R_i$ with such a pair $(\lambda_i(r_i), u_i(r_i))$. If $\lambda_i(r_i) = (\lambda_i^1, ..., \lambda_i^K)$, we define $\text{supp}(\lambda_i(r_i)) = \cup_k \text{supp}(\lambda_i^k)$. Let $R_{-i}(r_i)$ be the projection of $\text{supp}(\lambda_i(r_i))$ on R_{-i}, that is, $R_{-i}(r_i)$ is the set of opponents' types that r_i deems possible. We say that type r_i is *cautious* if $\text{supp}(\lambda_i(r_i)) = S_{-i} \times R_{-i}(r_i)$. Hence, a cautious type does not exclude any opponent's strategy.

For every player i and player j, let $\mathcal{F}(S_{-i} \times R_{-i}, Z| \ S_j \times T_j)$ be the set of acts on $S_{-i} \times R_{-i}$ which only depend upon the argument in $S_j \times R_j$. By $mrg(q_i(r_i)| \ S_j \times R_j)$ we denote the marginal of the preference relation $q_i(r_i)$ on the set of acts $\mathcal{F}(S_{-i} \times R_{-i}, Z| \ S_j \times R_j)$. As such, $mrg(q_i(r_i)| \ S_j \times R_j)$ can be identified with a preference relation on the set of acts $\mathcal{F}(S_j \times R_j, Z)$. Since $q_i(r_i)$ is given by a lexicographic probability distribution $\lambda_i(r_i)$ on $S_{-i} \times R_{-i}$ and some utility function u_i, it follows that $mrg(q_i(r_i)| \ S_j \times R_j)$ is given by some lexicographic probability distribution $\lambda_{ij}(r_i)$ on $S_j \times R_j$ and the same utility function u_i. Let $\lambda_{ij}(r_i| \ R_j)$ be the marginal of λ_{ij} on R_j, and let $R_j(r_i) = \text{supp}(\lambda_{ij}(r_i| \ R_j))$ be the set of player j types that r_i deems possible. We say that type r_i *respects the opponents' preferences* if for every player j, every type $r_j \in R_j(r_i)$ and all strategies s_j, s_j' such that r_j strictly prefers s_j over s_j', it holds that (s_j, r_j) is infinitely more likely than (s_j', r_j) in $\lambda_{ij}(r_i)$. Hence, player i should deem superior strategies infinitely more likely than inferior strategies.

Let r_i be some type for player i. For every opponent j, let

$$R_j^1(r_i) = R_j(r_i)$$

be the set of player j types that t_i directly attaches positive probability to. For $j = i$, we define $R_i^1(r_i) = \{r_i\}$. Let

$$R^1(r_i) = \cup_{j \in I} R_j^1(r_i)$$

For every $k \geq 2$, let $R^k(r_i)$ be defined recursively by

$$R^k(r_i) = \bigcup_{r \in R^{k-1}(r_i)} R^1(r).$$

By $R^\infty(r_i) = \cup_{k \in \mathbb{N}} R^k(r_i)$ we denote the set of types that r_i, directly or indirectly, attaches positive probability to. The cardinality of $R^\infty(r_i)$ is called the *complexity* of type r_i.Let $\tilde{R} \subseteq \times_{j \in I} R_j$ be an event. We say that r_i respects common belief about $\tilde{R}$ if $R^\infty(r_i) \subseteq \tilde{R}$.

Definition 5.1. *Let $\mathcal{S}$ be an extensive form structure and u a profile of utility functions. A type $r_i \in R_i$ is called* properly rationalizable *for $(\mathcal{S}, u)$ if r_i respects common belief about (1) the event that types have utility functions as specified by u, (2) the event that types are cautious, (3) the event that types respect the opponents' preferences, and (4) the event that types have finite complexity*[4]. *A strategy s_i is called* properly rationalizable *for $(\mathcal{S}, u)$ if there is a properly rationalizable type r_i for $(\mathcal{S}, u)$ such that s_i is optimal for r_i.*

[4]Asheim (2001) assumes from the beginning that the set of possible types for every player is finite, and hence the complexity of every type is always finite within his framework.

We shall now introduce a refinement of proper rationalizability, based on the additional requirement that types should have *non-increasing type supports.* By the latter, we mean the property that type r_i's k-th order belief should only assign positive probability to opponents' types that have already been assigned positive probability in his $(k-1)$-th order belief. Formally, let $\lambda_i^k(r_i) \in \Delta(S_{-i} \times R_{-i})$ be the k-th order belief held by type r_i, and let $\lambda_i^k(r_i|\ R_{-i})$ be the marginal of $\lambda_i^k(r_i)$ on the set of opponents' types.

Definition 5.2. *A type r_i in R_i is said to be with non-increasing type supports if supp $\lambda_i^k(r_i|\ R_{-i}) \subseteq$ supp $\lambda_i^{k-1}(r_i|\ R_{-i})$ for all $k \geq 2$. A type r_i in R_i is said to be properly rationalizable for $(\mathcal{S}, u)$ with non-increasing type suuports if it is properly rationalizable for $(\mathcal{S}, u)$ and respects common belief about the event that types are with non-increasing type supports. A strategy s_i is said to be properly rationalizable for $(\mathcal{S}, u)$ for types with non-increasing type supports if it can be chosen optimally by some type that is properly rationalizable for $(\mathcal{S}, u)$ with non-increasing type supports.*

Following Asheim (2001), it can be shown that every strategy s_i assigned positive probability in some mixed strategy proper equilibrium for $(\mathcal{S}, u)$ is properly rationalizable for $(\mathcal{S}, u)$ for types with non-increasing type supports. In fact, such strategies may be supported by properly rationalizable types that respect common belief about the event that all types assign positive probability to only one type for each opponent. Such types trivially respect common belief about the event that types are with non-increasing type supports. Therefore, we may conclude that properly rationalizable strategies with non-increasing type supports always exist for every $(\mathcal{S}, u)$.

5.2. The Main Result

We now prove that every properly rationalizable strategy for $(\mathcal{S}, u)$ for types with non-increasing type supports is persistently rationalizable for $(\mathcal{S}, u)$. In order to establish this result we need the following two technical lemmas. The first lemma provides a useful technical property of extensive form structures with observable deviators. For this lemma we need some additional notation. Let i and j be different players, $h_i \in H_i^*$ and $h_j \in H_j$. If h_j precedes h_i, let $A(h_j, h_i)$ be the set of actions at h_j which lead to the information set h_i, that is, $a \in A(h_j, h_i)$ if and only if there is some path from the root to h_i at which a is chosen at h_j. If h_j does not precede h_i, then define $A(h_j, h_i) = A(h_j)$. Recall that $S_j(h_i)$ is the set of player j strategies that do not avoid h_i. Let $Z_j(h_i)$ be the set of terminal nodes that can be reached if player j chooses a strategy in $S_j(h_i)$. For a given strategy s_j, let $H_j(s_j)$ be the collection of player j information sets that are not avoided by s_j.

Lemma 5.3. *Let $\mathcal{S}$ be an extensive form structure with observable deviators. Let i and j be different players, $h_i \in H_i^*$ and $h_j \in H_j$. Then, the following holds:*
(a) $s_j \in S_j(h_i)$ if and only if $s_j(h_j) \in A(h_j, h_i)$ for every $h_j \in H_j(s_j)$,
(b) $z \in Z_j(h_i)$ if and only if for every information set $h_j \in H_j$ on the path to z, the unique action at h_j leading to z belongs to $A(h_j, h_i)$.

The proof can be found in the appendix. The second lemma deals with the problem of transforming a type from the "proper rationalizability type space " to a type from the "persistent rationalizability type space " while preserving its "relevant properties". Such transformations are relevant for the problem at hand since, in order to prove that properly rationalizable strategies for types with non-increasing type supports are persistently rationalizable, we shall show that every properly rationalizable *type* can be transformed into a persistently rationalizable type, while preserving its "relevant properties".

