Birgit Heydenreich, Debasis Mishra,
Rudolf Müller, Marc Uetz

Optimal Mechanisms for Single Machine Scheduling

RM/08/033

JEL code: D44, D71, C69, C72

Maastricht research school of **E**conomics
of **TE**chnology and **OR**ganizations

Universiteit Maastricht
Faculty of Economics and Business Administration
P.O. Box 616
NL - 6200 MD Maastricht

phone : ++31 43 388 3830
fax : ++31 43 388 4873

Optimal Mechanisms for Single Machine Scheduling

Birgit Heydenreich[1,*] Debasis Mishra[2] Rudolf Müller[1]
Marc Uetz[3]

October 8, 2008

Abstract

We study the design of optimal mechanisms in a setting where job-agents compete for being processed by a service provider that can handle one job at a time. Each job has a processing time and incurs a waiting cost. Jobs need to be compensated for waiting. We consider two models, one where only the waiting costs of jobs are private information (1-d), and another where both waiting costs and processing times are private (2-d). Probability distributions represent the public common belief about private information. We consider discrete and continuous distributions. In this setting, an optimal mechanism minimizes the total expected expenses to compensate all jobs, while it has to be Bayes-Nash incentive compatible. We derive closed formulae for the optimal mechanism in the 1-d case and show that it is efficient for symmetric jobs. For non-symmetric jobs, we show that efficient mechanisms perform arbitrarily bad. For the 2-d discrete case, we prove that the optimal mechanism in general does not even satisfy IIA, the 'independent of irrelevant alternatives' condition. Hence any attempt along the lines of the classical auction setting is doomed to fail. In the 2-d discrete case, we also show that the optimal mechanism is not even efficient for symmetric agents.

[1]Maastricht University, Quantitative Economics, P.O.Box 616, 6200 MD Maastricht, The Netherlands. Email:{b.heydenreich,r.muller}@ke.unimaas.nl
*Supported by NWO grant 2004/03545/MaGW 'Local Decisions in Decentralised Planning Environments'.
[2]Indian Statistical Institute, Planning Unit, 7, S.J.S. Sansanwal Marg, New Delhi - 110 016, India. Email:dmishra@isid.ac.in
[3]University of Twente, Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email: m.uetz@utwente.nl

1 Introduction

The design of optimal auctions is recognized as an intriguing issue in auction theory; first studied by Myerson (1981) for the case of single item auctions. In that setting, the goal is to maximize the seller's revenue. We study the design of optimal auctions (or more precisely, mechanisms) in a setting where job-agents compete for being processed by a service provider that can only handle one job at a time. No job can be interrupted once started, and each job is characterized by service time and weight, the latter representing his disutility for waiting per unit time. It is well known that the total disutility of the jobs is minimized by a scheduling policy known as Smith's rule: schedule jobs in order of non-increasing ratios of weight over service time (Smith 1956).

Our Contribution. We consider different cases. In the *one-dimensional* (1-d) case, jobs' processing times are public information and a job's weight is only known to the job itself. We further distinguish between the discrete and continuous case. Publicly known probability distributions over a finite set of possible weights represent common beliefs about the weights in the discrete case. For the continuous case, we regard continuous probability distributions. In the *two-dimensional* (2-d) case, both weights and processing times are private information of the jobs. For all different settings, we aim at finding Bayes-Nash incentive compatible mechanisms that minimize the expected expenses of the service provider. Given jobs' reports about their private information, a mechanism determines both an order in which jobs are served, and for each job a payment that the job receives. The payment can be seen as a compensation for waiting. By a graph theoretic interpretation of the incentive compatibility constraints - as used e.g. by Rochet (1987) and in Malakhov and Vohra (2007) - we show how to derive optimal mechanisms. For the one-dimensional discrete and continuous case, we obtain closed formulae for modified job weights, and show that serving the jobs in the order of non-increasing ratios of these modified weights over service times is optimal for the service provider, as long as a certain regularity condition is fulfilled. It turns out that the optimal mechanism is not necessarily efficient, i.e., in general it does not maximize total utility. But it does so if e.g. all jobs are symmetric. For non-symmetric jobs with discrete weights, we show by example that the cost can be arbitrarily far from optimal if we insist on efficiency. We also compare our optimal mechanism to the generalized VCG mechanism and see that for discrete weights, expected payments differ even for the case of symmetric jobs. For continuous weights, however, revenue equivalence applies and the generalized VCG mechanism is an optimal mechanism for symmetric jobs. Furthermore, we analyze a mechanism in the continuous setting that corresponds to the first price auction and we show that this yields another optimal mechanism. For the two-dimensional discrete case, our main result is that the optimal mechanism generally does not satisfy a property called IIA, 'independence of irrelevant alternatives'. From that we conclude that the optimal mechanism cannot be expressed in terms of modified weights along the lines of the 1-d case. In fact, any kind of priority based list scheduling algorithm where the priorities of a job depend only on the characteristics of that job itself cannot in general be an optimal mechanism. We conclude that optimal mechanism design for the two-dimensional case is substantially more involved than two-dimensional mechanism design for auction settings, as studied in Malakhov and Vohra (2007). We also show that even for symmetric jobs, in the 2-d case the optimal mechanism is not efficient.

Related Work. Optimal mechanism design goes back to Myerson (1981). He studies optimal mechanisms for single item auctions and continuous 1-dimensional type spaces. Here, optimal auctions are modifications of efficient auctions, more specifically, modifications of the Vickrey auction. When regarding the seller as additional agent who bids zero in the original auction, his modified bid might be non-zero in the optimal auction yielding a reservation price. Malakhov and Vohra (2007) regard optimal mechanisms for an auction setting with discrete 2-dimensional type spaces. The derived optimal mechanisms again employ the efficient allocation rule with modified bids. As Malakhov and Vohra (2007), we follow Myerson's approach and analyze in how far it also works in a simple scheduling setting. We observe similarities and differences, see Section 3. Especially, we show that for 2-dimensional type spaces the traditional approach must fail to determine an optimal auction. The fact that multi-dimensional optimal mechanism design is harder than that for 1-dimensional types, is well-known. For example, Armstrong (2000) studies a multi-object auction model where valuations are additive and drawn from a binary distribution (i.e., high or low). He gives optimal auctions under specific conditions that reduce the type graph. From this paper it becomes evident that optimal mechanism design with multi-dimensional discrete types is difficult. For our model, we formalize this difficulty by showing that traditional approaches inevitably yield IIA-mechanisms and that in some cases none of these is optimal. For details, we refer to Section 4. In Hartline and Karlin (2007), the authors give an introduction to optimal mechanism design with 1-dimensional continuous types under dominant strategy incentive compatibility. Both Myerson's and our optimal allocation rules turn out to be dominant strategy implementable as well, while they yield optimal mechanisms in the larger class of Bayes-Nash incentive compatible mechanisms. Other scheduling models have been looked at from a different angle in the economic literature. See, e.g., Mitra (2001) for efficient and budget-balanced mechanism design in a 1-dimensional model and Moulin (2007) for mechanisms that prevent merging and splitting of jobs.

Organization. In Section 2, we study the 1-d discrete case and derive closed formulae for the optimal mechanism. We compare the optimal to efficient mechanisms in Section 3. In Section 4, we study the 2-d discrete case and show that known approaches are doomed to fail here. The continuous case is studied in Section 5 and standard auction formats for the continuous scheduling model are analyzed in Section 6. We conclude with Section 7.

2 Optimal Mechanisms for the 1-Dimensional Setting

2.1 Setting and Preliminaries

Consider a single machine which can handle one job at a time. Let $J = \{1, \dots, n\}$ denote the set of jobs. We regard jobs as selfish agents that act strategically. Each job j has a processing time p_j and a weight w_j. While p_j is publicly known, the actual w_j is private information to job j. We refer to the private information of a job as its type. Jobs share common beliefs about other jobs' types in terms of probability distributions. We assume discrete distribution of weights, that is, agent j's weight w_j follows a probability distribution over the discrete set $W_j = \{w_j^1, \dots, w_j^{m_j}\} \subset \mathbb{R}$, where $w_j^1 < \cdots < w_j^{m_j}$. Let φ_j be the probability distribution of w_j, that is, $\varphi_j(w_j^i)$ denotes the probability associated with w_j^i for $i = 1, \dots, m_j$. Let $\Phi_j(w_j^i) = \sum_{k=1}^{i} \varphi_j(w_j^k)$ be the cumulative probability up to w_j^i. Both φ_j and Φ_j are public information. We assume that jobs' weights are independently distributed. Let us denote by $W = \Pi_{j\in J} W_j$ the set of all type profiles. For any job j, let $W_{-j} = \Pi_{k\neq j} W_k$. Let φ be the joint probability distribution of $w = (w_1, \dots, w_n)$. Then $\varphi(w) = \Pi_{j=1}^{n} \varphi_j(w_j^{i_j})$ for $w = (w_1^{i_1}, \dots, w_n^{i_n}) \in W$. Let w_{-j} and φ_{-j} be defined analogously. For $w_j^i \in W_j$ and $w_{-j} \in W_{-j}$, we denote by (w_j^i, w_{-j}) the type profile where job j has type w_j^i and the types of all other jobs are w_{-j}.

A direct revelation mechanisms consists of an allocation rule f and a payment scheme π. Jobs have to report their weights and they might report untruthfully if it suits them. Depending on those reports, the allocation rule selects a *schedule*, i.e. an order in which jobs are processed on the machine. The payment scheme assigns a payment that is made to jobs in order to reimburse them for their waiting cost.

Let $\mathfrak{S} = \{\sigma \mid \sigma \text{ is a permutation of } (1, \dots, n)\}$ denote the set of all feasible schedules. Then the allocation rule is a mapping $f \colon W \to \mathfrak{S}$. For any schedule $\sigma \in \mathfrak{S}$, let σ_j be the position of job j in the ordering of jobs in σ. Then, by $S_j(\sigma) = \sum_{\sigma_k<\sigma_j} p_k$, we denote the start time or waiting time of job j in σ. If job j has waiting time S_j and actual weight

w_j^i, it encounters a valuation of $-w_j^i S_j$. If j additionally receives payment π_j, his total utility is $\pi_j - w_j^i S_j$, i.e., we assume quasi-linear utilities. Let us denote by $ES_j(f, w_j^i) := \sum_{w_{-j} \in W_{-j}} S_j(f(w_j^i, w_{-j}))\varphi_{-j}(w_{-j})$ the expected waiting time of job j if it reports weight w_j^i and allocation rule f is applied. Denote by $E\pi_j(w_j^i) := \sum_{w_{-j} \in W_{-j}} \pi_j(w_j^i, w_{-j})\varphi_{-j}(w_{-j})$ the expected payment to j. We assume that jobs aim at maximizing their expected utility.

Definition 1 *A mechanism (f, π) is* Bayes-Nash incentive compatible *if for every agent j and every two types $w_j^i, w_j^k \in W_j$*

$$E\pi_j(w_j^i) - w_j^i ES_j(f, w_j^i) \geq E\pi_j(w_j^k) - w_j^i ES_j(f, w_j^k) \tag{1}$$

under the assumption that all agents apart from j report truthfully. If for allocation rule f there exists a payment scheme π such that (f, π) is Bayes-Nash incentive compatible, then f is called Bayes-Nash implementable*. The payment scheme π is referred to as an* incentive compatible *payment scheme.*

In order to account for individual rationality, we need to guarantee non-negative utilities for all agents that report their true weight. It will be convenient to ensure individual rationality by introducing a so-called dummy weight $w_j^{m_j+1}$, which we add to the type space W_j for every agent j. We assume $ES_j(f, w_j^{m_j+1}) = 0$ and $E\pi_j(w_j^{m_j+1}) = 0$ for all $j \in J$. Furthermore, we impose the incentive constraints $E\pi_j(w_j^i) - w_j^i ES_j(f, w_j^i) \geq E\pi_j(w_j^{m_j+1}) - w_j^i ES_j(f, w_j^{m_j+1})$, which imply that $E\pi_j(w_j^i) - w_j^i ES_j(f, w_j^i) \geq 0$ for any Bayes-Nash incentive compatible mechanism (f, π). Therefore, the dummy weights together with the mentioned assumptions guarantee that individual rationality is satisfied along with the incentive constraints. The dummy weight can be interpreted as an option for any job not to take part in the mechanism.

