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An Impossibility in Sequencing Problems*

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Abstract

A set of agents with different waiting costs have to receive a service of different length of time from a single provider which can serve only one agent at a time. One needs to form a queue and set up monetary transfers to compensate the agents who have to wait. We prove that no rule satisfies *efficiency of queues* and *coalitional strategy-proofness*.

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1 Introduction

A set of agents simultaneously arrive at a service facility that can only serve one agent at a time. Agents require service for different lengths of time. The waiting cost may vary from one agent to the other. Each agent is assigned a “consumption bundle” consisting of a position in the queue and a positive or negative transfer. Each agent has quasi-linear preferences over positions and transfers. For such a *sequencing problem*, a rule assigns each agent a position in the queue and a positive or negative transfer such that no two agents are assigned the same position, and the sum of the transfers is not positive.

Our objective is to identify rules that are well-behaved from the normative and strategic viewpoints. The first requirement is efficiency. It says that if an allocation is selected, there

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should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers. Since preferences are linear, *Pareto-efficiency* can be decomposed into two axioms: on the one hand, *efficiency of queues*, which says that a queue should minimize the total waiting cost, and on the other hand, *balancedness*, which says that transfers should sum up to zero. Second is a minimal fairness requirement, *no-envy*, which requires that no agent should prefer another agent’s assignment to her own. *No-envy* implies *efficiency of queues*. Third is immunity to strategic behavior. As unit waiting costs may not be known, the rule should provide agents the incentive to reveal these costs truthfully. *Strategy-proofness* requires that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it.¹ We are also concerned about possible manipulations by groups, and consider *coalitional strategy-proofness*: no group of agents should make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting their waiting costs. Finally is *non-bossiness*: if an agent’s change in her announcement does not affect her assignment, then it should not affect any other agent’s assignment.

For *sequencing problems*, *Pareto-efficiency* and *strategy-proofness* are compatible (Suijs, 1996). For the subdomain of *sequencing problems* in which agents require service for the same length of time², *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* are compatible. (Kayı and Ramaekers, 2007). We show that if we impose the stronger incentive property of *coalitional strategy-proofness* with *efficiency of queues*, we have an impossibility result. Independently, Mutuswami and Mitra (2006) show that *coalitional strategy-proofness*, *efficiency of queues* and *feasibility*—which says that transfers should sum up less or equal to zero—are not compatible. In our proof, we make use of *non-bossiness* and get some corollaries: We show that no rule satisfies *Pareto-efficiency* and *coalitional strategy-proofness*, no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness*, and no rule satisfies *no-envy*, *non-bossiness*, and *strategy-proofness*.

In Section 2, we formally introduce the model. In Section 3, we define the properties on rules. In Section 4, we give the impossibility result.

¹For an extensive survey on *strategy-proofness*, see Thomson, 2006.

²For *queueing problems*, see Mitra and Sen, 1998; Maniquet, 2003; Katta and Sethuraman, 2006; Chun 2006.

2 Model

There is a finite set of agents N . Let $|N| = n \geq 2$. For each agent $i \in N$, let $r_i \in \mathbb{R}_{++}$ be the *servicing time* and $c_i \in \mathbb{R}_+$ be the *unit waiting cost* of $i \in N$. Let $r = (r_i)_{i \in N} \in \mathbb{R}_{++}^N$ and $c = (c_i)_{i \in N} \in \mathbb{R}_+^N$ be the lists of the servicing times and unit waiting costs respectively. A queue is a bijection $\sigma : N \rightarrow \{1, 2, \dots, n\}$. Let \mathcal{Q} be the set of all queues. For each agent $i \in N$, we will denote $\sigma(i)$ as σ_i . Each agent $i \in N$ has to be assigned a *position* $\sigma_i \in \{1, 2, \dots, n\}$ in a queue σ and may receive a positive or negative monetary *transfer* $t_i \in \mathbb{R}$. Preferences are linear over $X \equiv \{1, 2, \dots, n\} \times \mathbb{R}$. For each queue $\sigma \in \mathcal{Q}$ and each $i \in N$, let $P_i(\sigma)$ be the set of agents *preceding* agent i in queue σ , i.e., $P_i(\sigma) \equiv \{j \in N \mid \sigma_j < \sigma_i\}$. If i is served σ_i -th, her *total waiting cost* is $\sum_{l \in P_i(\sigma)} r_l c_i$. Her preferences can be represented by the function u_i defined as follows: for each $(\sigma_i, t_i) \in X$, $u_i(\sigma_i, t_i) = -\sum_{l \in P_i(\sigma)} r_l c_i + t_i$. We use the following notational shortcut. If her waiting cost is c'_i , then her preferences are represented by the function u'_i , defined by $u'_i(\sigma_i, t_i) = -\sum_{l \in P_i(\sigma)} r_l c'_i + t_i$; if it is \tilde{c}_i , then we use $\tilde{u}_i(\sigma_i, t_i) = -\sum_{l \in P_i(\sigma)} r_l \tilde{c}_i + t_i$, and so on. A *sequencing problem* is defined as a list $s \equiv (r_i, c_i)_{i \in N} \in \mathbb{R}_{++}^N \times \mathbb{R}_+^N$. Let $\mathcal{S} \equiv \mathbb{R}_{++}^N \times \mathbb{R}_+^N$ be the class of all problems.