The following lemma formalizes what we mean by "relevant properties" and states that such "property preserving" transformations can always be carried out. Before stating the lemma, we need some additional definitions. Let R_i and T_i denote the set of player i types in the proper rationalizability model and persistent rationalizability model, respectively. In order to facilitate the exposition, we assume that R_i only contains types that respect common belief about the events that types are cautious and have finite complexity. For every two players i and j and information set $h_i \in H_i^*$, recall that $Z_j(h_i)$ denotes the set of terminal nodes that can be reached by some strategy in $S_j(h_i)$. Let the utility functions $(u_i)_{i\in I}$ be given. Define for every player i, every $h_i \in H_i^*$ and every opponent j the player j utility function $\tilde{u}_j(h_i) : Z \to \mathbb{R}$ by

$$\tilde{u}_j(h_i)(z) = \begin{cases} u_j(z), & \text{if } z \in Z_j(h_i), \\ u_j(z) - K_j(h_i), & \text{if } z \notin Z_j(h_i), \end{cases} \tag{5.1}$$

where the constant $K_j(h_i) > 0$ is chosen such that $u_j(z_1) > u_j(z_2) - K_j(h_i)$ for all $z_1 \in Z_j(h_i)$ and all $z_2 \notin Z_j(h_i)$. In the proof of the main result, $\tilde{u}_j(h_i)$ shall represent player i's belief about player j's utility function once information h_i has been reached.

For every type $r_i \in R_i$ and every information set $h_i \in H_i^*$, let $\lambda_i(r_i, h_i)$ be the marginal of the lexicographic probability distribution λ_i on $S_{-i}(h_i) \times R_{-i}$, and let $\mu_i(r_i, h_i) \in \Delta(S_{-i}(h_i) \times R_{-i})$ be the first-order belief of $\lambda_i(r_i, h_i)$. Recall that $\mu_i(t_i, h_i)$ is a probability distribution on $S_{-i}(h_i) \times T_{-i}$ for all $t_i \in T_i$ and h_i.

Lemma 5.4. *There is a transformation mapping t^* which to every type $r_i \in R_i$ and every opponent's information set h_l assigns some type $t^*(r_i, h_l)$ in T_i such that*
(a) $t^*(r_i, h_l)$ has utility function $\tilde{u}_i(h_l)$ for all r_i and h_l,
(b) $\mu_i(t^*(r_i, h_l), h_i)((s_j, t_j)_{j\neq i}) = \mu_i(r_i, h_i)(\{(s_j', r_j')_{j\neq i} |\ s_j' = s_j \text{ and } t^*(r_j', h_i) = t_j \text{ for all } j \neq i\})$ for all r_i, h_l and h_i, and all $(s_j, t_j)_{j\neq i}$ in $S_{-i}(h_i) \times T_{-i}$.

Here, $\mu_i(t^*(r_i, h_l), h_i)((s_j, t_j)_{j\neq i})$ denotes the probability that type $t^*(r_i, h_l)$ assigns at information set h_i to the profile $(s_j, t_j)_{j\neq i}$ of opponents' strategy-type pairs. The proof can be found in the appendix. We are now ready to prove the announced result.

Theorem 5.5. *Let $\mathcal{S}$ be an extensive form structure with observable deviators and $u = (u_i)_{i\in I}$ a profile of utility functions. Then, every properly rationalizable strategy for $(\mathcal{S}, u)$ for types with non-increasing type supports is persistently rationalizable for $(\mathcal{S}, u)$.*

Proof. Lemma 5.4 guarantees that there is some transformation mapping t^* which to every type $r_i \in R_i$ with finite complexity and information set h_l assigns some type $t^*(r_i, h_l)$ satisfying the properties (1) and (2) above. As a preliminary step, we first show that for every player i, every properly rationalizable type $r_i^* \in R_i$ with non-increasing type supports, every player $l \neq i$, every $h_l \in H_l^*$, the type $t^*(r_i^*, h_l)$ has the following properties: (a) it is updating consistent, (b) it satisfies proper belief revision, (c) it initially believes u and (d) it believes in sequential rationality.

(a) Fix a type $t^*(r_i^*, h_l)$, induced by a properly rationalizable type r_i^* with non-increasing type supports. First of all, it is easily verified that $t^*(r_i^*, h_l)$ is updating consistent since the vector of probability distributions $(\mu_i(t^*(r_i^*, h_l), h_i))_{h_i \in H_i^*}$ is induced by the cautious lexicographic probability distribution $\lambda_i(r_i^*)$ on $S_{-i} \times R_{-i}$.

(b) We now show that $t^*(r_i^*, h_l)$ satisfies proper belief revision. Let $h_i^1, h_i^2 \in H_i^*$ be such that h_i^2 follows h_i^1. Let $t_j^2 \in \mathrm{supp}\mu_i(t^*(r_i^*, h_l), h_i^2 | T_j)$. By Lemma 5.4 (b), it follows that $t_j^2 = t^*(r_j, h_i^2)$ for some r_j to which $\mu_i(r_i^*, h_i^2)$ assigns positive probability. Recall that $\mu_i(r_i^*, h_i^2)$ is the first-order probability distribution of the marginal of the lexicographic probability distribution $\lambda_i(r_i^*)$ on $S_{-i}(h_i^2) \times R_{-i}$. Similarly, $\mu_i(r_i^*, h_i^1)$ is the first-order probability distribution of the marginal of the lexicographic probability distribution $\lambda_i(r_i^*)$ on $S_{-i}(h_i^1) \times R_{-i}$. Since $S_{-i}(h_i^2) \subseteq S_{-i}(h_i^1)$, and since r_i^* is with non-increasing type supports, it follows that $\mu_i(r_i^*, h_i^1)$ assigns positive probability to r_j also. Let $t_j^1 = t^*(r_j, h_i^1)$. Then, by Lemma 5.4 (b), we know that $\mu_i(t^*(r_i^*, h_l), h_i^1)$ assigns positive probability to t_j^1. Now, choose some arbitrary information set $h_j \in H_j^*$. We prove that t_j^1 and t_j^2 at h_j hold the same preference relation on strategies in $S_j(h_j) \cap S_j(h_i^2)$, which would imply that $t^*(r_i^*, h_l)$ satisfies proper belief revision.

By definition we have that $t_j^1 = t^*(r_j, h_i^1)$ and $t_j^2 = t^*(r_j, h_i^2)$. It then follows from Lemma 5.4 (a) that t_j^1 has utility function $\tilde{u}_j(h_i^1)$ at h_j and t_j^2 has utility function $\tilde{u}_j(h_i^2)$ at h_j. Since $Z_j(h_i^2) \subseteq Z_j(h_i^1)$ we know, by definition, that $\tilde{u}_j(h_i^1)$ and $\tilde{u}_j(h_i^2)$ coincide at terminal nodes in $Z_j(h_i^2)$. Hence, at information set h_j, the types t_j^1 and t_j^2 hold the same utilities at $Z(h_j) \cap Z_j(h_i^2)$. Since $t_j^1 = t^*(r_j, h_i^1)$ and $t_j^2 = t^*(r_j, h_i^2)$, we may deduce from Lemma 5.4 (b) that t_j^1 and t_j^2 at information set h_j hold the same marginal probability distribution on the opponents' strategies in $S_{-j}(h_j)$. Together with the fact that t_j^1 and t_j^2 hold the same utilities at $Z(h_j) \cap Z_j(h_i^2)$, and the definition that $Z_j(h_i^2)$ is the set of terminal nodes that are possible if player j chooses a strategy in $S_j(h_i^2)$, it follows that t_j^1 and t_j^2 hold the same preference relation on strategies in $S_j(h_j) \cap S_j(h_i^2)$. Hence, $t^*(r_i^*, h_l)$ satisfies proper belief revision.

(c) By Lemma 5.4 (b), we may conclude that $\mu_i(t^*(r_i^*, h_l), h_0)$ assigns positive probability only to types t_j such that $t_j = t^*(r_j', h_0)$ for some r_j'. By Lemma 5.4 (a), every type $t^*(r_j', h_0)$ has utility function $\tilde{u}_j(h_0) = u_j$. Hence, $t^*(r_i^*, h_l)$ believes at h_0 that all opponents j hold utility function u_j, and hence $t^*(r_i^*, h_l)$ initially believes u.