We next define the notion of monotonicity w.r.t. weights, which is easily shown to be a necessary condition for Bayes-Nash implementability. In our setting, it is even a sufficient condition.

Definition 2 *An allocation rule f satisfies* monotonicity w.r.t. weights *or short* monotonicity *if for every agent $j \in J$, $w_j^i < w_j^k$ implies that $ES_j(f, w_j^i) \geq ES_j(f, w_j^k)$.*

Theorem 1 *An allocation rule f is Bayes-Nash incentive compatible if and only if it satisfies monotonicity w.r.t. weights.*

Before we give a proof of Theorem 1, we introduce the type graph for the Bayes-Nash setting. T_f has node set W_j and contains an arc from any node w_j^i to any other node w_j^k of length

$$\ell_{ik} = w_j^i[ES_j(f, w_j^k) - ES_j(f, w_j^i)].$$

Here, ℓ_{ik} represents the gain in expected valuation for agent j by truthfully reporting type w_j^i instead of lying type w_j^k. The incentive constraints for a Bayes-Nash incentive compatible mechanism (f, π) and job j can be read as

$$E\pi_j(w_j^k) \le E\pi_j(w_j^i) + w_j^i[ES_j(f, w_j^k) - ES_j(f, w_j^i)] = E\pi_j(w_j^i) + \ell_{ik}.$$

That is, the expected payments $E\pi_j(\cdot)$ constitute a node potential in T_f. According to Müller, Perea, and Wolf (2007), Bayes-Nash implementability of an allocation rule f is equivalent to the non-negative cycle property of the type graph T_f for any agent j. Monotonicity is equivalent to the fact that there is no negative cycle consisting of only two arcs in T_f. We call this property the *non-negative two-cycle property*. It follows from

$$\begin{aligned} \ell_{ik} + \ell_{ki} &= w_j^i[ES_j(f, w_j^k) - ES_j(f, w_j^i)] + w_j^k[ES_j(f, w_j^i) - ES_j(f, w_j^k)] \\ &= (w_j^i - w_j^k)[ES_j(f, w_j^k) - ES_j(f, w_j^i)]. \end{aligned}$$

The last term is non-negative for all jobs j and any two types w_j^i and w_j^k if and only if monotonicity holds.

Proof (*Theorem 1*). All that remains to show is that the non-negative two-cycle property implies the non-negative cycle property. We first show that the arc lengths satisfy a property called *decomposition monotonicity*, i.e., whenever $i < k < l$ then $\ell_{ik} + \ell_{kl} \le \ell_{il}$ and $\ell_{lk} + \ell_{ki} \le \ell_{li}$. From that property follows that the length of any cycle can be lower bounded by the lengths of a number of two cycles, which proves the theorem.

Decomposition monotonicity follows from

$$\begin{aligned} \ell_{ik} + \ell_{kl} &= w_j^i[ES_j(f, w_j^k) - ES_j(f, w_j^i)] + w_j^k[ES_j(f, w_j^l) - ES_j(f, w_j^k)] \\ &\le w_j^i[ES_j(f, w_j^k) - ES_j(f, w_j^i)] + w_j^i[ES_j(f, w_j^l) - ES_j(f, w_j^k)] \\ &= w_j^i[ES_j(f, w_j^l) - ES_j(f, w_j^i)] \\ &= \ell_{il}, \end{aligned}$$

where the inequality follows from monotonicity. Note that everything remains true if the dummy type is involved, i.e., if $l = m_j + 1$. The inequality $\ell_{lk} + \ell_{ki} \le \ell_{li}$ follows similarly.

In order to prove the second claim, consider a finite cycle c with nodes c_1 to c_k and rename the nodes such that $c_1 < c_2 < \cdots < c_k$. Replace every arc (c_u, c_v) with $u < v$

by arcs $(c_u, c_{u+1}), (c_{u+1}, c_{u+2}), \ldots, (c_{v-1}, c_v)$. Do the same for all arcs (c_v, c_u) with $u < v$. Call the resulting cycle c'. The cycle length of c' is less than or equal to the length of c, due to decomposition monotonicity. The new cycle c' consists only of two-cycles. Due to monotonicity, those have non-negative length. Hence, c has non-negative length as well. □

2.2 Optimal Mechanisms

Let us start by investigating the *efficient* allocation rule for the given setting, i.e., the allocation rule that maximizes the total valuation of agents. It is well known that scheduling in order of non-increasing weight over processing time ratios minimizes the sum of weighted start times $\sum_{j=1}^{n} w_j S_j(f(w))$ for any type profile $w \in W$, and therefore maximizes the total valuation of all agents. This allocation rule is known as Smith's rule (Smith 1956). The optimal mechanism that we derive deploys a slightly different allocation rule, namely Smith's rule with respect to certain modified weights.

Our goal is to set up a mechanism that is Bayes-Nash incentive compatible and among all such mechanisms minimizes the expected total payment that has to be made to the jobs. Given any Bayes-Nash incentive compatible mechanism (f, π), one can obviously substitute the payment scheme by its expected payment scheme yielding $(f, E\pi(\cdot))$ without loosing Bayes-Nash incentive compatibility. Moreover, the expected total payment to the agents remains unchanged under the substitution. Therefore, we restrict focus to mechanisms in which agents always receive a payment which is independent of the specific report of the other agents and of the actual allocation.

Note that, unlike e.g. in Myerson (1981), in the discrete setting considered here revenue equivalence does not hold. Therefore, there are possibly multiple payment schemes that make an allocation rule incentive compatible. Let f be an allocation rule and let $\pi^f(\cdot)$ be a payment scheme that minimizes expected expenses for the machine among all payment schemes that make f Bayes-Nash incentive compatible. More specifically, $\pi_j^f(w_j^i)$ denotes the payment to agent j declaring weight w_j^i under this optimal payment scheme. Let $P^{min}(f) = \sum_{j \in J} \sum_{w_j^i \in W_j} \varphi_j(w_j^i) \pi_j^f(w_j^i)$ be the minimum expected total expenses for allocation rule f. The following lemma specifies the optimal payment scheme for a given allocation rule.

Lemma 1 *For a Bayes-Nash implementable allocation rule f, the payment scheme defined by*

$$\pi_j^f(w_j^{m_j+1}) = 0, \quad \pi_j^f(w_j^i) = \sum_{k=i}^{m_j} w_j^k [ES_j(f, w_j^k) - ES_j(f, w_j^{k+1})] \ \ \textit{for } i = 1, \ldots, m_j$$

is incentive compatible, individually rational and minimizes the expected total payment made to agents. The corresponding expected total payment is given by

$$P^{min}(f) \quad = \quad \sum_{j\in J}\sum_{i=1}^{m_j} \varphi_j(w_j^i)\overline{w}_j^i ES_j(f, w_j^i),$$

where the modified weights $\overline{w}_j$ *are defined as follows*

$$\overline{w}_j^1 = w_j^1, \quad \overline{w}_j^i = w_j^i + (w_j^i - w_j^{i-1})\frac{\Phi_j(w_j^{i-1})}{\varphi_j(w_j^i)} \textit{ for } i = 2, \ldots, m_j.$$

Proof. Let $\mathfrak{p} = (w_j^i = a_0, a_1, \ldots, a_m = w_j^{m_j+1})$ denote a path from w_j^i to $w_j^{m_j+1}$ in the type graph T_f for agent j. Denote by $length(\mathfrak{p})$ the sum of its arc lengths. Let (f, π) be a Bayes-Nash incentive compatible mechanism. Adding up the incentive constraints

$$E\pi_j(a_i) \leq E\pi_j(a_{i-1}) + a_{i-1}[ES_j(f, a_i) - ES_j(f, a_{i-1})] = E\pi_j(a_{i-1}) + \ell_{a_{i-1}a_i}$$

for $i = 1, \ldots, m$ yields

$$E\pi_j(w_j^{m_j+1}) \leq E\pi_j(w_j^i) + length(\mathfrak{p}).$$

Assuming $E\pi_j(w_j^{m_j+1}) = 0$, this is equivalent to $-length(\mathfrak{p}) \leq E\pi_j(w_j^i)$. As f is Bayes-Nash implementable, T_f satisfies the non-negative cycle property. Consequently, we can compute shortest paths in T_f. With $dist(w_j^i, w_j^{m_j+1})$ being the length of a shortest path from w_j^i to $w_j^{m_j+1}$, the above yields $-dist(w_j^i, w_j^{m_j+1}) \leq E\pi_j(w_j^i)$. Therefore, $-dist(w_j^i, w_j^{m_j+1})$ is a lower bound on the expected payment for reporting w_j^i. On the other hand, since we have

$$dist(w_j^i, w_j^{m_j+1}) \leq \ell_{ik} + dist(w_j^k, w_j^{m_j+1})$$

for any two types w_j^i and w_j^k, it follows that

$$-dist(w_j^k, w_j^{m_j+1}) \leq -dist(w_j^i, w_j^{m_j+1}) + \ell_{ik}.$$

Consequently, $-dist(\cdot, w_j^{m_j+1})$ defines a node potential in T_f. Setting $\pi_j^f(w_j^i) = -dist(w_j^i, w_j^{m_j+1})$ therefore yields an incentive compatible payment scheme that minimizes the expected payment to every agent for any reported type of the agent. Consequently, this payment scheme also minimizes the expected total payment to agents. Recall that individual rationality is satisfied along with the incentive constraints.

Since arc lengths in T_f satisfy decomposition monotonicity, a shortest path from w_j^i to $w_j^{m_j+1}$ is the path that includes all intermediate nodes $w_j^{i+1}, \ldots, w^{m_j}$. Observing that

$-dist(w_j^{m_j+1}, w_j^{m_j+1}) = 0$ and $-dist(w_j^i, w_j^{m_j+1}) = \sum_{k=i}^{m_j} w_j^k [ES_j(f, w_j^k) - ES_j(f, w_j^{k+1})] \, \forall w_j^i \in W_j \setminus \{w_j^{m_j+1}\}$ proves the first claim.

Next, we compute the minimum expected total payment for allocation rule f.

$$\begin{aligned}
P^{min}(f) &= \sum_{j\in J}\sum_{i=1}^{m_j} \varphi_j(w_j^i)\pi_j^f(w_j^i) \\
&= \sum_{j\in J}\sum_{i=1}^{m_j} \varphi_j(w_j^i)\sum_{k=i}^{m_j} w_j^k[ES_j(f,w_j^k) - ES_j(f,w_j^{k+1})] \\
&= \sum_{j\in J}\sum_{i=1}^{m_j} \varphi_j(w_j^i)\left(\sum_{k=i}^{m_j} w_j^k ES_j(f,w_j^k) - \sum_{k=i+1}^{m_j} w_j^{k-1} ES_j(f,w_j^k)\right) \\
&= \sum_{j\in J}\sum_{i=1}^{m_j} \varphi_j(w_j^i)\left(w_j^i ES_j(f,w_j^i) + \sum_{k=i+1}^{m_j} ES_j(f,w_j^k)(w_j^k - w_j^{k-1})\right) \\
&= \sum_{j\in J} ES_j(f,w_j^1)w_j^1\varphi_j(w_j^1) \\
&\quad + \sum_{j\in J}\sum_{i=2}^{m_j} ES_j(f,w_j^i)\left(\varphi_j(w_j^i)w_j^i + (w_j^i - w_j^{i-1})\sum_{k=1}^{i-1}\varphi_j(w_j^k)\right) \\
&= \sum_{j\in J} ES_j(f,w_j^1)w_j^1\varphi_j(w_j^1) \\
&\quad + \sum_{j\in J}\sum_{i=2}^{m_j} ES_j(f,w_j^i)\left(\Phi_j(w_j^i)w_j^i - \Phi_j(w_j^{i-1})w_j^{i-1}\right)
\end{aligned}$$

Let us define modified weights $\overline{w}_j$ by setting $\overline{w}_j^1 = w_j^1$ and for $i = 2, \ldots, m_j$

$$\begin{aligned}
\overline{w}_j^i &= \frac{w_j^i\Phi_j(w_j^i) - w_j^{i-1}\Phi_j(w_j^{i-1})}{\varphi_j(w_j^i)} \\
&= \frac{w_j^i\varphi_j(w_j^i) + w_j^i\Phi_j(w_j^{i-1}) - w_j^{i-1}\Phi_j(w_j^{i-1})}{\varphi_j(w_j^i)} \\
&= w_j^i + (w_j^i - w_j^{i-1})\frac{\Phi_j(w_j^{i-1})}{\varphi_j(w_j^i)}.
\end{aligned}$$

This yields

$$P^{min}(f) = \sum_{j\in J}\sum_{i=1}^{m_j}\varphi_j(w_j^i)\overline{w}_j^i ES_j(f,w_j^i).$$

□

Given the minimum payments per allocation rule, we want to specify the allocation rule f which minimizes $P^{min}(f)$ among all Bayes-Nash implementable allocation rules.

