Approximate CAPM When Preferences Are CRRA*

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Abstract

In general equilibrium models of financial markets, the capital asset pricing formula does not hold when agents have von Neumann-Morgenstern utility with constant relative risk aversion. In this paper we examine under which conditions on endowments and dividends the pricing formula provides a good benchmark for equilibrium returns. While it is easy to construct examples where equilibrium returns are arbitrarily far from those predicted by CAPM, we show that there is a large class of economies where CAPM provides a very good approximation. Although the pricing formula does not hold exactly for the chosen specification, it turns out that pricing-errors are extremely small.

JEL codes: D52, D58, G11, G12.

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1 Introduction

The capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) predicts that equilibrium returns of assets are a linear function of their market β, the slope in the regression of a security's return on the market's return. This intuitively appealing result has long shaped the way practitioners think about average returns and risk. While the empirical validity of the model is very controversial (see for example Fama and French (1992)) it remains one of the central building blocks in financial economics.

However, in consumption based asset pricing models where agents choose portfolios in order to maximize von Neumann-Morgenstern utility over non-negative consumption processes, the CAPM pricing formula only holds if one assumes that all agents' utility functions are quadratic (see Geanakoplos and Shubik (1990), Berk (1997)). In applied general equilibrium, this is usually thought of as an unrealistic specification of preferences and it is assumed instead that agents' preferences exhibit constant relative risk aversion (CRRA). Under this assumption CAPM does not hold and there are no general results linking an asset's equilibrium return to its covariance with the market's return.

In this paper we investigate under which conditions equilibrium asset prices can be well approximated by the CAPM formula in economies where all agents have CRRA preferences.

It is easy to construct examples where assets' market β does not explain any variation in cross sectional returns. In these economies any econometric test would reject CAPM and empirical contradictions of CAPM might be explained by the fact that some agents do not have quadratic utility. For example, when all assets have the same price and each asset payoff has the same covariance with aggregate endowments CAPM, predicts that the excess returns must be identical across assets. But since higher moments matter when agents have CRRA preferences the equilibrium returns generally differ across assets. We give a simple example where these differences are quantitatively substantial.

On the other hand we show that for a large class of economies the CAPM pricing formula provides a very good prediction for actual equilibrium returns. If there is a single agent whose labor income is independent of all asset payoffs and if all asset payoffs are generated by a linear factor structure with iid factors and factor-weights adding up to one, the CAPM pricing formula holds independently of the specification of the agent's utility function. This simply follows from the absence of arbitrage and is itself of limited interest because of the strong assumptions on payoffs and the representative agent. The main contribution of the paper is to provide hundreds of examples which demonstrate that this relation is very robust with respect to the introduction of heterogeneous agents and variations in the factor structure. The CAPM provides an excellent approximation to equilibrium excess returns for a wide variety of dividends and individual endowments.

In the examples, we assume that there are three agents and $32,768$ states of nature and we examine the robustness of CAPM with respect to different specifications of preferences, payoffs and endowments. We assume that the agents have CRRA utility functions (but different degrees of risk aversion) and examine the following cases:

- Endowments and dividends are drawn from a uniform distribution. We randomly generate 100 economies with differ with respect to the support of the uniform distributions.
- Endowments and dividends are drawn from a log-normal distribution. We randomly generate 100 economies which differ with respect to the agents' coefficients of risk aversion.
- Endowments and dividends are determined by two factors and an idiosyncratic shock each of which are drawn from a log-normal distribution. We randomly generate 100 economies with differ with respect to the factor-loads.
- Endowments and dividends are drawn from a log-normal distribution and there is an option on one of the stocks. We randomly generate 100 economies which differ with respect to the strike-price of the option.

For all economies under consideration we compare the computed return on individual stocks to the return predicted by the CAPM-pricing formula. We find that in all 400 cases the average mean squared pricing errors (for returns) across stocks lie below 0.04 percent. The average error across all simulations is in the order of magnitude of 0.005 percent. In these economies standard statistical procedures would accept CAPM.

The paper is organized as follows. In Section 2 we introduce the general equilibrium model and summarize several results on CAPM in this setting. In Section 3 we give simple examples which demonstrate under which conditions CAPM pricing provides a good approximation for equilibrium returns when agents have CRRA preferences. In Section 4 we compute returns for several hundred examples and show the robustness of our results. Section 5 concludes.