(d) We finally show that $t^*(r_i^*, h_l)$ believes in sequential rationality. Hence, we must prove that $\mu_i(t^*(r_i^*, h_l), h_i)$ assigns probability one to the set of sequentially rational strategy-type pairs (s_j, t_j) for all players j and at all information sets $h_i \in H_i^*$. Let $t_i^* = t^*(r_i^*, h_l)$. Since r_i^* is properly rationalizable, we know r_i^* has finite complexity, and hence the probability distributions $\mu_i(r_i^*, h_i)$ on $S_{-i}(h_i) \times R_{-i}$ are discrete for all h_i. By property (b) in Lemma 5.4 we know that the probability distribution $\mu_i(t_i^*, h_i)$ on $S_{-i}(h_i) \times T_{-i}$ will be discrete as well for all h_i. Hence, it suffices to show that $\mu_i(t_i^*, h_i)$ assigns positive probability only to sequentially rational strategy-type pairs (s_j, t_j).

Fix an information set h_i^*. Suppose that (s_j, t_j) is a strategy-type pair in $S_j(h_i^*) \times T_j$ that is not sequentially rational. We prove that $\mu_i(t_i^*, h_i^*)$ puts probability zero on (s_j, t_j).

Suppose, on the contrary, that $\mu_i(t_i^*, h_i^*)$ assigns positive probability to (s_j, t_j). Since, by properties (a) and (b) in Lemma 5.4, t_i^* believes at h_i^* that player j holds utility function $\tilde{u}_j(h_i^*)$, we know that $u_j(t_j) = \tilde{u}_j(h_i^*)$. By property (b) in Lemma 5.4, we also know that t_j can be written as $t^*(r_j, h_i^*)$ for some $r_j \in R_j(r_i^*)$. Take now an arbitrary $r_j \in R_j(r_i^*)$ such that $t^*(r_j, h_i^*) = t_j$.

Since (s_j, t_j) is not sequentially rational, there exists some information set $h_j^* \in H_j^*(s_j)$ such that s_j is not optimal given the probability distribution $\mu_j(t_j, h_j^*)$ on $S_{-j}(h_j^*) \times T_{-j}$ and the utility function $\tilde{u}_j(h_i^*)$. Let $E_{\tilde{u}_j(h_i^*)}(s_j, \mu_j(t_j, h_j^*))$ be the expected utility for player j by choosing strategy s_j when having the subjective probability distribution $\mu_j(t_j, h_j^*)$ and utility function $\tilde{u}_j(h_i^*)$. Since s_j is not optimal at h_j^*, there is some other strategy $\tilde{s}_j \in S_j(h_j^*)$ such that

$$E_{\tilde{u}_j(h_i^*)}(s_j, \mu_j(t_j, h_j^*)) < E_{\tilde{u}_j(h_i^*)}(\tilde{s}_j, \mu_j(t_j, h_j^*)). \tag{5.2}$$

Let $H_j^*(h_j^*)$ be the collection of player j information sets $h_j \in H_j^*$ that weakly precede or weakly follow h_j^*, and let $\hat{s}_j$ be the strategy which coincides with $\tilde{s}_j$ at all information sets in $H_j^*(\tilde{s}_j) \cap H_j^*(h_j^*)$, and coincides with s_j at all information sets in $H_j^*(s_j) \backslash H_j^*(h_j^*)$. Since both s_j and $\hat{s}_j$ are in $S_j(h_j^*)$, it follows, by perfect recall, that s_j and $\hat{s}_j$ coincide on player j information sets preceding h_j^*. Hence, s_j and $\hat{s}_j$ only differ at player j information sets following h_j^*. Since the extensive form structure is with observable deviators, we have that $S(h_j^*) = S_j(h_j^*) \times S_{-j}(h_j^*)$. Since $s_j, \tilde{s}_j$ and $\hat{s}_j$ are all in $S_j(h_j^*)$, and the marginal of $\mu_j(t_j, h_j^*)$ on the space of opponents' strategies is a probability distribution on $S_{-j}(h_j^*)$, it follows that $\mu_j(t_j, h_j^*)$ together with each of the strategies $s_j, \tilde{s}_j$ and $\hat{s}_j$ induces a probability distribution on the terminal nodes which assigns all weight to $Z(h_j^*)$. Since $\tilde{s}_j$ and $\hat{s}_j$ coincide on all player j information sets that precede $Z(h_j^*)$, it follows from (5.2) that

$$E_{\tilde{u}_j(h_i^*)}(s_j, \mu_j(t_j, h_j^*)) < E_{\tilde{u}_j(h_i^*)}(\hat{s}_j, \mu_j(t_j, h_j^*)). \tag{5.3}$$

Since $s_j \in S_j(h_i^*)$, we know that s_j can only lead to terminal nodes in $Z_j(h_i^*)$, and hence $(s_j, \mu_j(t_j, h_j^*))$ induces a probability distribution on $Z_j(h_i^*)$. By (5.1), we know that $\tilde{u}_j(h_i^*)$ coincides with u_j on $Z_j(h_i^*)$, and hence

$$E_{\tilde{u}_j(h_i^*)}(s_j, \mu_j(t_j, h_j^*)) = E_{u_j}(s_j, \mu_j(t_j, h_j^*)). \tag{5.4}$$

Now, let

$$\hat{H}_j = \{h_j \in H_j^*(\hat{s}_j) | \ \hat{s}_j(h_j) \notin A(h_j, h_i^*)\}.$$

By definition of $A(h_j, h_i^*)$, we have that $a \in A(h_j) \backslash A(h_j, h_i^*)$ if and only if h_j precedes h_i^* and a avoids h_i^*. Hence, if $a \in A(h_j) \backslash A(h_j, h_i^*)$ and $\tilde{h}_j$ follows h_j and a, then $\tilde{h}_j$ cannot precede h_i, and hence $A(\tilde{h}_j, h_i^*) = A(\tilde{h}_j)$. Consequently, if h_j and $\tilde{h}_j$ are both in $\hat{H}_j$, then h_j cannot precede nor follow $\tilde{h}_j$. Note that every h_j in $\hat{H}_j$ follows h_j^*. Namely, we have seen that s_j and $\hat{s}_j$ can only differ at information sets following h_j^*. Since $s_j \in S_j(h_i^*)$, we have, by Lemma 5.3 (a), that $s_j(h_j) \in A(h_j, h_i^*)$ for all h_j. In particular, $s_j(h_j) \in A(h_j, h_i^*)$ at all information sets h_j not following h_j^*. Since $\hat{s}_j$ coincides with s_j on these information sets, it follows that $\hat{s}_j(h_j) \in A(h_j, h_i^*)$ at all information sets h_j not following h_j^*. Hence, $\hat{H}_j$ can only contain information sets following h_j^*.

Let s_j^* be some strategy which coincides with $\hat{s}_j$ on all information sets in $H_j^*(\hat{s}_j)\backslash\hat{H}_j$, and chooses some action in $A(h_j,h_i^*)$ at all information sets $h_j \in H_j^*(\hat{s}_j)\cap\hat{H}_j$. Then, by construction, $s_j^*(h_j) \in A(h_j,h_i^*)$ at all information sets $h_j \in H_j^*(s_j^*)$. By Lemma 5.3 (a), it then follows that $s_j^* \in S_j(h_i^*)$. By construction, s_j^* only differs from $\hat{s}_j$ at information sets $h_j \in H_j^*(s_j^*) \cap \hat{H}_j$. At these information sets h_j, the strategy s_j^* chooses some $a \in A(h_j,h_i^*)$, which eventually leads to $Z_j(h_i^*)$. At such information sets $h_j \in \hat{H}_j$, the strategy $\hat{s}_j$ chooses some action $a \notin A(h_j,h_i^*)$, eventually leading to $Z\backslash Z_j(h_i^*)$. The latter follows from Lemma 5.3 (b), stating that $Z_j(h_i^*)$ is exactly the set of terminal nodes z such that at every player j information set h_j preceding z it holds that the unique action at h_j leading to z belongs to $A(h_j,h_i^*)$. By (5.1), we know that

$$\tilde{u}_j(h_i^*)(z_1) > \tilde{u}_j(h_i^*)(z_2)$$

for all $z_1 \in Z_j(h_i^*)$ and all $z_2 \in Z\backslash Z_j(h_i^*)$, which, together with the observations above, implies that

$$E_{\tilde{u}_j(h_i^*)}(\hat{s}_j,\mu_j(t_j,h_j^*)) \leq E_{\tilde{u}_j(h_i^*)}(s_j^*,\mu_j(t_j,h_j^*)). \tag{5.5}$$

Moreover, since $s_j^* \in S_j(h_i^*)$, we may conclude, similarly to (5.4), that

$$E_{\tilde{u}_j(h_i^*)}(s_j^*,\mu_j(t_j,h_j^*)) = E_{u_j}(s_j^*,\mu_j(t_j,h_j^*)). \tag{5.6}$$

By combining (5.3), (5.4), (5.5) and (5.6), we obtain that

$$E_{u_j}(s_j,\mu_j(t_j,h_j^*)) < E_{u_j}(s_j^*,\mu_j(t_j,h_j^*)). \tag{5.7}$$

Note that s_j^* and s_j only differ at information sets following h_j^*. We have seen namely, that $\hat{s}_j$ only differs from s_j at information sets following h_j^*, while s_j^* only differs from $\hat{s}_j$ at information sets in $\hat{H}_j$. Since $\hat{H}_j$ only contains information sets following h_j^*, it follows that s_j^* and s_j only differ at information sets following h_j^*.

