

László Á. Kóczy

Voting Paradoxes and the Human Intuition

RM/06/048



Maastricht research school of **E**conomics  
of **T**Echnology and **O**Rganizations

Universiteit Maastricht  
Faculty of Economics and Business Administration  
P.O. Box 616  
NL - 6200 MD Maastricht

phone : ++31 43 388 3830  
fax : ++31 43 388 4873



# Voting Paradoxes and the Human Intuition\*

László Á. Kóczy<sup>†</sup>

October 2, 2006

## Abstract

Brams (2003) presents three paradoxes for power indices: some rather counter-intuitive behaviour that is exhibited by both the Shapley-Shubik and the Banzhaf indices. We show that the proportional index is free from such paradoxical behaviour. This result suggests that our intuition may be based on the proportional index and as such its use in evaluating power measures is limited.

**Keywords and phrases:** a priori voting power, paradox of large size, paradox of new members, paradox of quarrelling members, Gamson's Law.

## 1 Introduction

Since Shapley and Shubik (1954) adopted the Shapley value to measure a priori voting power game theory has contributed considerable literature to this question. This literature established theoretical underpinnings for the existing or rediscovered indices, introduced new ones, but the plethora of power indices already hints that there is no single best.

In the battle among power indices axiomatisation was one of the most powerful tools. Establishing a collection of elementary properties that uniquely determine a particular index has become a sport often ignoring the fact that the declared axioms are not less ad hoc than the indices themselves (Laruelle and Valenciano, 2005, pp37-38). Felsenthal and Machover (1995) find that a

---

\*The author thanks the funding of the Netherlands Organisation for Scientific Research and the Marie Curie Intra-European Fellowship (MEIF-CT-2004-011537) of the European Union as well as the hospitality of the Economic Research Institute of the Hungarian Academy of Sciences in the summer 2005.

<sup>†</sup>Department of Economics, Maastricht University, P.O.Box 616, NL-6200MD Maastricht. Email: [l.koczy@algec.unimaas.nl](mailto:l.koczy@algec.unimaas.nl). Fax: +31-43-3884878.

set of desirable properties or postulates characterises the indices much better, although so far no index satisfies their postulates.

We focus on certain properties known as *paradoxes of voting power*. Felsenthal and Machover (1998) discuss these paradoxes at great length: in their words a paradox is a true statement that is absurd. Paradoxes of voting power are statements concerning power measures, the best known of which are the three paradoxes by Brams (2003); they deal with some basic changes of the voting game and show that the power as measured by some of the power measures might change counter-intuitively. In particular the Shapley-Shubik and (normalised) Banzhaf indices exhibit these paradoxes.

We show that the index that coincides with the voting shares is free from these paradoxes. It is not our intention to promote this index – it even fails to recognise that holding only part of the votes is often sufficient to have all the power, – but to use it as a stepping stone towards the origin of Brams’s paradoxes.

While we can dismiss this index quickly, as recently as in 1957 for the first Council of Ministers of the European Economic Community or even later for the Nassau County Board (Banzhaf, 1965) creating dummy voters caused no uproar suggesting that the human intuition is not necessarily correct in perceiving voting power. We would like our result to be seen as a contribution to the literature that investigates the use of postulates and paradoxes in measuring voting power (see Laruelle and Valenciano (2005)).

The structure of the paper is as follows. In the following we define the aforementioned power indices, elaborate on the paradoxes and show that the trivial index is immune to the paradoxes.

## 2 The ‘trivial’ index is free from paradoxes

We study *weighted voting games*. Let  $N$  be a collection of  $n$  voters having  $w_1, w_2, \dots, w_n$  votes;  $w = \sum_{i=1}^n w_i$ . A quota of  $w \geq q > w/2$  is *required* to pass a bill. For more on weighted voting games see Straffin (1994).

A power index is a function  $k$  that assigns to each weighted voting game and each player  $i \in N$  a value  $k_i \in \mathbb{R}$ .

We define the *proportional index*  $\alpha$  by  $\alpha_i = \frac{w_i}{w}$ .

This measure is popularly known in political science as Gamson’s Law: “Any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute” (Gamson, 1961). The index is not only simple, but in the following we show that it satisfies a group of appealing properties listed by Brams (2003) and Felsenthal and Machover (1998).

In the following we describe some paradoxes of voting power; stated as postulates, that is, in their positive, “intuitive” form. A paradox occurs if such a postulate is not satisfied. We begin with three natural properties due to Brams (2003) that nevertheless fail for some of the well-known indices.

**Postulate of (large) size** Let  $G = (N, (w_i)_{i \in N}, q)$  be a voting game and  $k$  a power index. Define  $G'$  by the merger of players  $i$  and  $j$ . The resulting party  $ij$  has a weight  $w_{ij} = w_i + w_j$ . The postulate requires  $k_i(G) + k_j(G) \leq k_{ij}(G')$ .

