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Inefficiency of equilibria in query auctions with continuous valuations

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Summary. We show that, when bidders have continuous valuations, any ex post equilibrium in an ex post individually rational query auction can only be ex post efficient when the running time of the auction is infinite for almost all realizations of valuations of the bidders. We also show that this result applies to the general class of bisection auctions. In contrast we show that, when we allow for inefficient allocations with arbitrarily small probability, there is a query auction (to be more specific, a bisection auction) that attains this level of approximate efficiency in equilibrium, while additionally the running time of the auction in equilibrium is finite for all realizations of valuations.

Keywords and Phrases: Query auctions, ex post equilibrium, efficiency.

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1 Introduction

This paper concerns the sales of a single indivisible item to n buyers by means of an auction, each buyer having a private valuation v_i for the item. [1] This setting has been thoroughly analyzed and is very well understood. (see for example Krishna (2002)). The sealed-bid second price auction by Vickrey (1961), and the ascending clock version of the English auction are two strategically equivalent designs that solve the allocation problem in dominant strategies. In the dominant strategy equilibrium both auctions allocate the item efficiently –they both award the item to the bidder with the highest valuation. The Vickrey auction collects sealed bids, allocates to the bidder with the highest bid (if more than one, using a lottery), and sets the price equal to the second-highest bid. The ascending clock version of the English auction continuously increases the price on a price clock, and bidders step out if the price becomes larger than their willingness to pay. The clock stops when the second last bidder stepped out. If at some point all remaining bidders step out simultaneously, the item is assigned by a lottery among them at the current price.

This is in many cases a satisfactory way to auction a single indivisible item. It is however not always feasible to execute either the Vickrey auction or the ascending clock auction, or for that matter any other auction that is strategically equivalent to these. Our first main result identifies such an environment –continuous valuations combined with a multi round query auction– in which in equilibrium necessarily the item will be assigned inefficiently with positive probability, unless one accepts the unrealistic phenomenon that the auction will last indefinitely –i.e. the auction runs an infinite number of rounds– for almost all realizations of valuations. In other words, *any* implementation of the Vickrey auction by means of a query auction in a model with continuous valuations will necessarily have an infinite running time for almost all realizations of valuations, and is hence not a feasible option in any practical sense.

We will now turn to a more detailed description of the exact environment in which and the exact conditions under which our inefficiency result holds.

Continuous valuations. In many applications, there is nothing wrong with assuming discrete valuations. It is however sometimes desirable to be able to run an auction without an a-priori agreement on the discretization of bids. The leading example from which we draw our motivation is a computerized bidding environment in multi-agent systems. In such an environment the precision with which bidding agents represent their valuations might be unknown,

[1] We assume quasi-linear utilities throughout the paper.

and therefore preferably be left unspecified. One way to model this is to allow valuations to take on continuous values. Also time constraints –an issue of some importance in combinatorial auctions like the UMTS auctions– can thus be captured, since time restrictions might force an auctioneer to determine valuations only up to a level of precision that is not of the same order of magnitude in which bidders do, or would like to, express valuations.

Query auctions. The computerized bidding environment is modeled as a query auction. In a query auction the auctioneer sequentially offers bidders the opportunity to take one of a finite set of actions. Such an offer is referred to as a query. [2] During the course of the auction each bidder may be, and usually will be, queried more than once. Typically an "action" takes the form of an answer to a query regarding the valuation of the bidder in question, such as "Is your valuation larger than 15?" to which the response can be either a *yes* or a *no*.

Determination of winner and payment in a query auction are based exclusively on the actions taken by the bidders in response to the queries of the auctioneer. The auction ends as soon as both winner and payment are determined. The number of times a particular bidder is queried during the auction is not assumed to be bounded, and the auction may thus potentially take an infinite number of query rounds. We will only consider ex post individually rational query auctions, meaning that each bidder, given his valuation v_i, has a plan of action in the auction that guarantees him a non-negative payoff, regardless of the behavior of the other bidders.

Inefficiency of ex post equilibrium. In this paper we investigate the efficiency of ex post equilibria in query auctions. An ex post equilibrium is a strategy profile such that, given any realization of valuations, the plan of action prescribed to a bidder in the auction by his strategy is a best response to the plans of action prescribed by the strategies of the other bidders given their valuations. The first main result in this paper can now be precisely formulated as follows. An ex post equilibrium is called sometimes finite if the set of realizations of valuations for which in equilibrium the auction ends in finite time has positive Lebesgue measure. The result is that, given any ex post individually rational query auction, any ex post equilibrium in that auction that is sometimes finite cannot be ex post efficient.

Existence of ex post equilibrium in bisection auctions. Still, ex post equilibrium need

[2] Each separate query by the auctioneer could be thought of as a round in the auction because the action taken by the queried bidder is, at least in our setting, supposed to be publicly observable. Only actions whose effects can only be observed at the same moment in time by other bidders are usually considered to be taken in the same round. Rounds typically differ from each other in terms of the information available to bidders. In that sense each query could be counted as a round. In this paper though we deviate slightly from this standard interpretation. The order in which bidders are queried is usually fixed, and a round is a sequence of queries in which each bidder is queried exactly once.

not always exist, which would render our first result useless. Therefore we show that for a very wide class of query auctions, namely the so-called bisection auctions, an ex post equilibrium exists. We show that a bisection auction is indeed ex post individually rational, and that the ex post equilibrium in a bisection auction is indeed sometimes finite. Thus our first result applies to this class of auctions. Moreover, under a mild assumption (namely that the price in the auction can in principle be driven up to exceed any possible valuation of any bidder, an assumption that is met by all existing auctions) the equilibrium can even be shown to be finite for any realization of valuations.

The bisection mechanism works as follows. Valuations of bidders are assumed to be drawn from an interval $I = [\alpha, \beta)$. Before the auction starts, an order of the bidders in the auction is chosen randomly. We assume that this ordering is $1 \prec 2 \prec 3 \prec \cdots \prec n-1 \prec n$.

The auction runs for an a priori indefinite number of rounds. In each round there is a specific payment P to be made by a bidder if he wins in this round. In every round there is also a query price Q which is higher than the current payment and an upper bound H on future payments that is higher than the query price. Initially the payment and the upper bound are set as $P = \alpha$ and $H = \beta$, and all bidders are active. In every round the auctioneer asks the bidders that are active in that round whether they would be willing to pay the query price. Bidders are queried openly in increasing order.

If only one bidder is willing to pay the query price, he becomes the winner of the auction. He has to pay the current payment (not the query price). If more than 1 bidder is willing to pay the query price, the auction proceeds into the next round. Only those bidders who agreed to pay the query price stay active. The query price becomes the payment, and the new query price is raised to a level strictly above the old query price, but still below the upper bound. If no bidder is willing to pay the query price all bidders stay active, the payment stays the same, the old query price becomes the new upper bound, and the new query price is set between the new payment and the new upper bound. In case no winner is found, i.e., should the auction run indefinitely, then among the bidders who are still active the one with highest ranking wins and he pays the lowest price that is still higher than or equal to any of the payments that were announced while the auction was running.

Effectively a bisection auction is a variation of the bisection auction presented in Grigorieva et al. (2002), the main two differences being that in the present paper the auction may last indefinitely, and that the auction stops as soon as the winner is found. The bisection auction in

Grigorieva et al. (2002) was designed to handle the situation in which bidders have discrete valuations. The present definition of a bisection auction is specifically designed to handle continuous valuations.

The second result of this paper is that each bisection auction has an ex post equilibrium. Given a bisection auction we construct a specific equilibrium, called the bluff equilibrium, for that auction. The bluff equilibrium requires each bidder to act as follows. When there still is an active bidder with a lower rank in the ordering, the bidder stays in the auction until the query price exceeds his valuation. As soon as he becomes the active bidder with the lowest rank, he stays in the auction until the lower bound exceeds his valuation (effectively a bluff since he will say *yes* to a query price exceeding his valuation). We show that this strategy is ex post individually rational, and that the resulting profile where each bidder uses this strategy is an ex post equilibrium. We also show that the bluff equilibrium is sometimes finite –and hence not ex post efficient according to our first result.

