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Stable Allocations of Risk*

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Abstract

The measurement and the allocation of risk are fundamental problems of portfolio management. Coherent measures of risk provide an axiomatic approach to the former problem. In an environment given by a coherent measure of risk and the various portfolios' realization vectors, risk allocation games aim at solving the second problem: How to distribute the diversification benefits of the various portfolios? Understanding these cooperative games helps us to find stable, efficient, and fair allocations of risk.

We show that the class of risk allocation and totally balanced games coincide hence a stable allocation of risk is always possible. When the aggregate portfolio is riskless: risk is limited to subportfolios, the class of risk allocation games coincides with the class of exact games. As in exact games any subcoalition may be subject to marginalization even in core allocations, our result further emphasizes the responsibility in allocating risk.

Keywords: Coherent Measures of Risk, Risk Allocation Games, Totally Balanced Games, Exact Games

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1 Introduction

The value of an investment portfolio is subject to all kinds of uncertain events. The fact that its future value can be less than in the most favorable state of the world could be seen as a *risk* investors face. Firms, banks, or insurance companies (which can also be seen as portfolios) face risk and regulators may require them to hold cash reserves as a cushion against default – this rather unfavorable state of nature – with the precise amounts determined by a *measure of risk*.

The literature knows of numerous possible ways to measure risk; lately interest shifted to *coherent measures of risk* (Artzner, Delbaen, Eber, and Heath, 1999) defined by four axioms: monotonicity, subadditivity, positive homogeneity, and translation invariance. These axioms have been shown to be compatible with a natural general equilibrium approach to measure risk (Csóka, Herings, and Kóczy, 2007b).

Of these axioms, subadditivity expresses that the risk of an aggregate portfolio should not exceed the total risk of the individual portfolios. *Risk allocation* then addresses the distribution of the diversification benefits; *risk allocation games* (Denault, 2001) are transferable utility games defined to this purpose. We separate the *risk environment* specifying the individual portfolios' realization vectors of discrete random variables and a coherent measure of risk, a real valued function on the realization vectors, from the derived cooperative game that we call *risk allocation game*. We identify risk allocation games with a very special class of TU-games: totally balanced games.

A *totally balanced game* is a cooperative game having a non-empty core in all of its subgames. Totally balanced games arise from a wide range of applications. They coincide with market games (Shapley and Shubik, 1969); also with a special case of market games with a continuum of indivisible commodities: cooperation in fair division (Legut, 1990); they are equivalent to a class of maximum flow problems (Kalai and Zemel, 1982a); and also to permutation games of less than four players (Tijs, Parthasarathy, Potters, and Prasad, 1984). Moreover, totally balanced games are generated by linear production games (Owen, 1975), generalized network problems (Kalai and Zemel, 1982b), and controlled mathematical programming problems (Dubey and Shapley, 1984).

We show that the class of risk allocation games coincides with the class of totally balanced games, that is all risk allocation games are totally balanced and all totally balanced games can be generated by a risk allocation game with a properly specified risk environment. This result ensures that a regulator can always allocate risk in a stable way. No matter how the risk

environment changes, there is always a core element.

We next provide a linear program such that its optimal objective value can be used to determine whether a given cooperative game is a risk allocation game or not. If the game is a risk allocation game, then an optimal solution to the linear program yields a risk environment that generates the game. We then show how to use the linear program to characterize all risk environments that generate a given totally balanced game.

At last, we focus on games where only the *distribution* of values is uncertain, while the value of the aggregate portfolio is constant over all states of nature. This case is relevant for situations where the risk of the aggregate portfolio is low compared to the risk involved in the individual portfolios. We show that the class of risk allocation games with no aggregate uncertainty coincides with the class of *exact games* (Schmeidler, 1972). As evidenced by the previous paragraphs, there are many applications giving rise to the class of totally balanced games; to the best of our knowledge, risk allocation with no aggregate uncertainty is the first application that leads to the class of exact games.

The fact that each risk allocation game is exact implies that for each coalition there is a core element such that the coalition only gets its stand-alone value. This means that in the case of no aggregate uncertainty, this coalition does not necessarily benefit from the diversification opportunities offered by the aggregate portfolio. As a consequence, the regulator has a high level of discretion in allocating the risk to the individual portfolios.

The structure of the paper is as follows. First we introduce coherent measures of risk, transferable utility games, and risk allocation games. In Section 3 we prove that the class of risk allocation games coincides with the class of totally balanced games and investigate our constructive proof by linear programming. In Section 4 we show that the class of risk allocation games with no aggregate uncertainty coincides with the class of exact games. In Section 5 we conclude.

2 Preliminaries

2.1 Coherent Measures of Risk

Consider the set \mathbb{R}^S of realization vectors, where S denotes the number of states of nature. State of nature s occurs with probability $p_s > 0$ and $\sum_{s=1}^S p_s = 1$. The vector $X \in \mathbb{R}^S$ represents a portfolio's (firm's, insurance company's, bank's, etc.) possible profit and loss realizations on a common chosen future time horizon, say at $t = 1$. The amount X_s is the portfolio's

payoff in state of nature s . Negative values of X_s correspond to losses. The inequality $Y \geq X$ means that $Y_s \geq X_s$ for all $s = 1, \dots, S$.

A *measure of risk* is a function $\rho : \mathbb{R}^S \rightarrow \mathbb{R}$ measuring the risk of a portfolio from the perspective of the present ($t = 0$). It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a *reference instrument* today, such that it ensures that the risk involved in the portfolio is acceptable to the regulator. We assume that the reference instrument has payoff 1 in each state of nature at $t = 1$, thus its realization vector is $1^S = (1, \dots, 1)^\top$. The reference instrument is riskless in the “classical sense,” having no uncertainty in its payoffs. It is most natural to think of it as a zero coupon bond. The price of the reference instrument is denoted by $\delta \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. We adjust the definition of coherent measures of risk to the discrete case with realization vectors as follows.

