Bettina Klaus

The Role of Replication-Invariance: Two Answers Concerning the Problem of Fair Division when Preferences are Single-Peaked

RM/07/029

JEL code: D63, D71

Maastricht research school of **E**conomics
of **TE**chnology and **OR**ganizations

Universiteit Maastricht
Faculty of Economics and Business Administration
P.O. Box 616
NL - 6200 MD Maastricht

phone : ++31 43 388 3830
fax : ++31 43 388 4873

The Role of Replication-Invariance: Two Answers Concerning the Problem of Fair Division when Preferences are Single-Peaked*

Bettina Klaus†

July 2007

Abstract

We consider the problem of allocating an infinitely divisible commodity among a group of agents with single-peaked preferences. A rule that has played a central role in the previous analysis of the problem is the so-called uniform rule. Thomson (1995a) proved that the uniform rule is the only rule satisfying *Pareto optimality*, *no-envy*, *one-sided population-monotonicity*, and *replication-invariance*. Replacing *one-sided population-monotonicity* by *one-sided replacement-domination* yields another characterization of the uniform rule (Thomson, 1997a). Until now, the independence of *replication-invariance* from the other properties in these characterizations was an open problem. In this note we prove this independence by means of a single example.

Keywords: Fair allocation, single-peaked preferences, population-monotonicity, replacement-domination, replication-invariance.

JEL classification: D63, D71.

1 Introduction

We consider the division of some perfectly divisible commodity among a group of agents with single-peaked preferences. This means that each agent has a most preferred amount below which and above which his welfare is decreasing. A typical example is rationing in a two-good exchange economy when prices are in disequilibrium (*e.g.,* Benassy, 1982): if the preferences of the agents over the two-dimensional space of bundles are strictly convex, then the restrictions of these preferences to the budget lines are single-peaked. In this context Benassy (1982) considered the uniform rationing scheme. For the more general class of division problems with single-peaked preferences, this solution is known as the uniform rule. Sprumont (1991) initiated the axiomatic analysis of this class of problems and proved that the uniform rule is the only rule that satisfies *Pareto optimality*, *no-envy*, and *strategy-proofness*. Since then a wide literature has been concerned with the search for and analysis of rules with appealing properties. We refer the interested reader to a survey of fair allocation when preferences are single-peaked by Thomson (1997b).

*The first version of this note was written while the author was visiting the University of Rochester in 1997. I acknowledge the hospitality of the Department of Economics at the University of Rochester and wish to thank William Thomson for many helpful comments. Furthermore, I thank the Netherlands Organisation for Scientific Research (NWO) for its support under grant VIDI-452-06-013.

†Department of Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands; e-mail: B.Klaus@algec.unimaas.nl

In this note we answer some questions asked by William Thomson (1995a,1997a). Thomson (1995a) proves that the uniform rule is the only rule satisfying *Pareto optimality*, *no-envy*, *replication-invariance*, and *one-sided population-monotonicity*. Replacing *one-sided population-monotonicity* by *one-sided replacement-domination* yields another characterization of the uniform rule (Thomson, 1997a). Until now, the independence of *replication-invariance* from the other properties in the characterizations mentioned above was an open problem. In this note we prove this independence by means of a single rule that satisfies the properties named in the characterizations but not *replication-invariance*.

2 The Model

In this section we introduce the problem of fair division when preferences are single-peaked and the properties for rules that will play a central role in this paper.

There is an infinite population of potential agents, indexed by the natural numbers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous and single-peaked preference relation R_i defined over the non-negative real numbers $\mathbb{R}_+$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}_+$, called *agent i's peak amount*, with the following property: for all $x, y \in \mathbb{R}_+$ with $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, we have $y \, P_i \, x$.[1] Each preference relation R_i can be described in terms of the *indifference function* $r_i\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ that is defined as follows. If $x \leq p(R_i)$, then $r_i(x) \geq p(R_i)$ and either $r_i(x) \, I_i \, x$ (if such a point exists) or $r_i(x) = \infty$. If $x \geq p(R_i)$, then $r_i(x) \leq p(R_i)$ and $r_i(x) \, I_i \, x$ (if such a point exists) or $r_i(x) = 0$.

By $\mathcal{R}$ we denote the class of all continuous, single-peaked preference relations over $\mathbb{R}_+$ and by $\mathcal{R}_b \subsetneq \mathcal{R}$ the subclass of preferences $R_i \in \mathcal{R}_b$ such that the corresponding indifference function r_i is *b*ounded, *i.e.*, $r_i(0) < \infty$. By $\mathcal{N}$ we denote the class of non-empty and finite subsets of $\mathbb{N}$. For $N \in \mathcal{N}$, $\mathcal{R}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$; $\mathcal{R}_b^N$ has a similar meaning.