Definition 3 *If* $f \in \arg\min\{P^{min}(f) \mid f : W \to \mathfrak{S},\ f$ *Bayes-Nash implementable*$\}$*, then we call the mechanism* (f, π^f) *an* optimal mechanism*.*

We will need the following regularity condition that ensures Bayes-Nash implementability of the allocation rule in our optimal mechanism.

Definition 4 *We say that* regularity *is satisfied if for every agent* j *and* $i = 2, \ldots, m_j - 1$

$$w_j^i + (w_j^i - w_j^{i-1})\frac{\Phi_j(w_j^{i-1})}{\varphi_j(w_j^i)} \le w_j^{i+1} + (w_j^{i+1} - w_j^i)\frac{\Phi_j(w_j^i)}{\varphi_j(w_j^{i+1})}.$$

This implies that $\overline{w}_j^i < \overline{w}_j^k$ *whenever* $w_j^i < w_j^k$*.*

Note that regularity is satisfied e.g. if the differences $w_j^i - w_j^{i-1}$ are constant and the distribution has a non-increasing reverse hazard rate[4].

Theorem 2 *Let the modified weights be defined as in Lemma 1. Let* f *be the allocation rule that schedules jobs in order of non-increasing ratios* $\overline{w}_j/p_j$*. If regularity holds, then* (f, π^f) *is an optimal mechanism.*

Proof. We show that f is Bayes-Nash implementable and minimizes $P^{min}(f)$ among all Bayes-Nash implementable allocation rules. For any allocation rule f, we can rewrite $P^{min}(f)$ as follows, using independence of weight distributions. Let $W_j' = W_j \setminus \{w_j^{m_j+1}\}$ and $W' = \Pi_{j \in J} W_j'$.

$$\begin{aligned}
P^{min}(f) &= \sum_{j \in J} \sum_{w_j^i \in W_j'} \varphi_j(w_j^i)\overline{w}_j^i ES_j(f, w_j^i) \\
&= \sum_{j \in J} \sum_{w_j^i \in W_j'} \varphi_j(w_j^i)\overline{w}_j^i \sum_{w_{-j} \in W_{-j}} S_j(f(w_j^i, w_{-j}))\varphi_{-j}(w_{-j}) \\
&= \sum_{j \in J} \sum_{(w_j^i, w_{-j}) \in W'} \varphi(w_j^i,\, w_{-j})\overline{w}_j^i S_j(f(w_j^i, w_{-j})) \\
&= \sum_{w \in W'} \varphi(w) \sum_{j \in J} \overline{w}_j S_j(f(w)).
\end{aligned}$$

[4]The reverse hazard rate of the distribution with pdf φ and cdf Φ is defined as $\varphi(x)/\Phi(x)$, see e.g. Krishna (2002).

Thus, $P^{min}(f)$ can be minimized by minimizing $\sum_{j\in J} \overline{w}_j S_j(f(w))$ for every reported type profile w. This is achieved by using Smith's rule with respect to modified weights, i.e., scheduling in order of non-increasing ratios $\overline{w}_j/p_j$. Under Smith's rule, the expected start time $ES_j(w_j)$ is clearly non-increasing in the modified weight $\overline{w}_j$. The regularity condition ensures that it is non-increasing in the original weights w_j. Therefore, Smith's rule with respect to modified weights satisfies monotonicity and is hence Bayes-Nash implementable by Theorem 1. This completes the proof. □

3 Optimality versus Efficiency

For symmetric agents the optimal and the efficient allocation coincide.

Corollary 1 *If agents are symmetric, i.e.* $W_1 = \cdots = W_n$, $\varphi_1 = \cdots = \varphi_n$ *and* $p_1 = \cdots = p_n$ *and if distributions are such that regularity holds, then the optimal mechanism is efficient.*

Proof. If $W_1 = \cdots = W_n = \{w^1, \ldots, w^m\}$ and $\varphi_1 = \cdots = \varphi_n$, then for any two agents j_1 and j_2, and $i = 1, \ldots, m$, the modified weights are equal, i.e. $\overline{w}^i_{j_1} = \overline{w}^i_{j_2}$. Since processing times are also equal and since regularity guarantees that modified weights are increasing in the original weights, scheduling jobs in order of their non-increasing ratios w_j/p_j is equivalent to scheduling them in order of their non-increasing ratios $\overline{w}_j/p_j$. That is, the efficient allocation rule and the allocation rule from the optimal mechanism in Theorem 2 coincide. □

If weight distributions differ among agents or if agents have different processing times, then the optimal mechanism is in general not efficient. In fact, when restricting to efficient mechanisms, the total expected payment can be arbitrarily bad in comparison to the optimal one. This is illustrated by the following two examples.

Example 1 *Let there be two jobs 1 and 2 with* $W_1 = \{M + 1\}$ *and* $W_2 = \{1, M\}$ *for some constant* M. *Let* $\varphi_2(1) = 1 - 1/M$, $\varphi_2(M) = 1/M$ *and* $p_1 = p_2 = 1$. *Let Eff be the efficient and Opt be the optimal allocation rule. Then the ratio* $P^{min}(\mathit{Eff})/P^{min}(\mathit{Opt})$ *goes to infinity as* M *goes to infinity.*

Proof. The efficient allocation rule, Smith's rule, always allocates job 1 first. So the optimal payment for Smith's rule is to pay 0 to job 1 and to pay M to job 2, irrespective of its type. The minimum expected total payment is hence $P^{min}(Eff) = M$.

For the optimal allocation, we compute modified weights after Lemma 1: $\overline{w}^1_1 = w^1_1 = M + 1$, $\overline{w}^1_2 = w^1_2 = 1$ and $\overline{w}^2_2 = M + (M - 1)(1 - 1/M)/(1/M) = M^2 - M + 1$. The latter is larger than $M+1$ if $M > 2$. Therefore, job 2 is scheduled in front of job 1 if he has weight M and behind if he has weight 1. The expected start times for job 2 are $ES_2(Opt, 1) = 1$ and $ES_2(Opt, M) = 0$, respectively. Optimal payments according to Lemma 1 are $\pi_2^{Opt}(1) = 1$ and $\pi_2^{Opt}(M) = 0$. For job 1, the expected start time is $ES_1(Opt, M+1) = 1/M$ and the expected payment $\pi_1^{Opt}(M+1) = 1+1/M$. Hence, $P^{min}(Opt) = 1+1/M+1\cdot(1-1/M) = 2$.

Consequently, $P^{min}(Eff)/P^{min}(Opt) = M/2$, which tends to infinity if M goes to infinity.

□

Remark 1 *In the above, the ratio of the expected payments of the efficient versus the optimal allocation rule is analyzed. Similarly, we can derive that the expected ratio of the payments tends to infinity as M approaches infinity. The latter is slightly more technical.*

Example 2 *Let there be two jobs 1 and 2 with the same weight distribution $W_1 = W_2 = \{1, M\}$, $\varphi_j(1) = 1-1/M$, $\varphi_j(M) = 1/M$ for $j = 1, 2$. Let $p_1 = 1/2$ and $p_2 = M/2+1$. Let Eff be the efficient and Opt be the optimal allocation rule. Then the ratio $P^{min}(Eff)/P^{min}(Opt)$ goes to infinity as M goes to infinity.*

Proof. The efficient allocation rule always schedules job 1 first, since $1/(1/2) = 2 > 2M/(M+2) = M/(M/2+1)$. Therefore, the expected start time of job 1 is 0 and that of job 2 is 1/2. Optimal payments according to Lemma 1 are $\pi_1^{Eff}(1) = \pi_1^{Eff}(M) = 0$ and $\pi_2^{Eff}(1) = \pi_2^{Eff}(1) = M/2$. Hence, $P^{min}(Eff) = M/2$.

For the optimal mechanism, we compute modified weights as $\overline{w}_1^1 = \overline{w}_2^1 = 1$ and $\overline{w}_1^2 = \overline{w}_2^2 = M^2 - M + 1$. Job 1 is scheduled first, whenever both jobs have the same weight or job 1 has a larger weight than job 2. In the case where job 1 has (modified) weight 1 and job 2 has modified weight $M^2 - M + 1$, job 2 is scheduled first for $M > 2$, since $1/(1/2) < (M^2 - M + 1)/(M/2+1)$. The resulting expected start times and payments are given below:

$$
\begin{array}{ll}
ES_1(Opt, 1) = 1/2 + 1/M & \pi_1^{Opt}(1) = 1/2 + 1/M \\
ES_1(Opt, M) = 0 & \pi_1^{Opt}(M) = 0 \\
ES_2(Opt, 1) = 1/2 & \pi_2^{Opt}(1) = 1 - 1/(2M) \\
ES_2(Opt, M) = 1/(2M) & \pi_2^{Opt}(M) = 1/2.
\end{array}
$$

Hence,

$$
\begin{aligned}
P^{min}(Opt) &= (\frac{1}{2} + \frac{1}{M})(1 - \frac{1}{M}) + (1 - \frac{1}{2M})(1 - \frac{1}{M}) + \frac{1}{2} \cdot \frac{1}{M} \\
&= (1 - \frac{1}{M})(\frac{3}{2} + \frac{1}{2M}) + \frac{1}{2} \cdot \frac{1}{M}.
\end{aligned}
$$

Thus, the ratio $P^{min}(Eff)/P^{min}(Opt)$ tends to infinity if M tends to infinity. □

Remark 2 *As in the first example, it can be shown that also that the expected ratio of the payments tends to infinity as M approaches infinity.*

Comparison to Myerson's result. For the single item auction and continuous type spaces, Myerson (1981) has made similar observations: in his setting, the Vickrey auction is an efficient auction. The optimal auction can be seen as a modified Vickrey auction with the seller submitting a bit himself. In our setting also, the allocation in the optimal mechanism is equivalent to the efficient allocation rule with respect to modified data. Nevertheless, in Myerson (1981) the optimal and the efficient mechanism may differ. For the single item auction this can be due to the seller keeping the item (even in the symmetric case) or because a bidder that has not submitted the highest bid can get the item in the asymmetric case. In our setting, the optimal and the efficient mechanism can only differ if agents are asymmetric, see Corollary 1 and Examples 1 and 2.

On the generalized VCG Mechanism. The VCG mechanism is due to Vickrey (1961), Clarke (1971) and Groves (1973). The allocation rule is the efficient one. In our setting this means scheduling in order of non-increasing ratios w_j/p_j. The payment scheme pays to agent j an amount that is equal to an appropriate constant (possibly depending on other agents' types, but not on j's type) minus the total loss in valuation of the other agents due to j's presence. For agent j with processing time p_j, the total loss in valuation of the other agents is equal to the product of p_j and the total weight of all agents processed after j. In order to ensure individual rationality, we have to add p_j times the total weight of all agents except j. Therefore, the resulting payment to j for reported type profile w and efficient schedule σ is equal to

$$\pi_j^{VCG}(w) = p_j \sum_{\substack{k \in J \\ \sigma_k < \sigma_j}} w_k.$$

As illustrated by examples 1 and 2, the allocation of the VCG mechanism can differ from the allocation of the optimal mechanism if agents are not symmetric. Moreover, if agents are symmetric, the VCG mechanism still can be non-optimal in terms of payments. This is illustrated by the following example.