Definition 2.1. A function $\rho : \mathbb{R}^S \rightarrow \mathbb{R}$ is called a *coherent measure of risk* (Artzner et al., 1999) if it satisfies the following axioms.

1. *Monotonicity:* for all $X, Y \in \mathbb{R}^S$ such that $Y \geq X$, we have $\rho(Y) \leq \rho(X)$.
2. *Subadditivity:* for all $X, Y \in \mathbb{R}^S$, we have $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
3. *Positive homogeneity:* for all $X \in \mathbb{R}^S$ and $h \in \mathbb{R}_+$, we have $\rho(hX) = h\rho(X)$.
4. *Translation invariance:* for all $X \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we have $\rho(X + a1^S) = \rho(X) - \delta a$.

Acerbi (2002) treats a subclass of coherent measures of risk: *spectral measures of risk*. The definition of spectral measures of risk with equiprobable outcomes, i.e. $p_1 = \dots = p_S = 1/S$, is as follows.

Let us introduce the ordered statistics $X_{s:S}$ given by the ordered values of the S -tuple X_1, \dots, X_S , that is $\{X_{1:S}, \dots, X_{S:S}\} = \{X_1, \dots, X_S\}$ and $X_{1:S} \leq X_{2:S} \leq \dots \leq X_{S:S}$.

Definition 2.2. Let the outcomes be equiprobable. Consider a vector $\phi \in \mathbb{R}^S$. The measure of risk $M_\phi : \mathbb{R}^S \rightarrow \mathbb{R}$ defined by

$$M_\phi(X) = -\delta \sum_{s=1}^S \phi_s X_{s:S} \tag{1}$$

is a *spectral measure of risk* if $\phi \in \mathbb{R}^S$ satisfies the conditions

1. Nonnegativity: $\phi_s \geq 0$ for all $s = 1, \dots, S$,
2. Normalization: $\sum_{s=1}^S \phi_s = 1$,
3. Monotonicity : ϕ_s is non-increasing, that is $\phi_{s_1} \geq \phi_{s_2}$ if $s_1 < s_2$ and $s_1, s_2 \in \{1, \dots, S\}$.

Spectral measures of risk are discounted weighted average losses, with non-increasing weights, with the highest weight on the worst outcome. The weight vector ϕ is the so-called *risk spectrum*, the “attitude” toward risk. An important example of a spectral measure of risk is the k -expected shortfall.

Definition 2.3. Let the outcomes be equiprobable and let $k \in \{1, \dots, S\}$. The k -expected shortfall of the realization vector X is defined by

$$\text{ES}_k(X) = -\delta \sum_{s=1}^k \frac{1}{k} X_{s:S}. \quad (2)$$

The k -expected shortfall is the discounted average of the worst k outcomes. For a detailed discussion see Acerbi and Tasche (2002).

2.2 Transferable Utility Games

Let $N = \{1, \dots, n\}$ denote a finite set of players. A value function $v : 2^N \rightarrow \mathbb{R}$ with $v(\{\emptyset\}) = 0$ gives rise to a cooperative game with transferable utility (game, for short) (N, v) . Let Γ denote the set of games with n players. An allocation is a vector $x \in \mathbb{R}^n$, where x_i is the payoff of player $i \in N$. An allocation x yields payoff $x(C) = \sum_{i \in C} x_i$ to a coalition $C \in 2^N$. An allocation $x \in \mathbb{R}^n$ is called *efficient*, if $x(N) = v(N)$; *individually rational*, if $x_i \geq v(\{i\})$ for all $i \in N$, and *coalitionally rational* if $x(C) \geq v(C)$ for all $C \in 2^N$. The *core* is the set of efficient and coalitionally rational allocations.

For each $C \in 2^N$ let $a(C) \in \mathbb{R}^n$ be the membership vector, $a_i(C) = 1$ for $i \in C$ and $a_i(C) = 0$ otherwise.

Definition 2.4. A *balanced vector of weights* is a vector $(\lambda^C)_{C \in 2^N} \in \mathbb{R}_+^{2^N}$ such that $\sum_{C \in 2^N} \lambda^C a(C) = a(N)$. A game (N, v) is *balanced* if $\sum_{C \in 2^N} \lambda^C v(C) \leq v(N)$ for all balanced vectors of weights.

A well-known interpretation of balancedness is that the players can distribute one unit of working time to any coalition and if each coalition is active during a fraction λ^C of a unit of time then the players cannot generate more value than $v(N)$, the value of the grand coalition. Balancedness is a necessary and sufficient condition for non-emptiness of the core in a transferable

utility game (Bondareva, 1963; Shapley, 1967). See Predtetchinski and Herings (2004) for an extension of the concept of balancedness to be necessary and sufficient for non-emptiness of the core in non-transferable utility games.

For a game (N, v) and a coalition $C \in 2^N$, a *subgame* (C, v^C) is obtained by restricting v to subsets of C .

Definition 2.5. A game (N, v) is *totally balanced* if for every $D \in 2^N$ its subgame (D, v^D) is balanced, that is, if for all $D \in 2^N$ and for all vectors $(\lambda^C)_{C \in 2^D} \in \mathbb{R}_+^{2^D}$ satisfying $\sum_{C \in 2^D} \lambda^C a(C) = a(D)$, we have $\sum_{C \in 2^D} \lambda^C v(C) \leq v(D)$.

In a totally balanced game every subgame has a non-empty core. Let Γ_{tb} denote the family of totally balanced games with n players. An interesting subclass of totally balanced games is the class of *exact games* (Schmeidler, 1972).