In the second half of the paper we analyze exactly how (in)efficient ex post equilibria, in particular the bluff equilibrium, may be. In particular we show that approximate efficiency can be achieved within the family of fixed fraction auctions, a special class of bisection auctions.

Fixed fraction auctions. In a fixed fraction auction, given the payment P and upper bound H in any round, the query price in that round is given by $Q = (1-c)P + cH$, where c is a real number in the interval $(0, 1)$. Thus, the increment with which the payment P is increased is a fixed fraction c of the current price interval $[P, H)$ (the interval that contains all future payments, no matter what responses the bidders give to future queries).

As a measure of inefficiency we employ the probability of inefficient allocation. We assume that valuations of bidders are drawn independently from the uniform distribution on the interval $[\alpha, \beta)$. [3] The probability of inefficient allocation is the probability –according to the joint probability distribution on $[\alpha, \beta)^n$– of the set of realizations of valuations for which in equilibrium the item does not get assigned to a bidder with the highest valuation.

The main finding in this part of the paper is that for the fixed fraction auction with fraction c the probability of inefficient allocation is smaller than or equal to c, *no matter how many bidders participate in the auction.* Moreover, the running time of a fixed fraction auction in the bluff equilibrium is finite for *all* realizations of valuations. Thus, the minimum level of efficiency

[3] We use the uniform distribution merely for ease of exposition. Our results in the second part of the paper hold as soon as valuations are i.i.d. draws from an arbitrary continuous probability distribution on $[\alpha, \beta)$.

can be determined by the auctioneer before it is known how many bidders will participate in the auction by choosing the appropriate fraction c, and finite running time is guaranteed. However, we also show that the probability of inefficient allocation is bounded away from zero. In other words, given the fraction c, the probability of inefficient allocation does not converge to zero as the number of participants becomes large. This implies that c is the only tool available to the auctioneer to control the level of inefficiency, increasing the number of participants is not an appropriate method.

As a comment on the full generality of these statements, we stress again that the same conclusions can be obtained for *any* continuous probability distribution on $[\alpha, \beta)$ from which valuations are independently drawn. In this general setting we construct a bisection auction, not necessarily a fixed fraction auction, for which all the above claims hold as well.

These results alleviate the severity of our initial inefficiency result. The inefficiency result said that in our setting efficiency can only be achieved at the expense of an infinite running time of the auction for almost all realizations of valuations. The second part of the paper on the other hand shows that, by choosing the appropriate auction, we can have approximate efficiency (meaning that the probability of inefficient allocation can be made arbitrarily small) in equilibrium, while the running time of the auction in equilibrium is finite for all realizations of valuations. Moreover, given the desired level of efficiency, the particular choice of auction can be made independently of the number of bidders that will participate in the auction.

Related literature. Rothkopf and Harstad (1994) study a model with continuous valuations where bidders are obedient and can only bid on a finite number of bid levels. They derive bounds on the loss of welfare and show that, when valuations are uniformly distributed, for 2 bidders and m bid levels the evenly spaced bid level auction is revenue optimal as well as welfare optimal, with a loss of welfare of $\frac{1}{6m^2}$. In the same model David et al. (2005) argue that a multi-round version of the auction with discrete bid levels has truthful bidding as a dominant strategy equilibrium. They show that in equilibrium for more than 2 bidders the optimal auction has decreasing bid increments, and subsequently analyze the probability of efficient allocation for both evenly spaced bid levels and the optimal choice of bid levels. Also Blumrosen and Nisan (2002) and Blumrosen et al. (2003) study a model where bidders have continuous valuations and a finite set of bid levels. They show that in their model truthful bidding is a dominant strategy equilibrium and derive bounds on the loss of welfare in equilibrium. Parkes (2005) studies a model in which bidders are uncertain about their preferences and preference elicitation is costly. He shows that in such an environment ascending price query auctions can achieve better allocative efficiency than a sealed bid auction, using less elicitation of preferences.

Organization of the paper. Section 2 is a preliminary section where we collected most of the known results on auction design that we use in this paper. In section 3 we show that, when valuations are continuous, efficiency of ex post equilibria in query auctions can only be obtained at the expense of an infinite running time of the auction for almost all realizations of valuations. In section 4 we provide a family of query auctions, the bisection auctions, in which we prove the existence of an ex post individually rational ex post equilibrium, called the bluff equilibrium, that has a finite running time for a non-negligible set of valuations. Hence bluff equilibria are ex post inefficient. In section 5 we show that, when we allow for inefficient allocations with arbitrarily small –but positive– probability, there is a bisection auction that attains this level of inefficiency in equilibrium, while the running time is finite for every realization of valuations. Section 6 concludes. Appendix 1 is devoted to an elementary proof of the Theorem of Green and Laffont in our (simple) context. Appendix 2 contains proofs of statements that are used in section 4.

2 Preliminaries

We will briefly discuss the notions used in this paper. A single indivisible object is being sold to a set $N = \{1, \dots, n\}$ of bidders by means of a deterministic auction. The set of actions of bidder i is denoted by F_i. Write $F = \prod_i F_i$. The winner determination rule

$$w\colon F \to N$$

decides for each profile $f = (f_i)_{i \in N}$ of actions in F who the winner of the object is. The payment function

$$p\colon F \to \mathbb{R}$$

determines for each profile f of actions in F the amount $p(f)$ the winner $w(f)$ has to pay to the auctioneer. A triplet (F, w, p) is an auction.

STRATEGIC BEHAVIOR Each bidder has a valuation v_i for the item. Valuations are drawn from a non-degenerate interval I and are assumed to be private information. Bidders have to decide in advance which action to choose for each valuation they might possibly have. Thus, a strategy of bidder i is a function $s_i\colon I \to F_i$ stating that bidder i, when having valuation $v_i \in I$, will take action $s_i(v_i)$ in F_i. A vector $s = (s_i)_{i \in N}$ of strategies is called a strategy profile.

A realization $v = (v_i)_{i \in N}$ of valuations defines an ex post game (F, w, p, v) with action space F_i for bidder i and payoff function $u_i(v_i)\colon F \to \mathbb{R}$ defined by

$$u_i(v_i)(f) := \begin{cases} v_i - p(f) & \text{if } \ i = w(f); \\ 0 & \text{otherwise.} \end{cases}$$

Since the ex post game (F, w, p, v) is a game in normal form, the classical definition of Nash equilibrium applies. An action profile $f = (f_i)_{i \in N}$ is a Nash equilibrium of the ex post game (F, w, p, v) if for every bidder i and every action $g_i \in F_i$ of that bidder it holds that

$$u_i(v_i)(f) \geq u_i(v_i)(f \mid g_i)$$

where $(f \mid g_i)$ denotes the action profile where bidder i chooses g_i and every other bidder j chooses f_j. A strategy profile $s = (s_i)_{i \in N}$ is an ex post equilibrium of the auction (F, w, p) if for every realization $v = (v_i)_{i \in N}$ of valuations the action profile $s(v) := (s_i(v_i))_{i \in N}$ in F is a Nash equilibrium of the ex post game (F, w, p, v).

In the same way other notions also carry over to the setting of an auction. A strategy s_i of bidder i is dominant if for every realization v_i of the valuation of bidder i and any profile f of actions

$$u_i(v_i)(f \mid s_i(v_i)) \geq u_i(v_i)(f \mid g_i)$$

holds for any action $g_i \in F_i$. Given a strategy profile s, strategy s_i is a best response for player i to s if

$$u_i(v_i)(s(v)) \geq u_i(v_i)(s(v) \mid f_i)$$

for any realization $v = (v_i)_{i \in N}$ of valuations and any action $f_i \in F_i$. Strategy s_i is ex post individually rational if for every realization v_i of the valuation of bidder i and any profile f of actions,

$$u_i(v_i)(f \mid s_i(v_i)) \geq 0.$$

The auction (F, w, p) itself is called ex post individually rational if every bidder has an ex post individually rational strategy in it. A strategy profile s is called ex post efficient if for every realization $v = (v_i)_{i \in N}$ it holds that

$$w(s(v)) \in \arg\max\{v_i \mid i \in N\}.$$

Direct auctions An auction (F, w, p) is called direct if $F_i = I$ for each bidder i. In other words, the action a bidder has to take in the auction is to report a valuation (not necessarily his true valuation). Since in a direct auction it is clear what the action spaces are, we will simply write (w, p) to denote such an auction. A straightforward but important observation is that any strategy profile s in (F, w, p) automatically induces a direct auction $(w \circ s, p \circ s)$.