Example 3 *There are two symmetric agents with $W_1 = W_2 = \{w^1, w^2\}$, $w^1 < w^2$, and $\varphi_j(w^1) = \varphi_j(w^2) = 1/2$ for $j = 1, 2$. Processing times are equal and without loss of generality $p_1 = p_2 = 1$. Then the expected expenses of the VCG mechanism are strictly higher than those of the optimal mechanism.*

Proof. Regularity is trivially satisfied and therefore the allocation of the optimal mechanism from Section 2 is efficient. There are four possible type profiles, each occurring with probability 1/4: (w^1, w^1), (w^1, w^2), (w^2, w^1), (w^2, w^2). The resulting schedules are the same for the VCG and the optimal mechanism and schedule the job with the higher weight first or randomize uniformly in the case of equal weights, respectively. Let us first compute the expected total payment for the VCG mechanism. The VCG mechanism pays to the job that is scheduled last the weight of the job that is scheduled before him. Thus, the VCG mechanism has to spend w^1 in the first case, and w^2 in the second, third and fourth case, respectively. The total expected payment of the VCG mechanism is hence $(3w^2+w^1)/4$. Let (f, π^f) denote the optimal mechanism from Section 2. In the optimal mechanism, the expected payment to a job with weight w^1 is equal to $E\pi_j^f(w^1) = w^1[ES_j(f, w^1) - ES_j(f, w^2)] + w^2 ES_j(f, w^2) = w^1[3/4 - 1/4] + w^2[1/4] = w^1/2 + w^2/4$. The expected payment to a job with weight w^2 is $E\pi_j^f(w^2) = w^2 ES_j(f, w^2) = w^2/4$. The total expected payment for the optimal mechanism is thus $2 \cdot 1/2 \cdot (w^1/2 + w^2/4 + w^2/4) = (w^1 + w^2)/2$. Since $w^2 > w^1$, the expected expenses of the VCG mechanism are strictly higher than those of the optimal mechanism. Therefore, the VCG mechanism is not optimal. □

4 The 2-Dimensional Setting

4.1 Setting and Notation

In contrast to the 1-dimensional setting, both weight and processing time of a job are now private information of the job. Hence j's type is the tuple (w_j, p_j). We restrict attention to discrete type spaces, i.e., $(w_j, p_j) \in W_j \times P_j$, where $W_j = \{w_j^1, \ldots, w_j^{m_j}\}$ with $w_j^1 \le \cdots \le w_j^{m_j}$ and $P_j = \{p_j^1, \ldots, p_j^{q_j}\}$ with $p_j^1 \le \cdots \le p_j^{q_j}$. Let φ_j be the probability distribution of j's type, that is, $\varphi_j(w_j^i, p_j^k)$ denotes the probability associated with the type (w_j^i, p_j^k) for $i = 1, \ldots, m_j$ and $k = 1, \ldots, q_j$. Both φ_j and Φ_j are public. Distributions are independent between agents. Denote by $T = \Pi_{j \in J}(W_j \times P_j)$ the set of all type profiles. For any job j, let $T_{-j} = \Pi_{r \neq j}(W_r \times P_r)$ be the set of type profiles of all jobs except j. Let φ be the joint probability distribution of $(w_1, p_1, \ldots, w_n, p_n)$. Then for type profile $t = (w_1^{i_1}, p_1^{k_1}, \ldots, w_n^{i_n}, p_n^{k_n}) \in T$, $\varphi(t) = \Pi_{j=1}^n \varphi_j(w_j^{i_j}, p_j^{k_j})$. Let t_{-j} and φ_{-j} be defined analogously. For $(w_j^i, p_j^k) \in W_j \times P_j$ and $t_{-j} \in T_{-j}$, we denote by $((w_j^i, p_j^k), t_{-j})$ the type profile where job j has type (w_j^i, p_j^k) and the types of the other jobs are represented by t_{-j}. Denote by $ES_j(f, w_j^i, p_j^k) := \sum_{t_{-j} \in T_{-j}} S_j(f((w_j^i, p_j^k), t_{-j}))\varphi_{-j}(t_{-j})$ the expected waiting time of job j if he reports type (w_j^i, p_j^k) and allocation rule f is applied. Denote by $E\pi_j(w_j^i, p_j^k) :=$

$\sum_{t_{-j} \in T_{-j}} \pi_j((w_j^i, p_j^k), t_{-j})\varphi_{-j}(t_{-j})$ the expected payment to j.

We assume that an agent can only report a processing time that is not lower than his true processing time and that a job is processed for his reported processing time. This is a natural assumption, since a job can add unnecessary work to achieve a longer processing time, but reporting a shorter processing time can easily be punished by preempting the job after the declared processing time (before it is actually finished).

Note that by regarding the processing time as private information, we introduce informational externalities: job j has a different valuation for a schedule if the processing time (and hence the type) of a job scheduled before j changes. In this regard, our model differs from the 2-dimensional auction model studied in Malakhov and Vohra (2007).

4.2 Bayes-Nash Implementability and the Type Graph

Definition 5 *A mechanism* (f, π) *is called* Bayes-Nash incentive compatible *if for every agent* j *and every two types* $(w_j^{i_1}, p_j^{k_1})$ *and* $(w_j^{i_2}, p_j^{k_2})$ *with* $i_1, i_2 \in \{1, \ldots, m_j\}$, $k_1, k_2 \in \{1, \ldots, q_j\}$, $k_1 \leq k_2$,

$$E\pi_j(w_j^{i_1}, p_j^{k_1}) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^{k_1}) \geq E\pi_j(w_j^{i_2}, p_j^{k_2}) - w_j^{i^1} ES_j(f, w_j^{i_2}, p_j^{k_2}) \tag{2}$$

under the assumption that all agents apart from j *report truthfully.*

Note that by defining the incentive constraints only for $k_1 \leq k_2$, we account for the fact that agents can only overstate their processing time, but cannot understate it.

In order to ensure individual rationality, again add a dummy type t_j^d to the type space for every agent j, and let $ES_j(f, t_j^d) = 0$ and $E\pi_j(t_j^d) = 0$ for all $j \in J$. As in the 1-dimensional case, the dummy types together with the mentioned extra incentive constraints guarantee that individual rationality is satisfied along with the incentive constraints. Sometimes, it will be convenient to write $(w_j^{m_j+1}, p_j^k)$ for some $k \in \{1, \ldots, q_j\}$ instead of t_j^d.

In the 2-dimensional setting, the type graph T_f of agent j has node set $W_j \times P_j$ and contains an arc from any node $(w_j^{i_1}, p_j^{k_1})$ to every other node $(w_j^{i_2}, p_j^{k_2})$ with $i \in \{1, \ldots, m_j\}$, $i_2 \in \{1, \ldots, m_j + 1\}$, $k \in \{1, \ldots, q_j\}$, $k_1 \leq k_2$ of length

$$\ell_{(i_1 k_1)(i_2 k_2)} = w_j^{i_1}[ES_j(f, w_j^{i_2}, p_j^{k_2}) - ES_j(f, w_j^{i_1}, p_j^{k_1})].$$

Note that we have arcs only in direction of increasing processing times, since agents can only overstate their processing time. Furthermore, every node has an arc to the dummy type, but there are no outgoing arcs from the dummy type.

Similar as in Malakhov and Vohra (2007), one can show that for monotonic allocation rules some arcs in the type graph are not necessary, since the corresponding incentive constraints are implied by others. We first give the definition of monotonicity in the 2-dimensional setting and then formulate a lemma which reduces the set of necessary incentive constraints.

Definition 6 *An allocation rule f satisfies* monotonicity w.r.t. weights *if for every agent $j \in J$ and fixed $p_j^k \in P_j$, $w_j^{i_1} < w_j^{i_2}$ implies that $ES_j(f, w_j^{i_1}, p_j^k) \geq ES_j(f, w_j^{i_2}, p_j^k)$.*

Lemma 2 *Let f be an allocation rule satisfying monotonicity w.r.t. weights. For any agent j, the following constraints imply all other incentive constraints:*

$$\begin{aligned} E\pi_j(w_j^i, p_j^k) - w_j^i ES_j(f, w_j^i, p_j^k) &\geq E\pi_j(w_j^{i+1}, p_j^k) - w_j^i ES_j(f, w_j^{i+1}, p_j^k) && (3)\\ &\qquad \textit{for } i \in \{1, \ldots, m_j\}, k \in \{1, \ldots, q_j\} \\ E\pi_j(w_j^{i+1}, p_j^k) - w_j^{i+1} ES_j(f, w_j^{i+1}, p_j^k) &\geq E\pi_j(w_j^i, p_j^k) - w_j^{i+1} ES_j(f, w_j^i, p_j^k) && (4)\\ &\qquad \textit{for } i \in \{1, \ldots, m_j - 1\}, k \in \{1, \ldots, q_j\} \\ E\pi_j(w_j^i, p_j^k) - w_j^i ES_j(f, w_j^i, p_j^k) &\geq E\pi_j(w_j^i, p_j^{k+1}) - w_j^i ES_j(f, w_j^i, p_j^{k+1}) && (5)\\ &\qquad \textit{for } i \in \{1, \ldots, m_j\}, k \in \{1, \ldots, q_j - 1\} \end{aligned}$$

Proof. For any $i_1, i_2, i_3 \in \{1, \ldots, m_j + 1\}$,$i_1 < i_2 < i_3$, and any $k \in \{1, \ldots, q_j\}$ the constraint

$$E\pi_j(w_j^{i_1}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^k) \geq E\pi_j(w_j^{i_3}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_3}, p_j^k)$$

is implied by

$$E\pi_j(w_j^{i_1}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^k) \geq E\pi_j(w_j^{i_2}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^k)$$

and

$$E\pi_j(w_j^{i_2}, p_j^k) - w_j^{i_2} ES_j(f, w_j^{i_2}, p_j^k) \geq E\pi_j(w_j^{i_3}, p_j^k) - w_j^{i_2} ES_j(f, w_j^{i_3}, p_j^k).$$

In fact, adding up the latter two constraints yields

$$\begin{aligned} & E\pi_j(w_j^{i_1}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^k) \\ \geq\; & E\pi_j(w_j^{i_3}, p_j^k) + w_j^{i_2}(ES_j(f, w_j^{i_2}, p_j^k) - ES_j(f, w_j^{i_3}, p_j^k)) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^k) \\ \geq\; & E\pi_j(w_j^{i_3}, p_j^k) + w_j^{i_1}(ES_j(f, w_j^{i_2}, p_j^k) - ES_j(f, w_j^{i_3}, p_j^k)) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^k) \\ =\; & E\pi_j(w_j^{i_3}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_3}, p_j^k), \end{aligned}$$

where the second inequality follows from monotonicity and $w_j^{i_1} < w_j^{i_2}$. Note that everything remains true if the dummy type is involved, i.e., if $(w_j^{i_3}, p_j^k) = (w_j^{m_j+1}, p_j^k) = t_j^d$. These arguments imply that all constraints of the type

$$E\pi_j(w_j^{i_1}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^k) \geq E\pi_j(w_j^{i_2}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^k) \tag{6}$$

are implied by the subset of constraints where $i_2 = i_1 + 1$.

A similar effect can be shown for the "reverse" incentive constraints, i.e., the above constraints for $i_3 < i_2 < i_1$, where $i_1, i_2, i_3 \in \{1, \ldots, m_j\}$. Again, out of all constraints of the type

$$E\pi_j(w_j^{i_1}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^k) \geq E\pi_j(w_j^{i_2}, p_j^k) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^k), \tag{7}$$

only those with $i_2 = i_1 - 1$ are necessary.

Similarly, out of all constraints of the type

$$E\pi_j(w_j^i, p_j^{k_1}) - w_j^i ES_j(f, w_j^i, p_j^{k_1}) \geq E\pi_j(w_j^i, p_j^{k_2}) - w_j^i ES_j(f, w_j^i, p_j^{k_2}), \tag{8}$$

for $i \in \{1, \ldots, m_j\}$, $k_1, k_2 \in \{1, \ldots, q_j\}$, $k_1 < k_2$ only those with $k_2 = k_1 + 1$ are necessary.