Definition 2.6. A game (N, v) is *exact* if for each $C \in 2^N$ there exists a core allocation x such that $x(C) = v(C)$.

Csóka, Herings, and Kóczy (2007a) introduce the concept of overbalancedness.

Definition 2.7. An *overbalanced vector of weights* is a vector $(\lambda^C)_{C \in 2^N} \in \mathbb{R}_+^{2^N}$ such that $\sum_{C \in 2^N \setminus \{D, N\}} \lambda^C a(C) = a(N) + \lambda^D a(D)$ for some $D \in 2^N$. A game (N, v) is *overbalanced* if $\sum_{C \in 2^N \setminus \{D, N\}} \lambda^C v(C) \leq v(N) + \lambda^D v(D)$ for all overbalanced vectors of weights.

A natural interpretation of overbalancedness is that if one coalition is forced to exist for a non-positive amount of time ($-\lambda^D \leq 0$), which requires its players to spend at least one unit of time in the other coalitions, the players cannot generate more value than $v(N)$.

Let Γ_e denote the family of exact games with n players. Csóka, Herings, and Kóczy (2007a) give the following necessary and sufficient condition for a game to be exact.

Theorem 2.8. *The game (N, v) belongs to Γ_e if and only if it is totally balanced and overbalanced.*

Convex games are a subset of exact games (Schmeidler, 1972).

Definition 2.9. A game (N, v) is *convex* if for all $C, D \in 2^N$ we have that $v(C) + v(D) \leq v(C \cup D) + v(C \cap D)$.

Let Γ_c denote the family of convex games with n players. We have that $\Gamma \supseteq \Gamma_{\text{tb}} \supseteq \Gamma_e \supseteq \Gamma_c$.

2.3 Risk Allocation Games

Denault (2001) introduces *risk capital allocation problems*: Suppose a firm has n portfolios, the matrix of their realization vectors is given by $X \in \mathbb{R}^{S \times n}$.¹ The question is how the risk of the firm measured by a coherent measure of risk has to be allocated to the portfolios.

Let $X_{\cdot i}$ denote the i -th column of X , the realization vector of portfolio i . Let X_s denote the row of X corresponding to state of nature s , $X_{s,i}$ its element at row s and column i , and $(X_{s,i})_{i \in D}$ the row vector corresponding to state of nature s with elements $i \in D$. For a coalition of portfolios $C \in 2^N$, let $X(C) = \sum_{i \in C} X_{\cdot i}$ and $X_s(C) = \sum_{i \in C} X_{s,i}$.

Denault (2001) assumes that the n -th portfolio equals $b \in \mathbb{R}$ units of reference instrument: $X_{\cdot n} = b1^S$. We will consider the slightly more general setting, where $X_{\cdot n}$ can be any portfolio. Moreover, we make a distinction between the risk environment and the induced game.

Definition 2.10. A *risk environment* is a tuple (N, S, p, X, ρ) , where N is the set of portfolios, S indicates the number of states of nature, $p = (p_1, \dots, p_S)$ is the vector of realization probabilities of the various states, X is the matrix of realization vectors, and ρ is a coherent measure of risk.

Definition 2.11. Given a risk environment (N, S, p, X, ρ) a *risk allocation game* is a game (N, v) , where the value function $v : 2^N \rightarrow \mathbb{R}$ is defined by

$$v(C) = -\rho(X(C)) \text{ for all } C \in 2^N. \quad (3)$$

A risk allocation game with n players is induced by the number of states of nature, their probability of occurrence, n realization vectors and a coherent measure of risk. Let Γ_r denote the family of risk allocation games with n players. In such a game, according to Equation (3), the larger the risk of any subset of portfolios, the lower its value.

If the rows of a matrix of realization vectors sum up to the same number, then there is *no aggregate uncertainty*. Formally:

Definition 2.12. A matrix of realization vectors $X \in \mathbb{R}^{S \times n}$ has *no aggregate uncertainty* if there exists a number $\alpha \in \mathbb{R}$ such that $X(N) = \alpha 1^S$.

Let Γ_{rnau} denote the family of risk allocation games with n players with no aggregate uncertainty. Obviously, $\Gamma_{\text{rnau}} \subseteq \Gamma_r$. We first study risk allocation games in general, then with no aggregate uncertainty.

¹Denault (2001) uses continuously distributed random variables. We adjust his setting to the more tractable setup with discrete random variables, resulting in realization vectors.

3 Total Balancedness

3.1 Risk Allocation Games and Totally Balanced Games

Denault (2001, Theorem 4) shows that the family of risk capital allocation problems is balanced. As a subgame of a risk allocation game is also a risk allocation game, we can adjust his proof to show that risk allocation games are totally balanced.

Proposition 3.1. *All games $(N, v) \in \Gamma_r$ are totally balanced, $\Gamma_r \subseteq \Gamma_{tb}$.*

Proof. Consider a risk environment (N, S, p, X, ρ) inducing the game (N, v) . We show that for any $D \in 2^N$, the subgame (D, v^D) is balanced. Take any $(\lambda^C)_{C \in 2^D} \in \mathbb{R}_+^{2^D}$ such that $\sum_{C \in 2^D} \lambda^C a(C) = a(D)$. Then by Equation (3) and the positive homogeneity and subadditivity of ρ we have that

$$\begin{aligned} \sum_{C \in 2^D} \lambda^C v^D(C) &= - \sum_{C \in 2^D} \rho(\lambda^C X(C)) \leq -\rho\left(\sum_{C \in 2^D} \left(\sum_{i \in C} \lambda^C X_{.i}\right)\right) \\ &= -\rho\left(\sum_{i \in D} \left(\sum_{C \in 2^D, C \ni i} \lambda^C X_{.i}\right)\right) = -\rho\left(\sum_{i \in D} X_{.i}\right) = -\rho(X(D)) \\ &= v^D(D), \end{aligned}$$

where the last line follows from rearranging the summation and using the fact that we have a balanced vector of weights. Thus (D, v^D) is balanced. \square

Not only is it true that all risk allocation games are totally balanced, but also any totally balanced game can be generated by a risk allocation game. We illustrate Proposition 3.2 and its proof by Example 3.3.