An *allocation* for $s \in \mathcal{S}$ is a pair $(\sigma, t) \equiv (\sigma_i, t_i)_{i \in N} \in X^N$. An allocation $(\sigma, t) \in X^N$ is *feasible* for $s \in \mathcal{S}$ if no two agents are assigned the same position in σ , (i.e., for each $\{i, j\} \subseteq N$ with $i \neq j$, we have $\sigma_i \neq \sigma_j$), and the sum of the coordinates of t is non-positive, (i.e., $\sum_{i \in N} t_i \leq 0$). Let $Z(s)$ be the set of all feasible allocations for $s \in \mathcal{S}$. An (allocation) *rule* φ is a function that associates with each problem $s \in \mathcal{S}$ a feasible allocation $\varphi(s) \in Z(s)$.

Given $s \in \mathcal{S}$ and $S \subseteq N$, $r_S \equiv (r_l)_{l \in S}$ and $c_S \equiv (c_l)_{l \in S}$ are restrictions of servicing times r and unit waiting costs c to S respectively. Given $i \in N$, $r_{-i} \equiv (r_l)_{l \in N \setminus \{i\}}$ and $c_{-i} \equiv (c_l)_{l \in N \setminus \{i\}}$ are the restrictions of r and c to $N \setminus \{i\}$ respectively.

3 Properties of rules

In this section, we define properties of rules. Let φ be a rule. First, if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers.

Pareto-efficiency: *For each $s \in \mathcal{S}$ and each $(\sigma, t) \in \varphi(s)$, there is no $(\sigma', t') \in Z(s)$ such that for each $i \in N$, $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ and for at least one $j \in N$, $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$.*

If an allocation is *Pareto-efficient* for s , any other allocation at which the queue is the same is also *Pareto-efficient*. Therefore, it is meaningful to speak of the *efficiency of queues*. It requires to minimize the total waiting cost. Thus, an allocation (σ, t) is *Pareto-efficient* for s if and only if for each $\sigma' \in \mathcal{Q}$, we have $\sum_{i \in N} \sum_{l \in P_i(\sigma')} r_l c_i \geq \sum_{i \in N} \sum_{l \in P_i(\sigma)} r_l c_i$, i.e., σ is *efficient* for s and $\sum_{i \in N} t_i = 0$, i.e., t is *balanced* for s . Let $\mathcal{Q}^*(s)$ be the set of all *efficient* queues for s . For each $s \in \mathcal{S}$ and each $(\sigma, t) \in Z(s)$, we have $\sigma \in \mathcal{Q}^*(s)$ if and only if for each $\{i, j\} \subset N$ with $i \neq j$, if $\sigma_i < \sigma_j$, then $c_i/r_i \geq c_j/r_j$. (Smith, 1956) For simplicity, throughout the paper, we will assume that agents have unequal ratios of unit waiting costs and servicing time. Therefore, there is only one *efficient* queue.

Summarizing the discussion above, *Pareto-efficiency* can be decomposed into two axioms:

Efficiency of queues: For each $s \in \mathcal{S}$ and each $(\sigma, t) \in \varphi(s)$, we have $\sigma \in \mathcal{Q}^*(s)$.

Balancedness: For each $s \in \mathcal{S}$ and each $(\sigma, t) \in \varphi(s)$, we have $\sum_{i \in N} t_i = 0$.

The next requirement is necessary for no agent to prefer another agent's assignment to her own.

No-envy: For each $s \in \mathcal{S}$, each $(\sigma, t) \in \varphi(s)$, and each $i \in N$, there is no $j \in N \setminus \{i\}$ such that $u_i(\sigma_j, t_j) > u_i(\sigma_i, t_i)$.

The last requirements are motivated by strategic considerations. The planner may not know the agents' cost parameters. If agents behave strategically when announcing them, neither efficiency nor equity may be attained. Thus, we require that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it.