Let $\mu_j(t_j,h_j^*|\ S_{-j}(h_j^*))$ be the marginal of $\mu_j(t_j,h_j^*)$ on $S_{-j}(h_j^*)$. Since the expected utility $E_{u_j}(s_j,\mu_j(t_j,h_j^*))$ only depends upon $\mu_j(t_j,h_j^*|\ S_{-j}(h_j^*))$, it follows that

$$E_{u_j}(s_j,\mu_j(t_j,h_j^*|\ S_{-j}(h_j^*))) < E_{u_j}(s_j^*,\mu_j(t_j,h_j^*|\ S_{-j}(h_j^*))). \tag{5.8}$$

Recall that $t_j = t^*(r_j,h_i^*)$. By property (b) in Lemma 5.4 it follows that t_j and r_j induce, at every h_j, the same marginal probability distribution on the space of opponents' strategies, hence

$$\mu_j(r_j,h_j|\ S_{-j}(h_j)) = \mu_j(t_j,h_j|\ S_{-j}(h_j)) \tag{5.9}$$

for all $h_j \in H_j^*$. Hence, by (5.8) and (5.9) it follows that

$$E_{u_j}(s_j,\mu_j(r_j,h_j^*|\ S_{-j}(h_j^*))) < E_{u_j}(s_j^*,\mu_j(r_j,h_j^*|\ S_{-j}(h_j^*))). \tag{5.10}$$

Let r_j be given by a lexicographic probability distribution λ_j on $S_{-j} \times R_{-j}$. Let $\lambda_j(r_j|\ S_{-j})$ be the marginal of λ_j on S_{-j}. Suppose that $\lambda_j(r_j|\ S_{-j}) = (\lambda_j^1(r_j|\ S_{-j}),...,\lambda_j^L(r_j|\ S_{-j}))$, where $\lambda_j^l(r_j|\ S_{-j})$ is the l-th order probability distribution of $\lambda_j(r_j|\ S_{-j})$. Let l^* be the first order for which $\lambda_j^{l^*}(r_j|\ S_{-j})(S_{-j}(h_j^*)) > 0$. Hence, $\lambda_j^l(r_j|\ S_{-j})(S_{-j}(h_j^*)) = 0$ for all $l < l^*$. Since, by assumption, $S(h_j^*) = S_j(h_j^*) \times S_{-j}(h_j^*)$, it follows that both $(s_j, \lambda_j^l(r_j|\ S_{-j}))$ and $(s_j^*, \lambda_j^l(r_j|\ S_{-j}))$ reach h_j^* with probability zero for all $l < l^*$. We have seen above that s_j and s_j^* only differ at information sets following h_j^*, and hence

$$E_{u_j}(s_j, \lambda_j^l(r_j|\ S_{-j})) = E_{u_j}(s_j^*, \lambda_j^l(r_j|\ S_{-j})) \tag{5.11}$$

for all $l < l^*$. For $l = l^*$, we have that

$$\begin{aligned} E_{u_j}(s_j, \lambda_j^{l^*}(r_j|\ S_{-j})) &= \lambda_j^{l^*}(r_j|\ S_{-j})(S_{-j}(h_j^*))\ E_{u_j}(s_j, \mu_j(r_j, h_j^*|\ S_{-j}(h_j^*))) + \\ &\quad + \sum_{z \notin Z(h_j^*)} \mathbb{P}_{(s_j, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z)\ u_j(z) \\ &< \lambda_j^{l^*}(r_j|\ S_{-j})(S_{-j}(h_j^*))\ E_{u_j}(s_j^*, \mu_j(r_j, h_j^*|\ S_{-j}(h_j^*))) + \\ &\quad + \sum_{z \notin Z(h_j^*)} \mathbb{P}_{(s_j, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z)\ u_j(z) \\ &= \lambda_j^{l^*}(r_j|\ S_{-j})(S_{-j}(h_j^*))\ E_{u_j}(s_j^*, \mu_j(r_j, h_j^*|\ S_{-j}(h_j^*))) + \\ &\quad + \sum_{z \notin Z(h_j^*)} \mathbb{P}_{(s_j^*, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z)\ u_j(z) \\ &= E_{u_j}(s_j^*, \lambda_j^{l^*}(r_j|\ S_{-j})). \end{aligned} \tag{5.12}$$

Here, $\mathbb{P}_{(s_j, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z)$ denotes the probability of reaching terminal node z under $(s_j, \lambda_j^{l^*}(r_j|\ S_{-j}))$. The first equality in (5.12) follows from the observation that (1) (s_j, s_{-j}) leads to a terminal node in $Z(h_j^*)$ if and only if $s_{-j} \in S_{-j}(h_j^*)$, and (2) $\mu_j(r_j, h_j^*|\ S_{-j}(h_j^*))$ is the marginal of $\lambda_j^{l^*}(r_j|\ S_{-j})$ on $S_{-j}(h_j^*)$. The inequality follows from (5.10) and the assumption that $\lambda_j^{l^*}(r_j|\ S_{-j})(S_{-j}(h_j^*)) > 0$. The second equality follows from the fact that s_j and s_j^* only differ at information sets following h_j^*, and hence $\mathbb{P}_{(s_j, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z) = \mathbb{P}_{(s_j^*, \lambda_j^{l^*}(r_j|\ S_{-j}))}(z)$ for all $z \notin Z(h_j^*)$. The last equality follows from the same argument as used for the first equality.

By (5.11) and (5.12), we may conclude that type r_j strictly prefers strategy s_j^* above s_j. By assumption, r_i^* is properly rationalizable, and hence respects opponents' preferences. Since, by assumption, $r_j \in R_j(r_i^*)$, it follows that r_i^* deems (s_j^*, r_j) infinitely more likely that (s_j, r_j). Since both s_j and s_j^* are in $S_j(h_i^*)$, it follows that $\mu_i(r_i^*, h_i^*)$ assigns probability zero to (s_j, r_j).

By property (b) in Lemma 5.4, we have that

$$\begin{aligned} &\mu_i(t_i^*, h_i^*|\ S_j(h_i^*) \times T_j)(s_j, t_j) \\ = \ &\mu_i(r_i^*, h_i^*|\ S_j(h_i^*) \times R_j)(\{(s_j', r_j) \mid s_j' = s_j \text{ and } t^*(r_j, h_i^*) = t_j\}), \end{aligned} \tag{5.13}$$

where $\mu_i(t_i^*, h_i^*|\ S_j(h_i^*) \times T_j)$ and $\mu_i(r_i^*, h_i^*|\ S_j(h_i^*) \times R_j)$ denote the marginals on $S_j(h_i^*) \times T_j$ and $S_j(h_i^*) \times R_j$, respectively. We have shown above that for all $r_j \in R_j(r_i^*)$ with $t^*(r_j, h_i^*) = t_j$ it holds that $\mu_i(r_i^*, h_i^*)$ assigns probability zero to (s_j, r_j). For all $r_j \in R_j \backslash R_j(r_i^*)$ we have, by definition, that $\mu_i(r_i^*, h_i^*)$ assigns probability zero to r_j, and hence to (s_j, r_j). Consequently, we have shown that for all $r_j \in R_j$ with $t^*(r_j, h_i^*) = t_j$ it holds that $\mu_i(r_i^*, h_i^*)$ assigns probability zero to (s_j, r_j). But then, (5.13) implies that $\mu_i(t_i^*, h_i^*|\ S_j(h_i^*) \times T_j)(s_j, t_j) = 0$. Hence, we may conclude that $\mu_i(t_i^*, h_i^*)$ assigns probability zero to all strategy-type pairs (s_j, t_j) in $S_j(h_i^*) \times T_j$ that are not sequentially rational. This implies that t_i^* believes in sequential rationality.