Proposition 3.2. *Each game $(N, v) \in \Gamma_{tb}$ is induced by some risk environment (N, S, p, X, ρ) , so $\Gamma_{tb} \subseteq \Gamma_r$.*

Proof. Take any game $(N, v) \in \Gamma_{tb}$. The zero-normalized value function v_0 corresponding to v is defined by

$$v_0(C) = v(C) - \sum_{i \in C} v(\{i\}), \quad C \in 2^N. \quad (4)$$

It is well-known that $(N, v_0) \in \Gamma_{tb}$. Using the singletons with weights 1 it follows from the total balancedness of v_0 that for any $C \in 2^N$

$$0 = \sum_{i \in C} v_0(\{i\}) \leq v_0(C). \quad (5)$$

Moreover, any $C \in 2^N$ partitions N into C and $N \setminus C$, and using weights 1 on C and $N \setminus C$ leads to

$$v_0(C) + v_0(N \setminus C) \leq v_0(N). \quad (6)$$

Using Equations (5) and (6) we obtain that for any $C \in 2^N$

$$0 \leq v_0(C) \leq v_0(N). \quad (7)$$

The remainder of the proof is constructive. We specify the risk environment (N, S, p, X^0, ρ) as follows. We introduce a state of nature for all non-empty coalitions of N , so $S = 2^N - 1$. We label states of nature by $C, D \in 2^N \setminus \{\emptyset\}$. We consider equiprobably outcomes, $p_1 = \dots = p_S = 1/S$, and we let ρ be the risk measure equal to the 1-expected shortfall (Definition 2.3) with $\delta = 1$. For each state of nature $C \in 2^N \setminus \{\emptyset\}$, let the row vector X_C^0 be such that

$$(X_{C,i}^0)_{i \in C} \text{ belongs to the core of } (C, v_0^C), \quad (8)$$

$$X_{C,i}^0 = v_0(N), \quad i \in N \setminus C. \quad (9)$$

The risk environment (N, S, p, X^0, ρ) induces the game (N, \bar{v}_0) . We will show that $\bar{v}_0 = v_0$.

By the definition of 1-expected shortfall, we have

$$\bar{v}_0(C) = -\rho(X^0(C)) = \min_{D \in 2^N \setminus \{\emptyset\}} X_D^0(C), \quad C \in 2^N. \quad (10)$$

The definition of a subgame, Equation (8), and the efficiency of a core element imply

$$v_0^C(C) = v_0(C) = X_C^0(C), \quad C \in 2^N \setminus \{\emptyset\}. \quad (11)$$

We show next that

$$X_C^0(C) \leq X_D^0(C), \quad C, D \in 2^N \setminus \{\emptyset\}. \quad (12)$$

Indeed, if $D \supseteq C$ then Inequality (12) follows from (8) as we have for a core element $(X_{D,i}^0)_{i \in D}$ in subgame (D, v_D^0) that

$$X_C^0(C) = v_0(C) \leq X_D^0(C). \quad (13)$$

If $D \not\supseteq C$ then one of the components of $(X_{D,i}^0)_{i \in C}$ is $v_0(N)$, and using Equation (7) Inequality (12) follows immediately. Combining Equations (11) and (12) with Equation (10) we obtain that $\bar{v}_0 = v_0$.

By using the matrix of realization vectors X defined by $X_{.i} = X_{.i}^0 + v(\{i\})1^S, i \in N$, we obtain a risk environment that induces the game (N, v) . \square

Example 3.3. We illustrate the construction used in the proof of Proposition 3.2 in an example with 3 players. Table 1 presents the value function v of a totally balanced game, as well as the zero-normalized value function v_0 corresponding to v . Note that Inequality (7) is satisfied by v_0 .

| C | $v(C)$ | $v_0(C)$ |
|---------------|--------|----------|
| $\{1\}$ | -10 | 0 |
| $\{2\}$ | 3 | 0 |
| $\{3\}$ | -2 | 0 |
| $\{1, 2\}$ | -4 | 3 |
| $\{1, 3\}$ | -6 | 6 |
| $\{2, 3\}$ | 2 | 1 |
| $\{1, 2, 3\}$ | -1 | 8 |

Table 1: A totally balanced game and its zero-normalized game.

In Table 2 we have specified the matrix of realization vectors X^0 according to requirements (8) and (9). For instance, for $C = \{1, 2\}$ we have that $(X_{\{1,2\},1}^0, X_{\{1,2\},2}^0) = (1, 2)$ is a point in the core of the subgame with players 1 and 2, and $X_{\{1,2\},3}^0 = 8 = v_0(N)$.

It is easy to check that the risk environment specified by X^0 and the risk measure equal to the 1-expected shortfall with $\delta = 1$ generate v_0 .