A direct auction (w, p) is called a Vickrey auction if for every profile $r = (r_i)_{i \in N}$ of reported valuations in I^N it holds that

$$w(r) \in \arg\max\{r_i \mid i \in N\} \quad \text{and} \quad p(r) := \max\{r_i \mid i \neq w(r)\}.$$

It is very well known that in a Vickrey auction bidding your valuation is a dominant strategy. If every bidder bids according to this strategy, the outcome is ex post efficient, and the winner pays an amount equal to the second-highest valuation.

A THEOREM OF GREEN AND LAFFONT In the next section we will use the following version of a Theorem by Green and Laffont, which shows under precisely which conditions a direct auction is a Vickrey auction (see Green and Laffont (1977)). For completeness a proof is given in Appendix 7.1.

Theorem 2.1 *A direct auction $(w \circ s, p \circ s)$ is a Vickrey auction if the following three conditions hold*
(a) the auction (F, w, p) is ex post individually rational;
(b) the strategy profile s is an ex post equilibrium of (F, w, p);
(c) the strategy profile s is ex post efficient in (F, w, p).

QUERY AUCTIONS Query auctions are already verbally defined in the introduction. We will give formal definitions of the two specific types of query auctions that are used in this paper, namely bisection auctions and fixed fraction auctions. However, for our purposes the verbal description of a general query auction that is given in the introduction suffices.

3 Efficient query equilibria are almost always infinite

Suppose we are given a query auction (F, w, p) together with an ex post equilibrium $s = (s_i)_{i \in N}$ in this auction. Such an equilibrium is called a query equilibrium. Let Z be the set of valuations $v = (v_i)_{i \in N}$ for which in the action profile $s(v) := (s_i(v_i))_{i \in N}$ the auctioneer asks a finite number of queries before the auction ends. We will assume that Z is measurable, and that moreover $w \circ s$ is also measurable. When Z has Lebesgue measure equal to zero, we say that the query equilibrium s is almost always infinite. When Z has Lebesgue measure greater than zero, we say that s is sometimes finite.

Theorem 3.1 *Let s be a query equilibrium in (F, w, p) and suppose that s is sometimes finite. Then the corresponding direct auction $(w \circ s, p \circ s)$ is not a Vickrey auction.*

Proof. Define

$$O(Z) := \{(w(s(v)), p(s(v))) \mid v \in Z\}.$$

Let Z^k be the set of valuations $v \in Z$ for which the auction ends after k queries given the profile of actions $s(v)$. Then the cardinality of the set

$$O(Z^k) := \{(w(s(v)), p(s(v)) \mid v \in Z^k\}$$

is finite since each player only has a finite number of possible responses to each query and the determination of winner and payment is based exclusively on the responses of the bidders to the queries of the auctioneer. Thus $O(Z) = \cup_{k=1}^{\infty} O(Z^k)$ is a countable set.

Now suppose that the corresponding direct auction $(w \circ s, p \circ s)$ is a Vickrey auction. Note that

$$O(Z) = \{((w \circ s)(v), (p \circ s)(v)) \mid v \in Z\}.$$

Define $Z_i := \{v \in Z \mid (w \circ s)(v) = i\}$. Since Z and $w \circ s$ are measurable by assumption, also each Z_i is measurable. So, since the Z_i's partition Z and the Lebesgue measure of Z is larger than zero, we know that at least one Z_i must have Lebesgue measure larger than zero as well. Take such a Z_i. Define for each $p^* \in \mathbb{R}$

$$Z_i(p^*) := \{v \in Z_i \mid (p \circ s)(v) = p^*\}.$$

Since $(w \circ s, p \circ s)$ is a Vickrey auction we know that each set $Z_i(p^*)$ is a subset of the set

$$\left\{v \in I^N \mid \max\{v_i \mid i \neq (w \circ s)(v)\} = p^*\right\}$$

which has Lebesgue measure zero. Thus, each $Z_i(p^*)$ itself is measurable and has Lebesgue measure zero. Hence, the set

$$P_i := \{p^* \mid Z_i(p^*) \neq \phi\}$$

must be uncountable, because $Z_i = \cup_{p^* \in P_i} Z_i(p^*)$ and Z_i has Lebesgue measure larger than zero. The set $O(Z)$ must have a cardinality that is at least as large as the cardinality of P_i because $p^* \mapsto (i, p^*)$ is an injective function from P_i to $O(Z)$, so $O(Z)$ is uncountable. This contradicts our earlier conclusion that $O(Z)$ is a countable set. Hence, the direct auction $(w \circ s, p \circ s)$ cannot be a Vickrey auction. ∎

Theorem 3.2 *Any ex post efficient ex post equilibrium in an ex post individually rational query auction is almost always infinite.*

Proof. Consider an ex post efficient ex post equilibrium s in an ex post individually rational query auction (F, w, p). Theorem 2.1 states that the corresponding direct auction $(w \circ s, p \circ s)$ is a Vickrey auction. However, if the equilibrium were sometimes finite, Theorem 3.1 states that the corresponding direct auction $(w \circ s, p \circ s)$ is not a Vickrey auction. Hence, the equilibrium cannot be sometimes finite. Since Z is measurable by assumption, the equilibrium must be almost always infinite. ∎

4 Bisection auctions

Thus, in a setting with continuous valuations, any (measurable) ex post equilibrium in an ex post individually rational query auction that ends with positive probability in finite time will necessarily be inefficient. In this section we describe a family of query auctions, called bisection auctions, to which this inefficiency result applies. The main characteristics of the family of bisection auctions are that they have a fixed order in which active bidders are queried, they all use a binary search algorithm to determine the price, and they all stop as soon as the winner is found. We show that each bisection auction has an ex post individually rational ex post equilibrium, called the bluff equilibrium. We also show that the bluff equilibrium is sometimes finite, and even, under a mild condition (namely that the price in the auction can be driven up to exceed any possible valuation of any bidder, an assumption that is met by all existing auctions), has a running time that is finite for all realizations of valuations. Hence, these equilibria will be inefficient. In the next section we discuss in more detail the extent to which bluff equilibria are, or are not, efficient.

4.1 Formal definition of a bisection auction

A bisection auction in the form we describe it here is specifically designed to handle the case in which valuations are drawn from an interval $I = [\alpha, \beta)$. We will however first give a description of a bisection auction that is free from any reference to valuations of the bidders.

We represent a bisection auction as an extensive form game on a complete binary decision tree. We will describe this tree first and subsequently discuss the winner determination rule W and the payment scheme P to give a complete description of the auction.

The playing field

We provide a complete description of the game tree. This tree is the same for the entire family of bisection auctions. The nodes of the tree are given in (1) and the directed edges are defined in (2). The game board has perfect information, meaning that each node in the tree will be a decision node for one of the players. Hence information sets are obsolete.

The set of players that can participate in the game is $N = \{1, \dots, n\}$. The response set for each player in every query node is $R = \{yes, no\}$. This reflects the fact that in each round of the auction each player will be faced with a binary query regarding his valuation. The precise nature of this query is explained in the next subsection.

(1) A node a in the game tree is represented by the history of responses players have to give in order to reach this particular node. Formally, $a = (a_k)_{k=1}^r$ with $r \in \mathbb{N}$ where $a_k = (a_{k,i})_{i=1}^n$ for $k < r$ and $a_r = (a_{r,i})_{i=1}^j$ for some $j \leq n$. Here $a_{k,i}$ is a particular response in the set R chosen by player i in round k.