We thus have shown that for every properly rationalizable type r_i for $(\mathcal{S}, u)$ with non-increasing type supports, and every information set h_l, the induced type $t^*(r_i, h_l)$ (a) is updating consistent, (b) satisfies proper belief revision, (c) initially believes u and (d) believes in sequential rationality.

By construction, every properly rationalizable type r_i for $(\mathcal{S}, u)$ with non-increasing type supports is such that, at every information set h_i, the probability distribution $\mu_i(r_i, h_i)$ on $S_{-i}(h_i) \times R_{-i}$ assigns positive probability only to properly rationalizable types for $(\mathcal{S}, u)$ with non-increasing type supports. Combining this insight with property (b) in Lemma 5.4 leads to the observation that for every properly rationalizable type r_i for $(\mathcal{S}, u)$ with non-increasing type supports, and information set h_l, the induced type $t^*(r_i, h_l)$ assigns, at every information set h_i, only positive probability to opponents' types t_j that can be written as $t_j = t^*(r_j, h_i)$ for some *properly rationalizable* type r_j for $(\mathcal{S}, u)$ with non-increasing type supports. Since every such $t^*(r_j, h_i)$ satisfies the properties (a) to (d) above, we have that every type $t^*(r_i, h_l)$ believes, at every h_i, that opponent types satisfy properties (a) to (d). By applying this argument recursively, it follows that type $t^*(r_i, h_l)$ respects common belief about the event that types satisfy properties (a) to (d). However, this implies that every type $t^*(r_i, h_l)$ induced by a properly rationalizable type r_i for $(\mathcal{S}, u)$ with non-increasing type supports, is persistently rationalizable and respects common belief about the event that types initially believe u.

Now, let s_i^* be a properly rationalizable strategy for $(\mathcal{S}, u)$ for types with non-increasing type supports. Then, there is some properly rationalizable type r_i^* for $(\mathcal{S}, u)$ with non-increasing type supports such that s_i^* is optimal for r_i^*. Let $t_i^* = t^*(r_i^*, h_0)$. Then, by property (b) in Lemma 5.4, t_i^* holds utility function $\tilde{u}_i(h_0) = u_i$. Since we have seen above that t_i^* is persistently rationalizable and respects common belief about the event that types initially believe u, it follows that t_i^* is persistently rationalizable for $(\mathcal{S}, u)$.

Since s_i^* is optimal for r_i^*, and since the lexicographic probability distribution $\lambda_i(r_i^*)$ has full support on S_{-i}, it follows that s_i^* is optimal with respect to $\mu_i(r_i^*, h_i|\ S_{-i}(h_i))$ at every information set $h_i \in H_i(s_i^*)$. Here, $\mu_i(r_i^*, h_i|\ S_{-i}(h_i))$ denotes the marginal of the probability distribution $\mu_i(r_i^*, h_i)$ on $S_{-i}(h_i)$. By property (b) in Lemma 5.4, we know that $\mu_i(r_i^*, h_i|\ S_{-i}(h_i)) = \mu_i(t_i^*, h_i|\ S_{-i}(h_i))$ for all h_i. Hence, s_i^* is optimal with respect to $\mu_i(t_i^*, h_i|\ S_{-i}(h_i))$ for all $h_i \in H_i(s_i^*)$. This implies that s_i^* is sequentially rational for t_i^*, and hence s_i^* is persistently rationalizable for $(\mathcal{S}, u)$. We thus have shown that every properly rationalizable strategy for $(\mathcal{S}, u)$ for types with non-increasing type supports is persistently rationalizable for $(\mathcal{S}, u)$. This completes the proof of this theorem. ■

6. Appendix

Proof of Lemma 3.3. Let f_i be the homeomorphism from P_i^c to $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$ discussed above. Let the sets $P_i^c(h_i)$ be such that $P_i^c = \times_{h_i \in H_i^*} P_i^c(h_i)$, and for every $h_i \in H_i^*$ let $f_i(h_i)$ be the corresponding homeomorphism from $P_i^c(h_i)$ to $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$. Let $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$ be the set of preference relations $(\mu_i(h_i), u_i(h_i)) \in \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i))$ for which $\mu_i(h_i)(S_{-i}(h_i) \times P_{-i}^{c,\infty}) = 1$. Let the sets

$P_i^{c,\infty}(h_i)$ be such that $P_i^{c,\infty} = \times_{h_i \in H_i^*} P_i^{c,\infty}(h_i)$. We prove that $f_i(h_i)(P_i^{c,\infty}(h_i)) = \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$. Since $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$ is homeomorphic to $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}^{c,\infty}, Z(h_i))$, this would imply that $P_i^{c,\infty}$ is homeomorphic to $\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}^{c,\infty}, Z(h_i))$.

We start by showing that $f_i(h_i)(P_i^{c,\infty}(h_i)) \subseteq \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$. Let $p_i(h_i) \in P_i^{c,\infty}(h_i)$. Then, by definition, $\mu_i(p_i, h_i)(S_{-i}(h_i) \times P_{-i}^{c,k}) = 1$ for every $k \in \mathbb{N}$ and hence $\mu_i(p_i, h_i)(S_{-i}(h_i) \times P_{-i}^{c,\infty}) = 1$, which implies that $f_i(h_i)(p_i(h_i)) \in \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$.

We next show that $f_i(h_i)^{-1}(\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})) \subseteq P_i^{c,\infty}(h_i)$, which would imply that $\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty}) \subseteq f_i(h_i)(P_i^{c,\infty}(h_i))$. Assume now that $p_i(h_i) \in f_i(h_i)^{-1}(\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty}))$, that is, $f_i(h_i)(p_i(h_i)) \in \mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})$. This means that $\mu_i(p_i, h_i)(S_{-i}(h_i) \times P_{-i}^{c,\infty}) = 1$. As such, $\mu_i(p_i, h_i)(S_{-i}(h_i) \times P_{-i}^{c,k-1}) = 1$ for all k, and hence $p_i(h_i) \in P_i^{c,k}(h_i)$ for all k. Here, the sets $P_i^{c,k}(h_i)$ are defined such that $P_i^{c,k} = \times_{h_i \in H_i^*} P_i^{c,k}(h_i)$. This implies that $p_i(h_i) \in \cap_{k \in \mathbb{N}} P_i^{c,k}(h_i) = P_i^{c,\infty}(h_i)$. Hence, $f_i(h_i)^{-1}(\mathcal{P}^{eu}(S_{-i}(h_i) \times P_{-i}, Z(h_i)|\ S_{-i}(h_i) \times P_{-i}^{c,\infty})) \subseteq P_i^{c,\infty}(h_i)$, which completes the proof. ■

Proof of Lemma 5.3. (a) Let $s_j \in S_j(h_i)$. Suppose that there is some $h_j \in H_j(s_j)$ with $s_j(h_j) \notin A(h_j, h_i)$. Then, necessarily, h_j precedes h_i. Hence, by the definition of $A(h_j, h_i)$, the action $s_j(h_j)$ avoids h_i. On the other hand, since h_j precedes h_i, there is some node $x \in h_j$ which leads to h_i. By perfect recall, there is some strategy profile s_{-j} such that (s_j, s_{-j}) reaches x. Hence, there is some strategy profile $(\tilde{s}_j, \tilde{s}_{-j})$ such that $(\tilde{s}_j, \tilde{s}_{-j})$ reaches x and h_i. Since $(\tilde{s}_j, \tilde{s}_{-j}) \in S(h_i)$ and $S(h_i) = \times_{k \in I} S_k(h_i)$, it follows that $\tilde{s}_{-j} \in \times_{k \neq j} S_k(h_i)$. Since $(\tilde{s}_j, \tilde{s}_{-j})$ reaches $x \in h_j$, we know, by perfect recall, that $\tilde{s}_j$ coincides with s_j on the player j information sets preceding h_j. Hence, $(s_j, \tilde{s}_{-j})$ reaches h_j. Since $s_j(h_j)$ avoids h_i, we have that $(s_j, \tilde{s}_{-j})$ does not reach h_i, and hence $(s_j, \tilde{s}_{-j}) \notin S(h_i)$. Since $S(h_i) = \times_{k \in I} S_k(h_i)$ and $\tilde{s}_{-j} \in \times_{k \neq j} S_k(h_i)$, it thus follows that $s_j \notin S_j(h_i)$, which is a contradiction. We may thus conclude that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$.

