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The family of c-bisection auctions: efficiency and running time

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Summary. In this paper we analyze the performance of a recently proposed sequential auction, called the c-bisection auction, that can be used for a sale of a single indivisible object. We discuss the running time and the efficiency in the ex-post equilibrium of the auction. We show that by changing the parameter c of the auction we can trade off efficiency against running time. Moreover, we show that the auction that gives the desired level of efficiency in expectation takes the same number of rounds for any number of players.

Keywords and Phrases: query auction, efficiency, running time

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1 Introduction

A central issue in auction design is to set the rules of the auction so that some economic goals are achieved despite the fact that agents act based on self-interest. The economic goals are often expressed as functions of agents' preferences. Therefore auction design needs to elicit information on agent's preferences.

However requiring elicitation of full and exact preference information is undesirable for several reasons. First, agents may prefer not to reveal information on their valuations for reasons of privacy or long-term competitiveness [13]. Second, determining one's valuation with a precision up to the last digit can be computationally demanding [9, 12, 14]. Finally, the full revelation of agents' preferences may require a prohibitive amount of communication [5, 11].

Such considerations lead to an interest in auctions where agents need not reveal their information entirely but only partially. It has been recognized that multi-round mechanisms can reduce revelation and associated with it computation and communication, compared to single-round mechanisms advocated by the revelation principle [1, 3]. One class of multi-round mechanisms are query auctions. In a query auction the auctioneer sequentially queries the agents about specific aspects of their preferences. As an answer to the query an agent can chose one of a finite set of actions. Through incremental querying, the auctioneer gradually collects the information on agents' valuations. By using a query strategy in which previously revealed information guides the selection of subsequent queries, elicitation is focused on pertinent information. Incremental querying has been applied in different settings ([7, 2]) and it has been shown that only a small fraction of agents' valuation information needs to be revealed before the (approximately) optimal allocation can be determined [5, 8].

When evaluating the effectiveness of elicitation we may generally care about the running time expressed in the number of queries required to determine an optimal (according to the specific objective of the designer) allocation [15]. Since information about agents' valuation becomes more refined with each query, a higher number of queries leads to a better allocation. This has prompted researchers to examine the trade-off between the running time of query auctions and the level of allocation optimality. For example, in [4] the issue is considered for English auctions when restricting queries to discrete levels. The authors analyze how the choice of query levels in the English auction affects the expected revenue and the expected duration (measured in terms of the number of levels that the query price has been raised through).

The motivation for the topic of research in this paper is as follows. In [6] we study the uses and limitations of query auctions regarding the objective of economic efficiency maximization. In particular, we prove that in a setting with valuations distributed according to a continuous density function any ex-post equilibrium in an ex-post individual rational query auction that ends with positive probability after a finite number of queries, can not be fully efficient. This result implies that in the setting of continuous valuations full efficiency can only be achieved at the expense of an infinite running time of a query auction for almost all realizations of valuations. So the question that arises is: what price (in terms of running time) has to be paid for getting a desired level of approximate efficiency.

Thus, in this paper we are concerned with the issue of the trade-off between running time and allocation efficiency in the recently proposed c-bisection auction [6]. The proposed auction is a query mechanism in which players submit their bids in sequence with full knowledge of all previous bids of other players. The auction is characterized by a parameter c which, together with the distribution from which valuations of players are drawn, determines a sequence of query prices. In [6] equilibrium properties of this auction are analyzed. It is shown that the auction has an ex-post equilibrium, called the bluff equilibrium, and under this equilibrium the auction ends in finite time, regardless of the realization of players' valuation. Due to the result mentioned above we know that inefficiency of the bluff equilibrium for some realizations of valuations is inevitable.

In this paper we discuss in detail the performance of the c-bisection auction under the bluff equilibrium. In particular, we study how the choice of parameter c and the number of participating players affect the running time of the auction and its (in)efficiency.

First, we investigate the running time of the auction according to two measures, namely the expected number of rounds and the expected number of queries performed in the auction.[5] For both measures we derive first a recursive formula and give then an upper bound for the function defined by this formula. We prove that for a fixed c the expected number of rounds is bounded by a function that is logarithmic in the number of players while the expected number of queries is bounded by a function that is linear in the number of players.

Second, we analyze the level of inefficiency of the auction. As measures of inefficiency we employ the probability of inefficient allocation and the expected loss of welfare. For the probability of inefficient allocation we derive a recursive formula and prove that it's not more than c for any number of players. We show that when valuations are uniformly drawn from $[0, 1)$ the expected loss of welfare is bounded from above by c^2 for any number of players. It means that by choosing the appropriate c, the minimum level of efficiency can be determined by the auctioneer before it is known how many players will participate

[5] As a query we consider each separate question of the auctioneer to an active player. As a round we consider a sequence of queries in which each active player is asked to act exactly once.

in the auction. We also give a (more) precise estimate of the expected loss of welfare by using computer simulation.

Furthermore we show that for a fixed number of players there is a trade-off between efficiency and running time: for the increasing efficiency of the auction we have to pay by an increasing number of rounds and an increasing number of queries. By simulation it turns out that the trade-off curves, which show the relation between the expected number of rounds and the expected loss of inefficiency, constructed for different numbers of players coincide with each other. Thus, in expectation the number of rounds of the auction that obtains a desired level of efficiency is independent of the number of players.

The paper is organized as follows. Section 2 introduces the rules of the c-bisection auction. In Section 3 the running time of the auction is analyzed. Section 4 is devoted to the analysis of the efficiency of the auction. Concluding remarks about the trade-off are reported in Section 5. The Appendix contains proofs of some statements and tables with computational results.

2 The c-bisection auction

Suppose a single indivisible object is auctioned to a set $N = \{1, \ldots, n\}$ of players. The players have independent private valuations, v_i, drawn from a common continuous probability distribution with cumulative density $F(v)$, within the range $[\alpha, \beta)$. We assume quasi-linear utilities. Valuations of players are private information, i.e. each player knows only his own valuation but not the valuations of the others. Before the start of the auction there is a lottery that determines the order of the players. W.l.o.g. we assume that this ordering is $1 \prec 2 \prec 3 \prec \cdots \prec n-1 \prec n$. A player with a lower ranking is called a predecessor. So e.g. player 5 is a predecessor of player 7.

The auction runs for an a priori indefinite number of rounds. Each round is characterized by payment p_r, query price q_r, upper bound u_r and a set of active players A_r. The payment specifies the price to be payed if a player wins in this round. The query price is used by the auctioneer to ask an active player whether his valuation is larger than or equal to the query price. Players are queried openly in increasing order, so that a player can observe the bids of his predecessors. In each round the query price q_r is chosen from the open interval (p_r, u_r).