Now, an economy can be formalized as follows. Let $\Omega \in \mathbb{R}_+$ be the amount of an infinitely divisible commodity, the *(social) endowment,* that has to be distributed among a group of agents $N \in \mathcal{N}$ with profile $R \in \mathcal{R}^N$.[2] We call a pair $e = (R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+$ an *economy.* Let $\mathcal{E}^N = \mathcal{R}^N \times \mathbb{R}_+$ be the class of all economies for $N \in \mathcal{N}$. Similarly, let $\mathcal{E}_b^N = \mathcal{R}_b^N \times \mathbb{R}_+$. A *feasible allocation* for $e = (R, \Omega) \in \mathcal{E}^N$ is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \Omega$. A *(allocation) rule* is a function φ that assigns to every $N \in \mathcal{N}$ and every $e \in \mathcal{E}^N$ a feasible allocation, denoted $\varphi(e)$. Given $i \in N$, we call $\varphi_i(e)$ the *allotment* of agent i.

A standard requirement is *Pareto optimality*: an allocation assigned by the rule cannot be changed in such a way that no agent is worse off and some agent is better off.

Pareto optimality: For all $N \in \mathcal{N}$ and all $e \in \mathcal{E}^N$, there is no feasible allocation $x \in \mathbb{R}_+^N$ such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$ and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

By the single-peakedness of the preferences it is easy to show that a rule is *Pareto optimal* if and only if it is *same-sided*, that is: for all $N \in \mathcal{N}$ and all $e \in \mathcal{E}^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$] or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$]. Sprumont (1991) uses same-sidedness as definition of *Pareto optimality*.

[1] P_i denotes the strict preference relation associated with R_i and I_i the indifference relation.

[2] Note that free disposal of the commodity is not allowed.

The following well-known property of *no-envy* states that no agent strictly prefers the allotment of another agent to his own allotment.

No-envy: For all $N \in \mathcal{N}$, all $e \in \mathcal{E}^N$, and all $i, j \in N$, $\varphi_i(e)\, R_i\, \varphi_j(e)$.

Next, we discuss so-called "solidarity properties" that describe the effect of certain changes in a single parameter of the economy while the other parameters are kept fixed, *e.g.,* the population or the preferences.

Thomson (1995a,1997a) shows that these properties are generally incompatible with the properties we introduced above. However, he shows that these incompatibilities only occur when the change in the parameter is such that it turns an economy in which there is "too much" to divide into an economy in which there is "too little" to divide, or *vice versa*. We call a change where this does not occur *one-sided:* let $N, \bar{N} \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$. If either $[\sum_N p(R_i) \geq \Omega$ and $\sum_{\bar{N}} p(\bar{R}_i) \geq \bar{\Omega}]$ or $[\sum_N p(R_i) \leq \Omega$ and $\sum_{\bar{N}} p(\bar{R}_i) \leq \bar{\Omega}]$, then $\bar{e}$ is a *one-sided change of* e.

Thomson shows that a wide class of rules satisfies one or even several "one-sided" versions of the solidarity properties, *i.e.,* solidarity is only required for one-sided changes.

First, we consider arrivals of new agents such that the implied change in the initial economy is one-sided, keeping the preferences of the remaining agents and the endowment fixed (Thomson, 1995a). *One-sided population-monotonicity* states that either all agents initially present (weakly) lose or all (weakly) gain.[3]

Let $N, M \in \mathcal{N}$, $N \subseteq M$, and $R \in \mathcal{R}^M$. Then, the *restriction* $(R_i)_{i \in N} \in \mathcal{R}^N$ *of* R *to* N is denoted by R_N.

One-sided population-monotonicity: For all $N, \bar{N} \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^{\bar{N}}$, if $N \subseteq \bar{N}$, $R = \bar{R}_N$, and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

Next, we consider a one-sided change of one agent's preference relation, keeping the preferences of the remaining agents, the set of agents, and the endowment fixed (Thomson, 1997a). *Welfare-domination under preference-replacement*, or *replacement-domination* for short, states that either all remaining agents (weakly) lose or all (weakly) gain.[4]

For $N, M \in \mathcal{N}$ with $N \subseteq M$ let $M \backslash N := \{i \in M \mid i \notin N\}$. Let $N \in \mathcal{N}$, $R, \bar{R} \in \mathcal{R}^N$, and $j \in N$. If $R_{N \backslash \{j\}} = \bar{R}_{N \backslash \{j\}}$ and $R_j \neq \bar{R}_j$, then we call $\bar{R}$ a *j-deviation from* R.