For any types $(w_j^{i_1}, p_j^{k^1})$,$(w_j^{i_2}, p_j^{k^2})$ with $i_1 < i_2$ and $k_1 < k_2$ the corresponding "diagonal" constraint

$$E\pi_j(w_j^{i_1}, p_j^{k_1}) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^{k_1}) \geq E\pi_j(w_j^{i_2}, p_j^{k_2}) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^{k_2})$$

follows by adding up the corresponding constraints of type (8) and (6)

$$E\pi_j(w_j^{i_1}, p_j^{k_1}) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^{k_1}) \geq E\pi_j(w_j^{i_1}, p_j^{k_2}) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^{k_2})$$

and

$$E\pi_j(w_j^{i_1}, p_j^{k_2}) - w_j^{i_1} ES_j(f, w_j^{i_1}, p_j^{k_2}) \geq E\pi_j(w_j^{i_2}, p_j^{k_2}) - w_j^{i_1} ES_j(f, w_j^{i_2}, p_j^{k_2}).$$

For any $(w_j^{i_1}, p_j^{k^1})$,$(w_j^{i_2}, p_j^{k^2})$ with $i_2 < i_1$ and $k_1 < k_2$, the corresponding "diagonal" constraint follows by adding up the appropriate constraints of type (8) and (7). □

Lemma 2 is in fact a generalization of decomposition monotonicity as discussed for the 1-dimensional case.

We define the reduced type graph of agent j, which contains only arcs that are necessary in the sense of Lemma 2. These arcs are:

- an arc from type (w_j^i, p_j^k) to (w_j^{i+1}, p_j^k) for all $i \in \{1, \ldots, m_j\}$ and $k \in \{1, \ldots, q_j\}$
- an arc from type (w_j^{i+1}, p_j^k) to (w_j^i, p_j^k) for all $i \in \{1, \ldots, m_j - 1\}$ and $k \in \{1, \ldots, q_j\}$

- an arc from type (w_j^i, p_j^k) to (w_j^i, p_j^{k+1}) for all $i \in \{1, \ldots, m_j\}$ and $k \in \{1, \ldots, q_j - 1\}$.

A sketch of the reduced type graph is given in Figure 1. Expected payments correspond to node potentials in the reduced type graph. Whenever we refer to the type graph T_f for a monotonic allocation rule f in the following, the reduced type graph is meant. The reduced type graph comes handy particularly when considering our (counter) examples in the next subsection.

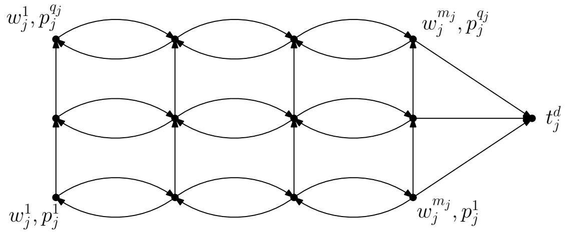


Figure 1: reduced type graph

We finally give the characterization of Bayes-Nash incentive compatible allocation rules for the 2-dimensional setting.

Theorem 3 *An allocation rule f is Bayes-Nash incentive compatible in the 2-dimensional setting if and only if it satisfies monotonicity with respect to weights.*

Proof. Implementability implies monotonicity as before. The claim reduces to showing that in the (reduced) type graph of any agent j the non-negative cycle property is equivalent to the non-negative two-cycle property. After the reduction, every cycle in T_f consists of a finite number of two-cycles. Hence the non-negative cycle property is equivalent to the non-negative two-cycle property.

□

4.3 On Optimal Mechanisms

We start by reviewing an approach to two-dimensional optimal mechanism design studied in Malakhov and Vohra (2007). Here, the authors regard a multi-item auction, where each agent's type (i, j) is given by a marginal valuation i per item and a capacity j. Above that capacity, the agent has zero valuation for each additional item. Agents can only overstate their capacity. The goal is revenue maximization. Bayes-Nash implementability is equivalent to the expected amount of items allocated to an agent being monotone in his reported value for i. Malakhov and Vohra (2007) use the type graph approach as follows.

First, they regard a subset of all allocation rules - namely those that are monotone in j as well. It turns out that all those rules have the same shortest path tree, namely the "up-first-then-right" tree (see Figure 2 for a 3×3 example).

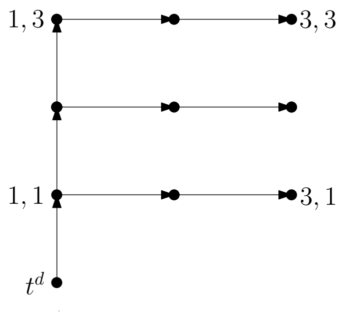


Figure 2: up-first-then-right tree

Second, the path lengths in this tree yield optimal payments to every job for every type. From that, the optimal revenue for a particular allocation rule is obtained as closed formula in terms of modified marginal valuations.

Third, the obtained expression for the revenue is maximized over *all* allocation rules. The resulting allocation rule is a modification of the efficient allocation rule. In addition, this rule turns out to be monotone in j, similar as in the proof of Theorem 2. Hence, its shortest path tree is the up-first-then-right tree.

In the last step, the monotonicity assumption in j is relaxed as follows. For any allocation rule – not necessarily monotone in j – the up-first-then-right tree yields an individual upper bound on the revenue for that specific allocation rule. By maximizing the individual upper bounds over all allocation rules, a global upper bound for the revenue is achieved. But this upper bound is assumed by the modified efficient allocation rule derived before, which yields hence an optimal mechanism.

It turns out that the described approach is doomed to fail in our setting. Especially, one cannot find any tree $B \subseteq T_f$ – as e.g. the up-first-then-right tree above – such that the allocation rule optimizing the expected total payment computed on the basis of B in turn has B as a shortest path tree. Note that the approach described above and also our approach for the 1-dimensional setting focus on one agent and the corresponding type graph. Hence any allocation rule derived by the described approach is necessarily a modified Smith's rule with modified weights that can be computed from the characteristics (type report and distribution) of the agent itself similar as in Lemma 1. Such an allocation rule satisfies the following IIA property.

Definition 7 *We say that an allocation rule f satisfies* independence of irrelevant alternatives *(IIA) if the relative order of any two jobs j_1 and j_2 is the same in the schedules $f(t_1)$ and $f(t_2)$ for any two type profiles $t_1, t_2 \in T$ that differ only in the types of agents from $J \setminus \{j_1, j_2\}$.*

In other words, the relative order of two jobs is independent of all other jobs. For the 2-d setting, this is not necessarily the case for optimal mechanisms.

Theorem 4 *The optimal allocation rule for the 2-dimensional setting does in general not satisfy IIA.*

Proof. Consider the following instance with three jobs. Job 1 has type $(1,1)$, job 2 has type $(2,2)$ and job 3 has type space $\{1.9, 2\} \times \{1, 2\}$. The probabilities for job 3's types are $\varphi_3(1.9, 1) = 0.8$, $\varphi_3(2, 2) = 0.2$ and $\varphi_3(1.9, 2) = \varphi_3(2, 1) = 0$ respectively. We will show that the best allocation rule that satisfies IIA achieves a minimum expected total payment of at least 5.6, whereas there exists an allocation rule – violating IIA – with an expected total payment of 4.88. The following argumentation would still work if we assumed small positive probabilities for types $(1.9, 2)$ and $(2, 1)$ as well, but everything would become much more technical.

There are six possible schedules for three jobs, where we denote e.g. by 312 the schedule where job 3 comes first and job 2 last. There are only two cases that occur with positive probability: job 3 has type $(1.9, 1)$, which we refer to as case a, and job 3 has type $(2, 2)$, which we refer to as case b. An allocation rule that satisfies IIA must schedule job 1 and 2 in the same relative order in case a and b. Therefore, any such rule must either choose a schedule from $\{123, 132, 312\}$ or from $\{213, 231, 321\}$ in both cases. As an example, we compute a lower bound on the optimal payment $P^{min}(f)$ for the case where f chooses schedule 123 in case a and schedule 132 in case b. Since there is only one possible type for job 1 and 2, only individual rationality matters for the optimal payments to those jobs and hence $\pi_1^f(1, 1) = 0$ and $\pi_2^f(2, 2) = 2(0.8 \cdot 1 + 0.2 \cdot (1 + 2)) = 2.8$. For job 3, we take individual rationality into account as well as the incentive constraint $\pi_3^f(1.9, 1) - 1.9 \cdot ES_3(1.9, 1) \geq \pi_3^f(2, 2) - 1.9 \cdot ES_3(2, 2)$. While individual rationality requires $\pi_3^f(1.9, 1) \geq 1.9 \cdot 3 = 5.7$ and $\pi_3^f(2, 2) \geq 2$, the latter is equivalent to $\pi_3^f(1.9, 1) \geq \pi_3^f(2, 2) + 3.8$. Therefore, $\pi_3^f(2, 2) \geq 2$ and $\pi_3^f(1.9, 1) \geq 5.8$. Hence $P^{min}(f) \geq 2.8 + 0.8 \cdot 5.8 + 0.2 \cdot 2 = 7.84$. Note that this is only a lower bound, since for the exact value of $P^{min}(f)$, we must additionally consider the incentive constraints that result from the two types $(1.9, 2)$ and $(2, 1)$, which have zero probability, but are in the type space of job 3.

In total, there are 18 allocation rules that satisfy IIA. We list the corresponding lower bounds (LB) on $P^{min}(f)$ in the following table.

$f(a)$	$f(b)$	π_1^f	π_2^f	LB $\pi_3^f(1.9, 1)$	LB $\pi_3^f(2, 2)$	LB $P^{min}(f)$
123	123	0	2	6	6	8
123	132	0	2.8	5.8	2	7.84
123	312	0.4	2.8	5.7	0	7.76
132	123	0	3.6	2.2	6	6.56
132	132	0	4.4	2	2	6.4
132	312	0.4	4.4	1.9	0	6.32
312	123	0.8	3.6	0.3	6	5.84
312	132	0.8	4.4	0.1	2	5.68
312	312	1.2	4.4	0	0	5.6
123	123	2	0	6	6	8
123	123	2.4	0	5.9	4	7.92
123	123	2.4	0.8	5.7	0	7.76
123	123	2.8	0	4.1	6	7.28
123	123	3.2	0	4	4	7.2
123	123	3.2	0.8	3.8	0	7.04
123	123	2.8	1.6	0.3	6	5.84
123	123	3.2	1.6	0.2	4	5.76
123	123	3.2	2.4	0	0	5.6

Hence, 5.6 is a lower bound for the expected total payment made by any IIA mechanism. On the other hand, regard the allocation rule that chooses schedule 132 in case a and schedule 231 in case b. We extend the allocation rule to the zero probability type such that it chooses schedule 132 for type $(2, 1)$ and schedule 231 for type $(1.9, 2)$. Clearly, this allocation rule violates IIA. The optimal payments to job 1 and 2 are $\pi_1^f(1, 1) = 0.8$ and $\pi_2^f(2, 2) = 1.6$ respectively. For the optimal payment to job 3, we depict the type graph with associated arc lengths in Figure 3. The shortest path lengths from $(1.9, 1)$ and $(2, 2)$ to the dummy node are -2.1 and -4, respectively. Hence, $\pi_3^f(1.9, 1) = 2.1$ and $\pi_3^f(2, 2) = 4$. Consequently, $P^{min}(f) = 0.8 + 1.6 + 0.8 \cdot 2.1 + 0.2 \cdot 4 = 4.88$. This proves the claim. □

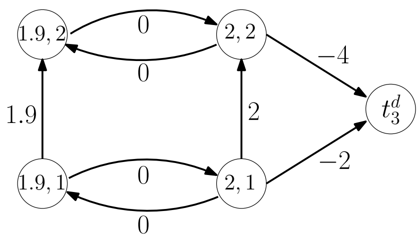


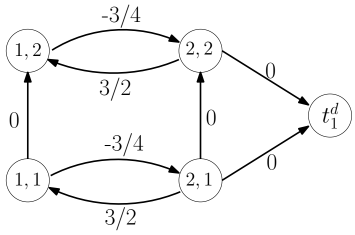
Figure 3: type graph job 3

Theorem 4 shows that any kind of priority based algorithm or list scheduling algorithm where the priority of a job can be computed from the characteristics of the job itself cannot be optimal in general. Moreover, the type graph approach must fail, since it focusses on a single agent. Hence, optimal mechanism design for our 2-dimensional setting is considerably more complicated than for the 1-dimensional setting and for traditional auction settings as described in Myerson (1981) and Malakhov and Vohra (2007). One explanation for this complication may lie in the fact that the 2-d setting considered here in fact entails informational externalities, as opposed to the auction setting in (Malakhov and Vohra 2007). On the other hand, the informational externalities introduced by private processing times are not the only cause for complications in the 2-dimensional setting: Consider the 1-dimensional setting, where only the processing times are private, but the weights are public information. It turns out that all allocation rules are implementable, even when we allow that jobs understate their processing times. The optimal payment to a job j that reports processing time p_j^k is equal to $w_j ES_j(f, p_j^k)$, and therefore the total payment to jobs for allocation rule f is equal to $P^{min}(f) = \sum_{j \in J} \sum_{k=1}^{q_j} \varphi_j(p_j^k) w_j ES_j(f, p_j^k)$. This is minimized by Smith's rule.