2 The Two-Period Finance Economy

The finance version of the GEI-model describes an economy over two periods of time, $t = 0, 1$, with uncertainty over the state of nature resolving in period $t = 1$. We describe the model, introduce the necessary notation and discuss the CAPM. For a thorough description of the GEI-model see for example Magill and Quinzii (1996).

2.1 The Model

There are $S+1$ states in the economy; at time $t=0$ the economy is in state $s=0$, at time $t=1$ one state of nature s out of S possible states of nature realizes. In each state $s=0,\ldots,S$, there is a single nondurable consumption good.

There are H agents, indexed by $h=1,\ldots,H$, that participate in the economy. An agent h is characterized by initial endowments (the initial income stream) $e^h=(e_0^h,e_1^h,\ldots,e_S^h)^\top \in \mathbb{R}_{++}^{S+1}$ and his preferences over consumption bundles (income streams available for consumption) $c^h=(c_0^h,c_1^h,\ldots,c_S^h)^\top \in \mathbb{R}_{+}^{S+1}$. To distinguish between first-period consumption and the random second-period consumption, we define $\widetilde{x}=(x_1,\ldots,x_S)^\top$ for any vector $x=(x_0,x_1,\ldots,x_S)^\top$. Aggregate endowments (aggregate incomes) are $e=\sum_{h=1}^H e^h$. Each agents' preferences are represented by a von Neumann-Morgenstern utility function

$$u^h(c^h)=v^h(c_0^h)+\delta\sum_{s=1}^{S}\rho_s v^h(c_s^h),$$

where probabilities $\rho_1,\ldots,\rho_S>0$, $\sum_{s=1}^S \rho_s=1$, and the discount factor $\delta>0$ are identical across agents, and where the Bernoulli function $v^h:\mathbb{R}_+\to\mathbb{R}$ is assumed to be strictly increasing and strictly concave.

There are J assets. Asset j pays dividends at date $t=1$ which we denote by $d_j\in\mathbb{R}^S$. The price of asset j at time $t=0$ is q_j. Without loss of generality we assume in this section that the assets are in zero net supply and we collect all assets' dividends in a pay-off matrix

$$A=(d_1,....,d_J)\in\mathbb{R}^{S\times J}.$$

Without loss of generality, we assume A to have full column rank. At time $t=0$ agent h chooses a portfolio-holding $\theta^h\in\mathbb{R}^J$ which uniquely defines the agents' consumption by $\widetilde{c}^h=\widetilde{e}^h+A\theta^h$ and $c_0^h=e_0^h-\theta^h\cdot q$. The net demand of agent h, $\widetilde{c}^h-\widetilde{e}^h$, belongs to the marketed subspace $\langle A\rangle=\{z\in\mathbb{R}^S \mid \exists\theta\in\mathbb{R}^J,\ z=A\theta\}$.

The exogenous parameters defining a finance economy $\mathcal{E}=((u^h,e^h)_{h=1,\ldots,H};A)$ are agents' utility functions and endowments, and the pay-off matrix.

We define asset prices to be arbitrage free if it is not possible to achieve a positive income stream in all states by trading in the available assets. It is well known that a price system $q\in\mathbb{R}^J$ precludes arbitrage if and only if there exists a state price vector $\pi\in\mathbb{R}_{++}^S$ such that $q=\pi^\top A$.

Definition 1 (Competitive Equilibrium): A competitive equilibrium for an economy $\mathcal{E}$ is a collection of portfolio-holdings $\theta^*=(\theta^{1*},\ldots,\theta^{H*})\in\mathbb{R}^{HJ}$, consumptions $c^*=(c^{1*},\ldots,c^{H*})$ and asset prices $q^*\in\mathbb{R}^J$ that satisfy the following conditions:

(1) $(c^{h*},\theta^{h*})\in\arg\max_{c^h,\theta^h}\ u^h(c^h)\ s.t.\ c^h=e^h+\begin{pmatrix}-q^{*\top}\\ A\end{pmatrix}\theta^h$ and $c^h\in\mathbb{R}_+^{S+1}$, $\quad h=1,\ldots,H$;

(2) $\sum_{h=1}^{H} \theta^{h*} = 0$.

Existence of an equilibrium follows from the results of Geanakoplos and Polemarchakis (1986).