The initial set of active players is $A_1 = N$. The auction starts with $p_1 = \alpha$ and $u_1 = \beta$ and q_1 is a point in (α, β). Given the current set A_r, the payment p_r, the query price q_r, the upper bound u_r and the bids of players in round r the characteristics of the next round $r+1$ are defined as follows. If all players submit a *no* bid they all remain active, i.e. $A_{r+1} = A_r$. The payment remains the same and the upper bound is set to the previous query price, i.e. $p_{r+1} = p_r$ and $u_{r+1} = q_r$. If at least two players submit a *yes* bid, all players that said *yes* remain active. The upper bound remains the same and the payment is set equal to the previous query price, i.e. $u_{r+1} = u_r$ and $p_{r+1} = q_r$. The new bounds determine a new query price q_{r+1} in (p_{r+1}, u_{r+1}). If only one player submits a *yes* bid the auction stops, this player wins the auction and pays p_r. If such a moment doesn't occur, i.e. at least two players remain always active, the winner is determined according to the order of players: among those players who remain active the player with the highest ranking wins. The price the winner pays is equal to the limit of the sequence of the payments that occurred in the subsequent rounds in the auction. Since the sequence of payments is increasing this limit is equal to the supremum of the payments.

The query price is defined as follows. For a given $c \in (0, 1)$ and any continuous probability distribution with cumulative density $F(v)$ from which valuations of players are drawn, in any round r, given the payment p_r and the upper bound u_r, the query price q_r is chosen such that

$$\frac{F(q_r) - F(p_r)}{F(u_r) - F(p_r)} = c,$$

i.e. interval $[p_r, q_r)$ contains a fraction c of the measure of $[p_r, u_r)$. For example, for uninform distribution the query price bisects the interval $[p_r, u_r)$ in fractions c and $1 - c$ so that $q_r = p_r + c(u_r - p_r)$.

The following strategy, called the bluff strategy, constitutes a symmetric ex-post equilibrium in the c-bisection auction (this result is proven in [6]). An ex-post equilibrium is a strategy profile such that, given any realization of valuations, the plan of actions prescribed by the strategy to a player is a best response to the plans of actions prescribed by the strategies of the other players. Under the bluff strategy an active player i having valuation v_i says *yes* in round r whenever:

1. $v_i \geq q_r$, or

2. $p_r \leq v_i < q_r$ and no active predecessor of him said *yes* in this round.

The following example illustrates how the auction proceeds under the bluff equilibrium.

Example. Suppose five players with valuations uniformly distributed on $[0, 1)$ participate in the c-bisection auction with c equal to 0.5. Suppose that according to the lottery the ordering of players is $A \prec B \prec C \prec D \prec E$. Players have the following private valuations respectively: 0.43, 0.71, 0.38, 0.79, and 0.86. The auction proceeds as follows:

Round r	Payment p_r	Query price q_r	Set of active players A_r	Player A	Player B	Player C	Player D	Player E
1	0	0.5	ABCDE	yes	yes	no	yes	yes
2	0.5	0.75	ABDE	no	yes	-	yes	yes
3	0.75	0.875	BDE	-	no	-	yes	no

In the first round player A, having no predecessor and valuation larger than p_1 says *yes*. Every other player, having predecessor A with *yes* decision, says *yes* iff his valuation is larger than $q_1 = 0.5$. All players except C say *yes* and therefore remain active. The payment and the query price increase to 0.5 and 0.75, respectively. Since $v_A < p_2$ player A says *no* in the second round. Now player B has no predecessor with *yes* decision and since $v_B > p_2$ he says *yes*. Players D and E say *yes* since their valuations are larger than $q_2 = 0.75$. Again the payment and the query price increase. In the third round player B says *no*, player D, having now no predecessors with *yes* decision, says *yes* and player E says *no*. So there is only one *yes* decision meaning that the auction ends. Player D wins the auction and pays 0.75.

Notice that the outcome in the example is not efficient - the winner is not the player with the highest valuation. But as we have already pointed out inefficiency for some realizations of valuations is inevitable. Later in the paper we investigate how inefficient this auction is by analyzing the probability of inefficient allocation and the expected loss of welfare.

Probability distribution of player actions. In the remaining part of the paper we focus on auction performance in expectation. In order to analyze the expected performance we need to know the probability of particular actions of players. Namely, we need to know the probability of saying *yes* and *no* by an active player under the bluff strategy.

Recall that in any round r of the c-bisection auction the query price q_r is determined in such a way that, given that the valuation of a player is in $[p_r, u_r)$, the probability that his valuation is in $[p_r, q_r)$ is equal to c. Write $i_r := \min\{i \mid i \in A_r\}$ - among the active players in round r the one with the lowest ranking; $j_r := \min\{i \mid i \in A_r, i \neq i_r\}$ - among the active players in round r the one with the second lowest ranking.

First let's observe that when player i_r says *no* for the first time, player j_r says *yes* with certainty. Indeed, in all previous rounds player i_r said *yes* and since j_r is active in round r also he said *yes* in those rounds. Both the payment and the query price increased so that $p_r = q_{r-1}$. Since player j_r follows the bluff strategy his previous *yes* decision implies that $v_{j_r} \geq q_{r-1} = p_r$. If in round r player i_r says *no*, player j_r is in the situation where he doesn't have any active predecessor with *yes* decision and therefore says *yes* whenever his valuation is not smaller than p_r, that is with certainty. It follows that after round r player i_r drops out so that $i_{r+1} = j_r$. Notice that in the equilibrium in every round r either player i_r or player j_r (or both) say *yes*. It means that in the equilibrium only players with *yes* decision remain active. This in turn implies that the upper bound always remains the same so that $u_r = \beta$ for any r, and the payment and the query price increase, so that $p_r = q_{r-1}$ for any $r > 1$.

Second, we need to know the probability that player i_r says *yes* in round r. Having no active predecessor player i_r says *yes* iff $v_{i_r} \geq p_r$. Since $p_1 = \alpha$ player i_1 in round 1 says *yes* with certainty. Now let's show that for any $r > 1$ the probability that player i_r says *yes* in round r equals $1 - c$. Regarding the identity of player i_r there are two possibilities - either $i_r = i_{r-1}$ (happens if decision of i_{r-1} was *yes*) or $i_r = j_{r-1}$ (happens if decision of i_{r-1} was *no* and consequently decision of j_{r-1} was *yes*). In both cases the decision of player i_r in round $r-1$ was *yes* implying that $v_{i_r} \geq p_{r-1}$. Thus, $P(v_{i_r} \geq p_r \mid v_{i_r} \geq p_{r-1}) = P(v_{i_r} \geq q_{r-1} \mid v_{i_r} \geq p_{r-1}) = 1 - c$. The last equality holds because q_{r-1} divides the interval (p_{r-1}, β) exactly in such a way that this conditional probability is equal to $1 - c$.