**Proposition 1.** *The index  $\alpha$  satisfies the Postulate of (large) size.*

*Proof.* Using the notation in the definition, the merged member’s weight is simply  $w_{ij} = w_i + w_j$ , the merger does not alter the total weight, hence  $\alpha_{ij}(G') = \frac{w_{ij}}{w} = \frac{w_i + w_j}{w} = \alpha_i(G) + \alpha_j(G)$ .  $\square$

**Postulate of new members** Now define  $G''$  as an extension of  $G$  by parties  $n + 1, \dots, m$  and weights  $w_{n+1}, \dots, w_m$  and a  $q''$  to meet the requirements. The postulate requires  $k_i(G) \geq k_i(G'')$ , that is, the introduction of new members should not increase a party’s power.

**Proposition 2.** *The index  $\alpha$  satisfies the Postulate of new members.*

*Proof.* An ‘old’ member’s weight remains  $w'_i = w_i$ , while the total increases to  $w' > w$  due to the new members, hence  $\alpha_i(G') = \frac{w_i}{w'} = \frac{w_i}{w} > \frac{w_i}{w} = \alpha_i(G)$ .  $\square$

Before we turn to Brams’s last paradox we mention three listed by Felsenthal and Machover (1998) that hold for the proportional index by definition.

**Postulate of redistribution** Let  $G = (N, (w_i)_{i \in N}, q)$  and  $G' = (N, (w'_i)_{i \in N}, q)$  be two weighted voting games. The postulate requires that for all  $i \in N$  we have  $k_i(G) < k_i(G')$  if and only if  $w_i < w'_i$ .

**Donation Postulate** is then the special case where  $w'_i = w_i + \delta$  and  $w'_j = w_j - \delta$  for some  $i, j \in N$  and  $w'_k = w_k$  for all  $k \notin \{i, j\}$ .

**Dominance Postulate** A power index satisfies dominance in a weighted voting game  $G$  if  $w_i < w_j$  implies  $k_i(G) < k_j(G)$ .

**Postulate of quarrelling members** If two parties refuse to vote together, this should not increase their total power.

The postulate of quarrelling members has been criticised for it allows an action (quarrelling) for the players that is beyond the specification of the game. Therefore when we study this postulate we must also specify our game better.

A priori voting indices have been used for national parliaments, (van Deemen and Rusinowska, 2003) where voting weights are given by the number of representatives each party has. Irrational voting weights can be produced as limits of such games. We motivate our further discussion by assuming that the players of the game are such parties.

Shapley and Shubik (1954) consider voting situations where parties throw their support at a motion in a random order until a winning coalition is reached. The last, *pivotal* party gets all the credit for the success of the motion. A party's Shapley-Shubik index is then the proportion of orderings where it is pivotal.

$$\phi_i = \frac{\# \text{ times } i \text{ is pivotal}}{n!} \quad (1)$$

The index of Shapley and Shubik is supported by a story that relies a particular assumption: parties form indivisible blocks, they act and move together. This model is particularly fit where the voting is a true weighted voting in the sense that there are  $n$  voters and the decision is simply a weighted result of these votes.

In other situations this is not the case. In a ballot, representatives may or may not follow party directives, in fact the party's view is no more than a collection of the members' views. In this case we cannot expect that parties will vote as blocks, especially if the order in which they throw their support at some case is also important. Instead, we must treat each representative separately and the power of a party is the probability that one of its representatives is pivotal.

Formally, we consider parties with  $w_i \in \mathbb{N}$  representatives. Voters arrive in an order determined by their keenness to approve the issue at hand, and order which is different from issue to issue and that is hence arbitrary. The voter whose support turns the coalition into a winning one, that is, into a coalition whose size exceeds the quota  $q$  is given all the credit, which is, as usual, normalised to 1. This credit then goes to the party he was representing. Hence we have

$$\hat{\phi}_i = \frac{\# \text{ times a representative of } i \text{ is pivotal}}{w!}. \quad (2)$$

Note the change of the numerator.

Also note that the index is determined by the behaviour of representatives and not parties. Therefore we do not discuss whether a player is not pivotal *because* its party does not support a proposal or *because* he or she happened to be less keen on this issue.

Now recall that a coalition is winning if it has size  $q$  or greater. Therefore a player is pivotal and gets credit if he or she is the  $q$ th to vote for. A party's power index is the probability that the  $q$ th voter is one of its representatives. However this probability is the same as that its representative is the first or any  $k$ th voter,  $\frac{w_i}{w}$ . This yields the following result:

**Proposition 3.** *For a voting game  $(N, (w_i)_{i \in N}, q)$  we have  $\hat{\phi}_i = \frac{w_i}{w} = \alpha_i$ , that is, the party's share of the votes.*

Therefore the modification of the Shapley-Shubik story indeed yields the proportional index.

**Proposition 4.** *The index  $\alpha$  satisfies the Postulate of quarrelling members.*

*Proof.* Here the proof cannot follow the same pattern, in fact this property cannot really be addressed within this model, but one has to modify the game is played. If players  $i$  and  $j$  quarrel and some representatives of  $i$  have agreed to a proposal, a representative of  $j$ , upon arriving will start a fight with  $i$  instead of casting a vote. Further voting is impossible and the following day a totally different coalition may form.