Now, let s_j be such that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$. We prove that $s_j \in S_j(h_i)$. We distinguish two cases. Suppose first that there is no player j information set preceding h_i. Then, obviously, $s_j \in S_j(h_i)$. Suppose now that there is some player j information set preceding h_i. Let $h_j \in H_j(s_j)$ be a player j information set preceding h_i such that there is no other player j information set in $H_j(s_j)$ between h_j and h_i. By assumption, $s_j(h_j) \in A(h_j, h_i)$, hence there exists a node $x \in h_j$ such that h_i can be reached through x via action $s_j(h_j)$. By perfect recall, there is some strategy profile $\tilde{s}_{-j}$ for the opponents such that $(s_j, \tilde{s}_{-j})$ reaches x. Since there is no $h_j' \in H_j(s_j)$ between h_j and h_i, and since h_i can be reached through x via $s_j(h_j)$, we can choose $\tilde{s}_{-j}$ such that $(s_j, \tilde{s}_{-j})$ reaches h_i. But then, by definition, $s_j \in S_j(h_i)$. This completes the proof of part (a).

(b) Suppose that $z \in Z_j(h_i)$ and that $h_j \in H_j$ is a player j information set on the path to z. Then, obviously, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Suppose, on the other hand, that the terminal node z is such that for every player j information set h_j on the path to z, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Let s_j be a strategy such that at every information set $h_j \in H_j(s_j)$ on the path to z, the strategy s_j chooses the unique action at h_j leading to z, and at every other information set $h_j \in H_j(s_j)$ the strategy s_j chooses some action in $A(h_j, h_i)$. Then, $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$, and hence, by part (a), $s_j \in S_j(h_i)$. Since z can be reached by strategy s_j, it follows that $z \in Z_j(h_i)$. This completes the proof. ■

Proof of Lemma 5.4. We need the following definitions. Let X_{-i} be some set and λ_i a cautious lexicographic probability distribution on $S_{-i} \times X_{-i}$, that is, $\text{supp}\lambda_i = S_{-i} \times \tilde{X}_{-i}$ for some $\tilde{X}_{-i} \subseteq X_{-i}$. For every information set h_i, let $\lambda_i(h_i)$ be the marginal of λ_i on $S_{-i}(h_i) \times X_{-i}$ and let $\mu_i(\lambda_i, h_i) \in \Delta(S_{-i}(h_i) \times X_{-i})$ be the first-order probability distribution of $\lambda_i(h_i)$. For a given vector of probability distributions $\mu_i = (\mu_i(h_i))_{h_i \in H_i^*}$ in $\times_{h_i \in H_i^*} \Delta(S_{-i}(h_i) \times X_{-i})$, we say that μ_i is *consistent* if there is some cautious lexicographic probability distribution λ_i on $S_{-i} \times X_{-i}$ such that $\mu_i(h_i) = \mu_i(\lambda_i, h_i)$ for all $h_i \in H_i^*$. In this case, we say that μ_i is induced by the lexicographic probability distribution λ_i.

By $\mathcal{P}^{cleu}(S_{-i} \times X_{-i}, u_i)$ we denote the set of preference relations on the set of acts $\mathcal{F}(S_{-i} \times X_{-i}, Z)$ that can be represented by a cautious lexicographic probability distribution on $S_{-i} \times X_{-i}$ and the utility function u_i. Let $(\times_{h_i \in H_i^*} \mathcal{P}^{eu}(S_{-i}(h_i) \times X_{-i}, u_i))^c$ be the set of vectors of expected utility preference relations on $\mathcal{F}(S_{-i}(h_i) \times X_{-i}, Z(h_i))$ such that (1) the corresponding vector of probability distributions $(\mu_i(h_i))_{h_i \in H_i^*}$ in $\times_{h_i \in H_i^*} \Delta(S_{-i}(h_i) \times X_{-i})$ is consistent, and (2) the utility function at h_i is given by $u_i|_{Z(h_i)}$.

Within the proper rationalizability framework, let the uncertainty spaces $Y_i^1, Y_i^2, \ldots$ be given by

$$Y_i^1 = S_{-i}$$

for all players i, and

$$Y_i^k = Y_i^{k-1} \times (\times_{j \neq i} \mathcal{P}^{cleu}(Y_j^{k-1}, u_j))$$

for $k \geq 2$ and all players i. Let $\tilde{Y}_i^1 = S_{-i}$, and for every $k \geq 2$ let

$$\tilde{Y}_i^k = \times_{j \neq i} \mathcal{P}^{cleu}(Y_j^{k-1}, u_j).$$

Then, $Y_i^k = \tilde{Y}_i^1 \times \ldots \times \tilde{Y}_i^k$ for all k, and $S_{-i} \times R_{-i}$ is homeomorphic to $\times_{k \in \mathbb{N}} \tilde{Y}_i^k$. Let $Y_i^k(h_i) = S_{-i}(h_i) \times \tilde{Y}_i^2 \times \ldots \times \tilde{Y}_i^k$ for all h_i.

Within the persistent rationalizability framework, let the uncertainty spaces $X_i^1(h_i), X_i^2(h_i), \ldots$ be given by

$$X_i^1(h_i) = S_{-i}(h_i)$$

for all players i and information sets $h_i \in H_i^*$, and

$$X_i^k(h_i) = X_i^{k-1}(h_i) \times (\times_{j \neq i} (\times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), \tilde{u}_j(h_i)))^c)$$

for $k \geq 2$, all players i and information sets $h_i \in H_i^*$. Here, $\tilde{u}_j(h_i)$ is the utility function as defined in (5.1). Let $\tilde{X}_i^1(h_i) = S_{-i}(h_i)$, and for every $k \geq 2$, let

$$\tilde{X}_i^k(h_i) = \times_{j \neq i} (\times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), \tilde{u}_j(h_i)))^c.$$

Then, $X_i^k(h_i) = \tilde{X}_i^1(h_i) \ldots \times \tilde{X}_i^k(h_i)$ for all k.

For every player i, information set h_i, opponent l and information set h_l, we define onto mappings

$$\begin{array}{rcl} f_i^k(h_i) & : & Y_i^k(h_i) \to X_i^k(h_i) \text{ for all } k, \\ \tilde{f}_i^1(h_i) & : & S_{-i}(h_i) \to \tilde{X}_i^1(h_i) \\ \tilde{f}_i^k(h_i) & : & \tilde{Y}_i^k \to \tilde{X}_i^k(h_i) \text{ for all } k \geq 2, \text{ and} \\ g_i^k(h_l) & : & \mathcal{P}^{cleu}(Y_i^k, u_i) \to (\times_{h_i \in H_i^*} \mathcal{P}^{eu}(X_i^k(h_i), \tilde{u}_i(h_l)))^c \text{ for all } k. \end{array}$$

We construct these mappings by induction on k. For $k = 1$ we have that $Y_i^1(h_i) = X_i^1(h_i) = \tilde{X}_i^1(h_i) = S_{-i}(h_i)$. Let $f_i^1(h_i)$ and $\tilde{f}_i^1(h_i)$ be the identity mapping from $S_{-i}(h_i)$ to $S_{-i}(h_i)$. Let the preference relation q_i^1 in $\mathcal{P}^{cleu}(Y_i^1, u_i)$ be given by a cautious lexicographic probability distribution λ_i^1 on Y_i^1. Let μ_i^1 be the corresponding vector of probability distributions in $(\times_{h_i \in H_i^*} \Delta(Y_i^1(h_i))$ induced by λ_i^1, and let $g_i^1(h_l)(q_i^1)$ be the consistent vector of expected utility preference relations in $(\times_{h_i \in H_i^*} \mathcal{P}^{eu}(Y_i^1(h_i), \tilde{u}_i(h_l)))^c$ induced by μ_i^1 and the utility function $\tilde{u}_i(h_l)$. Since $X_i^1(h_i) = Y_i^1(h_i)$, it follows that $g_i^1(h_l)(q_i^1)$ belongs to $(\times_{h_i \in H_i^*} \mathcal{P}^{eu}(X_i^1(h_i), \tilde{u}_i(h_l)))^c$. It is easily seen that $g_i^1(h_l)$ is onto.