The length of a node is defined as $l(a) = (r-1)n + j$. The initial node $a_0 = (\ )$ has length zero. This node corresponds to the first round where the first player has to respond. The nodes with length greater than or equal to $(r-1)n$ but less than rn correspond to round r, where the nodes with length $(r-1)n$ are referred to as the start of round r. The set of nodes corresponding to round r is denoted by X_r. For a node $a \in X_r$, the node $a^* \in X_r$ is the node of length $(r-1)n$ for which $a^*_{k,i} = a_{k,i}$ for all $k < r$ and all $i \in N$.

A node a that has a length of $l(a) = (r-1)n + j - 1$ for some $r > 0$ is a decision node of player j in round r. [4] Let D_j denote the collection of all decision nodes of player j and D_{jr} denote the collection of player j's decision nodes in round r. A predecessor of a decision node a of player j is a node from D_j that player j encounters when moving from the initial node a_0 to node a. We denote by $a(k) \in D_{jk}$ the predecessor of a in round k. Conversely, a is a successor of $a(k)$.

(2) There is a directed edge from node a to node b if $l(b) = l(a) + 1$, and for all j and k for which $a_{k,j}$ is defined, $a_{k,j} = b_{k,j}$. So, there is an edge between two nodes if in the second node one player has given an additional response in comparison with the first node.

Winner determination and payment rules

The next ingredient of the description of the game is the determination of the winner of the item and the specification of payments. As we allow bisection auctions to last indefinitely, we will define the winner and the payment on infinite sequences of actions, which we call endnodes.

[4] Notice that this implies that the bidders are queried according to the fixed ordering $1 \prec 2 \prec 3 \prec \cdots \prec n-1 \prec n$.

Thus, an endnode of the game is an infinite sequence $h = (h^{r,i})_{r\in\mathbb{N},i\in N}$ of nodes in the game tree such that (1) $h^{1,1} = a_0$ (the initial node is the first element of this sequence), and (2) there is an edge from node $h^{r,i}$ to node $h^{r,i+1}$ for any $i < n$, and from node $h^{r,n}$ to node $h^{r+1,1}$ for any $r \in \mathbb{N}$.

Each endnode $(h^{r,i})_{r\in\mathbb{N},i\in N}$ may be viewed as a history of infinite length such that its upper part of length $(r-1)n + i - 1$ coincides with node $h^{r,i}$. Thus, the set of all possible endnodes of the game is order isomorphic to the set $2^{\mathbb{N}}$.

WINNER DETERMINATION For an endnode $h = (h^{r,i})_{r\in\mathbb{N},i\in N}$, we denote by $A(h)$ the set of players who remain active throughout the play, that is $A(h) = \cap_r A(h^{r,1})$, where for a node $a \in X_1$ we define the set of active players by $A(a) := N$ and for a node $a \in X_{r+1}$ for some $r \geq 1$ we define the set $A(a)$ of active players iteratively by

$$A(a) := \begin{cases} A(a(r)) & \text{if } a_{r,i} = no \;\; \forall i \in A(a(r)); \\ \{i \in A(a(r)) : a_{r,i} = yes\} & \text{otherwise.} \end{cases}$$

Notice that $|A(h)| \geq 1$ will always hold. The winner of the game in endnode h is

$$W(h) := \max\,\{i \mid i \in A(h)\}.$$

PAYMENT RULE The difference between different bisection auctions is in the payment rule. Suppose that $I = [\alpha, \beta)$. The price the winner pays depends on when it became known that he is the winner. We will first provide a recursive description of the way the payment rule is constructed.

First we associate with each node $a \in X_r$ of length $(r-1)n$ a current price interval $[P(a), H(a))$ and a query price $Q(a)$ in the interior of this half-open interval. The query price $Q(a)$ bisects the current interval $[P(a), H(a))$ into two intervals $[P(a), Q(a))$ and $[Q(a), H(a))$ of smaller size. For this reason we call this auction a bisection auction.

Once the game has reached node $a \in X_r$, the price $P(a^*)$ is the minimum amount the winner, whoever it may be, will have to pay, regardless of what happens from now on. In the same way $H(a^*)$ is a hard upper bound on the payment of the winner. The query in round r associated with the query price $Q(a^*)$ is

Is your valuation greater than or equal to the query price $Q(a^*)$?

The answer to this query is an element of the response set $R = \{yes, no\}$, and only the responses of players that are currently active (in round r that is) can influence the outcome of the auction.

Formally $P(a_0) := \alpha$ and $H(a_0) := \beta$, and $Q(a_0)$ is an element of the interval $(P(a_0), H(a_0))$. For a node $a \in X_r$ of length $(r-1)n$ with $r > 1$ we recursively define

$$P(a) := \begin{cases} P(a(r-1)) & \text{if } |\{i \in A(a(r-1)) : a_{r-1,i} = yes\}| \leq 1 \\ Q(a(r-1)) & \text{otherwise.} \end{cases}$$

and

$$H(a) := \begin{cases} Q(a(r-1)) & \text{if } |\{i \in A(a(r-1)) : a_{r-1,i} = yes\}| \leq 1 \\ H(a(r-1)) & \text{otherwise.} \end{cases}$$

Finally, we again choose $Q(a)$ in the interval $(P(a), H(a))$.

Now, for an endnode $h = (h^{r,i})_{r\in\mathbb{N}, i\in N}$ with $|A(h)| = 1$ we define the running time of the auction by $T(h) = \min\{r \in \mathbb{N} : |A(h^{r,1})| = 1\}$. Otherwise we define the running time by $T(h) = \infty$. The price the winner in endnode h pays is

$$P(h) := \sup \{P(h^{r,1}) \mid r \leq T(h)\}.$$

All other players pay zero. The resulting payoff in endnode h for player j having valuation v_j is given by

$$U_j(v_j)(h) := \begin{cases} v_j - P(h) & \text{if } j = W(h) \\ 0 & \text{otherwise.} \end{cases}$$

This completes the description of a bisection auction in its representation as a query auction. Notice that effectively a bisection auction is completely characterized by the choices of α and β, and the choices of the query price $Q(a)$ for every node a that has a length $(r-1)n$ for some $r \geq 1$. The price bounds $P(a)$ and $H(a)$ as well as winner determination and payment specification are uniquely determined by the choices of α, β, and the queries $Q(a)$.

One-shot representation of bisection auctions

A bisection auction is a query auction, meaning that the auction has multiple rounds and in each round the players can give several (two in this case) responses to the queries of the auctioneer. We will briefly discuss how the one-shot representation (F, w, p) of such an auction looks like in the terminology of Section 2.

A plan of action of player j is a function f_j that assigns to each decision node $a \in D_j$ a response $f_j(a)$ in R. The action set F_j is the collection of all plans of action of player j. For the profile of plans of action $f = (f_i)_{i\in N}$ in $F := \prod_i F_i$ the winner $w(f)$ and payment $p(f)$ are now defined as follows. The realization of f is the endnode $h = (h^{r,i})_{r\in\mathbb{N}, i\in N}$ where $h^{1,1} = a_0$, $h^{r,i+1} = (h^{r,i}, f_i(h^{r,i}))$ for any $i < n$ and $h^{r+1,1} = (h^{r,n}, f_n(h^{r,n}))$ for any $r \in \mathbb{N}$. Then $w(f) := W(h)$ and $p(f) := P(h)$. Automatically $u_j(v_j)(f) = U_j(v_j)(h)$.

This auction will be ex post individually rational as long as valuations are larger than or equal to α. Indeed, in this case always saying *no* guarantees a player a non-negative payoff. If a player does so he can win only if all other players also keep on saying *no*, in which case the payment for the winner is α.

4.2 Ex post equilibrium

In this section we will introduce for each bisection auction a strategy profile that constitutes an individually rational ex-post Nash equilibrium in the given bisection auction. We will also show that in equilibrium there is a set of valuations whose Lebesgue measure is larger than zero for which the auction ends in finite time. Consequently, in equilibrium, the allocation is not ex post efficient.