Further, we need to know the probability of saying *yes* in round r for any player $i \neq i_r$, $i \in A_r$. We can distinguish two cases. First, consider the case where player i says *yes* in round r. From the fact that $i \in A_r$ follows that player i_r said *yes* in round $r-1$ and thus $v_i \geq q_{r-1}$. In round r he says *yes* iff $v_i \geq q_r$. Thus, $P(v_i \geq q_r \mid v_i \geq q_{r-1}) = P(v_i \geq q_r \mid v_i \geq p_r) = 1 - c$. The last equality holds because q_r divides the interval (p_r, β) exactly in such a way that this conditional probability is equal to $1 - c$. Secondly, consider the case where player i_r says *no* in round r. As we described above player j_r says *yes* with certainty. For any other player i the situation is the same as in the previous case and thus also here player $i \in N \backslash \{i_r, j_r\}$ says *yes* with probability $1 - c$.

Now notice that the analysis above was done without specifying the distribution function from which valuations of players are drawn. Due to the price setting rule of the c-fraction auction the obtained probability results hold regardless of the distribution function of valuations. Thus, in the remaining part of the paper, namely in Section 4, for simplicity of argumentation we focus on the setting where valuations of players are independently drawn from the uniform distribution in $[0, 1)$. Moreover, it could be seen from the analysis above that the probability of saying *yes* or *no* by an active player doesn't depend on the round. It enables us to derive recursive formulas (in the number of active players) for the expected number of rounds and the expected number of queries performed in the auction.

3 Running time of the auction

In this section we investigate the expected running time of the c-bisection auction if the bluff strategies are played. We analyze two measures, namely the expected number of rounds and the expected number of queries performed in the auction before the winner is found. As a query we consider each separate question of the auctioneer to an active player. As a round we consider a sequence of queries in which each active player is asked to act exactly once. For both measures we derive first a recursive formula and give then an upper bound for the function defined by this formula.

3.1 The expected number of rounds

Let $e_c(k)$ be the expected number of rounds of the auction with k active players, given that the decision of the active player with the lowest ranking is *yes* in the current round; and $e_c^*(k)$ be the expected number of rounds given that this decision is *no*. Consider round r with n active players and suppose that the decision of player i_r in the current round is *yes*. The current round contributes 1 to $e_c(n)$. Now let's compute the expected number of remaining rounds. If all active players apart from player i_r say *no*, the auction stops after this round. If k (for some $1 \leq k \leq n-1$) active players apart from player i_r say *yes*, then the auction continues with $k+1$ active players. The probability of this situation given the *yes* decision of player i_r is $\binom{n-1}{k}(1-c)^k c^{n-1-k}$ (since when player i_r says *yes* all other players say *yes* and *no* with probabilities $1-c$ and c respectively). In the next round player $i_{r+1} = i_r$ says *yes* or *no* with probabilities $1-c$ and c respectively. Thus if k active players apart from player i_r say *yes* in the round r, the expected number of remaining rounds is equal to $(1-c)e_c(k+1) + ce_c^*(k+1)$. Hence, for any $n \geq 2$

$$e_c(n) = 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \Big[(1-c)e_c(k+1) + ce_c^*(k+1)\Big]. \tag{1}$$

Now recall that if player i_{r+1} says *no* player j_{r+1} says *yes* with certainty, which causes player i_{r+1} to drop out of the auction. Thus, $e_c^*(2) = 1$ and $e_c^*(k+1) = e_c(k)$ for any $k > 1$. These observations are used in Appendix to rewrite the above recursive relation to

$$\Big[1 - (1-c)^n\Big]\ e_c(n) = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} e_c(k). \tag{2}$$

This formula is valid for any $n \geq 2$.

Now notice that since in the first round player i_1 says *yes* with certainty, the expected number of rounds of the auction of n players is equal to $e_c(n)$. Thus using formula 2 we can compute the expected number of rounds in the auction of n players. Plugging in $n = 2$ yields $e_c(2) = \frac{1+c(1-c)}{c(2-c)}$. All other values can be determined recursively. Table 1 in Appendix presents the computational results for different values of c in the auction with up to 100 players (data is within an accuracy of 0.001). Figure 1(a) shows how for a fixed value of c the expected number of rounds increases in the number of players who participate in the auction. Furthermore, Figure 1(b) demonstrates how for a fixed number of players the expected number of rounds decreases as c increases.

Generally we show that the expected number of rounds of the auction is bounded from above by a function that is logarithmic in the number of players. To prove this, first we introduce several notations and lemmas.

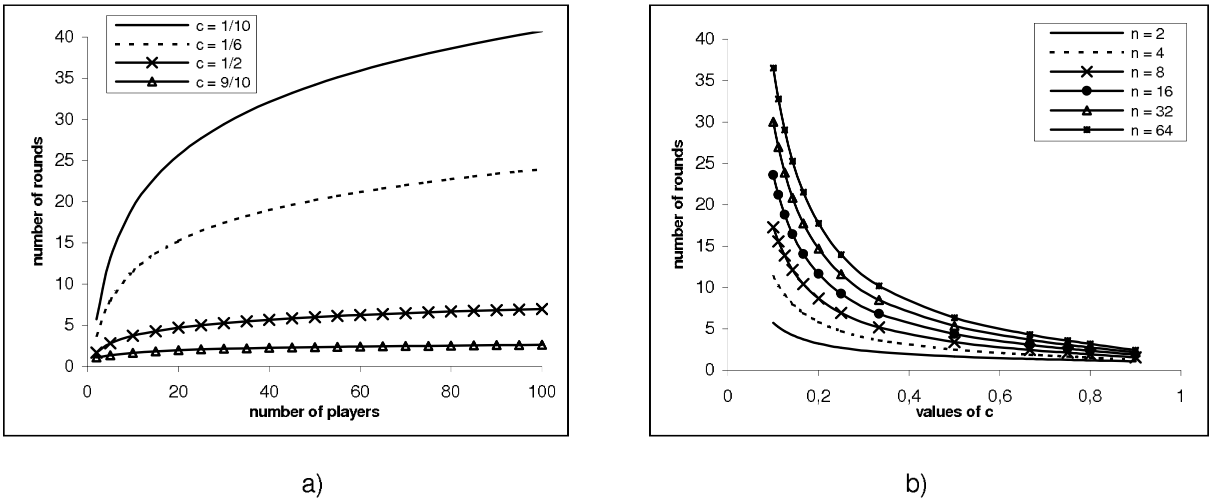


Figure 1: The expected number of rounds (a) for different fixed values of c; (b) for different fixed numbers of players.

Define $D_n = \prod_{k=1}^{n} \frac{1}{1-(1-c)^k}$ for any $n \geq 2$.
Also define $E_2 = \frac{1+c(1-c)}{c(2-c)}$ and for any $n > 2$

$$E_n = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} E_k.$$

Lemma 3.1 *For any $n \geq 2$, $e(n) \leq E_n \cdot D_n$.*

Proof. The proof is by induction on n. The basis of the induction is trivial since $e_c(2) = E_2$ and $D_2 > 1$. Suppose that $e_c(k) \leq E_k \cdot D_k$ is true for any $2 \leq k \leq n-1$. Notice that $D_n \geq D_{n-1} \geq \ldots \geq D_2 > 1$. Thus, using the recursive formula for $e_c(n)$ and the induction hypothesis,

$$\begin{aligned}
\Big[1-(1-c)^n\Big]\, e_c(n) &= 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} e_c(k) \\
&\leq 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} E_k D_k \\
&\leq 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} E_k D_{n-1} \\
&\leq D_{n-1}\Big[1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} E_k\Big] \\
&= E_n \cdot D_{n-1},
\end{aligned}$$

which completes the proof. ■

Now we find bounds on D_n and E_n.