Now, assume that $k \geq 2$ and suppose that the onto functions $f_i^{k-1}(h_i)$, $\tilde{f}_i^{k-1}(h_i)$ and $g_i^{k-1}(h_l)$ have already been defined for all i, h_i, j and h_l. Choose a player i and information set h_i. By construction,

$$\tilde{Y}_i^k = \times_{j \neq i} \mathcal{P}^{cleu}(Y_j^{k-1}, u_j)$$

and

$$\tilde{X}_i^k(h_i) = \times_{j \neq i} (\times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), \tilde{u}_j(h_i)))^c.$$

By assumption, $g_j^{k-1}(h_i)$ maps $\mathcal{P}^{cleu}(Y_j^{k-1}, u_j)$ onto the space $(\times_{h_j \in H_j^*} \mathcal{P}^{eu}(X_j^{k-1}(h_j), \tilde{u}_j(h_i)))^c$ for every player j. Let the function $\tilde{f}_i^k(h_i) : \tilde{Y}_i^k \to \tilde{X}_i^k(h_i)$ be given by

$$\tilde{f}_i^k(h_i) = \times_{j \neq i} g_j^{k-1}(h_i). \tag{6.1}$$

Then, $\tilde{f}_i^k(h_i)$ maps $\tilde{Y}_i^k$ onto $\tilde{X}_i^k(h_i)$.

By construction, $Y_i^k(h_i) = S_{-i}(h_i) \times \tilde{Y}_i^2 \times ... \times \tilde{Y}_i^k$ and $X_i^k(h_i) = \tilde{X}_i^1(h_i) \times ... \times \tilde{X}_i^k(h_i)$. Let the function $f_i^k(h_i) : Y_i^k(h_i) \to X_i^k(h_i)$ be given by

$$f_i^k(h_i) = \tilde{f}_i^1(h_i) \times ... \times \tilde{f}_i^k(h_i). \tag{6.2}$$

Then, $f_i^k(h_i)$ maps $Y_i^k(h_i)$ onto $X_i^k(h_i)$.

Let now the preference relation q_i^k in $\mathcal{P}^{cleu}(Y_i^k, u_i)$ be given, induced by a cautious lexicographic probability distribution λ_i^k on Y_i^k. Let $\mu_i^k(q_i^k) = (\mu_i^k(q_i^k, h_i))_{h_i \in H_i^*}$ be the induced vector of probability distributions in $\times_{h_i \in H_i^*} \Delta(Y_i^k(h_i))$. We now transform every $\mu_i^k(q_i^k, h_i) \in \Delta(Y_i^k(h_i))$ into some probability distribution $\mu_i^k(h_i) \in \Delta(X_i^k(h_i))$ as follows. For every event $E \subseteq X_i^k(h_i)$, define

$$\mu_i^k(h_i)(E) = \mu_i^k(q_i^k, h_i)(\{y_i^k \in Y_i^k(h_i) |\ f_i^k(h_i)(y_i^k) \in E\}). \tag{6.3}$$

Since $f_i^k(h_i)$ is onto, we have that $\mu_i^k(h_i)$ is indeed a probability distribution on $X_i^k(h_i)$. Moreover, it is easily checked that the vector $(\mu_i^k(h_i))_{h_i \in H_i^*}$ is consistent. Let $g_i^k(h_l)(q_i^k)$ be the vector of expected utility preference relations in $(\times_{h_i\in H_i^*}\mathcal{P}^{eu}(X_i^k(h_i),\tilde{u}_i(h_l)))^c$ given by the above constructed probability distributions $\mu_i^k(h_i)$ on $X_i^k(h_i)$ and the utility function $\tilde{u}_i(h_l)$. Then, $g_i^k(h_l)$ is onto. By induction on k, it thus follows that the mappings $f_i^k(h_i)$, $\tilde{f}_i^k(h_i)$ and $g_i^k(h_l)$ are onto for all k.

Recall that we assume that every type-space R_i contains only types that respect common belief about the event that types are cautious. It then follows that every type-space R_i is homeomorphic to $\mathcal{P}^{cleu}(S_{-i}\times R_{-i},u_i)$. Now, let $r_i\in R_i$ be a type with finite complexity. Since $S_{-i}\times R_{-i}$ is homeomorphic to $\times_{k\in\mathbb{N}}\tilde{Y}_i^k$ and $Y_i^k=\tilde{Y}_i^1\times...\times\tilde{Y}_i^k$ for all k, it follows that r_i induces a coherent preference hierarchy $(q_i^k(r_i))_{k\in\mathbb{N}}$ with $q_i^k\in\mathcal{P}^{cleu}(Y_i^k,u_i)$ for all $k\in\mathbb{N}$. For every opponent's information set $h_l\in H_l^*$ and every k, let

$$g_i^k(h_l)(q_i^k(r_i))\in(\times_{h_i\in H_i^*}\mathcal{P}^{eu}(X_i^k(h_i),\tilde{u}_i(h_l)))^c$$

be the induced vector of expected utility preference relations as defined above. Let

$$t^*(r_i,h_l)=(g_i^k(h_l)(q_i^k(r_i)))_{k\in\mathbb{N}} \tag{6.4}$$

be the corresponding preference hierarchy in $\times_{k\in\mathbb{N}}(\times_{h_i\in H_i^*}\mathcal{P}^{eu}(X_i^k(h_i),\tilde{u}_i(h_l)))^c$. By construction, we have that

$$\times_{k\in\mathbb{N}}(\times_{h_i\in H_i^*}\mathcal{P}^{eu}(X_i^k(h_i),\tilde{u}_i(h_l)))^c\subseteq T_i$$

and hence $t^*(r_i,h_l)$ is a type in T_i for all r_i and h_l.

Let t^* be the mapping which assigns to every type $r_i\in R_i$ with finite complexity and every h_l such a type $t^*(r_i,h_l)\in T_i$. We prove that t^* satisfies properties (a) and (b) in Lemma 5.4. Property (a) follows immediately from the construction of $t^*(r_i,h_l)$. In order to prove property (b), fix a type r_i with finite complexity, an information set h_l and let $t_i=t^*(r_i,h_l)$. Then, r_i induces for every $k\in\mathbb{N}$ a lexicographic probability distribution $\lambda_i^k(r_i)$ on Y_i^k which, in turn, induces a vector $(\mu_i^k(r_i,h_i))_{h_i\in H_i^*}$ of first-order probability distributions in $\times_{h_i\in H_i^*}\Delta(Y_i^k(h_i))$. Since r_i has finite complexity, it follows that $\mu_i^k(r_i,h_i)$ is a probability distribution with finite support for every k and h_i.

We have seen above that t_i is in $\times_{k\in\mathbb{N}}(\times_{h_i\in H_i^*}\mathcal{P}^{eu}(X_i^k(h_i),\tilde{u}_i(h_l)))^c$. Hence, t_i induces a consistent vector $(\mu_i^k(t_i,h_i))_{h_i\in H_i^*}$ of probability distributions in $\times_{h_i\in H_i^*}\Delta(X_i^k(h_i))$. By (6.3), we then know that

$$\mu_i^k(t_i,h_i)(E_i^k)=\mu_i^k(r_i,h_i)(\{y_i^k\in Y_i^k(h_i)|\ f_i^k(h_i)(y_i^k)\in E_i^k\}) \tag{6.5}$$

for all h_i and events $E_i^k\subseteq X_i^k(h_i)$. We have seen that $\mu_i^k(r_i,h_i)$ has finite support. From (6.5) it thus follows that $\mu_i^k(t_i,h_i)$ has finite support as well. Hence, in (6.5) it suffices to concentrate on single-point events $E_i^k=\{x_i^k\}$.

Since $t_i\in T_i$, we have that t_i induces for every information set h_i some probability distribution $\mu_i(t_i,h_i)\in\Delta(S_{-i}(h_i)\times T_{-i})$. Let $\mu_i(t_i,h_i|\ X_i^k(h_i))$ denote the marginal of $\mu_i(t_i,h_i)$ on $X_i^k(h_i)\subseteq S_{-i}(h_i)\times T_{-i}$. By construction, we have that $\mu_i(t_i,h_i|\ X_i^k(h_i))=\mu_i^k(t_i,h_i)$. Let $\mu_i(r_i,h_i|\ Y_i^k(h_i))$ denote the marginal of $\mu_i(r_i,h_i)$ on $Y_i^k(h_i)\subseteq S_{-i}(h_i)\times R_{-i}$. Then, by construction, $\mu_i(r_i,h_i|\ Y_i^k(h_i))=\mu_i^k(r_i,h_i)$. By (6.5) we then obtain that

$$\mu_i(t_i,h_i|\ X_i^k(h_i))(x_i^k)=\mu_i(r_i,h_i|\ Y_i^k(h_i))(\{y_i^k\in Y_i^k(h_i)|\ f_i^k(h_i)(y_i^k)=x_i^k\}) \tag{6.6}$$

for all k and all $x_i^k \in X_i^k(h_i)$.