When there are only two agents present, then IIA is trivially satisfied. Recall that in the 1-dimensional case the optimal mechanism is efficient for symmetric agents and regular distributions and that the uniform distribution is regular. This is contrasted by the following theorem.

Theorem 5 *Even for two symmetric agents,* 2×2*-type spaces and uniform probability distributions, the optimal mechanism is not efficient.*

Proof. Consider the following example with two jobs, $W_1 = W_2 = \{1, 2\}$ and $P_1 = P_2 = \{1, 2\}$. We assume that $\varphi_1(i, k) = \varphi_2(i, k) = \frac{1}{4}$ for $i, k \in \{1, 2\}$. On one hand, consider the efficient allocation rule f_e, which schedules the job with higher weight over processing time ratio first. On the other hand, regard the so-called w-rule, f_w, that schedules the job with



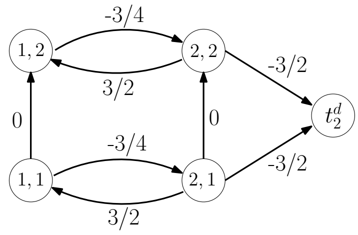


Figure 4: type graphs for the w-rule for jobs 1 and 2

the higher weight first. In case of ties, both rules schedule job 1 first. The expected start times are listed below.

$$ES_1(f_w,1,1) = ES_1(f_w,1,2) = 3/4 \qquad ES_2(f_w,1,1) = ES_2(f_w,1,2) = 3/2$$
$$ES_1(f_w,2,1) = ES_1(f_w,2,2) = 0 \qquad ES_2(f_w,2,1) = ES_2(f_w,2,2) = 3/4$$

$$ES_1(f_e,1,1) = ES_1(f_e,2,2) = 1/4, \qquad ES_2(f_e,1,1) = ES_2(f_e,2,2) = 1,$$
$$ES_1(f_e,1,2) = 1, \qquad ES_2(f_e,1,2) = 3/2,$$
$$ES_1(f_e,2,1) = 0, \qquad ES_2(f_e,2,1) = 1/4.$$

The type graphs corresponding to f_w for job 1 and 2 respectively are shown in Figure 4. From this, the optimal payments can be computed as:

$$\pi_1^{f_w}(2,1) = \pi_1^{f_w}(2,2) = 0, \qquad \pi_2^{f_w}(2,1) = \pi_2^{f_w}(2,2) = 3/2,$$
$$\pi_1^{f_w}(1,1) = \pi_1^{f_w}(1,2) = 3/4, \qquad \pi_2^{f_w}(1,1) = \pi_2^{f_w}(1,2) = 9/4.$$

Hence the (minimum) total expected payment for the w-rule is:

$$P^{min}(f_w) = \frac{1}{4}\sum_j \sum_{(i,k)} \pi_j^{f_w}(i,k) = 9/4.$$

The type graphs corresponding to f_e for agent 1 and 2 respectively are shown in Figure 5. From this, the node potentials that minimize payment can be computed as:

Figure 5: type graphs for the efficient rule for job 1 and 2

$$\begin{aligned}\pi_1^{f_e}(1,1) &= \pi_1^{f_e}(2,2) = 1/2,\\ \pi_1^{f_e}(2,1) &= 0\\ \pi_1^{f_e}(1,2) &= 5/4,\end{aligned}$$

$$\begin{aligned}\pi_2^{f_e}(1,1) &= \pi_2^{f_e}(2,2) = 2,\\ \pi_2^{f_e}(1,2) &= 5/2,\\ \pi_2^{f_e}(2,1) &= 1/2.\end{aligned}$$

Hence the (minimum) total expected payment in the efficient rule is:

$$P^{min}(f_e) = \frac{1}{4}\sum_j \sum_{(i,k)} \pi^j(i,k) = 37/16.$$

Hence, $P^{min}(f_e) > P^{min}(f_w)$. This is even true if we break ties randomly. Thus, the efficient allocation is for some instances dominated by at least the w-rule and consequently does not correspond to the optimal mechanism even in the most symmetric case possible in this setting. □

5 Optimal Mechanisms for the Continuous Setting

For this section, we impose the following changes on the discrete setting described in the previous sections. For every job j, let the weight w_j be a continuous random variable with publicly known support $[m_j, M_j]$, probability density function φ_j, and cumulative distribution function Φ_j. Probability distributions are assumed to be independent between jobs. We will prove some results for general probability distributions and others for uniform distribution of weights. The latter has $\Phi_j(x) = (x - m_j)/(M_j - m_j)$ and $\varphi_j(x) = 1/(M_j - m_j)$ for all $j \in J$ and $x \in [m_j, M_j]$. Again, the actual weight is private information of an agent, whereas the processing time p_j of an agent j is fixed and common knowledge. We will refer to the definitions of Section 2, unless we give a new definition here.

In the following, we show that the characterization of Bayes-Nash implementable allocation rules from the previous section also applies to the continuous case. In addition, revenue equivalence holds. We show that Smith's rule with respect to certain modified weights and payments computed from the network approach is again an optimal mechanism under regularity. If the regularity condition is satisfied and agents are symmetric, then this mechanism is efficient, as before. The regularity condition is satisfied for instance by the uniform distribution. If $m_j = 0$ for $j = 1, \ldots, n$ and if the weights of all agents are distributed uniformly over their respective (not necessarily equal) intervals $[0, M_j]$, then this optimal mechanism is even efficient if the processing times differ among agents.

Hartline and Karlin (2007) discuss optimal mechanism design for a similar setting as the continuous setting at hand. They derive optimal mechanisms subject to dominant strategy implementability and thus mechanisms that are optimal in a more restricted class of mechanisms. The allocation rule of the optimal mechanism that we derive turns out to be dominant strategy implementable as well, but is optimal within the larger class of Bayes-Nash incentive compatible mechanisms. Strictly speaking, our results are therefore not implied by the results in Hartline and Karlin (2007). On the other hand, looking at the techniques described in Hartline and Karlin (2007), our optimal mechanism could be derived using these techniques, too. Although our optimal payments and regularity conditions differ from those in Hartline and Karlin (2007), these differences are completely due to the fact that in our case agents are paid by the mechanism and therefore individual rationality requires adding different constants.

5.1 Bayes-Nash Implementability and Revenue Equivalence

We make use of the type graph as before. Note that for continuous distribution of weights, the type graph has uncountably many nodes. We do not introduce an extra dummy node here, but we will account for individual rationality explicitly when deriving optimal mechanisms. In the continuous case, the following holds:

Theorem 6 *An allocation rule f is Bayes-Nash incentive compatible in the continuous setting if and only if it satisfies monotonicity.*

The proof of Theorem 6 is almost identical to the proof of Theorem 1. Note that even in an infinite type graph we only need to consider finite cycles. We do not repeat the proof here.

Theorem 7 *In the continuous setting, every Bayes-Nash implementable allocation rule f satisfies revenue equivalence.*

Proof. We use the characterization of revenue equivalence given in Heydenreich, Müller, Uetz, and Vohra (2008). Fix a Bayes-Nash implementable allocation rule f and agent j and consider the type graph T_f. Let $w, z \in [m_j, M_j]$ be two types of agent j. Using the same notation as before, we derive the following for the distance from w to z.

$$\begin{aligned}
dist(w,z) &= \inf_{(w=a_0,\dots,a_{k_{\mathfrak{p}}}=z)\in\mathcal{P}(w,z)} \sum_{i=0}^{k_{\mathfrak{p}}-1} \ell_{a_i a_{i+1}} \\
&= \inf_{(w=a_0<\dots<a_{k_{\mathfrak{p}}}=z)\in\mathcal{P}(w,z)} \sum_{i=0}^{k_{\mathfrak{p}}-1} \ell_{a_i a_{i+1}} \\
&= \inf_{(w=a_0<\dots<a_{k_{\mathfrak{p}}}=z)\in\mathcal{P}(w,z)} \sum_{i=0}^{k_{\mathfrak{p}}-1} a_i[ES_j(f,a_{i+1}) - ES_j(f,a_i)] \\
&= \inf_{(w=a_0<\dots<a_{k_{\mathfrak{p}}}=z)\in\mathcal{P}(w,z)} \left(-wES_j(f,w) + zES_j(f,z) + \sum_{i=2}^{k_{\mathfrak{p}}}(a_{i-1}-a_i)ES_j(f,a_i)\right) \\
&= -wES_j(f,w) + zES_j(f,z) - \int_w^z ES_j(f,x)dx.
\end{aligned}$$

Here, we use decomposition monotonicity and the nonnegative two-cycle property for the second equality. The last equality follows from decomposition monotonicity and the fact that $ES_j(f,\cdot)$ is a non-increasing function and therefore Riemann integrable. Similarly, we get

$$dist(z,w) = wES_j(f,w) - zES_j(f,z) - \int_z^w ES_j(f,x)dx,$$

and therefore $dist(w,z) = -dist(z,w)$. According to Theorem **??**, f satisfies revenue equivalence. □

5.2 Optimal Mechanisms

As in the discrete case, we design a mechanism which assigns the payments to agents only on the basis of their reports, no matter what the announced types of the other agents are and no matter how therefore the actual allocation looks like. The goal is to minimize the expected total payment made to jobs.

The following lemma gives payments that minimize the expected total payment made to jobs for a given allocation rule.

Lemma 3 *For a Bayes-Nash implementable allocation rule f, the payment scheme*

$$\pi_j^f(w_j) = w_j ES_j(f, w_j) + \int_{w_j}^{M_j} ES_j(f, x)\, dx \text{ for } j \in J,\ w_j \in [m_j, M_j]$$

is incentive compatible, individual rational and minimizes the expected total payment made to agents. The expected total payment is then given by

$$P^{min}(f) = \sum_{j \in J} \int_W S_j(f(w)) \overline{w}_j \varphi(w) dw,$$

where the modified weights $\overline{w}_j$ are defined as

$$\overline{w}_j := w_j + \frac{\Phi_j(w_j)}{\varphi_j(w_j)} \text{ for } w_j \in [m_j, M_j].$$

Proof. The given payment scheme is equal to

$$\pi_j^f(w_j) = -dist(w_j, M_j) + M_j ES_j(f, M_j) \text{ for } j \in J,\ w_j \in [m_j, M_j].$$

Similar to the previous section, it can easily be checked that this payment scheme satisfies the incentive constraints. For any allocation rule f, the expected payment to any agent $j \in J$ is fixed up to a constant due to Theorem 7. The constant must be chosen high enough such that individual rationality is satisfied, but also low enough, such that the total expected payment is minimized. Observe that the expected utility for type M_j is equal to $-M_j ES(f, M_j) + E\pi_j^f(M_j) = 0$, therefore adding a negative constant would violate individual rationality at type M_j. On the other hand, for any type $w_j \in [m_j, M_j]$, the expected utility is equal to $-w_j ES(f, w_j) + E\pi_j^f(w_j) = \int_{w_j}^{M_j} ES_j(f, x)\, dx \geq 0$, thus individual rationality is satisfied. Hence, adding a positive constant would make the expected payment non-minimum. Consequently, the above payment scheme is incentive compatible, individual rational and minimizes the expected payment for every type of every job. Hence, it also minimizes the expected total payment to agents.
Next, we derive an expression for the expected total payment.

$$\begin{aligned} P^{min}(f) &= \sum_{j \in J} \int_{m_j}^{M_j} \varphi_j(w_j) \pi_j^f(w_j)\, dw_j \\ &= \sum_{j \in J} \int_{m_j}^{M_j} \varphi_j(w_j) \left(-dist(w_j, M_j) + M_j ES_j(f, M_j)\right) dw_j \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in J}\int_{m_j}^{M_j}\varphi_j(w_j)\left(w_jES_j(f,w_j)+\int_{w_j}^{M_j}ES_j(f,x)\,dx\right)dw_j\\
&= \sum_{j\in J}\int_{m_j}^{M_j} w_jES_j(f,w_j)\varphi_j(w_j)\,dw_j+\sum_{j\in J}\int_{m_j}^{M_j}\int_{w_j}^{M_j}ES_j(f,x)\varphi_j(w_j)\,dx\,dw_j
\end{aligned}
$$