Lemma 3.2 *For any $n \geq 2$, $D_n \leq e^{\frac{1-c}{c^2}}$.*

Proof. It's enough to show that $\ln D_n \leq \frac{1-c}{c^2}$. Let's define $\lambda = \frac{1}{1-c}$. Notice that since $0 < c < 1$ it holds that $\lambda > 1$.

We have

$$\begin{aligned}\ln D_n &= \ln\left(\prod_{k=1}^{n} \frac{\lambda^k}{\lambda^k - 1}\right) = \sum_{k=1}^{n}\Big[\ln \lambda^k - \ln(\lambda^k - 1)\Big] \leq \sum_{k=1}^{n}\Big(\ln x\Big)'_{|x=\lambda^k-1} = \sum_{k=1}^{n} \frac{1}{\lambda^k - 1} \\ &\leq \sum_{k=1}^{n} \frac{1}{\lambda^k - \lambda^{k-1}} = \frac{1}{\lambda - 1}\sum_{k=1}^{n} \frac{1}{\lambda^{k-1}} \leq \frac{1}{\lambda - 1}\sum_{k=0}^{\infty}\frac{1}{\lambda^k} = \frac{\lambda}{(\lambda-1)^2} = \frac{1-c}{c^2}.\end{aligned}$$

■

Lemma 3.3 *For any $n \geq 2$ and any $c \leq \frac{1}{2}$, $E_n \leq 1 + \log_a n$, with base $a = \frac{1}{1-c}$.*

Proof. The proof is by induction on n. The basis of induction holds since $\frac{1+c(1-c)}{c(2-c)} \leq \log_a 2 + 1$ for any $c \leq \frac{1}{2}$. Suppose $E_k \leq 1 + \log_a k$ for any $2 \leq k \leq n-1$. Using the induction hypothesis,

$$\begin{aligned}E_n &= 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k \\ &\leq 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}(\log_a k + 1) \\ &\leq 1 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k + \sum_{k=1}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} \\ &\leq 2 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k.\end{aligned}$$

Since the logarithm with base $a = \frac{1}{1-c}$ is concave, we know that if $\lambda_k \geq 0$ and $\sum_{k=0}^{n}\lambda_k = 1$ then

$$\sum_{k=0}^{n}\lambda_k \log_a(x_k) \leq \log_a\Big(\sum_{k=0}^{n}\lambda_k x_k\Big).$$

So let's take $\lambda_k = \binom{n}{k}(1-c)^k c^{n-k}$ for all k and take $x_0 = x_n = 1$, $x_k = k$ for any $1 \leq k \leq n-1$. Then

$$\begin{aligned}E_n &\leq 2 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k \\ &= 2 + \sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k}\log_a(x_k) \\ &\leq 2 + \log_a\left[\sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k} x_k\right] \\ &= 2 + \log_a\left[\sum_{k=1}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} k + c^n + (1-c)^n\right]\end{aligned}$$

$$\begin{aligned} &\leq 2 + \log_a \left[\sum_{k=1}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} k + n(1-c)^n \right] \\ &= 2 + \log_a \left[\sum_{k=0}^{n} \binom{n}{k} (1-c)^k c^{n-k} k \right] \\ &= 2 + \log_a [(1-c)n] \\ &= 1 + \log_a n. \end{aligned}$$

The last inequality holds since for any $c \leq \frac{1}{2}$ and any $n \geq 2$ it holds that $c^n + (1-c)^n \leq 2(1-c)^n \leq n(1-c)^n$. ■

A final immediate consequence of Lemmas 3.1 – 3.3 is the following theorem.

Theorem 3.4 *For any $c \leq \frac{1}{2}$ and any $n \geq 2$, $e_c(n) \leq e^{\frac{1-c}{c^2}} \left(\log_{\frac{1}{1-c}} n + 1 \right)$.*

Remark: Since $e_c(n) > e_{\overline{c}}(n)$ when $c < \overline{c}$, the upper bound for $c = \frac{1}{2}$ is also valid for any $c > \frac{1}{2}$.

We showed that the expected number of rounds of the c-bisection auction is bounded from above by a function that is logarithmic in the number of players. A comparison of the bound with the computed results suggests that this bound is not tight. It can be easily checked that for a fixed value of c the ratio between the bound and the computed result is approximately constant (as a function of n), implying that the bound is likely to have the correct order of magnitude.

3.2 The expected number of queries

Let $b_c(k)$ be the expected number of queries of the auction with k active players, given that the decision of the active player with the lowest ranking is *yes* in the current round; $b_c^*(k)$ be the expected number of queries given that this decision is *no*. Notice that in a round with k active players k queries are performed. Following the same argumentation as we used for determining the formula for the expected number of rounds we find that for any $n \geq 2$

$$b_c(n) = n + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \Big[(1-c) b_c(k+1) + c b_c^*(k+1) \Big]. \quad (3)$$

Again, notice that when player $i_{r+1} = i_r$ says *no* player j_{r+1} says *yes* with certainty, which causes player i_{r+1} to drop out of the auction. Thus, $b_c^*(2) = 2$ and for all $k > 1$ it holds that $b_c^*(k+1) = 1 + b_c(k)$. This is used in Appendix to derive the following recursive relation. For any $n \geq 2$

$$\Big[1 - (1-c)^n \Big] \, b_c(n) = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} b_c(k). \quad (4)$$

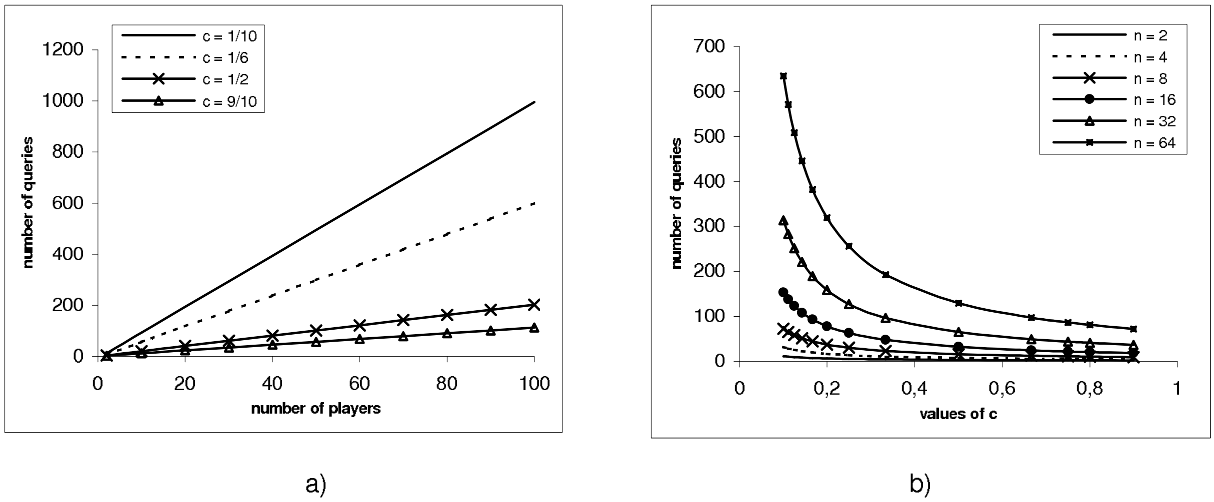


Figure 2: The expected number of queries (a) for different fixed values of c; (b) for different fixed numbers of players.