Consider the bisection auction. Define the set of players who have ranking less than j and are active in node $a \in D_j$ by

$$A_j(a) := \{i \in A(a) : i < j\}.$$

These are the players whose decisions in the current round are observable for player j when he has to make a decision in node a. Let

$$D_j^1 := \cup_r \{a \in D_{jr} \mid \exists i \in A_j(a) : a_{r,i} = yes\}$$

be the set of decision nodes of player j such that there is at least one active predecessor of player j whose action in the current round was *yes*. Thus, $D_j^2 := D_j \backslash D_j^1$ is the set of decision nodes of player j such that all active predecessors of player j took decision *no*.

As before, a strategy for player j in the bisection auction is a function s_j that assigns to each possible valuation $v_j \in [\alpha, \beta)$ a plan of action $s_j(v_j)$ in F_j. Thus, for each decision node $a \in D_j$ of player j, $s_j(v_j)$ specifies a response $s_j(v_j)(a)$ in R.

Definition 4.1 Let v_j be a valuation of player j and let $a \in D_j$ be a decision node of player j. The bluff strategy b_j of player j is defined by

$$b_j(v_j)(a) := \begin{cases} yes & \text{if } \ a \in D_j^1 \text{ and } \ Q(a^*) \leq v_j \\ yes & \text{if } \ a \in D_j^2 \text{ and } \ P(a^*) \leq v_j \\ no & \text{otherwise.} \end{cases}$$

This strategy has a bluffing component with regard to the query "Is your valuation greater than the current query price?". Indeed, in D_j^1 player j compares his valuation v_j with the current query price and in any node from D_j^2 with the current payment. So in nodes from D_j^2 when his valuation is greater than the payment even if it is smaller than the query he replies *yes* and thus deceives the auctioneer by pretending to have higher valuation than he really has. Therefore one can think of nodes from D_j^1 as truthful nodes and nodes from D_j^2 as bluff nodes.

Now we will show that the profile $b = (b_i)_{i \in N}$ of bluff strategies constitutes an ex post individually rational ex post equilibrium.

Proposition 4.2 *A bluff strategy is ex post individually rational. Hence, as said before, bisection auctions are ex post individually rational.*

Proof. Suppose that player j follows his bluff strategy and due to the plans of action chosen by the other players endnode h is realized. If $j \neq W(h)$ then $u_j(v_j)(h) = 0$. So, suppose that $j = W(h)$. We consider two cases. Case 1: if $T(h) < \infty$. Let a be the decision node of player j in round $T(h)$. Then all players in $A_j(a)$ said *no*, while player j said *yes*. Therefore $v_j \geq P(a^*)$, and since $P(a^*)$ is the price to be paid by j, he has a non-negative payoff. Case 2: if $T(h) = \infty$. Since $j = W(h)$, Lemma 7.2 implies that player j said *yes* in every round. Then, by definition of the bluff strategy, $P(h^{r,1}) \leq v_j$ for every r. Hence, also $P(h) \leq v_j$. ■

Theorem 4.3 *The strategy profile $b = (b_i)_{i \in N}$ is an ex post Nash equilibrium.*

Proof. Let $(v_i)_{i \in N}$ be a realization of valuations and $f_j \in F_j$ be a plan of action of player j. Let a be the first decision node at which player j following f_j deviates from $b_j(v_j)$. In case player j is not active in node a both $b_j(v_j)$ and f_j yield payoff 0 and we are done. So suppose that player j is active in node a. We consider two cases.

Case 1. In case $a \in D_j^1$. If $f_j(a) = no$ and $b_j(v_j)(a) = yes$, the payoff of playing f_j is 0, while according to Proposition 4.2 the payoff of playing $b_j(v_j)$ is at least 0. Consider the case where $f_j(a) = yes$ and $b_j(v_j)(a) = no$. When player j says *yes* in a, there are at least two players who say *yes* in the current round by definition of D_j^1. So, the winning payment will be at least $Q(a^*)$. Further, $v_j < Q(a^*)$ because $b_j(v_j)(a) = no$. Hence, the payoff of playing f_j is non-positive while the payoff of playing $b_j(v_j)$ is 0.

Case 2. In case $a \in D_j^2$. If $f_j(a) = yes$ and $b_j(v_j)(a) = no$ we know that $P(a^*) > v_j$. Since the payment of the winner is at least $P(a^*)$, playing f_j has non-positive payoff, while playing $b_j(v_j)$ guarantees non-negative payoff. Consider the case where $f_j(a) = no$ and $b_j(v_j)(a) = yes$. Suppose that $b(v) = b_i(v_i)_{i \in N}$ is such that all successors of j say *no* if j says *yes*. Then, following $b_j(v_j)$ player j wins at price $P(a^*)$ while following f_j he might win at price at least $P(a^*)$. Now suppose that there is a successor i of j that plays *yes* if player j says *yes*. Since player i plays according to $b_i(v_i)$, he will also say *yes* when player j switches to *no*. But then the payoff of playing f_j would be 0 while the payoff of playing $b_j(v_j)$ is non-negative. ■

The following example shows that bidders that do not have the highest ranking do not have a dominant strategy. Hence the previous result cannot be strengthened much further. It can be shown that the bluff strategy is a dominant strategy for bidder n.

Example. Consider a game with two players and suppose that player 2 has the following strategy: if in the first round player 1 says *yes* then play *yes* in the first round and *no* in all other rounds; otherwise play *no* in all rounds. Then *any* best response of player 1 against this strategy chooses *no* in the first round (and *yes* in some later round) whenever the valuation of player 1 is strictly larger than zero. Now, consider another strategy of player 2: if in the first round player 1 says *yes* then play *no* in all rounds; otherwise play *yes* in the first round and *no* in all other rounds. In this case *any* best response of player 1 against this strategy chooses *yes* in the first round (in decision node a_0) whenever the valuation of player 1 is strictly larger than zero. It follows that there is no strategy of player 1 which is a best response against both strategies of player 2. Consequently player 1 doesn't have a dominant strategy. ■

Theorem 4.4 *The strategy profile $b = (b_i)_{i \in N}$ is sometimes finite. Consequently, the allocation under b is not ex post efficient.*

Proof. Consider the set V of realizations $v = (v_i)_{i \in N}$ of valuations for which $\alpha \leq v_i < Q(a_0)$ for all $i \in N$. Then for each $v \in V$ in round 1 bidder 1 says *yes* and all other bidders say *no*. Thus, the auction ends after this first round, and the Lebesgue measure of the set of valuations for which the auction ends after one round is at least $\big(Q(a_0) - \alpha\big)^n > 0$. Consequently, by Theorems 3.2, 4.2, and 4.3, the allocation under b is not ex post efficient. ■

As in the above proof, consider the set V of realizations $v = (v_i)_{i \in N}$ of valuations for which $\alpha \leq v_i < Q(a_0)$ for all $i \in N$. A direct way to conclude that allocation under b is not ex post efficient is via the observation that for each $v \in V$ bidder 1 wins the item (for a price of α). So, if we take for example $v_1 = \alpha$, and $v_i = \frac{Q(a_0)+\alpha}{2}$ for all $i \neq 1$, then the allocation is not ex post efficient.

4.3 Finite running time

Not every bisection auction will have a finite running time under the bluff equilibrium for any realization of valuations. If we take for example $\alpha = 0$, $\beta = 1$, and for each $a \in X_r$

$$Q(a^*) = (1 - \frac{1}{(r+1)^2})P(a^*) + \frac{1}{(r+1)^2}H(a^*)$$

we get a bisection auction for which for each endnode h we have that $P(h) \leq \frac{3}{4}$. [5] It is clear that the running time in equilibrium is not finite as soon as at least two players have a valuation larger than $\frac{3}{4}$. In order to exclude such pathological cases, consider the quantity

$$P^* := \sup\{P(a) \mid a \text{ is of length } (r-1)n \text{ for some } r \in \mathbb{N}\}.$$

This quantity is the price the winner has to pay in the bisection auction when in any round there are at least two players who say *yes* to their query in that round.