One-sided replacement-domination: For all $N \in \mathcal{N}$, all $e = (R, \Omega)$, $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $\bar{e}$ is a one-sided change of e, then either [for all $i \in N \backslash \{j\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N \backslash \{j\}$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

As a last property we introduce *replication-invariance* (Thomson, 1995a,1997a). *Replication-invariance* states that if an economy is replicated, *i.e.,* the amount to divide and the preference profile are replicated, then the replica of the allocation assigned by the rule for the initial economy equals the allocation assigned by the rule for the replicated economy.

[3]For a survey on *population-monotonicity* we refer to Thomson (1995b).

[4]*Replacement-domination* has been studied in a variety of settings and we refer the interested reader to a recent review of the literature by Thomson (1999).

Since *replication-invariance* is a well-known property and its formal description is somewhat cumbersome, for a formal statement we refer to Thomson (1995a,1997a). We call an economy $\bar{e}$, obtained by replication of an economy e, a *replica of* e. Similarly, we call an allocation $\bar{x}$ obtained by replication of an allocation x, a *replica of* x.

Replication-invariance: For all $N, \bar{N} \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$, if $\bar{e}$ is a replica of e, then $\varphi(\bar{e})$ is a replica of $\varphi(e)$.

3 Replication-Invariance and the Uniform Rule

In this section we first introduce the uniform rule and the characterizations of this rule that lead to the question whether *replication-invariance* in these characterizations is independent from the other properties.

The uniform rule

The following rule, known as the uniform rule, has played a central role in the literature of fair division when preferences are single-peaked.

Uniform rule *U*: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all $j \in N$,

$$U_j(e) := \begin{cases} \min\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \geq \Omega, \\ \max\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \leq \Omega, \end{cases}$$

where λ solves $\sum_N U_i(e) = \Omega$.

So, in case of *excess demand*, *i.e.*, $\sum_N p(R_i) \geq \Omega$, each agent either receives his peak amount or his allotment is greater than or equal to the allotment of each other agent. Similarly, in case of *excess supply*, *i.e.*, $\sum_N p(R_i) \leq \Omega$, each agent either receives his peak amount or his allotment is smaller than or equal to the allotment of each other agent.

Another interpretation of this rule is the following "Walrasian interpretation". All agents are asked to maximize their preferences subject to a common upper or lower "budget bound" that is chosen such that feasibility is obtained for this list of maximizers.[5]

Two characterizations of the uniform rule and two questions

Thomson establishes the following characterization of the uniform rule (Thomson, 1995a, Theorem 4).

Theorem 1 (***Thomson, 1995a***).
On the domain $\bigcup_{N\in\mathcal{N}} \mathcal{E}_b^N$, the uniform rule is the only rule that satisfies Pareto optimality, no-envy, one-sided population-monotonicity, and replication-invariance.

Question 1: Following the proof of this Theorem, Thomson asks whether *replication-invariance* is independent from the other characterizing properties (see Thomson, 1995a,

[5] I wish to thank William Thomson for bringing this "Walrasian interpretation"of the uniform rule to my attention.

page 242).[6]

The next characterization of the uniform rule is due to Thomson (1997a, Theorem 1).

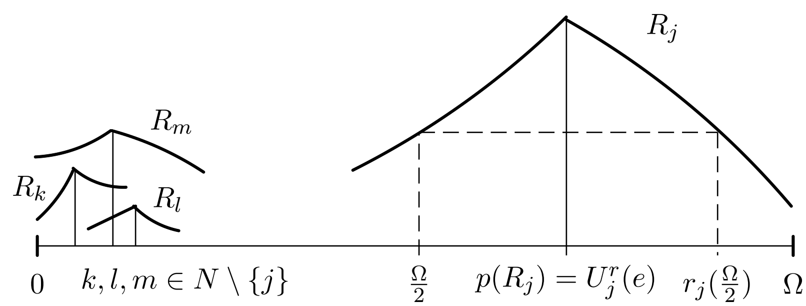
Theorem 2 (*Thomson, 1997a*).
On the domain $\bigcup_{N\in\mathcal{N}}\mathcal{E}^N$, the uniform rule is the only rule that satisfies Pareto optimality, no-envy, one-sided replacement-domination, and replication-invariance.

Question 2: Following the proof of this Theorem, Thomson asks whether *replication-invariance* is independent from the other characterizing properties (see Thomson, 1997a, page 161).