Strategy-proofness: For each $s \in \mathcal{S}$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $(\sigma, t) = \varphi(s)$ and $(\sigma', t') = \varphi(r, c'_i, c_{-i})$, then $u_i(\sigma_i, t_i) \geq u_i(\sigma'_i, t'_i)$.

We also consider the requirement that no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting its members' waiting costs.

Coalitional strategy-proofness: For each $s \in \mathcal{S}$ and each $S \subseteq N$, there is no $c'_S \in \mathbb{R}_+^S$ such that if $(\sigma, t) = \varphi(s)$ and $(\sigma', t') = \varphi(r, c'_S, c_{N \setminus S})$, then for each $i \in S$, we have $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ and for some $j \in S$, we have $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$.

The next requirement is that if an agent's change in her announcement does not affect her assignment, then it should not affect any other agent's assignment.

Non-bossiness: For each $s \in \mathcal{S}$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $\varphi_i(s) = \varphi_i(r, c'_i, c_{-i})$, then $\varphi(s) = \varphi(r, c'_i, c_{-i})$.

4 Result

First, we establish a relationship between *efficiency of queues* and *coalitional strategy-proofness*, and *non-bossiness*.

Proposition 1. *Let φ be a rule satisfying efficiency of queues and coalitional strategy-proofness, then it satisfies non-bossiness.*

Proof. Let φ be a rule satisfying the axioms of Proposition 1. We need to show that for each $s \in \mathcal{S}$, each $i \in N$, each $c'_i \in \mathbb{R}_+$, if $z = \varphi(s)$ and $z' = \varphi(r, (c'_i, c_{-i}))$ are such that $z_i = z'_i$, then $z = z'$. Indeed, let $s = (r, c) \in \mathcal{S}$, $i \in N$, $c'_i \in \mathbb{R}_+$, $z = \varphi(s)$, and $z' = \varphi(r, (c'_i, c_{-i}))$ be such that $z_i = z'_i$. By contradiction, suppose that there is $j \in N$ such that $z_j \neq z'_j$. Then, since $z_i = z'_i$, we have $u_i(\sigma, t_i) = u_i(\sigma', t'_i)$. By *efficiency of queues*, $\sigma_j = \sigma'_j$. Since $z_j \neq z'_j$, we have $t_j \neq t'_j$. First, suppose $t_j > t'_j$. Then, $u_j(\sigma, t_j) > u_j(\sigma', t'_j)$ and there is $(c'_i, c_j) \in \mathcal{R}_+^{\{i,j\}}$ such that $u_i(\sigma, t_i) = u_i(\sigma', t'_i)$ and $u_j(\sigma, t_j) > u_j(\sigma', t'_j)$, contradicting *coalitional strategy-proofness*. Second, suppose $t_j < t'_j$. Then, $u_j(\sigma, t_j) < u_j(\sigma', t'_j)$ and there is $(c_i, c_j) \in \mathcal{R}_+^{\{i,j\}}$ such that $u_i(\sigma, t_i) = u_i(\sigma', t'_i)$ and $u_j(\sigma', t'_j) > u_j(\sigma, t_j)$, contradicting *coalitional strategy-proofness*. \square

Then, we prove our main result.

Theorem 1. *No rule satisfies efficiency of queues and coalitional strategy-proofness.*

Proof. By contradiction, let φ be a rule satisfying the axioms of Theorem 1. Then, by Proposition 1, φ satisfies *non-bossiness*. Now, assume that for each $i \in N$, we have $r_i = 1$. We establish two claims:

Claim 1: For each $s \in \mathcal{S}$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$,

if $(\sigma, t) = \varphi(r, c)$ and $(\sigma', t') = \varphi(r, (c'_i, c_{-i}))$ are such that $\sigma_i = \sigma'_i$, then $(\sigma, t) = (\sigma', t')$.

Let $s \in \mathcal{S}$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(r, c)$ and $(\sigma', t') = \varphi(r, (c'_i, c_{-i}))$ are such that $\sigma_i = \sigma'_i$. By *strategy-proofness*, $-\sum_{l \in P_i(\sigma)} r_l c_l + t_i \geq -\sum_{l \in P_i(\sigma')} r_l c_l + t'_i$ and $-\sum_{l \in P_i(\sigma)} r_l c'_l + t_i \leq -\sum_{l \in P_i(\sigma')} r_l c'_l + t'_i$. By assumption, for each $i \in N$, we have $r_i = 1$. Thus, $-(\sigma_i - 1)c_i + t_i \geq$

$-(\sigma'_i - 1)c_i + t'_i$ and $-(\sigma_i - 1)c'_i + t_i \leq -(\sigma'_i - 1)c'_i + t'_i$. Thus, as $\sigma_i = \sigma'_i$, we have $t_i = t'_i$. By *non-bossiness*, $(\sigma, t) = (\sigma', t')$.