We say that a bisection auction is *regular* if $P^* = \beta$. Regularity implies for example that the price can in principle be driven up by the bidding process to a level where a bidder makes a loss if he becomes the winner.

As an immediate consequence of Lemma 7.3, the bluff equilibrium of a regular bisection auction guarantees a finite running time for *any* realization of valuations. Notice that this statement is stronger than just saying that we have a finite running time almost surely. Furthermore, from Lemma 7.1 it immediately follows that, when every bidder plays according to his bluff strategy, in any round of the auction there is at least one player that says *yes*. Thus we have established the following Theorem.

Theorem 4.5 *In a regular bisection auction the profile of bluff strategies has a finite running time for any realization of valuations. Moreover, the query price increases from round to round up to the moment where the winner is found.*

5 Approximate efficiency in bisection auctions

From Theorem 4.4 we know that the bluff equilibrium of a bisection auction cannot be ex post efficient. In the next two sections we investigate in more detail the level of inefficiency of the bluff equilibrium. The measure of efficiency (or inefficiency) we use is the probability that the auction allocates efficiently (inefficiently). We show that, with an appropriate choice of auction, the probability of inefficient allocation can be made as small as we like, irrespective of the number of bidders that will participate in the auction.

For ease of exposition we initially do this in the context of a special type of bisection auction, called fixed fraction auctions, and for the uniform distribution on the interval $[0, 1)$. In subsection 5.3 we will argue that these results can also be obtained for an arbitrary continuous probability distribution on the interval $[\alpha, \beta)$.

[5] Because $Q(a^*) \leq \sum_{r=1}^{\infty} \frac{1}{(r+1)^2} \leq \frac{1}{4} + \int_{r=2}^{\infty} \frac{1}{s^2} ds$. Equivalently, use the fact that $\frac{1}{(r+1)^2} \leq \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$ for every $r \geq 2$.

We start with the definition of a fixed fraction auction. In the fixed fraction auction with fixed fraction $c \in (0,1)$, the query price $Q(a^*)$ in node $a \in X_r$ is given by

$$Q(a^*) := (1-c)P(a^*) + cH(a^*)$$

in the current price interval $[P(a^*), H(a^*))$. Thus the query price $Q(a^*)$ is $P(a^*)$ plus an increment equal to a fixed fraction c of the size of the current price interval. It is straightforward to check that fixed fraction auctions are indeed regular. Hence, by Theorem 4.5, under the bluff equilibrium a fixed fraction auction ends in finite time regardless of the realization of valuations.

5.1 The associated direct auction

In the next subsection we compute the probability of inefficient allocation of fixed fraction auctions when the equilibrium $b = (b_i)_{i \in N}$ is played. To make these computations easier to understand, we first provide a concrete description of the direct auction associated with the bluff equilibrium of a fixed fraction auction. This description is used in the next subsection.

For the moment we assume that private valuations of players are independently drawn from the uniform distribution on the interval $I = [0,1)$. Given the fixed fraction auction with fraction c, consider the following associated direct auction (w_{direct}, p_{direct}). For $r \in \mathbb{N}$, write $I_r := [1-(1-c)^{r-1}, 1-(1-c)^r)$. Note that the intervals $I_1, I_2, I_3, \dots$ partition the unit interval $[0,1)$ from which valuations are drawn. Now let $v = (v_i)_{i \in N}$ be a profile of valuations. Write $I_r(v) := I_r \cap \{v_i \mid i \in N\}$, the set of valuations that are in I_r. Let $r(v)$ be the highest natural number r for which $I_r(v)$ is not empty. Then w_{direct} is defined by

$$w_{direct}(v) := \min\{i \in N \mid v_i \in I_{r(v)}\}.$$

Let $s(v)$ be the highest natural number r for which $I_r \cap \{v_i \mid i \in N, i \neq w_{direct}(v)\}$ is not empty. The payment function p_{direct} is defined by

$$p_{direct}(v) := \begin{cases} 1-(1-c)^{s(v)-1} & \text{if } i > w_{direct}(v) \text{ for all } i \in I_{s(v)}(v) \\ 1-(1-c)^{s(v)} & \text{else.} \end{cases}$$

We will now show that (w_{direct}, p_{direct}) equals the direct auction $(w \circ b, p \circ b)$ where w and p are defined as in Section 4 and $b = (b_i)_{i \in N}$ is the bluff equilibrium. We call this auction the direct fixed fraction auction.

Theorem 5.1 *For any realization $v = (v_i)_{i \in N}$ of valuations it holds that $w_{direct}(v) = (w \circ b)(v)$ and $p_{direct}(v) = (p \circ b)(v)$. Consequently, truthful bidding is a dominant strategy in the direct auction (w_{direct}, p_{direct}).*

Proof. Let $v = (v_i)_{i \in N}$ be a realization of valuations. By Theorem 4.5 we know that the price will always increase. Consider the round $s(v)$ in which the price is equal to $1-(1-c)^{s(v)-1}$ and the query price is equal to $1-(1-c)^{s(v)}$. The active bidders in round $s(v)$ are $(w \circ b)(v)$, all bidders i with $v_i \in I_{s(v)}$, and –possibly– one more bidder i^* with $v_{i^*} \in I_{s(v)-1}$ who happened to be the bidder with the lowest ranking number among those bidders that were active in the previous round and said *yes* in that round. We distinguish three cases.

Case 1. If i^* exists. Bidder i^* will say *no* in this round $s(v)$, and the next active bidder, say j, in the bidding order will say *yes*. If $j = (w \circ b)(v)$ then all other active bidders say *no*. So, $(w \circ b)(v) = \min\{i \in N \mid v_i \in I_{r(v)}\}$ and $(p \circ b)(v) = 1-(1-c)^{s(v)-1}$. If $j \neq (w \circ b)(v)$ then both j and $(w \circ b)(v)$ say *yes* in this round, while all other active bidders say *no*. In the next round though j will say *no* and $(w \circ b)(v)$ says *yes*. Hence in this case $(w \circ b)(v) = \min\{i \in N \mid v_i \in I_{r(v)}\}$ and $(p \circ b)(v) = 1-(1-c)^{s(v)}$.

Case 2. If i^* does not exist and $i > (w \circ b)(v)$ for all $i \in I_{s(v)}(v)$. In this case $(w \circ b)(v)$ says *yes* in round $s(v)$ while all other active bidders say *no*. Hence $(w \circ b)(v) = \min\{i \in N \mid v_i \in I_{r(v)}\}$ and $(p \circ b)(v) = 1-(1-c)^{s(v)-1}$.

Case 3. When not in Case 1 or 2. Then an active bidder $j \neq (w \circ b)(v)$ says *yes* in round $s(v)$ together with $(w \circ b)(v)$, while all other bidders say *no*. In round $s(v)+1$ bidder j says *no* and bidder $(w \circ b)(v)$ says *yes*. Hence in this case $(w \circ b)(v) = \min\{i \in N \mid v_i \in I_{r(v)}\}$ and $(p \circ b)(v) = 1-(1-c)^{s(v)}$. ■

5.2 Performance results

In this subsection we consider how efficient a fixed fraction auction is when the equilibrium profile of bluff strategies is played. In particular we show that the probability of inefficient allocation in the bluff equilibrium for the fixed fraction auction with fraction c is less than c, independent of the number of bidders that participate in the auction. Thus, by choosing the appropriate fixed fraction auction, the probability of inefficient allocation can be made as small as we like, *independent of the number of bidders* !! We however also show that, for fixed c, the probability of inefficient allocation is larger than a certain positive constant no matter how many bidders participate in the auction. Thus, we can reduce this probability *only* by reducing the fraction c, not by increasing the number of participants.

Notice that by Theorem 5.1 the probability of inefficient allocation is the same for both a fixed fraction auction under the bluff equilibrium and for the corresponding direct auction under the truthtelling equilibrium. Thus it suffices to compute bounds on the probability of inefficient allocation for the corresponding direct fixed fraction auction under the truthtelling equilibrium.