We give the answers to Questions 1 and 2 by means of a single rule, unequal to the uniform rule, that satisfies *Pareto optimality*, *no-envy*, *one-sided population-monotonicity, one-sided replacement-domination*, but *not replication-invariance*.

Before we define the "absorbing agent" rule, we introduce some notation. Let $N\in\mathcal{N}$. Then, by $\mathcal{A}^N\subsetneq\mathcal{E}^N$ we denote the following subclass of economies: for $e=(R,\Omega)\in\mathcal{A}^N$ we have that (i) there is too much to distribute, *i.e.*, $z(e)<0$ and (ii) there is exactly one agent, namely agent $j\in N$, with a relatively large peak amount $p(R_j)>\frac{\Omega}{2}$ (see Figure 1). For such an economy $e\in\mathcal{A}^N$, we construct the allocation assigned at e as follows.

Figure 1: An economy $e\in\mathcal{A}^N$.



We start from the uniform allocation $U(e)$ and let agent j "absorb" some part of the excess supply the remaining agents, $i\in N\backslash\{j\}$, experience at their uniform allotments, denoted by the list $(U_i(e))_{i\in N\backslash\{j\}}$. We denote the *excess supply of the agents in $N\backslash\{j\}$ at the allocation* $U(e)$ by $s_j(e)=\sum_{N\backslash\{j\}}(U_i(e)-p(R_i))$.

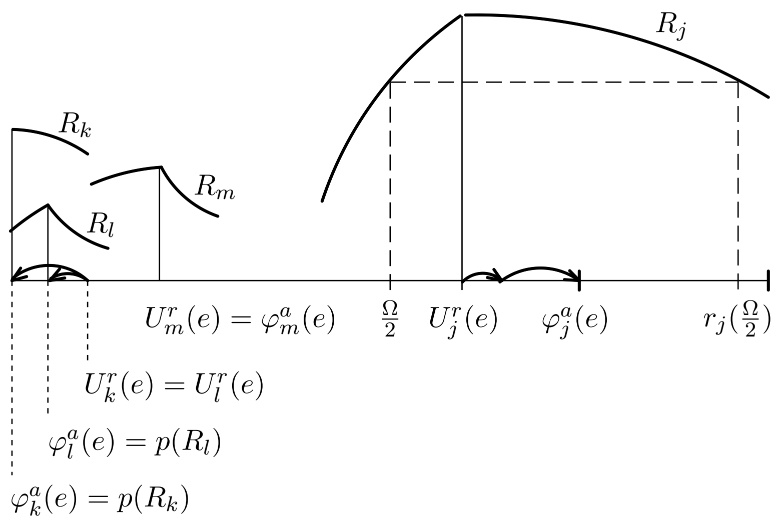
Note that in order to preserve *same-sidedness*, we let agent j absorb at most $s_j(e)$. However, when we let agent j absorb part, or even the whole, of the excess supply $s_j(e)$, we do not want him to envy the other agents. Therefore, we introduce the sufficiently small upper bound $r_j(\frac{\Omega}{2})$ for agent j's allotment. Recall that either $r_j(\frac{\Omega}{2})\,I_j\,\frac{\Omega}{2}$ or $r_j(\frac{\Omega}{2})=\infty$. Note that the allotments of all other agents are smaller than $\frac{\Omega}{2}$ and therefore, as long as agent j's allotment is between his peak amount $p(R_j)$ and $r_j(\frac{\Omega}{2})$, he does not envy the other agents.

[6] Thomson also asks the question whether the uniform rule is the only rule that satisfies *Pareto optimality*, *individual rationality from equal division*, *one-sided population-monotonicity*, and *replication-invariance*.

The answer is that there is a large class of rules satisfying the properties mentioned in the question. Examples are the "proportional reallocation rule" Pr^r operated from equal division and the "maximally satiating reallocation rule" operated from equal division (Klaus, 1998; Klaus, Peters, and Storcken, 1998).

Summarizing, if $N \in \mathcal{N}$ and $e = (R, \Omega) \in \mathcal{A}^N$, then we obtain the allocation assigned by the absorbing agent rule from $U(e)$ by letting agent j absorb the amount $a(e) := \min\{s_j(e), r_j(\frac{\Omega}{2}) - p(R_j)\}$ and subtracting this amount as equally as possible (with the agents' peak amount as lower bounds) from the uniform allotments of the other agents (see Figures 2 and 3). We denote this allocation by $\varphi^a(e)$. If $N \in \mathcal{N}$ and $e = (R, \Omega) \notin \mathcal{A}^N$, then we apply the uniform rule.

Figure 2: The rule φ^a for an economy $e \in \mathcal{A}^N$ with $a(e) = s_j(e)$.