To generate the value function v , we transform X^0 into X by specifying $X_i = X_i^0 + v(\{i\})1^S$ for all $i \in N$. The risk environment corresponding to X and the risk measure equal to the 1-expected shortfall with $\delta = 1$ can be verified to induce the game (N, v) .

| S | $X_{.1}^0$ | $X_{.2}^0$ | $X_{.3}^0$ | $X_{.1}$ | $X_{.2}$ | $X_{.3}$ |
|---------------|------------|------------|------------|----------|----------|----------|
| $\{1\}$ | 0 | 8 | 8 | -10 | 11 | 6 |
| $\{2\}$ | 8 | 0 | 8 | -2 | 3 | 6 |
| $\{3\}$ | 8 | 8 | 0 | -2 | 11 | -2 |
| $\{1, 2\}$ | 1 | 2 | 8 | -9 | 5 | 6 |
| $\{1, 3\}$ | 2 | 8 | 4 | -8 | 11 | 2 |
| $\{2, 3\}$ | 8 | 1 | 0 | -2 | 4 | -2 |
| $\{1, 2, 3\}$ | 2 | 1 | 5 | -8 | 4 | 3 |

Table 2: Payoff matrices for the zero normalized and the original games.

Note that in our constructive proof the statement of Proposition 3.2 is strengthened in the sense that the family of games induced by risk environ-

ments with $S \leq 2^n - 1$ and the risk measure equal to the 1-expected shortfall with $\delta = 1$ coincides with the family of totally balanced games with n players, that is any totally balanced game can be generated by a properly specified risk environment with the 1-expected shortfall and $2^n - 1$ states of nature. From Propositions 3.1 and 3.2 we derive the following theorem.

Theorem 3.4. *The class of risk allocation games coincides with the class of totally balanced games, $\Gamma_r = \Gamma_{\text{tb}}$.*

Kalai and Zemel (1982b) use a similar construction to show that a game is totally balanced if and only if it is the minimum game of a finite collection of *additive* games. A game (N, v) is called *additive* if there exists a set of real numbers b_1, \dots, b_n such that for every $C \in 2^N$, $v(C) = \sum_{i \in C} b_i$. For a finite collection of games $\{v_t\}_{t \in T}$ the *minimum game* is defined by $(\min v_t)(C) = \min_{t \in T} v_t(C)$. It is easy to see that the totally balanced game v in Table 1 is equal to the minimum game of the collection of additive games generated by $X_C, C \in 2^N \setminus \{\emptyset\}$, in Table 2.

3.2 Linear Programming Results

Consider a totally balanced game $(N, v) \in \Gamma_{\text{tb}}$. Throughout the subsection, we choose $S = 2^n - 1$, $p_1 = \dots = p_S = 1/S$, and ρ the risk measure equal to 1-expected shortfall with $\delta = 1$, just like in Proposition 3.2. Whenever we write v is generated by a matrix of realization vectors X we mean that the risk allocation game induced by the risk environment (N, S, p, X, ρ) equals (N, v) .

In the proof of Proposition 3.2 the matrix of realization vectors X generating v was constructed using the *core requirement*² (8): for every $C \in 2^N \setminus \{\emptyset\}$

$$(X_{C,i})_{i \in C} \text{ belongs to the core of } (C, v^C). \quad (14)$$

The other elements of X were chosen to be sufficiently large.

Let a game $(N, v) \in \Gamma$ be given. We develop a linear programming problem such that the optimal value of the objective function will exceed $\sum_{C \in 2^N} v(C)$ if $(N, v) \in \Gamma \setminus \Gamma_{\text{tb}}$ and is equal to $\sum_{C \in 2^N} v(C)$ whenever the game is totally balanced. Moreover, in the latter case the matrices derived from the linear program's optimal solutions generate v .

To do so, given a matrix X we define the vector $\hat{X} \in \mathbb{R}^{S^n}$ by juxtaposing the rows of $X \in \mathbb{R}^{S \times n}$, that is $\hat{X} = (X_1, X_2, \dots, X_S)^\top \in \mathbb{R}^{S^n}$, and the

²There we had a zero normalized game, but it is easy to see that after renormalizing the core requirement is still satisfied.

reverse operation transforms a vector $\hat{X} \in \mathbb{R}^{Sn}$ into a matrix $X \in \mathbb{R}^{S \times n}$. We will use the notations \hat{X} and X interchangeably.

Let $0^n = (0, 0, \dots, 0) \in \mathbb{R}^{1 \times n}$ be the n -dimensional row vector of zeros. For every $C \in 2^N \setminus \{\emptyset\}$, we define the matrices

$$A(C) = \begin{pmatrix} a(C)^\top & 0^n & & 0^n \\ 0^n & a(C)^\top & & 0^n \\ & & \ddots & \vdots \\ 0^n & 0^n & \dots & a(C)^\top \end{pmatrix} \in \mathbb{R}^{S \times Sn} \quad (15)$$

containing the membership vector $a(C)$ transposed along the “diagonal” and 0^n otherwise.

A matrix $X \in \mathbb{R}^{S \times n}$ generates v if and only if for every $C \in 2^N \setminus \{\emptyset\}$

$$v(C) = \min_{D \in 2^N \setminus \{\emptyset\}} X_D(C). \quad (16)$$

Equation (16) can be rewritten as

$$v(C) = \min_{D \in 2^N} A_D(C) \hat{X}, \quad (17)$$

where $A_D(C)$ denotes the D -th row of $A(C)$. It follows from Equation (17) that X generates v if and only if for every $C \in 2^N \setminus \{\emptyset\}$

$$A(C) \hat{X} \geq v(C) 1^S, \quad (18)$$

where for each C at least one inequality holds with equality.