Now notice that since in the first round player i_1 says *yes* with certainty, the expected number of queries in the auction of n players is equal to $b_c(n)$. Thus using formula 4 we can compute the expected number of queries performed in the auction of n players. Plugging in $n = 2$ yields $b_c(2) = \frac{2+2c(1-c)}{c(2-c)}$. All other values can be determined recursively. Table 2 in Appendix presents the computational results for different values of c in the auction with up to 100 players (data is within an accuracy of 0.001). Figure 2(a) demonstrates that for a fixed value of c the expected number of queries increases in the number of players participating in the auction. Figure 2(b) shows that for a fixed number of players the expected number of queries decreases as c becomes larger.

Generally we show that the expected number of queries is bounded from above by a function that is linear in the number of players. To prove this we introduce several notations and lemmas.

Define $B_2 = \frac{2+2c(1-c)}{c(2-c)}$ and

$$B_n = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} B_k$$

for any $n > 2$.
Recall that $D_n = \prod_{k=1}^{n} \frac{1}{1-(1-c)^k}$.

Lemma 3.5 *For any $n \geq 2$, $b_c(n) \leq B_n \cdot D_n$.*

Proof. The proof is identical to the proof of Lemma 3.1 if we replace $e_c(k)$ by $b_c(k)$ and E_k by B_k for all $2 \leq k \leq n$. ∎

From Lemma 3.2 we know that for any $n \geq 2$, $D_n \leq e^{\frac{1-c}{c^2}}$. Now we find a bound on B_n.

Lemma 3.6 *For any $n \geq 2$, $B_n \leq \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$.*

Proof. The proof is by induction on n. The basis of the induction holds since it can be easily shown that $B_2 < 3\left(\frac{2}{c} + \frac{1}{2}\right)$. Now suppose that $B_k \leq \left(\frac{2}{c} + \frac{1}{2}\right)(k+1)$ for any $2 \leq k \leq n-1$. Using the induction hypothesis,

$$\begin{aligned}
B_n &= n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} B_k \\
&\leq n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} \left(\frac{2}{c} + \frac{1}{2}\right)(k+1) \\
&\leq 2n + c + \left(\frac{2}{c} + \frac{1}{2}\right) \sum_{k=0}^{n} \binom{n}{k} (1-c)^k c^{n-k} k + \left(\frac{2}{c} + \frac{1}{2}\right) \sum_{k=0}^{n} \binom{n}{k} (1-c)^k c^{n-k} \\
&= 2n + c + \left(\frac{2}{c} + \frac{1}{2}\right)(1-c)n + \left(\frac{2}{c} + \frac{1}{2}\right) \\
&= \left(\frac{2}{c} + \frac{1}{2}\right)(n+1) + c\left(1 - \frac{n}{2}\right) \\
&\leq \left(\frac{2}{c} + \frac{1}{2}\right)(n+1).
\end{aligned}$$

The last inequality holds since $n \geq 2$. ■

A final immediate consequence of Lemmas 3.2, 3.5 and 3.6 is the following theorem.

Theorem 3.7 *For any integer $n \geq 2$, $b_c(n) \leq e^{\frac{1-c}{c^2}} \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$.*

We showed that the expected number of queries is bounded from above by a function that is linear in the number of players. Again, a comparison of the bound with the computed results suggests that this bound is not tight. It can be easily checked that for a fixed value of c the ratio between the bound and the computed result is approximately constant (as a function of n), implying that the bound is likely to have the correct order of magnitude.

4 Efficiency of the auction

In this section we investigate the efficiency of the c-bisection auction when the bluff equilibrium is played. In particular we compute the probability of inefficient allocation and the expected loss of welfare. Here for simplicity of argumentation we focus on the setting where valuations of players are independently drawn from the uniform distribution in $[0, 1)$.

In order to compute these measures of inefficiency it is convenient to consider the direct revelation mechanism associated with the bluff equilibrium. We construct a direct auction that mimics the bluff strategies of the c-bisection auction.

4.1 The direct c-bisection auction

Consider the following direct auction (w_d, p_d), called *the direct c-bisection auction.* For $r \in \mathbb{N}$, write $I_r := \Big[1 - (1-c)^{r-1}, 1 - (1-c)^r\Big)$. [6] Note that the intervals $I_1, I_2, \ldots$ partition the unit interval $[0, 1)$ from which valuations are drawn. Now let $v = (v_i)_{i \in N}$ be a profile of valuations. Write $I_r(v) := I_r \cap \{v_i \mid i \in N\}$ - the set of valuations that belong to the interval I_r. Let $r(v)$ be the highest natural number r for which $I_r(v)$ is not empty. Among players whose valuations belong to the interval $I_{r(v)}$ the one with the lowest ranking is declared to be the winner. So the winner w_d is defined by

$$w_d(v) := \min\{i \in N \mid v_i \in I_{r(v)}\}.$$

Let $s(v)$ be the highest natural number r for which $I_r \cap \{v_i \mid i \in N \backslash \{w_d(v)\}\}$ is not empty. The price the winner pays is equal to the lower bound of interval $I_{s(v)}$ if all players whose valuations belong to this interval have a ranking higher than the winner. Otherwise the price equals the upper bound of this interval. So the payment p_d is defined by

$$p_d(v) := \begin{cases} 1 - (1-c)^{s(v)-1} & \text{if } i > w_d(v) \text{ for all } i \in I_{s(v)}(v) \\ 1 - (1-c)^{s(v)} & \text{else.} \end{cases}$$

Notice that the first condition always holds if $|I_{r(v)}| > 1$, i.e. if $s(v) = r(v)$. If $I_{r(v)}$ contains only one valuation, the payment depends on the ranking of the players with valuations in $I_{s(v)}$.