The absorbing agent rule φ^a**:** For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{A}^N$, let $j \in N$ be such that $p(R_j) > \frac{\Omega}{2}$. [7] Then, for all $i \in N$,

$$\varphi^a_i(e) = \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R_{N \setminus \{j\}}, \Omega - \varphi^a_j(e)) & \text{if } i \in N \setminus \{j\}. \end{cases}$$

For all $N \in \mathcal{N}$ and all $e = (R, \Omega) \in \mathcal{E}^N \setminus \mathcal{A}^N$,

$$\varphi^a(e) = U(e).$$

The rule φ^a is by construction *same-sided* and therefore *Pareto optimal*. Since, we either assign the uniform allocation to an economy or change the uniform allocation in such a way that envy cannot arise, φ^a satisfies *no-envy*.[8]

The following remark implies that the rule φ^a does not satisfy *replication-invariance*.

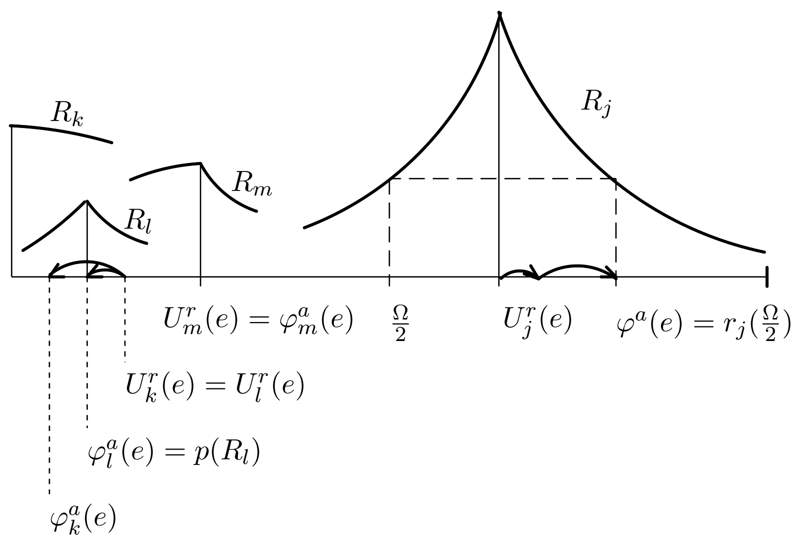
Remark 1. Note that for all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, all $k \in \mathbb{N}$, $k \geq 2$, and all k-replica $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathcal{E}^{\bar{N}}$ of e, if $e \in \mathcal{A}^N$, then $\bar{e} \notin \mathcal{A}^{\bar{N}}$ and $\varphi^a(\bar{e})$ is not a k-replica of $\varphi^a(e)$. □

Finally, by the following lemmas, we complete the proof that *replication-invariance* in Theorems 1 and 2 is independent from the other properties.

[7] Recall that the absorbing agent j is unique and that $a(e) := \min\{s_j(e), r_j(\frac{\Omega}{2}) - p(R_j)\}$.

[8] The absorbing agent rule φ^a satisfies other well-known properties, such as *anonymity* and *individual rationality from equal division*. It follows easily that φ^a satisfies none of the properties *strategy-proofness*, *peak-onliness*, *one-sided resource-monotonicity*, *consistency*, and *converse consistency*. For a more detailed discussion, we refer to Klaus (1998).

Figure 3: The rule φ^a for an economy $e \in \mathcal{A}^N$ with $a(e) = r_j(\frac{\Omega}{2}) - p(R_j)$.



Lemma 1. *The rule φ^a satisfies one-sided population-monotonicity.*

Lemma 2. *The rule φ^a satisfies one-sided replacement-domination.*

Before we prove Lemmas 1 and 2, we introduce the solidarity property *one-sided resource-monotonicity* (Thomson, 1994). *One-sided resource-monotonicity* states that after a one-sided change of an economy that is induced by a change in the social endowment, either all agents (weakly) lose or all (weakly) gain.

One-sided resource-monotonicity: For all $N \in \mathcal{N}$ and all $e = (R, \Omega)$, $\bar{e} = (R, \bar{\Omega}) \in \mathcal{E}^N$, if $\bar{e}$ is a one-sided change of e, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

The uniform rule satisfies *one-sided resource-monotonicity* (Thomson, 1994).

Proof of Lemma 1. Let $N, \bar{N} \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^{\bar{N}}$ be such that $N \subseteq \bar{N}$ and $R = \bar{R}_N$. Now, it is sufficient to show that, if $\bar{e}$ is a one-sided change of e, then either (i) [for all $i \in N$, $\varphi_i^a(e)\, R_i\, \varphi_i^a(\bar{e})$] or (ii) [for all $i \in N$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$]. Let $\bar{e}$ be a one-sided change of e.