Recall that $S_j(f(w_j, w_{-j}))$ denotes the start time of job j, when other jobs report w_{-j} and that $W = \Pi_{i=1}^n[m_j, M_j]$. The summands of the first sum can be written as

$$
\begin{aligned}
&\int_{m_j}^{M_j} w_jES_j(f,w_j)\varphi_j(w_j)\,dw_j\\
&= \int_W w_jS_j(f(w_j,w_{-j}))\varphi_1(w_1)\ldots\varphi_j(w_j)\cdots\varphi_n(w_n)dw_1\cdots dw_j\ldots dw_n\\
&= \int_W w_jS_j(f(w))\varphi(w)dw,
\end{aligned}
$$

where $\varphi(\cdot)$ is the joint distribution function of all agents. The summands in the second sum can be rewritten as follows

$$
\begin{aligned}
&\int_{m_j}^{M_j}\int_{w_j}^{M_j} ES_j(f,x)\varphi_j(w_j)\,dx\,dw_j\\
&= \int_{m_j}^{M_j}\int_{m_j}^{x} ES_j(f,x)\varphi_j(w_j)\,dw_j\,dx\\
&= \int_{m_j}^{M_j} ES_j(f,x)\Phi_j(x)dx\\
&= \int_{m_j}^{M_j} ES_j(f,w_j)\Phi_j(w_j)dw_j\\
&= \int_W S_j(f(w_j,w_{-j}))\Phi_j(w_j)\varphi_1(w_1)\ldots\varphi_{j-1}(w_{j-1})\varphi_{j+1}(w_{j+1})\ldots\varphi_n(w_n)dw_1\ldots dw_j\ldots dw_n\\
&= \int_W S_j(w)\frac{\Phi_j(w_j)}{\varphi_j(w_j)}\varphi(w)dw.
\end{aligned}
$$

Hence, we get for the total payment

$$
\begin{aligned}
P^{min}(f) &= \sum_{j\in J}\int_W S_j(f(w))\left(w_j+\frac{\Phi_j(w_j)}{\varphi_j(w_j)}\right)\varphi(w)dw\\
&= \sum_{j\in J}\int_W S_j(f(w))\overline{w}_j\varphi(w)dw,
\end{aligned}
$$

where $\overline{w}_j := w_j + \Phi_j(w_j)/\varphi_j(w_j)$ defines the modified weight for job j. □

As in the discrete case, $P^{min}(f)$ can be minimized for arbitrary distributions of weights by applying Smith's rule with respect to the modified weights. The resulting mechanism will be Bayes-Nash incentive compatible if the following regularity condition holds.

Definition 8 *The regularity condition holds in the continuous case if for $j \in J$ and $w, z \in [m_j, M_j]$, $w < z$:*

$$w + \frac{\Phi_j(w)}{\varphi_j(w)} \leq z + \frac{\Phi_j(z)}{\varphi_j(z)}.$$

We get the following result.

Theorem 8 *Let the modified weights and the payment scheme π^f be defined as in Lemma 3. Let f be the allocation rule that schedules jobs in order of non-increasing ratios $\overline{w}_j/p_j$. If regularity holds, then (f, π^f) is an optimal mechanism.*

Proof. As mentioned above, Smith's rule minimizes $\sum_{j\in J} S_j(f(w))\overline{w}_j$ for every type profile $w \in W$. Therefore, it also minimizes the total expected payment. As in the discrete case, the regularity condition ensure that modified weights be non-decreasing in the original weights. As $ES_j(w_j)$ is non-increasing in the modified weight $\overline{w}_j$ under Smith's rule with respect to modified weights, it is non-increasing in the original weight w_j for every $j \in J$ if regularity holds. Hence, under regularity weak monotonicity and consequently Bayes-Nash implementability is satisfied. □

The following theorem gives two important cases, when this optimal mechanism is efficient.

Theorem 9 *The optimal mechanism is efficient in the following two cases.*
1) Agents are symmetric, i.e., have identically distributed weights and equal processing times and the regularity condition holds for the distribution functions.
2) Agents' weights are distributed uniformly over $[0, M_j]$ for $j = 1, ..., n$. Processing times can be arbitrary.

Proof. 1) Smith's rule with respect to modified weights is equivalent to Smith's rule with respect to the original weights as in the discrete case. Regularity ensures weak monotonicity and hence Bayes-Nash incentive compatibility.

2) For the uniform distribution, we get for the virtual weights

$$w + \frac{\Phi_j(w)}{\varphi_j(w)} = w + \frac{w/M_j}{1/M_j} = 2w,$$

which is increasing and linear in w and the linear relationship does not depend on the agent. Hence, Smith's rule with respect to virtual weights is equivalent to Smith's rule with respect to original weights, no matter what the processing times are. □

6 Optimal Mechanisms via Standard Auction Formats

After having derived an optimal mechanism for the continuous case, we are interested whether standard auction formats also yield optimal mechanisms for our scheduling setting. We study the VCG mechanism and a mechanism that corresponds to the first price auction.

6.1 The Generalized VCG Mechanism

Recall that for the discrete setting, the generalized VCG mechanism was not optimal, even in cases when the optimal mechanism allocates efficiently. In the continuous setting, however, revenue equivalence implies that the expected payments to agent j in all Bayes-Nash incentive compatible mechanisms that allocate efficiently are the same up to a constant. As the optimal mechanism proposed in Section 5 allocates efficiently in the case of symmetric agents and regularity, also the VCG mechanism can be used in this case to derive an optimal mechanism by adding an appropriate constant to the payments of every agent.

Theorem 10 *For symmetric agents under regularity, the VCG mechanism with payments*

$$\pi_j^{VCG}(w) = p_j \sum_{\substack{k \in J \\ \sigma_k < \sigma_j}} w_k.$$

is optimal. Here, σ denotes the efficient schedule.

Proof. Assume symmetric agents with weights identically distributed over $[m, M]$ according to density function φ_1 and cumulative distribution function Φ_1. The distributions are assumed to satisfy regularity. Without loss of generality, let the processing times be equal to one. The result already follows from revenue equivalence and the fact that under the VCG mechanism, any agent with type equal to his maximum possible type M has expected start time equal to zero and hence zero expected utility, just as in the optimal mechanism from Section 5.

Nevertheless, we check the equality of the expected payments under the VCG and the optimal mechanism explicitly for illustrative purposes. Since jobs have equal processing times, the VCG mechanism allocates in order of non-increasing weights. The payment made to an agent j under the VCG mechanism is the sum of the weights of all agents processed before j. To derive the expected payment to agent j announcing type w_j, we notice that any other agent k is scheduled before j if k's weight x is larger than w_j. In this case, x is paid to j. The expected payment at type w_j is therefore

$$E\pi_j^{VCG}(w_j) = (n-1)\int_{w_j}^{M} x\varphi_1(x)dx.$$

In the optimal mechanism proposed in Section 5, the payment for type w_j and any w_{-j} is equal to

$$\pi_j^f(w_j, w_{-j}) = w_j ES_j(f, w_j) + \int_{w_j}^{M} ES_j(f, x)dx.$$

The start time when announcing type w_j is a binomially distributed random variable with parameters $n-1$ and $1-\Phi_1(w_j)$, as the placement of any of the $n-1$ other jobs in front of j can be seen as a binomial trial with success probability $1-\Phi_1(w_j)$. The start time counts the number of "successes". Therefore, $ES_j(f, w_j) = (n-1)(1-\Phi_1(w_j))$. We get for the payments

$$\begin{aligned}
E\pi_j^f(w_j) &= w_j(n-1)(1-\Phi_1(w_j)) + (n-1)\int_{w_j}^{M}(1-\Phi_1(x))dx \\
&= w_j(n-1)(1-\Phi_1(w_j)) + (n-1)\int_{w_j}^{M}\int_{x}^{M}\varphi_1(y)\,dy\,dx \\
&= w_j(n-1)(1-\Phi_1(w_j)) + (n-1)\int_{w_j}^{M}\int_{w_j}^{y}\varphi_1(y)\,dx\,dy \\
&= w_j(n-1)(1-\Phi_1(w_j)) + (n-1)\int_{w_j}^{M}\varphi_1(y)(y-w_j)\,dy \\
&= w_j(n-1)(1-\Phi_1(w_j)) - (n-1)w_j(1-\Phi_1(w_j)) + (n-1)\int_{w_j}^{M} y\varphi_1(y)\,dy \\
&= (n-1)\int_{w_j}^{M} x\varphi_1(x)\,dx \\
&= E\pi_j^{VCG}(w_j).
\end{aligned}$$

Hence, $E\pi_j^f(w_j) = E\pi_j^{VCG}(w_j)$ for all $j \in J$ and all types w_j. Therefore, the total expected payments of the optimal and the VCG mechanism are equal, too. Hence, the VCG mechanism is optimal. □

Remarkably, the payment under π^f depends only on the reported type of an agent and is constant over all reports of the other agents' and therefore over all allocations. In contrast, π^{VCG} depends only on the allocation and not on the specific report of the agent. Nevertheless, both yield the same expected payments.

6.2 The First-Price Equivalent

In the first price auction, the highest bidder wins the object and has to pay the amount of his bid. In this auction, truthful reporting does not necessarily maximize a bidder's expected utility. On the other hand, there is a strictly increasing and differentiable bidding function β such that bidding according to β for all agents is a Bayes-Nash equilibrium. This result can e.g. be found in Myerson (1991). Especially, for uniformly distributed valuations for the object, the bidding function β scales the true valuation down by a factor of $(n-1)/n$.

We do a similar analysis for the continuous case of our scheduling problem. For symmetric agents, we derive a strictly increasing and differentiable function β yielding a symmetric Bayes-Nash equilibrium in which all agents report according to β. From that, it is easy to construct another optimal mechanism for the continuous case. We furthermore show that for two agents with different processing times, there is no such function β.

The Mechanism for the Symmetric Case. Suppose, the jobs in J are symmetric and their weights are drawn independently and identically distributed from the interval $[0, M]$ with probability density function φ_1 and cumulative distribution function Φ_1. Suppose $\varphi_1(\cdot) > 0$ on $[0, M]$. Processing times are all equal to one. The proposed mechanism (f, π) works as follows. Schedule jobs in order of non-increasing weights and pay to each job an amount equal to his actual start time times his announced weight. A bidding function β is a function $\beta\colon [0, M] \to \mathbb{R}_+$. Recall the definition of a Bayes-Nash equilibrium.

Definition 9 *Reporting according to $\beta\colon [0, M] \to \mathbb{R}_+$ is a Bayes-Nash equilibrium if any agent j with weight w_j maximizes his expected utility by reporting $\beta(w_j)$ given that all other agents report according to β, too.*

Let us assume that there is a symmetric Bayes-Nash equilibrium in which agents report according to the same strictly increasing and differentiable bidding function β. We will first derive a functional form for β and then show that reporting according to β is a Bayes-Nash equilibrium.