We introduce some additional notation. Let

$$E = (a(\{1\})^\top, a(\{2\})^\top, \dots, a(N)^\top) \in \mathbb{R}^{1 \times Sn},$$

$$V = \begin{pmatrix} v(\{1\}) 1^S \\ v(\{2\}) 1^S \\ \vdots \\ v(N) 1^S \end{pmatrix} \in \mathbb{R}^{S^2}, \quad (19)$$

and

$$A = \begin{pmatrix} A(\{1\}) \\ A(\{2\}) \\ \vdots \\ A(N) \end{pmatrix} \in \mathbb{R}^{S^2 \times Sn}. \quad (20)$$

Consider the linear programming problem (P_v) :

$$\begin{aligned} \min E\hat{X} \\ \text{s. t.} \quad A\hat{X} &\geq V \\ \hat{X} &\in \mathbb{R}^{S^n} \end{aligned} \quad (P_v).$$

The objective function of (P_v) captures the constructive proof of Proposition 3.2, as it is minimizing exactly the sum of those elements of \hat{X} which are used in the core requirement (14). Using Equation (18) it can be seen that the feasibility constraints in the linear program are the necessary requirements for v to be generated by a feasible solution.

The set of optimal solutions of (P_v) is non-empty, since $\hat{X} = (k, \dots, k) \in \mathbb{R}^{S^n}$ is a feasible solution, where $k = \max_{C \in 2^N} v(C)$ and the set of feasible solutions is bounded from below. We denote the set of optimal solutions of (P_v) by \mathcal{X}_v^* and a particular optimal solution by $\hat{X}^* \in \mathbb{R}^{S^n}$.

Proposition 3.5. *Consider a game $(N, v) \in \Gamma$ and an optimal solution $\hat{X}^* \in \mathcal{X}_v^*$ of (P_v) . The optimal value of the objective function $E\hat{X}^*$ equals $\sum_{C \in 2^N} v(C)$ if and only if v is generated by X^* .*

Proof.

(\Rightarrow) By the feasibility constraints it holds that

$$A_D(C)\hat{X}^* \geq v(C), \quad C \in 2^N \setminus \{\emptyset\}, D \in 2^N \setminus \{\emptyset\}, \quad (21)$$

Since by assumption $E\hat{X}^* = \sum_{C \in 2^N} v(C)$, it follows that

$$A_C(C)\hat{X}^* = v(C), \quad C \in 2^N \setminus \{\emptyset\}. \quad (22)$$

It follows by Inequality (21) and Equation (22) that v is generated by X^* .

(\Leftarrow) We use a proof by contradiction. Suppose $E\hat{X}^* \neq \sum_{C \in 2^N} v(C)$. By the feasibility constraints it holds that

$$E\hat{X}^* > \sum_{C \in 2^N} v(C). \quad (23)$$

Note that $\min_{D \in 2^N \setminus \{\emptyset\}} X_D^*(C)$ is attained in row C of X^* , since otherwise we could decrease the objective function by substituting the row attaining the minimum for row C . Combining this with Equation (23) we obtain that there exists a coalition $C \in 2^N$ such that

$$\min_{D \in 2^N \setminus \{\emptyset\}} X_D^*(C) > v(C), \quad (24)$$

which implies that v is not generated by X^* , a contradiction. \square

Take any matrix of realization vectors $X \in \mathbb{R}^{z \times n}$, where z is a strictly positive integer. Let $Y(X) \in \mathbb{R}^{(2^N - 1) \times n}$ denote a matrix in which for all $C \in 2^N \setminus \{\emptyset\}$ we have that $Y_C(X) = X_{B.}$, where $B \in \arg \min_{D \in 2^N \setminus \{\emptyset\}} X_D(C)$. The following proposition claims that the realization matrix X generates v if and only if $\hat{Y}(X)$ is an optimal solution of (P_v) .

Proposition 3.6. *Consider a game $(N, v) \in \Gamma_{\text{tb}}$. The matrix of realization vectors $X \in \mathbb{R}^{z \times n}$ generates v if and only if $\hat{Y}(X) \in \mathcal{X}_v^*$.*

Proof.

(\Rightarrow) Since X generates v , for all $C \in 2^N \setminus \{\emptyset\}$ there exists a state of nature $s(C)$ such that

$$\begin{aligned} X_{s(C)}(C) &= v(C), \\ X_s(C) &\geq v(C), \quad s \in \{1, \dots, S\}. \end{aligned}$$

It follows that $\hat{Y}(X)$ is a feasible and optimal solution of (P_v) .

(\Leftarrow) Since the game (N, v) is totally balanced, according to Theorem 3.4 v is generated by some matrix of realization vectors. By Proposition 3.5, it follows that v is generated by all elements of \mathcal{X}_v^* , so $Y(X)$ generates v , and by construction X generates v . \square

The following result shows that any matrix of realization vectors X that generates v satisfies (14).

Proposition 3.7. *Consider a game $(N, v) \in \Gamma_{\text{tb}}$. Any optimal solution of (P_v) $\hat{X}^* \in \mathcal{X}_v^*$ satisfies the core requirement (14).*

Proof. Take any $\hat{X}^* \in \mathcal{X}_v^*$. Since by Proposition 3.2 all totally balanced games can be generated, we know by Proposition 3.5 that $E\hat{X}^* = \sum_{C \in 2^N} v(C)$. For every $C \in 2^N \setminus \{\emptyset\}$, feasibility requires that $A_C(C)\hat{X}^* \geq v(C)$, so

$$A_C(C)\hat{X}^* = v(C). \tag{25}$$

The equalities in (25) together with the feasibility constraints imply that the rows of X^* contain core allocations of the respective subgames. \square

Propositions 3.6 and 3.7 imply that if a game is generated by $X \in \mathbb{R}^{z \times n}$, then $Y(X)$ satisfies the core requirement (14). Thus to generate a given totally balanced game the rows of the matrix of realization vectors can be permuted and some of them can be combined, but essentially the core requirement (14) is satisfied in all of them.

4 Exactness

In this section we show that if there is *no aggregate uncertainty* in a risk environment, then the induced risk allocation game is an exact game, and conversely all exact games can be generated by a properly specified risk environment with no aggregate uncertainty. Proposition 4.1 claims that risk allocation games with no aggregate uncertainty are exact.