Case 1: $e \notin \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^{\bar{N}}$. Then, $\varphi^a(e) = U(e)$ and $\varphi^a(\bar{e}) = U(\bar{e})$. Since, the uniform rule is *one-sided population-monotonic*, either (i) or (ii) holds.

Case 2: $e \in \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^{\bar{N}}$. Hence, $z(\bar{e}) \leq 0$, $z(e) < 0$, and there exists $j \in N$ such that $p(R_j) > \frac{\Omega}{2}$. Since $\bar{e} \notin \mathcal{A}^{\bar{N}}$, we have $z(\bar{e}) = 0$. So, by *Pareto optimality*, for all $i \in N$, $\varphi_i^a(\bar{e}) = p(R_i)$. This implies (ii).

Case 3: $e \notin \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^{\bar{N}}$. Hence, $z(e) \leq z(\bar{e}) < 0$ and there exists $j \in \bar{N} \backslash N$ such that $p(\bar{R}_j) > \frac{\Omega}{2}$. Let $R' = \bar{R}_{\bar{N}\backslash\{j\}} \in \mathcal{R}^{\bar{N}\backslash\{j\}}$. Then, for all $i \in N$,

$$\varphi_i^a(e) = U_i(e) \text{ and } \varphi_i^a(\bar{e}) = U_i(R', \Omega - \varphi_j^a(\bar{e})). \quad (1)$$

Since the uniform rule is *one-sided resource-monotonic* and (R', Ω), $(R', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $\sum_{\bar{N}\backslash\{j\}} p(R'_i) \leq \Omega - \varphi_j^a(\bar{e}) < \Omega$, it follows that for all $i \in N \subseteq \bar{N}\backslash\{j\}$,

$$U_i(R', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega). \tag{2}$$

Since the uniform rule is *one-sided population-monotonic* and $e = (R, \Omega) \in \mathcal{E}^N$, $(R', \Omega) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $N \subseteq \bar{N}\backslash\{j\}$, $R = R'_N$, and $\sum_{\bar{N}\backslash\{j\}} p(R'_i) \leq \Omega$, it follows that for all $i \in N$,

$$U_i(R', \Omega)\, R_i\, U_i(e). \tag{3}$$

Thus, (1), (2), and (3) together imply that for all $i \in N$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$. Hence, (ii) holds.

Case 4: $e \in \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^{\bar{N}}$. Hence, $z(e) \leq z(\bar{e}) < 0$ and there exists $j \in N \subseteq \bar{N}$ such that $p(R_j) > \frac{\Omega}{2}$. Let $R' = R_{N\backslash\{j\}} \in \mathcal{R}^{N\backslash\{j\}}$ and $\bar{R}' = \bar{R}_{\bar{N}\backslash\{j\}} \in \mathcal{R}^{\bar{N}\backslash\{j\}}$. Then,

$$\begin{aligned} \varphi_i^a(e) &= \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R', \Omega - \varphi_j^a(e)) & \text{if } i \in N\backslash\{j\}, \end{cases} \\ \varphi_i^a(\bar{e}) &= \begin{cases} p(R_j) + a(\bar{e}) & \text{if } i = j, \\ U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e})) & \text{if } i \in \bar{N}\backslash\{j\}. \end{cases} \end{aligned} \tag{4}$$

Note that $U_j(e) = p(R_j) = U_j(\bar{e})$. Therefore, $\Omega - U_j(e) = \sum_{N\backslash\{j\}} U_i(e) = \sum_{\bar{N}\backslash\{j\}} U_i(\bar{e})$ and $s_j(e) \geq s_j(\bar{e})$. Hence, $a(e) \geq a(\bar{e})$ and $p(R_j) \leq \varphi_j^a(\bar{e}) \leq \varphi_j^a(e)$. Thus,

$$\varphi_j^a(\bar{e})\, R_j\, \varphi_j^a(e). \tag{5}$$

Now, we have either

(a) $a(\bar{e}) = s_j(\bar{e})$ or

(b) $a(\bar{e}) = r_j(\frac{\Omega}{2}) - p(R_j)$.

Suppose (a) holds. Then, for all $i \in N\backslash\{j\}$, $\varphi_i^a(\bar{e}) = p(R_i)$. Thus, for all $i \in N\backslash\{j\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{6}$$

Hence, (5) and (6) imply (ii).