Fix agent j with actual weight w_j and suppose that every other agent k with true weight w_k reports $\beta(w_k)$. Suppose, j reports some weight b_j. Then his expected utility is $(b_j - w_j)$

times his expected start time. If j bids $b_j \leq \beta(0)$ then he will get the last position with probability one and therefore has utility $(n-1)(b_j - w_j) \leq (n-1)(\beta(0) - w_j)$. This utility is maximized at $\beta(0)$ and j will never bid strictly less than $\beta(0)$. Reporting more than $\beta(M)$ leads to an expected start time of 0 for agent j and hence to an expected utility of 0. Reporting any $w_j \leq b_j \leq \beta(M)$ leads to a non-negative expected utility. Hence we can assume $b_j \leq \beta(M)$ without loss of generality. Consequently, $\beta(0) \leq b_j \leq \beta(M)$. As β is continuous and strictly increasing, we can compute $\beta^{-1}(b_j) =: \tilde{w}_j$. Scheduling in order of non-increasing reports $\beta(w_k)$ is equivalent to scheduling in order of non-increasing reports w_k, as β is increasing. Therefore, j's start time when reporting $b_j = \beta(\tilde{w}_j)$ is again a binomially distributed random variable with parameters $n-1$ and $1 - \Phi_1(\tilde{w}_j)$ and expected value

$$ES_j(f, b_j) = (n-1)(1 - \Phi_1(\tilde{w}_j)).$$

Job j's expected utility is then equal to

$$(b_j - w_j)(n-1)(1 - \Phi_1(\tilde{w}_j)) = (b_j - w_j)(n-1)(1 - \Phi_1(\beta^{-1}(b_j))).$$

Differentiating with respect to b_j yields

$$(n-1)(1 - \Phi_1(\beta^{-1}(b_j))) - (n-1)(b_j - w_j)\frac{\varphi_1(\beta^{-1}(b_j))}{\beta'(\beta^{-1}(b_j))}.$$

The expected utility should be maximized at $b_j = \beta(w_j)$. We apply the first order condition.

$$\begin{aligned} (1 - \Phi_1(w_j)) - (\beta(w_j) - w_j)\frac{\varphi_1(w_j)}{\beta'(w_j)} &= 0 \\ \Leftrightarrow \beta'(w_j)(1 - \Phi_1(w_j)) - \beta(w_j)\varphi_1(w_j) &= -w_j\varphi_1(w_j). \end{aligned}$$

This should be true for any true weight $x \in [0, M]$. Hence, we can write for $x \in [0, M]$

$$\frac{d}{dx}(\beta(x)(1 - \Phi_1(x))) = -x\varphi_1(x).$$

Integrating both sides from w_j to M yields

$$\beta(M)(1 - \Phi_1(M)) - \beta(w_j)(1 - \Phi_1(w_j)) = -\int_{w_j}^{M} x\varphi_1(x)dx$$

$$\Leftrightarrow \beta(w_j) \;=\; \frac{1}{1-\Phi_1(w_j)} \int_{w_j}^{M} x\varphi_1(x)dx.$$

The report $\beta(M)$ is obtained by taking the limit $\lim_{x\to M} \beta(x)$. Note that β is differentiable and strictly increasing if φ is strictly positive on $[0, M]$. Unlike in the first price auction, the bidding function is independent of the number of agents.

Next, we show that agents indeed maximize their expected utility by reporting according to β.

Theorem 11 *Let f be the allocation rule that schedules in order of non-increasing reported weights. Let π be such, that every agent gets a payment equal to his announced weight times his actual start time. Then, in the mechanism (f, π), reporting according to $\beta\colon [0, M] \to \mathbb{R}_+$, with*

$$\beta(w_j) = \frac{1}{1-\Phi_1(w_j)} \int_{w_j}^{M} x\varphi_1(x)dx.$$

is a Bayes-Nash equilibrium.

Proof. Fix agent j with true weight w_j and suppose that all other agents report according to β. We show that reporting $\beta(w_j)$ indeed maximizes j's expected utility. Suppose, j reports b_j. Let $Eu_j(b_j, w_j)$ be the expected utility for j when reporting b_j while having actual weight w_j. As we already have seen, there is no loss of generality in assuming $\beta(0) \le b_j \le \beta(M)$. Hence, $\beta(\tilde{w}_j) = b_j$ for some $\tilde{w}_j \in [0, M]$. The expected utility from reporting $\beta(\tilde{w}_j)$ is equal to

$$\begin{aligned}
Eu_j(b_j, w_j) &= (\beta(\tilde{w}_j) - w_j)(n-1)(1-\Phi_1(\tilde{w}_j)) \\
\Leftrightarrow \frac{1}{n-1} Eu_j(b_j, w_j) &= \int_{\tilde{w}_j}^{M} x\varphi_1(x)dx - w_j(1-\Phi_1(\tilde{w}_j)) \\
&= [x\Phi_1(x)]_{\tilde{w}_j}^{M} - \int_{\tilde{w}_j}^{M} \Phi_1(x)dx - w_j(1-\Phi_1(\tilde{w}_j)) \\
&= M - \tilde{w}_j\Phi_1(\tilde{w}_j) - \int_{\tilde{w}_j}^{M} \Phi_1(x)dx - w_j(1-\Phi_1(\tilde{w}_j)) \\
&= (M - w_j) + (w_j - \tilde{w}_j)\Phi_1(\tilde{w}_j) - \int_{\tilde{w}_j}^{M} \Phi_1(x)dx.
\end{aligned}$$

Hence,

$$\frac{1}{n-1}[Eu_j(\beta(w_j), w_j) - Eu_j(\beta(\tilde{w}_j), w_j)] = \int_{\tilde{w}_j}^{w_j} \Phi_1(x)dx - (w_j - \tilde{w}_j)\Phi_1(\tilde{w}_j) \ge 0.$$

This completes the proof. □

We give two examples of explicit bidding functions for the exponential and the uniform distribution.

Example 4 (Exponential distribution) *Let* $\Phi_1(w) = 1 - e^{-\lambda w}$ *for some* $\lambda > 0$. *The interval is* $[0, \infty)$. *Then*

$$\begin{aligned}\beta(w) &= \frac{1}{e^{-\lambda w}}\lambda \int_w^\infty x e^{-\lambda x} dx \\ &= \frac{1}{e^{-\lambda w}}\left([-xe^{-\lambda x}]_w^\infty + \int_w^\infty e^{-\lambda x} dx\right) \\ &= \frac{1}{e^{-\lambda w}}\left(we^{-\lambda w} + \left[-\frac{e^{-\lambda x}}{\lambda}\right]_w^\infty\right) \\ &= w + \frac{1}{\lambda}.\end{aligned}$$

Thus, if weights are exponentially distributed, an agent has to add the mean weight $1/\lambda$ to his actual weight in the equilibrium.

Example 5 (Uniform distribution) *Let agents' weights be uniformly distributed over* $[0, M]$. *That is* $\varphi_1(w) = 1/M$ *and* $\Phi_1(w) = w/M$. *Thus,*

$$\beta(w) = \frac{1}{1 - \frac{w}{M}} \int_w^M \frac{x}{M} dx = \frac{1}{M - w}\left[\frac{x^2}{2}\right]_w^M = \frac{M + w}{2}.$$

Taking the limit $w \to M$ *yields additionally* $\beta(M) = M$.

Hence, for uniform distributions, an agent reports the mean of his true weight and the maximum weight M.

From the above analysis, we get the following Bayes-Nash incentive compatible and optimal mechanism.

Theorem 12 *Allocating jobs in order of non-increasing reported weights and paying to job* j *with report* w_j *and realized start time* S_j *the payment* $S_j\beta(w_j)$ *is a Bayes-Nash incentive compatible and optimal mechanism.*

Proof. As β is increasing, scheduling jobs in order of non-increasing $\beta(w_j)$ is equivalent to scheduling in order of non-increasing w_j. If bidding according to β is a Bayes-Nash equilibrium in the mechanism where a job j bidding w_j is paid S_jw_j, then truthful bidding is a Bayes-Nash equilibrium in the mechanism where j is paid $S_j\beta(w_j)$. Therefore, the mechanism proposed in the theorem is Bayes-Nash incentive compatible. As the allocation is again efficient, expected payments for each type coincide up to a constant with the expected payments of the VCG mechanism and with those of the optimal mechanism described in the previous two sections. The constant is zero, as also in this mechanism, a job with maximum weight M has zero expected start time, zero payment and hence zero utility, just as in the VCG mechanism and the optimal mechanism from Section 5. □

A Negative Result for Unequal Processing Times. In the case with two agents that have unequal processing times, there is no bidding function β according to which *both* agents report in a Bayes-Nash equilibrium.

Theorem 13 *Suppose, there are two agents, whose weights are continuous random variables with equal support* $[0, M]$. *If* $p_1 \neq p_2$, *then there is no continuous bidding function* $\beta\colon [0, M] \to \mathbb{R}_+$ *according to which both agents report in a Bayes-Nash equilibrium. That is, there is no symmetric Bayes-Nash equilibrium.*

Note that the theorem holds for arbitrary continuous random weights with support $[0, M]$. Especially, we do not need to assume that weights are identically distributed.

Proof. Without loss of generality let $p_2 < p_1$. Assume β is a continuous equilibrium bidding function. Let agent 2 bid according to β and look at agent 1. Let

$$b_1 = \min\left\{\left(\frac{1}{2} + \frac{p_1}{2p_2}\right)\beta(0), \beta(M)\right\},$$

then $b_1 \in [\beta(0), \beta(M)]$. There exists $w_1 \in [0, M]$ with $\beta(w_1) = b_1$, as β is continuous. Then

$$\begin{aligned}
& \beta(w_1) = b_1 \leq \left(\frac{1}{2} + \frac{p_1}{2p_2}\right)\beta(0) \\
\Leftrightarrow\quad & p_2\beta(w_1) \leq \left(\frac{p_2}{2} + \frac{p_1}{2}\right)\beta(0) < p_1\beta(0) \\
\Leftrightarrow\quad & \frac{\beta(w_1)}{p_1} < \frac{\beta(0)}{p_2}.
\end{aligned}$$

Bidding any b with $b/p_1 \leq \beta(0)/p_2$ results in an expected start time of p_2 for agent 1. The expected utility is then $(b - w_1)p_2$ which is strictly larger at $b = (p_1/p_2)\beta(0)$ than at $b = \beta(w_1) < (p_1/p_2)\beta(0)$. Thus, β does not maximize the expected utility at w_1. □

7 Discussion

We have seen that the graph theoretic approach is an intuitive tool for optimal mechanism design and yields a closed formula for the optimal mechanism in the 1-dimensional case. The results parallel Myerson's results for single item auctions; although there are differences. It is not hard to see that the optimal allocation rule – Smith's rule with respect to modified weights – is even dominant strategy implementable, with the same total expected payment for the mechanism. In order to obtain a dominant strategy incentive compatible mechanism, only the payment scheme has to be defined appropriately for each reported type profile.

In the discrete case, efficient mechanisms can be arbitrarily bad with respect to the total payment made to agents. For symmetric agents, however, the optimal mechanism is efficient. Even so, the payments of the generalized VCG mechanism can still be non-optimal. In the continuous case, revenue equivalence holds and the generalized VCG mechanism as well as a mechanism derived from the first price auction are optimal in those cases where the optimal mechanisms allocates efficiently.

Moreover, we have seen that in the two-dimensional case the canonical approach does not work and that optimal mechanism design seems to be considerably more complicated than in the traditional auction models. We leave it as an open problem to identify (closed formulae for) optimal mechanisms for the 2-d case. It is conceivable, however, that closed formulae don't exist.

References

Armstrong, M. (2000). Optimal multi-object auctions. *Review of Economic Studies 67*, 455–481.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public Choice 11*(1), 17–33.

Groves, T. (1973). Incentives in teams. *Econometrica 41*, 617–631.

Hartline, J. and A. Karlin (2007). Profit maximization in mechanism design. In N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani (Eds.), *Algorithmic Game Theory*. Cambridge University Press.

Heydenreich, B., R. Müller, M. Uetz, and R. Vohra (2008). Characterization of revenue equivalence. *Econometrica*. to appear.

Krishna, V. (2002). *Auction Theory*. Academic Press.

Malakhov, A. and R. Vohra (2007). An optimal auction for capacity constrained bidders: a network perspective. *Economic Theory*.

Mitra, M. (2001). Mechanism design in queueing problems. *Economic Theory 17*(2), 277–305.

Moulin, H. (2007). On scheduling fees to prevent merging, splitting, and transferring of jobs. *Mathematics of Operations Research 32*, 266–283.

Müller, R., A. Perea, and S. Wolf (2007). Weak monotonicity and Bayes-Nash incentive compatibility. *Games and Economic Behavior 61*(2), 344–358.

Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research 6*(1), 58–73.

Myerson, R. (1991). *Game Theory: Analysis of Conflict*. Harvard University Press.

Rochet, J.-C. (1987). A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics 16*(2), 191–200.

Smith, W. (1956). Various optimizers for single stage production. *Naval Research Logistics Quarterly 3*, 59–66.

Vickrey, W. (1961). Counterspeculation, auctions and competitive sealed tenders. *Journal of Finance 16*, 8–37.