Example. Consider the same example as in Section 2 with players $A \prec B \prec C \prec D \prec E$ whose valuations are 0.43, 0.71, 0.38, 0.79 and 0.86, respectively. Suppose that in the direct c-bisection auction with $c = 1/2$ the players truthfully report their valuations. Then $r(v) = s(v) = 3$. Players with valuation in I_3 are players D and E. Player D has ranking lower than player E so he is the winner. The price he pays for the object is equal to the lower bound of I_3, namely 0.75. So we get the same outcome as the one we found in Section 2.

Generally, it is shown in [6] that for any realization of valuations $v = (v_i)_{i \in N}$ the outcome $\Big(w_d(v), p_d(v)\Big)$ equals the outcome of the c-bisection auction when players, having these valuations, follow the bluff strategies. Consequently, by the revelation principle [10] truth telling is a dominant strategy in the direct c-bisection auction. Due to this result the efficiency performance of both the c-bisection query

[6] In case of a general density function $F(v)$ in $[\alpha, \beta)$ intervals I_r are defined recursively as follows. Write $I_r = [\alpha_r, \beta_r)$ where $\alpha_1 = \alpha$, $\alpha_r = \beta_{r-1}$ and β_r is chosen such that $\frac{F(\beta_r) - F(\alpha_r)}{F(\beta) - F(\alpha_r)} = c$.

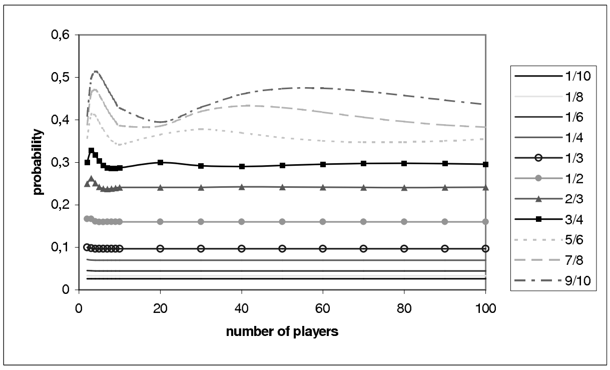


Figure 3: The probability of inefficient allocation.

auction under the bluff equilibrium and the direct c-bisection auction under the truth telling equilibrium are the same. Thus it suffices to compute the probability of inefficient allocation and the expected loss of welfare for the direct c-bisection auction under the truth telling equilibrium.

4.2 The probability of inefficient allocation

We derive a recursive formula for the probability of inefficient allocation and give an upper bound for the function defined by this formula. First notice that the direct c-bisection auction restricted to the interval $[c, 1)$ with k players having valuations uniformly drawn from this interval has identical form and structure as the original direct auction with k players having valuations uniformly drawn from $[0, 1)$.

Let's denote by P_n the probability that the auction with n players terminates in an inefficient allocation. First, consider the case where the valuations of all players are smaller than c. The probability of this event is c^n. In this case the auction is only efficient if the player with the lowest ranking has the highest valuation. By symmetry this happens with probability $\frac{1}{n}$. Thus this case contributes $\frac{n-1}{n}c^n$ to P_n. Next consider the case where k players have valuations larger than or equal to c and $n - k$ players have valuations smaller than c. It happens with probability $\binom{n}{k}c^{n-k}(1-c)^k$. For $k = 1$ the auction is efficient, so this case adds zero to P_n. For $k > 1$ the auction can be inefficient and due to the structural similarity, inefficiency takes place with probability P_k. Hence,

$$P_n = \frac{n-1}{n}c^n + \sum_{k=2}^{n} \binom{n}{k} c^{n-k}(1-c)^k P_k.$$

This can be rewritten to the following recursive relation, $P_2 = \frac{1}{2} \cdot \frac{c}{2-c}$ and for $n > 2$:

$$\Big[1 - (1-c)^n\Big] P_n = \frac{n-1}{n}c^n + \sum_{k=2}^{n-1} \binom{n}{k} c^{n-k}(1-c)^k P_k. \tag{5}$$

Direct computation of this expression for different combinations of n and c gives the values that are plotted in Figure 3. In general, we show the following upper bound on P_n.

Theorem 4.1 *For all $n \geq 2$, $P_n \leq c$.*

Proof. The proof is by induction on n. The basis of induction holds since $P_2 = \frac{1}{2} \cdot \frac{c}{2-c} \leq c$. Suppose that $P_k \leq c$ for all $2 \leq k \leq n-1$. Then

$$\begin{aligned}
P_n &= \frac{1}{1-(1-c)^n}\left[\frac{n-1}{n}\cdot c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\cdot P_k\right] \\
&\leq \frac{1}{1-(1-c)^n}\left[c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\cdot c\right] \\
&= \frac{1}{1-(1-c)^n}\left[c^n + c\Big(1-c^n-n(1-c)c^{n-1}-(1-c)^n\Big)\right] \\
&= \frac{c(1-(1-c)^n)}{1-(1-c)^n} + \frac{c^n - c^{n+1} - n(1-c)c^n}{1-(1-c)^n} \\
&= c + \frac{c^n(1-c)(1-n)}{1-(1-c)^n} \\
&\leq c.
\end{aligned}$$

The first inequality holds by the induction assumption and the fact that $\frac{n-1}{n} < 1$. The last inequality holds since $n \geq 2$. ■

Moreover, in the same way for $c \leq \frac{1}{2}$ it can be shown that $P_n \leq \frac{1}{2}c$ for all $n \geq 2$. This theorem shows in particular that by choosing an appropriate fraction c in the auction we can make the probability of inefficiency as small as we like, *independent of the number of players!*

4.3 The expected loss of welfare

The welfare of an auction is equal to the valuation of the winner. Thus given a realization of valuations $v = (v_i)_{i \in N}$, the welfare achieved by the auction is the valuation of $w_d(v) := \min\{i \mid v_i \in I_{r(v)}\}$. The maximum welfare, given v, is $\max\{v_i \mid i \in N\} = \max\{v_i \mid v_i \in I_{r(v)}\}$. Thus, the loss $L(v)$ of welfare is

$$L(v) = \max\{v_i \mid v_i \in I_{r(v)}\} - v_{w_d(v)}.$$

The expected loss of welfare, denoted by L_n, is the expected value of this difference. To estimate the value of L_n we simulated the direct c-bisection auction and ran it for valuations uniformly and independently drawn from the interval $[0, 1)$. For each combination of value c and number of players n we ran 10000 trials. Figure 4 shows the 99% confidence interval for the expected loss of welfare. It is interesting to notice that the maximum expected loss doesn't arrive at the minimum number of players.

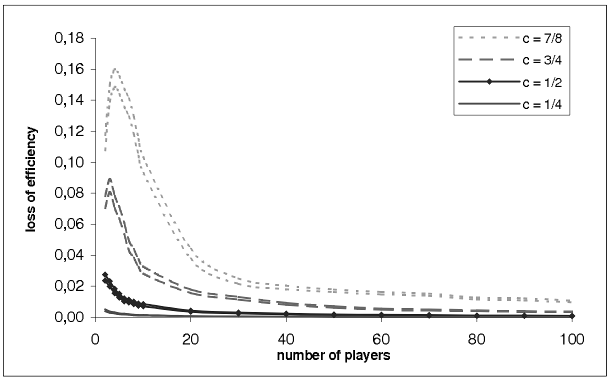


Figure 4: The expected loss of welfare, 99% confidence interval.