Suppose (b) holds. Then, $a(\bar{e}) = a(e)$ and $\Omega - \varphi_j^a(\bar{e}) = \Omega - \varphi_j^a(e)$. Since the uniform rule is *one-sided population-monotonic* and $(R', \Omega - \varphi_j^a(e)) \in \mathcal{E}^{N\backslash\{j\}}$, $(\bar{R}', \Omega - \varphi_j^a(\bar{e})) \in \mathcal{E}^{\bar{N}\backslash\{j\}}$ are such that $N\backslash\{j\} \subseteq \bar{N}\backslash\{j\}$, $R' = \bar{R}'_{N\backslash\{j\}}$, and $\sum_{\bar{N}\backslash\{j\}} p(\bar{R}_i) \leq \Omega - \varphi_j^a(e)$, it follows that for all $i \in N\backslash\{j\}$,

$$U_i(\bar{R}', \Omega - \varphi_j^a(\bar{e}))\, R_i\, U_i(R', \Omega - \varphi_j^a(e)). \tag{7}$$

Thus, by (4), for all $i \in N\backslash\{j\}$,

$$\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e). \tag{8}$$

Hence, (5) and (8) imply (ii). □

Proof of Lemma 2. Let $N \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, $\bar{e} = (\bar{R}, \Omega) \in \mathcal{E}^N$, and $k \in N$ be such that $\bar{R}$ is a k-deviation from R. It is sufficient to show that if $\bar{e}$ is a one-sided change of e, then either (i) [for all $i \in N\backslash\{k\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or (ii) [for all $i \in N\backslash\{k\}$, $\varphi_i^a(\bar{e})\, R_i\, \varphi_i^a(e)$]. Let $\bar{e}$ be a one-sided change of e.

Case 1: $e \notin \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^N$. Then, $\varphi^a(e) = U(e)$ and $\varphi^a(\bar{e}) = U(\bar{e})$. Since, the uniform rule satisfies *one-sided replacement-domination*, either (i) or (ii) holds.

Case 2: $e \notin \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^N$. Hence, $\bar{R}$ and R are such that $p(R_k) \leq \frac{\Omega}{2}$, $p(\bar{R}_k) > \frac{\Omega}{2}$, and $z(e) \leq z(\bar{e}) < 0$. Thus, $U_k(e) \leq \frac{\Omega}{2} < p(\bar{R}_k) \leq \varphi^a_k(\bar{R}, \Omega)$. Let $R' = R_{N\backslash\{k\}} \in \mathcal{R}^{N\backslash\{k\}}$. Then, for all $i \in N\backslash\{k\}$,

$$\varphi^a_i(e) = U_i(e) = U_i(R', \Omega - U_k(e)) \text{ and } \varphi^a_i(\bar{e}) = U_i(R', \Omega - \varphi^a_k(\bar{e})). \tag{9}$$

Since the uniform rule is *one-sided resource-monotonic* and $(R', \Omega - U_k(e))$, $(R', \Omega - \varphi^a_k(\bar{e})) \in \mathcal{E}^{N\backslash\{k\}}$ are such that $\sum_{N\backslash\{k\}} p(R_i) \leq \Omega - \varphi^a_k(\bar{e}) \leq \Omega - U_k(e)$, it follows that for all $i \in N\backslash\{k\}$,

$$U_i(R', \Omega - \varphi^a_k(\bar{e}))\, R_i\, U_i(R', \Omega - U_k(e)). \tag{10}$$

Hence, (9) and (10) imply (ii).

Case 3: $e \in \mathcal{A}^N$, $\bar{e} \notin \mathcal{A}^N$. Similar to Case 2 by interchanging the roles of $\bar{R}$ and R.

Case 4: $e \in \mathcal{A}^N$, $\bar{e} \in \mathcal{A}^N$. Without loss of generality we assume that $\bar{R}$ is a k-deviation from R such that $p(\bar{R}_k) \geq p(R_k)$. Hence, $z(e) \leq z(\bar{e}) < 0$ and there exists $j \in N$ such that $p(R_j) > \frac{\Omega}{2}$ and $p(\bar{R}_j) > \frac{\Omega}{2}$. Let $R' = R_{N\backslash\{j\}} \in \mathcal{R}^{N\backslash\{j\}}$ and $\bar{R}' = \bar{R}_{N\backslash\{j\}} \in \mathcal{R}^{N\backslash\{j\}}$. Then,

$$\varphi^a_i(e) = \begin{cases} p(R_j) + a(e) & \text{if } i = j, \\ U_i(R', \Omega - \varphi^a_j(e)) & \text{if } i \in N\backslash\{j\}, \end{cases} \tag{11}$$

$$\varphi^a_i(\bar{e}) = \begin{cases} p(\bar{R}_j) + a(\bar{e}) & \text{if } i = j, \\ U_i(\bar{R}', \Omega - \varphi^a_j(\bar{e})) & \text{if } i \in N\backslash\{j\}. \end{cases}$$