Let c be the fraction. For the direct auction (w_{direct}, p_{direct}) we derive a recurrent relation for the probability $P_n(c)$ that the auction with n bidders terminates in an inefficient allocation given that all bidders bid truthfully. Let $v = (v_i)_{i \in N}$ be a realization of valuations. First consider the case where all valuations are smaller than c. The probability of this event is c^n. Further, in this event the auction is only efficient when bidder 1, the winner of the auction in this case, has the highest valuation. By symmetry this happens with probability $\frac{1}{n}$. Thus this event contributes a term $c^n \frac{n-1}{n}$ to the total probability of inefficient allocation $P_n(c)$. Next consider the case where $k \geq 1$ bidders have a valuation greater than or equal to c and $n-k$ bidders have a valuation smaller than c. If $k = 1$, the direct bisection auction is efficient, so this case adds zero probability to $P_n(c)$. The event $k \geq 2$ happens with probability $\binom{n}{k} c^{n-k} (1-c)^k$. The probability of inefficient allocation in this case is equal to $P_k(c)$ since the direct bisection auction restricted to the interval $[c, 1)$ with k bidders is isomorphic to the original direct bisection auction with k bidders having valuations uniformly drawn from $[0, 1)$. Hence

$$P_n(c) = c^n \frac{n-1}{n} + \sum_{k=2}^{n} \binom{n}{k} c^{n-k} (1-c)^k P_k(c)$$

for all $n \geq 2$. This can be rewritten to

$$\Big(1 - (1-c)^n\Big) P_n(c) = \frac{n-1}{n} \cdot c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} P_k(c).$$

In particular, $P_2(c) = \frac{1}{2} \cdot \frac{c}{2-c}$. We will show that the probability of inefficiency $P_n(c)$ is smaller than c, and also bounded away from zero. In order to simplify computations, take $\lambda := 1 - c$. Write

$$Z(c) := \sum_{k=3}^{\infty} \frac{c^k + k\lambda c^{k-1}}{1 - \lambda^k - k\lambda c^{k-1} - c^k}.$$

Theorem 5.2 *For all $n \geq 2$, $P_n(c) \leq c$ and $P_n(c) \geq e^{-Z(c)} P_2(c)$.*

Proof. First we show that $P_n(c) \leq c$. Since $P_2(c) = \frac{1}{2} \cdot \frac{c}{2-c} \leq c$, we know that our claim is true for $n = 2$. Suppose that $P_k(c) \leq c$ for all $2 \leq k \leq n-1$. Then

$$\begin{aligned} P_n(c) &= \frac{1}{1-(1-c)^n}\Big[\frac{n-1}{n}\cdot c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\cdot P_k(c)\Big] \\ &\leq \frac{1}{1-(1-c)^n}\Big[c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\cdot c\Big] \\ &= \frac{1}{1-(1-c)^n}\Big[c^n + c\Big(1-c^n-n(1-c)c^{n-1}-(1-c)^n\Big)\Big] \\ &= \frac{c(1-(1-c)^n)}{1-(1-c)^n} + \frac{c^n - c^{n+1} - n(1-c)c^n}{1-(1-c)^n} \\ &= c + \frac{c^n(1-c)(1-n)}{1-(1-c)^n} \leq c, \end{aligned}$$

which concludes the proof for the upper bound on $P_n(c)$.

Now we show that $e^{-Z(c)}P_2(c) \leq P_n(c)$. Define $B_n(c)$ by $B_2(c) := P_2(c)$ and

$$B_n(c) := \frac{1}{1-\lambda^n}\left[\sum_{k=2}^{n-1}\binom{n}{k}\lambda^k c^{n-k} B_k(c)\right]$$

for $n \geq 3$. A simple induction argument shows that $B_n(c) \leq P_n(c)$ for all n. We will show that $B_n(c) \geq e^{-Z}P_2(c)$ for all n. Define $Q_2(c) := 1$ and for $n \geq 3$

$$Q_n(c) := \frac{1-\lambda^n - n\lambda c^{n-1} - c^n}{1-\lambda^n}\cdot Q_{n-1}(c) = \prod_{k=3}^{n}\frac{1-\lambda^k - k\lambda c^{k-1} - c^k}{1-\lambda^k}.$$

We will first show that for all $n \geq 2$, $B_k(c) \geq Q_n(c)B_2(c)$ holds for all $2 \leq k \leq n$. Clearly this holds for $n = 2$. Take $n \geq 3$. Assume that for all $2 \leq k \leq n-1$ we have that $B_k(c) \geq Q_{n-1}(c)B_2(c)$. Since $0 < Q_n(c) \leq Q_{n-1}(c)$ we have that $B_k(c) \geq Q_n(c)B_2(c)$ for all $2 \leq k \leq n-1$. For $k = n$,

$$\begin{aligned} B_n(c) &= \frac{1}{1-\lambda^n}\left[\sum_{k=2}^{n-1}\binom{n}{k}\lambda^k c^{n-k} B_k(c)\right] \\ &\geq \frac{1}{1-\lambda^n}\left[\sum_{k=2}^{n-1}\binom{n}{k}\lambda^k c^{n-k} Q_{n-1}(c)B_2(c)\right] \\ &= \frac{1}{1-\lambda^n}\cdot Q_{n-1}(c)\cdot B_2(c)\cdot\left[\sum_{k=2}^{n-1}\binom{n}{k}\lambda^k c^{n-k}\right] \\ &= \frac{1-\lambda^n - n\lambda c^{n-1} - c^n}{1-\lambda^n}\cdot Q_{n-1}(c)\cdot B_2(c) = Q_n(c)B_2(c) \end{aligned}$$

which shows that $B_n(c) \geq Q_n(c)B_2(c)$. Now notice that $Z(c) > 0$. So, $Q_2(c) = 1 \geq e^{-Z(c)}$, while for $n \geq 3$

$$\begin{aligned}\log(Q_n(c)) &= \sum_{k=3}^{n} \left(\log\left(1-\lambda^k - k\lambda c^{k-1} - c^k\right) - \log\left(1-\lambda^k\right)\right) \\ &= -\sum_{k=3}^{n} \left(\log\left(1-\lambda^k\right) - \log\left(1-\lambda^k - k\lambda c^{k-1} - c^k\right)\right) \\ &\geq -\sum_{k=3}^{n} \frac{k\lambda c^{k-1} + c^k}{1-\lambda^k - k\lambda c^{k-1} - c^k} \geq -Z(c),\end{aligned}$$

where the first inequality follows from the fact that $\log y - \log x \leq \frac{y-x}{x}$ for $y > x$. Hence, since $B_2(c) := P_2(c)$, $B_n(c) \geq Q_n(c) \cdot B_2(c) \geq e^{-Z(c)} \cdot P_2(c)$ for all $n \geq 2$. ■

5.3 The generality of the approximate efficiency result

As said before, for *any* continuous probability distribution G on $[\alpha, \beta)$ from which valuations are independently drawn and for any $c \in (0,1)$, we can construct a bisection auction for which under the bluff equilibrium the probability of inefficient allocation is smaller than or equal to c.

A bisection auction that attains a level of inefficiency less than or equal to c can be constructed as follows. For any decision node a with length $(r-1)n$ for some $r \geq 1$, given the lower bound of $P(a)$ and the upper bound of $H(a)$, choose the query price $Q(a)$ in such a way that, given that the valuation of a bidder is an element of $[P(a), H(a))$, the probability that his valuation is in $[P(a), Q(a))$ is equal to (or less than) c. In other words, choose $Q(a)$ in such a way that

$$\frac{G(Q(a)) - G(P(a))}{G(H(a)) - G(P(a))} \leq c.$$

This auction will be regular as long as we choose $Q(a)$ sufficiently far from $P(a)$. This can be done in many ways, but one of them is to choose $Q(a)$ in such a way that

$$\frac{G(Q(a)) - G(P(a))}{G(H(a)) - G(P(a))} = c.$$

Continuity of the probability distribution is merely required to guarantee that choices can be made in this way. However, less demanding conditions would clearly suffice as well.