In general, we show the following statement. [7]

Theorem 4.2 *For all* $n \geq 2$, $L_n \leq c^2$.

Proof. Let $v = (v_i)_{i \in N}$ be a realization of valuations for which the allocation in the direct c-bisection auction is not efficient. In other words, $\max\{v_i \mid v_i \in I_{r(v)}\} > v_{w_d(v)}$. Since the valuation of $w_d(v)$ is an element of $I_{r(v)}$ we get that

$$L(v) \leq \text{ length } (I_{r(v)}) \leq c.$$

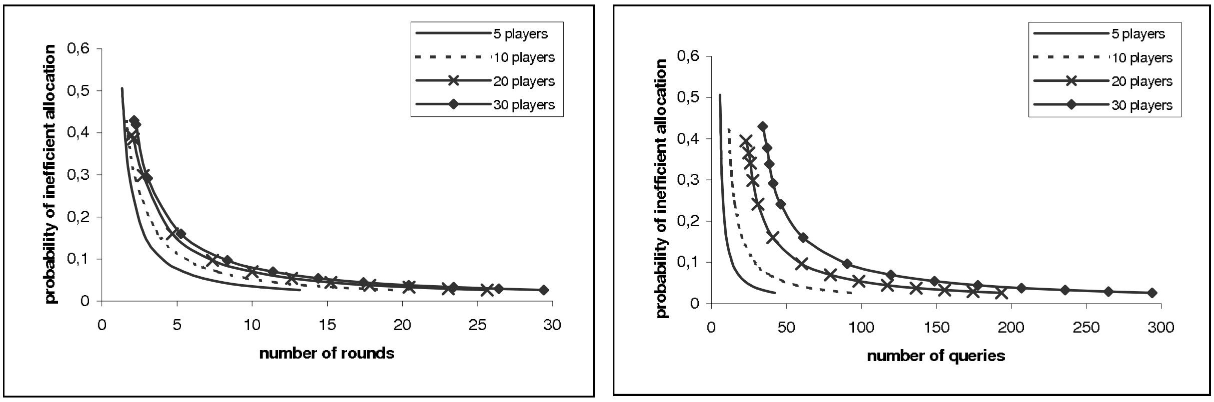
Hence, $L_n \leq cP_n$. Applying the result of Theorem 4.1 completes the proof. ■

As for probability of inefficient allocation, by choosing an appropriate fraction c in the auction we can limit the expected loss of welfare to an arbitrary chosen level, *independent of the number of players.*

5 Concluding remarks: trade-off between efficiency and running time

From the analysis above we derive the following relation between the value of c, the level of efficiency and the running time. For a fixed number of players, a smaller fraction c leads to a lower expected loss of welfare and lower probability of inefficient allocation. But at the same time it leads to a higher expected number of rounds and queries. Thus, increasing running time is a price that we have to pay for increasing efficiency. Depending on the priorities of the auctioneer he may trade off efficiency against running time. Figure 5 shows, for some fixed n, the relation between the expected running time and the probability of inefficient allocation. These relations are built on computational results based on recursive formulas 2, 4 and 5. Figure 6 shows, for some fixed n, the relation between the expected running time

[7] This result can not be generalized to an arbitrary density function $F(v)$ since it's based on the lengths of intervals I_r which entirely depend on distribution of valuations.



a) b)

Figure 5: The trade-off between (a) the probability of inefficient allocation and the number of rounds; (b) the probability of inefficient allocation and the number of queries.

and the expected loss of inefficiency. Because we don't have exact values for the expected loss of welfare we estimated the values by taking the middle point of the 99% confidence interval from the simulation results reported above. Notice that in Figure 6(a) the trade-off curves drawn for different numbers of players almost coincide with each other. It means that in order to get the desired level of efficiency we need to run the auction that in expectation takes the same number of rounds *for any number of players participating in the auction* (of course, the choice of c to be used in this auction will depend on the number of players). This explains why for the same level of efficiency more players require more queries to be asked, which is demonstrated in Figure 6(b).

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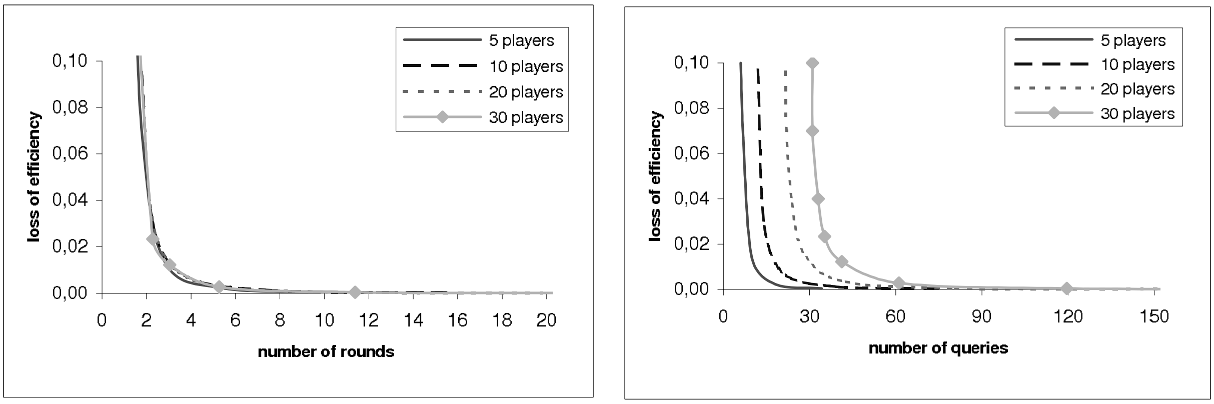

a) b)

Figure 6: The trade-off between (a) the loss of efficiency and the number of rounds; (b) the loss of efficiency and the number of queries.