Proposition 4.1. *All games with $(N, v) \in \Gamma_{\text{rnau}}$ are exact, $\Gamma_{\text{rnau}} \subseteq \Gamma_e$.*

Proof. Consider the risk environment (N, S, p, X, ρ) , where X has no aggregate uncertainty. We show that the induced risk allocation game is totally balanced and overbalanced (Definition 2.7), so exact by Theorem 2.8. Total balancedness follows from Proposition 3.1. For overbalancedness take any vector $(\lambda^C)_{C \in 2^N} \in \mathbb{R}_+^{2^N}$ such that $\sum_{C \in 2^N \setminus \{D, N\}} \lambda^C a(C) = a(N) + \lambda^D a(D)$. Then by Equation (3), the positive homogeneity, and subadditivity of ρ we have that

$$\begin{aligned}
 \sum_{C \in 2^N \setminus \{D, N\}} \lambda^C v(C) &= - \sum_{C \in 2^N \setminus \{D, N\}} \rho\left(\sum_{i \in C} \lambda^C X_i\right) \\
 &\leq -\rho\left(\sum_{C \in 2^N \setminus \{D, N\}} \left(\sum_{i \in C} \lambda^C X_i\right)\right) \\
 &= -\rho\left(\sum_{i \in N} \left(\sum_{C \ni i, C \in 2^N \setminus \{D, N\}} \lambda^C X_i\right)\right) \\
 &= -\rho\left(\sum_{i \in N} X_i + \lambda^D \sum_{i \in D} X_i\right), \tag{26}
 \end{aligned}$$

where the last two lines follow from rearranging the summation and using the fact that we have an overbalanced vector of weights, thus if $i \in D$ then $\sum_{C \ni i, C \in 2^N \setminus \{D, N\}} \lambda^C = 1 + \lambda^D$, and if $i \notin D$ then $\sum_{C \ni i, C \in 2^N \setminus \{D, N\}} \lambda^C = 1$. Using translation invariance and positive homogeneity, Inequality (26) can be rewritten as

$$\begin{aligned}
 \sum_{C \in 2^N \setminus \{D, N\}} \lambda^C v(C) &\leq -\rho\left(\sum_{i \in N} X_i + \lambda^D \sum_{i \in D} X_i\right) \\
 &= -\rho(X(N)) - \rho(\lambda^D X(D)) \\
 &= -\rho(X(N)) - \lambda^D \rho(X(D)) \\
 &= v(N) + \lambda^D v(D), \tag{27}
 \end{aligned}$$

thus we have an overbalanced game. \square

Proposition 4.2 show that each exact game is generated by some risk environment with no aggregate uncertainty.

Proposition 4.2. *Each game $(N, v) \in \Gamma_e$ is induced by some risk environment (N, S, p, X, ρ) such that X has no aggregate uncertainty, $\Gamma_e \subseteq \Gamma_{\text{rnau}}$.*

Proof. Consider the exact game $(N, v) \in \Gamma_e$. We specify the risk environment (N, S, p, X, ρ) as follows. We introduce a state of nature for all proper non-empty subcoalitions of N , thus $S = 2^n - 2$. Let $p_1 = \dots = p_S = 1/S$, and let ρ be the 1-expected shortfall with $\delta = 1$. For all $C \in 2^N \setminus \{\emptyset\}$ there exist a core element x_C such that $x_C(C) = v(C)$ since (N, v) is exact. Construct $X \in \mathbb{R}^{S \times n}$ as follows. We define, for all $C \in 2^N \setminus \{\emptyset, N\}$, $X_C = x_C$. Since x_C is a core element, it holds that $X_C(N) = v(N)$, thus X has no aggregate uncertainty. We denote the game induced by the risk environment by \bar{v} . Now we have for every $C \in 2^N \setminus \{\emptyset\}$ that

$$\bar{v}(C) = \min_{D \in 2^N \setminus \{\emptyset, N\}} X_D(C) = v(C), \quad (28)$$

thus $\bar{v} = v$. □

Note that in the proof of Proposition 4.2 the sum of the entries in each row of X is equal to $v(N)$. That is why we need only $2^n - 2$ states of nature.

Combining Propositions 4.1 and 4.2 we have the following theorem.

Theorem 4.3. *The class of risk allocation games with no aggregate uncertainty coincides with the class of exact games, $\Gamma_{\text{rnau}} = \Gamma_e$.*

Csóka, Herings, and Kóczy (2007a) show that if there are less than four players then the class of exact games coincides with the class of convex games. Using this result Theorem 4.3 can be reformulated as follows.

Theorem 4.4. *Let $n \in \mathbb{N}$ be such that $n < 4$. Then the class of risk allocation games with n portfolios and no aggregate uncertainty coincides with the class of convex games with n players, $\Gamma_{\text{rnau}} = \Gamma_c$.*

Theorem 4.4 is illustrated by the following example.

Example 4.5. In this example we show how a 3-player convex game is generated by a risk allocation game with no aggregate uncertainty. Note that the game in Table 1 of Example 3.3 is not convex since $v(\{1, 2\}) + v(\{1, 3\}) = -4 - 6 = -10 > v(\{1\}) + v(\{1, 2, 3\}) = -10 - 1 = -11$. However, by changing $v(\{1, 2\})$ to -5 we get the convex game displayed in Table 3. This game is generated by the risk environment with matrix of realization vectors X depicted in Table 4 and the risk measure of 1-expected shortfall with $\delta = 1$.

| C | $v(C)$ |
|---------------|--------|
| $\{1\}$ | -10 |
| $\{2\}$ | 3 |
| $\{3\}$ | -2 |
| $\{1, 2\}$ | -5 |
| $\{1, 3\}$ | -6 |
| $\{2, 3\}$ | 2 |
| $\{1, 2, 3\}$ | -1 |

Table 3: The value function of a convex game v .

| S | $X_{.1}$ | $X_{.2}$ | $X_{.3}$ | $\sum_{i \in N} X_{s,i}$ |
|-----|----------|----------|----------|--------------------------|
| 1 | -3 | 4 | -2 | -1 |
| 2 | -7 | 3 | 3 | -1 |
| 3 | -10 | 5 | 4 | -1 |

Table 4: A matrix of realization vectors generating v .