Claim 2: For each $s \in \mathcal{S}$ such that for each $j, k \in N$, we have $c_j \neq c_k$ if and only if $j \neq k$, for each $i \in N$, and each $c'_i \in \mathbb{R}_+$ such that for each $j \in N \setminus \{i\}$, we have $c'_i > c_j$ if and only if $c_i > c_j$, if $(\sigma, t) = \varphi(s)$, then $(\sigma, t) = \varphi(r, (c'_i, c_{-i}))$.

Let $s \in \mathcal{S}$, $i \in N$, $c'_i \in \mathbb{R}_+$ be such that for each $j \in N \setminus \{i\}$, we have $c'_i \neq c_j$ and $c'_i > c_j$ if and only if $c_i > c_j$, and $(\sigma, t) = \varphi(s)$, $(\sigma', t') = \varphi(r, (c'_i, c_{-i}))$. By *efficiency of queues*, $\sigma'_i = \sigma_i$. Thus, by Claim 1, $(\sigma, t) = \varphi(r, (c'_i, c_{-i}))$.

Claims 1 to 2 being proved, we now come to a contradiction. Without loss of generality, suppose $N = \{1, 2, \dots, n\}$. Let $\{c, c'\} \subseteq \mathbb{R}_+^N$ be such that

- (i) $c_1 > c_2 > c_3 \dots > c_n$,
- (ii) $c'_2 > c'_1 > c'_3 > \dots > c'_n$, and
- (iii) for each $i \in N \setminus \{1\}$, $c'_i = c_i$.

Let $(\sigma, t) = \varphi(s)$ and $(\sigma', t') = \varphi(r, c')$. By *efficiency of queues*, for each $i \in N$, we have $\sigma_i = i$, whereas $\sigma'_1 = 2$, $\sigma'_2 = 1$, and for each $i \in N \setminus \{1, 2\}$, we have $\sigma_i = \sigma'_i = i$. Thus, $(\sigma, t) \neq (\sigma', t')$. By *strategy-proofness*, $u_1(\sigma_1, t_1) = t_1 \geq -c_1 + t'_1 = u_1(\sigma'_1, t'_1)$ and $u'_1(\sigma'_1, t'_1) = -c'_1 + t'_1 \geq t_1 = u_1(\sigma_1, t_1)$. That is, $t'_1 \in [t_1 + c'_1, t_1 + c_1]$. Thus, agent 1's transfer depends either on a constant, i.e., $\bar{t}_1 = t_1 + c$ with $c \in [c_1, c'_1]$, or on its own announcement, i.e., $\bar{t}_1 = t_1 + f(c'_1, c_1)$ with $f(c'_1, c_1) \in [c_1, c'_1]$. Clearly, this contradicts *strategy-proofness*. \square

The following paragraphs establish examples of rules that satisfy only one of the axioms in Theorem 1.

- (i) Equally Distributed Pairwise Pivotal rule (Kayı and Ramaekers, 2007 and Suijs, 1996) satisfies *efficiency of queues* but not *coalitional strategy-proofness*.
- (ii) Any rule that selects the same arbitrary queue and sets the transfer to each agent equal to zero satisfies *coalitional strategy-proofness*, but not *efficiency of queues*.

Finally, by using the implication of properties, we have the following corollaries.

Corollary 1.

1. No rule satisfies Pareto-efficiency and coalitional strategy-proofness.
2. No rule satisfies Pareto-efficiency, non-bossiness, and strategy-proofness.

3. No rule satisfies no-envy, non-bossiness, and strategy-proofness.³

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³1. Since *Pareto-efficiency* implies *efficiency of queues* and by Theorem 1, we have the result. 2. Since the contradiction at the end of Theorem 1 is to *strategy-proofness*, the result follow from Proposition 1. 3. Assume that φ satisfies *no-envy*. Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi(c)$, $\{i, j\} \subset N$, with $i \neq j$ be such that $c_i > c_j$ but $\sigma_i > \sigma_j$. By *no-envy*, we have $u_i(\sigma_i, t_i) \geq u_i(\sigma_j, t_j)$ and $u_j(\sigma_j, t_j) \geq u_j(\sigma_i, t_i)$. Then, $(\sigma_i - \sigma_j)c_i + t_j \leq t_i \leq (\sigma_i - \sigma_j)c_i$ that contradicts $c_i > c_j$. It follows that *no-envy* implies *efficiency of queues*. The result follows from Theorem 1.