In the following we calculate the powers of players  $i$  and  $j$  in case they quarrel. Observe that a winning coalition has always size  $q$ . Now suppose that all we know is that player  $i$  has  $a$  representatives in the coalition. Then the conditional probability that  $i$  gets credit is simply  $\frac{a}{q}$ . We can therefore write  $i$ 's power as

$$\alpha_i(\mathbf{G}) = \frac{\sum_{a=0}^{w_i} \frac{a}{q} \binom{q}{a} \binom{w-q}{w_i-a} w_i! (w-w_i)!}{w!} = \frac{w_i}{w}. \quad (3)$$

Similarly we can condition over the number  $b$  of representatives of  $j$  in  $q$  and calculate powers. The probability that  $i$  or  $j$  win having respectively  $a$  and  $b$  representatives in the winning coalition is

$$\frac{a+b}{q} \binom{q}{a,b} \binom{w-q}{w_i-a, w_j-b} w_i! w_j! (w-w_i-w_j)!$$

In case  $i$  and  $j$  quarrel either  $a$  or  $b$  must be 0 (hence  $ab = 0$ ). After some trivial simplifications the inequality to be shown is the following:

$$\alpha_i(\mathbf{G}''') + \alpha_j(\mathbf{G}''') = \frac{\sum_{a=0}^{w_i} \sum_{b=0, ab=0}^{w_j} \frac{a+b}{q} \binom{q}{a,b} \binom{w-q}{w_i-a, w_j-b}}{\sum_{a=0}^{w_i} \sum_{b=0, ab=0}^{w_j} \binom{q}{a,b} \binom{w-q}{w_i-a, w_j-b}} < \frac{w_i + w_j}{w} = \alpha_i(\mathbf{G}) + \alpha_j(\mathbf{G}) \quad (4)$$

Equivalent transformations lead to

$$w_i + w_j < w_i \frac{\binom{w-w_i-1}{q-1}}{\binom{w-w_i-w_j-1}{q-1}} + w_j \frac{\binom{w-w_j-1}{q-1}}{\binom{w-w_i-w_j-1}{q-1}}. \quad (5)$$

This inequality holds term-by-term.  $\square$

### 3 Discussion

It is remarkable that such a trivial and in many ways faulty index satisfies all of Brams's postulates. Diermeier and Merlo (2004), Gelman, Katz, and Bafumi (2004) Fréchette, Kagel, and Morelli (2005) (and references therein), find that proportional distribution of power behaves surprisingly well both in empirical tests and coalition formation models, suggesting that assigning power proportionally is a common practice if not the standard.

If proportionality is natural, so must be its properties, including the postulates we looked at here. This is only one step away from calling something that does not satisfy these properties paradoxical. We see the origin of Brams's paradoxes in the proportional index, in its widespread use and in the fact that the principle of proportionality might somehow still appeal even to academics working with power indices.

### References

- BANZHAF, III., J. F. (1965): "Weighted voting doesn't work: A mathematical analysis," *Rutgers Law Review*, 19(2), 317–343.
- BRAMS, S. J. (2003): *Game theory and politics*. Dover, Mineola, N.Y., 2nd edn., First published in 1975.
- DIERMEIER, D., AND A. MERLO (2004): "An Empirical Investigation of coalitional bargaining procedures," *Journal of Public Economics*, 88(3-4), 783–797.
- FELSENTHAL, D. S., AND M. MACHOVER (1995): "Postulates and paradoxes of relative voting power - A critical re-appraisal," *Theory and Decision*, 38(2), 195–229.
- (1998): *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Edward Elgar, Cheltenham.



- FRÉCHETTE, G. R., J. H. KAGEL, AND M. MORELLI (2005): “Gamson’s Law versus non-cooperative bargaining theory,” *Games and Economic Behavior*, 51(2), 365–390.
- GAMSON, W. A. (1961): “A theory of coalition formation,” *American Sociological Review*, 26(3), 373–382.
- GELMAN, A., J. N. KATZ, AND J. BAFUMI (2004): “Standard Voting Power Indexes Do Not Work: An Empirical Analysis,” *British Journal of Political Science*, 34, 657–674.
- LARUELLE, A., AND F. VALENCIANO (2005): “A critical reappraisal of some voting paradoxes,” *Public Choice*, 125(1), 14–41.
- SHAPLEY, L. S., AND M. SHUBIK (1954): “A method for evaluating the distribution of power in a committee system,” *American Political Science Review*, 48(3), 787–792.
- STRAFFIN, JR., P. D. (1994): “Power and Stability in Politics,” in *Handbook of Game Theory with Economic Applications*, ed. by R. J. Aumann, and S. Hart, vol. II, chap. 32, pp. 1127–1151. Elsevier, Amsterdam.
- VAN DEEMEN, A., AND A. RUSINOWSKA (2003): “Paradoxes of Voting Power in Dutch Politics,” *Public Choice*, 115(1-2), 109–137.