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Appendix

Derivation of formula 2

Let's denote by $P_k^n = \binom{n}{k}(1-c)^k c^{n-k}$. Using the facts that $e_c^*(2) = 1$ and $e_c^*(k+1) = e_c(k)$ for all $k \geq 2$ we can rewrite formula 1 as follows:

$$\begin{aligned}
e_c(n) &= 1 + \sum_{k=1}^{n-1} P_k^{n-1}\Big[(1-c)e_c(k+1) + ce_c^*(k+1)\Big] \\
&= 1 + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)P_{n-1}^{n-1} e_c(n) + c\sum_{k=2}^{n-1} P_k^{n-1} e_c^*(k+1) + cP_1^{n-1} e_c^*(2) \\
&= 1 + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)^n e_c(n) + c\sum_{k=2}^{n-1} P_k^{n-1} e_c(k) + (n-1)(1-c)c^{n-1} \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + (1-c)\sum_{k=2}^{n-1} P_{k-1}^{n-1} e_c(k) + c\sum_{k=2}^{n-1} P_k^{n-1} e_c(k) \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\Big[(1-c)P_{k-1}^{n-1} + cP_k^{n-1}\Big]\ e_c(k) \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} P_k^n e_c(k).
\end{aligned}$$

This can be rewritten to

$$\Big[1 - (1-c)^n\Big]\ e_c(n) = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} e_c(k).$$

Derivation of formula 4

Recall that $P_k^n = \binom{n}{k}(1-c)^k c^{n-k}$ Using the facts that $b^*(2) = 2$ and $b^*(k+1) = b(k)+1$ for all $k \geq 2$, we get from formula 3 that

$$\begin{aligned}
b_c(n) &= n + \sum_{k=1}^{n-1} P_k^{n-1} \Big[(1-c)b_c(k+1) + cb_c^*(k+1)\Big] \\
&= n + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} b_c(k+1) + (1-c)P_{n-1}^{n-1} b_c(n) + c\sum_{k=2}^{n-1} P_k^{n-1} b_c^*(k+1) + cP_1^{n-1} b_c^*(2) \\
&= n + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} b_c(k+1) + (1-c)^n b_c(n) + c\sum_{k=2}^{n-1} P_k^{n-1}\Big[b_c(k)+1\Big] + 2(n-1)(1-c)c^{n-1} \\
&= n + (1-c)^n b_c(n) + 2(n-1)(1-c)c^{n-1} + (1-c)\sum_{k=2}^{n-1} P_{k-1}^{n-1} b_c(k) + c\sum_{k=2}^{n-1} P_k^{n-1} b_c(k) + c\sum_{k=2}^{n-1} P_k^{n-1} \\
&= n + (1-c)^n b_c(n) + 2(n-1)(1-c)c^{n-1} + c - c^n - (n-1)(1-c)c^{n-1} + \\
&\quad \sum_{k=2}^{n-1}\Big[(1-c)P_{k-1}^{n-1} + cP_k^{n-1}\Big]\ b_c(k) \\
&= n + (1-c)^n b_c(n) + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n} P_k^n b_c(k).
\end{aligned}$$

Rewriting yields, for any $n \geq 2$,

$$\Big[1-(1-c)^n\Big]\ b_c(n) = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} b_c(k).$$

Table 1: The expected number of rounds $e_c(n)$ in the c-bisection auction.

$n \diagdown c$	1/10	1/8	1/6	1/4	1/3	1/2	2/3	3/4	5/6	7/8	9/10
2	5.737	4.733	3.727	2.714	2.200	1.667	1.375	1.267	1.171	1.127	1.101
3	8.901	7.230	5.555	3.873	3.021	2.143	1.663	1.483	1.319	1.240	1.193
4	11.273	9.102	6.927	4.742	3.638	2.505	1.891	1.660	1.446	1.341	1.277
5	13.172	10.600	8.024	5.437	4.131	2.794	2.076	1.807	1.557	1.431	1.353
6	14.753	11.848	8.938	6.016	4.542	3.035	2.230	1.931	1.654	1.512	1.423
7	16.109	12.918	9.721	6.513	4.895	3.241	2.361	2.037	1.738	1.584	1.486
8	17.296	13.854	10.407	6.947	5.203	3.421	2.475	2.129	1.813	1.650	1.545
9	18.350	14.686	11.016	7.334	5.477	3.581	2.576	2.211	1.879	1.709	1.598
10	19.299	15.435	11.565	7.681	5.724	3.726	2.667	2.283	1.939	1.762	1.647
20	25.647	20.443	15.233	10.006	7.373	4.690	3.275	2.760	2.312	2.102	1.971
30	29.417	23.418	17.412	11.387	8.353	5.264	3.637	3.048	2.524	2.281	2.140
40	32.109	25.541	18.967	12.372	9.052	5.673	3.894	3.255	2.681	2.406	2.249
50	34.203	27.194	20.177	13.140	9.596	5.991	4.095	3.414	2.808	2.508	2.333
60	35.918	28.547	21.168	13.768	10.042	6.252	4.260	3.543	2.913	2.595	2.405
70	37.370	29.693	22.007	14.299	10.419	6.472	4.399	3.652	3.002	2.672	2.469
80	38.628	30.686	22.735	14.760	10.746	6.664	4.520	3.747	3.077	2.740	2.527
90	39.740	31.563	23.377	15.167	11.035	6.833	4.627	3.831	3.143	2.801	2.580
100	40.735	32.348	23.952	15.532	11.294	6.984	4.722	3.907	3.201	2.855	2.629

Table 2: The expected number of queries $b_c(n)$ in the c-bisection auction.

$n \diagdown c$	1/10	1/8	1/6	1/4	1/3	1/2	2/3	3/4	5/6	7/8	9/10
2	11.474	9.467	7.455	5.429	4.400	3.333	2.750	2.533	2.343	2.254	2.202
3	21.790	17.779	13.759	9.718	7.674	5.571	4.442	4.029	3.666	3.496	3.396
4	32.027	26.013	19.988	13.935	10.879	7.752	6.094	5.495	4.972	4.727	4.582
5	42.217	34.200	26.171	18.109	14.044	9.897	7.717	6.938	6.264	5.949	5.762
6	52.375	42.356	32.323	22.254	17.181	12.017	9.320	8.365	7.545	7.162	6.936
7	62.511	50.490	38.454	26.378	20.298	14.120	10.908	9.778	8.815	8.368	8.104
8	72.630	58.607	44.568	30.487	23.401	16.211	12.484	11.180	10.078	9.568	9.268
9	82.735	66.711	50.669	34.583	26.492	18.291	14.051	12.575	11.333	10.763	10.427
10	92.830	74.804	56.761	38.670	29.575	20.363	15.611	13.962	12.582	11.952	11.582
20	193.465	155.430	117.372	79.251	60.124	40.845	31.016	27.653	24.893	23.679	22.985
30	293.842	235.802	177.735	119.597	90.451	61.132	46.258	41.203	37.070	35.264	34.248
40	394.111	316.068	237.995	159.843	120.684	81.336	61.430	54.691	49.201	46.802	45.457
50	494.320	396.274	298.196	200.035	150.865	101.495	76.563	68.144	61.307	58.319	56.644
60	594.492	476.443	358.361	240.192	181.014	121.626	91.673	81.574	73.394	69.825	67.820
70	694.637	556.587	418.501	280.325	211.140	141.736	106.766	94.989	85.468	81.320	78.989
80	794.763	636.711	478.622	320.440	241.249	161.832	121.847	108.394	97.531	92.809	90.152
90	894.874	716.820	538.729	360.542	271.345	181.916	136.918	121.790	109.586	104.290	101.311
100	994.973	796.918	598.825	400.633	301.431	201.992	151.981	135.180	121.634	115.766	112.466