Notice that the rows of X correspond to appropriately chosen marginal contribution vectors. For instance, in the first row of X we have the marginal contribution vector corresponding to the permutation $(3,2,1)$: $v(\{3\}) - v(\{\emptyset\}) = -2 - 0 = -2$, $v(\{2, 3\}) - v(\{2\}) = 2 - (-2) = 4$, and $v(\{1, 2, 3\}) - v(\{2, 3\}) = -1 - 2 = -3$. At any marginal contribution vector, there are n coalitions that exactly receive their value. Thus to generate a convex game fewer states of nature are required than $2^n - 1$. In the example 3 states of nature suffice. Also note that all rows of X sum up to -1, since the sum of the marginal contributions is always the value of the grand coalition, and there is no aggregate uncertainty.

Similarly to Proposition 3.6 we can characterize all the risk environments that generate a given exact game.

Proposition 4.6. *Consider a game $(N, v) \in \Gamma_e$. The matrix of realization vectors $X \in \mathbb{R}^{z \times n}$ without aggregate uncertainty generates v if and only if $\hat{Y}(X) \in \mathcal{X}_v^*$.*

Proof. Proposition 3.6 characterizes all the matrices that generate a given totally balanced game. Since by Proposition 4.1 only exact games can be generated with matrices satisfying no aggregate uncertainty, the proof is straightforward. \square

5 Conclusion

In this paper we have discussed transferable utility cooperative games derived from a risk environment: risk allocation games. We have shown that the class of risk allocation games coincides with the class of totally balanced games. This result makes sure that a regulator or performance evaluator can always allocate risk in a stable way: there will always be a core element, no matter how the risk environment is changing.

We have also studied the case when the aggregate portfolio has the same payoff in all states of nature. We proved that if there is no aggregate uncertainty then the class of risk allocation games equals the class of exact games, where for each coalition there is a core element such that the coalition gets only its stand-alone value. This means that if there is no aggregate uncertainty, then not necessarily everybody benefits from the diversification effects in a stable allocation of risk. The regulator or performance evaluator has much discretionary power in allocating risk, since for each coalition there is always a stable allocation of risk such that the coalition gets its stand-alone value.

We have characterized all the matrices of realization vectors that generate a given totally balanced game or a given exact game. In both cases the vectors derived from the matrices by juxtaposing their rows are related to the optimal solutions of a linear programming problem.

Denault (2001) shows that if a risk allocation game for an arbitrary matrix of realization vectors is convex then the risk measure by which it is induced is necessarily additive, thus the generated risk allocation game is also additive. However, by imposing some structure on the matrix of realization vectors we have proven the following theorem: If there are less than four players and the matrix of realization vectors has no aggregate uncertainty, then the generated risk allocation game is convex, and any convex game can be generated by such a risk environment.

References

- Acerbi, C., 2002. Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking and Finance* 26, 1505–1518.
- Acerbi, C., Tasche, D., 2002. On the coherence of expected shortfall. *Journal of Banking and Finance* 26, 1487–1504.
- Artzner, Ph. F., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. *Mathematical Finance* 9, 203–228.

- Bondareva, Olga N., 1963. Some applications of linear programming methods to the theory of cooperative games (in Russian). *Problemy Kybernetiki* 10, 119–139.
- Csóka, P., Herings, P. J. J., Kóczy, L. Á., 2007a. Balancedness conditions for exact games. *METEOR Research Memorandum RM07/040*, 1–13.
- Csóka, P., Herings, P. J. J., Kóczy, L. Á., 2007b. Coherent measures of risk from a general equilibrium perspective. *Journal of Banking and Finance* 31, 2517–2534.
- Denault, M., 2001. Coherent allocation of risk capital. *Journal of Risk* 4, 1–34.
- Dubey, P., Shapley, L. S., 1984. Totally balanced games arising from controlled programming problems. *Mathematical Programming* 29, 245–267.
- Kalai, E., Zemel, E., 1982a. Generalized network problems yielding totally balanced games. *Operations Research* 30, 998–1008.
- Kalai, E., Zemel, E., 1982b. Totally balanced games and games of flow. *Mathematics of Operations Research* 7, 476–478.
- Legut, J., 1990. On totally balanced games arising from cooperation in fair division. *Games and Economic Behavior* 2, 47–60.
- Owen, G., 1975. On the core of linear production games. *Mathematical Programming* 9, 358–370.
- Predtetchinski, A., Herings, P. J. J., 2004. A necessary and sufficient condition for the non-emptiness of the core of a non-transferable utility game. *Journal of Economic Theory* 116, 84–92.
- Schmeidler, D., 1972. Cores of exact games. *Journal of Mathematical Analysis and Applications* 40, 214–225.
- Shapley, L. S., 1967. On balanced sets and cores. *Naval Research Logistics Quarterly* 14, 453–460.
- Shapley, L. S., Shubik, M., 1969. On market games. *Journal of Economic Theory* 1, 9–25.
- Tijs, S., Parthasarathy, T., Potters, J., Prasad, V. Rajendra, 1984. Permutation games: Another class of totally balanced games. *OR Spektrum* 6, 119–123.