The bisection auction thus constructed is in general not a fixed fraction auction. It is clear though that when G is the uniform distribution the fixed fraction auction with fraction c fits the above description. Moreover, since the conditional probabilities generated by the fixed fraction c are the only pieces of information that we use in the proofs in the above subsections, it is clear that all the above results, and Theorem 5.2 in particular, extend immediately to the general setting sketched in this subsection.

6 Conclusions

We have shown that, in a setting where bidders have continuous valuations, ex post efficiency –allocating the item to a bidder with the highest valuation– in a query auction can only be obtained at the price of an infinite running time of the auction for almost all realizations of valuations. We also showed that this negative result applies to a wide class of query auctions, in particular bisection auctions.

Nevertheless, to alleviate this result, we also show that, for any continuous probability distribution from which valuations are independently drawn, when we allow the allocation to be inefficient with an arbitrarily small but strictly positive probability, there is a bisection auction that attains this level of efficiency, independent of the number of bidders that participate in the auction, and with a finite running time for all realizations of valuations.

7 Appendices

This section contains the proof of the Theorem of Green and Laffont, as well as proofs of lemmata that are used in Section 4.

7.1 Appendix 1. Proof of the Theorem of Green and Laffont

Proof. Assume that (a), (b) and (c) hold. We will show that $(w \circ s, p \circ s)$ is a Vickrey auction. To this end, let $v = (v_i)_{i \in N}$ be a profile of valuations in I^N. Since s is ex post efficient, we know that

$$(w \circ s)(v) = w(s_i(v_i)_{i \in N}) \in \arg\max\{v_i \mid i \in N\}.$$

So we only have to show that $(p \circ s)(v) = v_{sec}$, where

$$v_{sec} := \max\{v_i \mid i \neq (w \circ s)(v)\}.$$

Write $i^* := (w \circ s)(v)$. Moreover, denote the profile of realizations of valuations $((v_j)_{j \neq i^*}, r)$ by $(v \mid r)$, and the profile of actions $(s_j(v_j)_{j \neq i^*} \mid s_{i^*}(r))$ by $s(v \mid r)$. The proof is in two steps.

I. First we will show that $(p \circ s)(v) \leq v_{sec}$. To this end, take a valuation $r \in I$ with $r > v_{sec}$. We show that $(p \circ s)(v) \leq r$.

Since the strategy profile s is an ex post equilibrium of (F, w, p) we know that $s(v)$ is a Nash equilibrium in the ex post game (F, w, p, v). Because $i^* = (w \circ s)(v)$, we know that

$$u_{i^*}(v_{i^*})(s(v \mid r)) \leq u_{i^*}(v_{i^*})(s(v)) = v_{i^*} - (p \circ s)(v). \tag{1}$$

Now suppose bidder i^* happens to have valuation r. Since $r > v_{sec}$, ex post efficiency of the strategy profile s in (F, w, p) implies that $i^* = (w \circ s)(v \mid r)$. Moreover, since the strategy profile s is an ex post equilibrium of (F, w, p) we know that $s(v \mid r)$ is a Nash equilibrium in the ex post game $(F, w, p, (v \mid r))$. Hence, by ex post individual rationality

$$u_{i^*}(b)(s(v \mid r)) = b - p(s(v \mid r)) = b - (p \circ s)(v \mid r) \geq 0.$$

The last inequality implies that $r \geq (p \circ s)(v \mid r)$.

Now, suppose that bidder i^* chose action $s_{i^*}(r)$ while having valuation v_{i^*}. Again, since $r > v_{sec}$, ex post efficiency of the strategy profile s in (F, w, p) implies that $i^* = (w \circ s)(v \mid r)$. So

$$u_{i^*}(v_{i^*})(s(v \mid r)) = v_{i^*} - (p \circ s)(v \mid r) \geq v_{i^*} - r \tag{2}$$

where the inequality follows from the result that $r \geq (p \circ s)(v \mid r)$.

Combination of the inequalities (1) and (2) yields $v_{i^*} - (p \circ s)(v) \geq v_{i^*} - r$. Hence, $(p \circ s)(v) \leq r$.

II. Secondly we will show that $(p \circ s)(v) \geq v_{sec}$. To this end, take an $r \in I$ with $r < v_{sec}$. We show that $(p \circ s)(v) \geq r$.

Suppose bidder i^* happens to have valuation r. Since the strategy profile s is ex post efficient in (F, w, p) and $r < v_{sec}$, we know that $i^* \neq (w \circ s)(v \mid r)$. Hence

$$u_{i^*}(r)(s(v \mid r)) = 0. \tag{3}$$

However, since $s(v \mid r)$ is a Nash equilibrium in $(F, w, p, (v \mid r))$, we also know that

$$u_{i^*}(r)(s(v \mid r)) \geq u_{i^*}(r)((s(v)) = r - (p \circ s)(v). \tag{4}$$

Combining equality (3) and inequality (4) yields that $(p \circ s)(v) \geq r$. ∎

7.2 Appendix 2. Proofs for Section 4

We say that a node from D_j^2 is in MD_j^2 if none of the predecessors of this node is in D_j^2. We analyze what happens in such a node when a player uses his bluff strategy.

Lemma 7.1 *Suppose player j has valuation v_j and follows the plan of action $b_j(v_j)$. If a node a is an element of MD_j^2 and if player j is active in this node, then $b_j(v_j)(a) = yes$.*

Proof. Let r be the round to which a belongs. If $r = 1$, then $P(a) = \alpha$, thus $v_j \geq P(a)$ and j says yes. If $r > 1$, by definition of MD_j^2 there was an active predecessor in the previous round who said yes. Since j is still active, j also said yes. Hence $v_j \geq Q(a(r-1)) = P(a)$. ∎

As a consequence after round r player j either is the winner (in case no other player who is active in round r and has ranking greater than j says *yes*) or he is active in round $r+1$ (otherwise). In any case all players with ranking less than j are nonactive from then on.

Lemma 7.2 *Suppose player j follows his bluff strategy, and a is a decision node of player j in round r. Suppose that player j is active in a and says no for the first time. Then as long as player j will stay active, his actions will be no.*

Proof. If $a \in D_j^1$, then player j is not active in any future round. Suppose then $a \in D_j^2$, then $v_j < P(a) < Q(a)$. This relation remains valid in any successor node of a. ■

The previous lemma states that, if player j follows his bluff strategy and says *yes* in node a where he is active, then all previous actions of him were *yes* as well.

Given the realization of valuations $(v_i)_{i \in N}$ suppose that player j follows his plan of action $b_j(v_j)$ while all other players follow $s_{-j}(v_{-j})$ –the profile of plans of action corresponding to an arbitrary profile of strategies s_{-j}. Let h be the realization of the game if the profile $(b_j(v_j), s_{-j}(v_{-j}))$ is played.

Lemma 7.3 *Suppose that in a regular bisection auction player j follows his bluff strategy and $j = W(h)$. Then $T(h) < \infty$.*

Proof. Suppose that $T(h) = \infty$. It implies that $|A(h)| > 1$ and, by definition of $W(h)$, for all $i \in A(h)$ it holds that $i \leq j$. First we argue that no player in $A(h)$ can say *yes* indefinitely. If a player in $A(h)$ would say *yes* indefinitely, then, because $j = W(h)$, player j must do so as well. However, since player j follows his bluff strategy, this implies that $\beta = P^* \leq v_j$ by the regularity of the auction. This contradicts the assumption that v_j is drawn from $[\alpha, \beta)$.

Consider the round in history h where for the first time a player i from $A(h)$ says *no*. Let a be the decision node that player j reaches in this round. If any other player from $A_j(a)$ says *yes* in this round then i is not active in the next round, which contradicts the fact that $i \in A(h)$. If all players from $A_j(a)$ say *no* in this round then, according to Lemma 7.1, player j says *yes* and, again, i is not active anymore after this round. Since $T(h) = \infty$, this means that any player from $A(h)$ says *yes* indefinitely. Contradiction. Hence, we conclude that $T(h) < \infty$. ■

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