Recall that $Y_i^k(h_i) = S_{-i}(h_i)\times\tilde{Y}_i^2\times...\times\tilde{Y}_i^k$, and that $X_i^k(h_i) = S_{-i}(h_i)\times\tilde{X}_i^2(h_i)\times...\times\tilde{X}_i^k(h_i)$. By (6.2) we know that the mapping $f_i^k(h_i)$ is equal to $\tilde{f}_i^1(h_i) \times ... \times \tilde{f}_i^k(h_i)$. That is,

$$f_i^k(h_i)(s_{-i}, \tilde{y}_i^2, ..., \tilde{y}_i^k) = (s_{-i}, \tilde{f}_i^2(h_i)(\tilde{y}_i^2), ..., \tilde{f}_i^k(h_i)(\tilde{y}_i^k)) \tag{6.7}$$

for all $(s_{-i}, \tilde{y}_i^2, ..., \tilde{y}_i^k) \in S_{-i}(h_i)\times\tilde{Y}_i^2\times...\times\tilde{Y}_i^k = Y_i^k(h_i)$. Note that $(s_{-i}, \tilde{f}_i^2(h_i)(\tilde{y}_i^2), ..., \tilde{f}_i^k(h_i)(\tilde{y}_i^k)) \in S_{-i}(h_i) \times \tilde{X}_i^2(h_i) \times ... \times \tilde{X}_i^k(h_i) = X_i^k(h_i)$. By (6.6) and (6.7) it thus follows that

$$\begin{aligned} & \mu_i(t_i, h_i|\ X_i^k(h_i))(s_{-i}, \tilde{x}_i^2, ..., \tilde{x}_i^k) & (6.8)\\ = \ & \mu_i(r_i, h_i|\ Y_i^k(h_i))(\{s'_{-i}, \tilde{y}_i^2, ..., \tilde{y}_i^k)|\ s'_{-i} = s_{-i},\ \tilde{f}_i^2(h_i)(\tilde{y}_i^2) = \tilde{x}_i^2, ...,\ \tilde{f}_i^k(h_i)(\tilde{y}_i^k) = \tilde{x}_i^k\}) \end{aligned}$$

for all $(s_{-i}, \tilde{x}_i^2, ..., \tilde{x}_i^k) \in X_i^k(h_i)$ and all $k \geq 2$.

Recall that $\mu_i(t_i, h_i) \in \Delta(S_{-i}(h_i)\times T_{-i})$ and $\mu_i(r_i, h_i) \in \Delta(S_{-i}(h_i)\times R_{-i})$. Since $S_{-i}(h_i)\times R_{-i}$ is homeomorphic to $S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{Y}_i^k)$ and $S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{X}_i^k(h_i))$ is homeomorphic to a subspace of $S_{-i}(h_i) \times T_{-i}$, we may conclude from (6.8) that

$$\begin{aligned} & \mu_i(t_i, h_i)(s_{-i}, \tilde{x}_i^2, \tilde{x}_i^3, ...) & (6.9)\\ = \ & \mu_i(r_i, h_i)(\{s'_{-i}, \tilde{y}_i^2, \tilde{y}_i^3, ...\})|\ s'_{-i} = s_{-i},\ \tilde{f}_i^2(h_i)(\tilde{y}_i^2) = \tilde{x}_i^2,\ \tilde{f}_i^3(h_i)(\tilde{y}_i^3) = \tilde{x}_i^3, ...\}) \end{aligned}$$

for all $(s_{-i}, \tilde{x}_i^2, \tilde{x}_i^3, ...)$ in $S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{X}_i^k)$.

Now, let $f_i(h_i)$ be the mapping from $S_{-i}(h_i) \times R_{-i}$ to $S_{-i}(h_i) \times T_{-i}$ given by

$$f_i(h_i)(s_{-i}, \tilde{y}_i^2, \tilde{y}_i^3, ...) = (s_{-i}, \tilde{f}_i^2(h_i)(\tilde{y}_i^2), \tilde{f}_i^3(h_i)(\tilde{y}_i^3), ...)$$

for all $(s_{-i}, \tilde{y}_i^2, \tilde{y}_i^3, ...) \in S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{Y}_i^k)$, where the latter space may be identified with $S_{-i}(h_i) \times R_{-i}$. Then, from (6.9) it follows that

$$\begin{aligned} & \mu_i(t_i, h_i)(s_{-i}, \tilde{x}_i^2, \tilde{x}_i^3, ...)\\ = \ & \mu_i(r_i, h_i)(\{s'_{-i}, \tilde{y}_i^2, \tilde{y}_i^3, ...\})|\ f_i(h_i)(s'_{-i}, \tilde{y}_i^2, \tilde{y}_i^3, ...) = (s_{-i}, \tilde{x}_i^2, \tilde{x}_i^3, ...)\}). & (6.10) \end{aligned}$$

By construction, we have that

$$f_i(h_i) = id \times (\times_{k\geq 2}\tilde{f}_i^k(h_i)) \tag{6.11}$$

where id is the identity mapping from $S_{-i}(h_i)$ to $S_{-i}(h_i)$. By (6.1) may then conclude that

$$\begin{aligned} f_i(h_i) &= id \times (\times_{k\geq 2} \times_{j\neq i} g_j^{k-1}(h_i)) = id \times (\times_{j\neq i} \times_{k\geq 2} g_j^{k-1}(h_i))\\ &= id \times (\times_{j\neq i} \times_{k\in\mathbb{N}} g_j^k(h_i)). & (6.12) \end{aligned}$$

By definition,

$$\times_{k\in\mathbb{N}} g_j^k(h_i) : \times_{k\in\mathbb{N}}\mathcal{P}^{cleu}(Y_j^k, u_j) \to \times_{k\in\mathbb{N}}(\times_{h_j\in H_j^*}\mathcal{P}^{eu}(X_j^k(h_j), \tilde{u}_j(h_i)))^c$$

with

$$(\times_{k\in\mathbb{N}} g_j^k(h_i))((q_j^k(r_j))_{k\in\mathbb{N}}) = ((g_j^k(h_i)(q_j^k(r_j)))_{k\in\mathbb{N}} = t^*(r_j, h_i)$$

for all $r_j \in R_j$. Hence, the function $\times_{k\in\mathbb{N}} g_j^k(h_i)$ maps every $r_j \in R_j$ to $t^*(r_j, h_i) \in T_j$. Combining this insight with (6.12) leads to the conclusion that $f_i(h_i)$ is a mapping from $S_{-i}(h_i) \times R_{-i}$ to $S_{-i}(h_i) \times T_{-i}$ with

$$f_i(h_i)((s_j, r_j)_{j\neq i}) = ((s_j, t^*(r_j, h_i)_{j\neq i}) \tag{6.13}$$

for all $(s_j, r_j)_{j\neq i}$ in $S_{-i}(h_i) \times T_{-i}$. Substituting (6.13) in (6.10), and letting $t_i = t^*(r_i, h_l)$, we obtain that

$$\begin{aligned} & \mu_i(t^*(r_i, h_l), h_i)((s_j, t_j)_{j\neq i}) \\ = \ & \mu_i(r_i, h_i)(\{(s'_j, r'_j)_{j\neq i} |\ s'_j = s_j \text{ and } t^*(r'_j, h_i) = t_j \text{ for all } j \neq i\}). \end{aligned}$$

Here, we used the fact that $S_{-i}(h_i) \times R_{-i}$ may be identified with $S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{Y}_i^k)$, and that $S_{-i}(h_i) \times (\times_{k\geq 2}\tilde{X}_i^k(h_i))$ is homeomorphic to a subspace of $S_{-i}(h_i) \times T_{-i}$. This establishes property (b) in Lemma 5.4 and hence the proof is complete. ■

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