Case 4.1: $k = j$. Since the uniform rule is *one-sided resource-monotonic* and $(R', \Omega - \varphi^a_j(e))$, $(\bar{R}', \Omega - \varphi^a_j(\bar{e})) \in \mathcal{E}^{N\backslash\{j\}}$ are such that $R' = \bar{R}'$ and $z(R', \Omega - \varphi^a_j(e)) \leq 0$ and $z((\bar{R}', \Omega - \varphi^a_j(\bar{e})) \leq 0$, it follows that for all $i \in N\backslash\{j\}$, either (i) or (ii).

Case 4.2: $k \neq j$. Since $p(\bar{R}_k) \geq p(R_k)$, it follows that $s_j(e) \geq s_j(\bar{e})$. Hence, $a(e) \geq a(\bar{e})$ and $\varphi^a_j(\bar{e}) \leq \varphi^a_j(e)$. Thus,

$$\varphi^a_j(\bar{e})\, R_j\, \varphi^a_j(e). \tag{12}$$

Now, we have either

(a) $a(\bar{e}) = s_j(\bar{e})$ or

(b) $a(\bar{e}) = r_j(\frac{\Omega}{2}) - p(R_j)$.

Suppose (a) holds. Hence, for all $i \in N\backslash\{j\}$, $\varphi^a_i(\bar{e}) = p(R_i)$. Thus, for all $i \in N\backslash\{j, k\}$,

$$\varphi^a_i(\bar{e})\, R_i\, \varphi^a_i(e). \tag{13}$$

Hence, (12) and (13) imply (ii).

Suppose (b) holds. Hence, $a(\bar{e}) = a(e)$ and $\Omega - \varphi^a_j(\bar{e}) = \Omega - \varphi^a_j(e)$. Since the uniform rule satisfies *one-sided replacement-domination* and $(R', \Omega - \varphi^a_j(e))$, $(\bar{R}', \Omega - \varphi^a_j(\bar{e})) \in \mathcal{E}^{N\backslash\{j\}}$, $k \in N$, are such that $p(\bar{R}_k) \geq p(R_k)$ and $R'_{N'\backslash\{k\}} = \bar{R}'_{N'\backslash\{k\}}$, it follows that for all $i \in N'\backslash\{k\} = N\backslash\{j, k\}$,

$$U_i(\bar{R}', \Omega - \varphi^a_j(\bar{e}))\, R_i\, U_i(R', \Omega - \varphi^a_j(e)). \tag{14}$$

Thus, by (11) and (14), for all $i \in N\backslash\{j, k\}$,

$$\varphi^a_i(\bar{e})\, R_i\, \varphi^a_i(e). \tag{15}$$

Hence, (12) and (15) imply (ii). □

Now, Remark 1, together with Lemmas 1 and 2 establish the answers to Questions 1 and 2.

4 References

Benassy JP (1982) *The Economics of Market Equilibrium.* Academic Press, San Diego

Klaus B (1998) *Fair Allocation and Reallocation: An Axiomatic Study.* Ph.D. Thesis, Maastricht University, The Netherlands, Unigraphic, Maastricht

Klaus B, Peters H, Storcken T (1998) Strategy-Proof Division with Single-Peaked Preferences and Individual Endowments. *Social Choice and Welfare* 15: 297–311

Sprumont Y (1991) The Division Problem with Single-Peaked Preferences: A characterization of the Uniform Allocation Rule. *Econometrica* 59: 509–519

Thomson W(1994) Resource-Monotonic Solutions to the Problem of Fair Division when Preferences are Single-Peaked. *Social Choice and Welfare* 11: 205–223

Thomson W (1995a) Population-Monotonic Solutions to the Problem of Fair Division when Preferences are Single-Peaked. *Economic Theory* 5: 229–246

Thomson W (1995b) Population monotonic allocation rules. In: Barnett WA, Moulin H, Salles M, Schofield N (eds.) Social choice, welfare and ethics. Cambridge University Press.

Thomson W (1997a) The Replacement Principle in Economies with Single-Peaked Preferences. *Journal of Economic Theory* 76: 145–168

Thomson W (1997b) *The Theory of Fair Allocation.* Princeton University Press, forthcoming

Thomson W (1999) Welfare-domination under preference-replacement: A survey and open questions. *Social Choice and Welfare* 16:373–394