2.2 The Capital Asset Pricing Model

Under the assumption that all agents are mean-variance optimizers Sharpe (1964) and Lintner (1965) derive a closed-form solution for equilibrium returns, the so-called β-pricing formula. This formula relates the return of a risky asset to the return of the market portfolio by the covariance of that asset with the market. It is well known that the β-pricing formula can be derived in the finance GEI-model, see Geanakoplos and Shubik (1990). We summarize the findings in the literature - Geanakoplos and Shubik (1990), Magill and Quinzii (1996), Oh (1996), and Willen (1997).

We denote by $1_n = (1, \ldots, 1)^\top \in \mathbb{R}^n$ the vector of all ones. We assume that there exist objective probabilities ρ_s, $s = 1, \ldots, S$, over the possible states of nature in period 1. Moreover, asset 1 is a riskless bond, $d_1 = 1_S$. For a random variable $x \in \mathbb{R}^S$, we define its expected value $\mathrm{E}(x) = \sum_{s=1}^{S} \rho_s x_s$, for two random variables $x, y \in \mathbb{R}^S$, we define the covariance as $\mathrm{cov}(x, y) = \sum_{s=1}^{S} \rho_s x_s y_s - \mathrm{E}(x)\mathrm{E}(y)$. The variance of a random variable $x \in \mathbb{R}^S$ is given by $\mathrm{var}(x) = \mathrm{cov}(x, x)$. Finally, we define $x \cdot_\rho y = \sum_{s=1}^{S} \rho_s x_s y_s$ for vectors $x, y \in \mathbb{R}^S$.

For any competitive equilibrium (θ^*, q^*), there exists a unique state price vector in the marketed subspace $\pi_A^* \in \langle A \rangle$ such that, for all assets j, $q_j^* = \pi_A^* \cdot_\rho d_j$. Using the definitions of variance and covariance, this implies

$$q_j^* = \mathrm{E}(\pi_A^*)\mathrm{E}(d_j) + \mathrm{cov}(\pi_A^*, d_j). \tag{1}$$

We define the return of a portfolio $\theta \in \mathbb{R}^J$ with $q^* \cdot \theta \neq 0$ by $r_\theta = \frac{A\theta}{q^* \cdot \theta}$ and we denote the return of the riskless bond by $R^{\mathrm{f}} = \frac{1}{q_1^*}$. With this we rewrite equation (1) as

$$q_j^* = \frac{1}{R^{\mathrm{f}}}\mathrm{E}(d_j) + \mathrm{cov}(\pi_A^*, d_j).$$

We define the pricing portfolio as the unique portfolio θ_A^* which solves $A\theta_A^* = \pi_A^*$. Notice that

$$q^* \cdot \theta_A^* = \pi_A^* \cdot_\rho A\theta_A^* = \pi_A^* \cdot_\rho \pi_A^* > 0,$$

where $\pi_A^* \neq 0$ follows from $\mathrm{E}(\pi_A^*) = q_1^* > 0$.

Since the return of the pricing portfolio satisfies $r_{\theta^*_A} = \frac{A\theta^*_A}{q^* \cdot \theta^*_A} = \frac{\pi^*_A}{\pi^*_A \cdot_\rho \pi^*_A}$ we can rewrite equation (1) as

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \frac{\mathrm{cov}(r_\theta, r_{\theta^*_A})}{\mathrm{var}(r_{\theta^*_A})}(\mathrm{E}(r_{\theta^*_A}) - R^{\mathrm{f}}). \tag{2}$$

While equation (2) relates the prices of the risky assets and looks similar to the CAPM pricing formula, this formula is rather useless if we have no further information on π^*_A. Note that so far all formulas followed simply from the absence of arbitrage. It is well known that under the assumption that one agent h's utility function is differentiable and that in an equilibrium with individual consumption $(c^{h*})_{h \in H}$, agent h's utility maximization problem has an interior solution, π^*_A can be characterized as

$$\pi^*_A \;=\; \mathrm{proj}_{\langle A \rangle}\left(\frac{\partial_{c^h_1} u^h(c^{h*})/\rho_1}{\partial_{c^h_0} u^h(c^{h*})}, \ldots, \frac{\partial_{c^h_S} u^h(c^{h*})/\rho_S}{\partial_{c^h_0} u^h(c^{h*})} \right),$$

where $\mathrm{proj}_{\langle A \rangle}$ denotes the projection on $\langle A \rangle$ under the inner product ρ.

Agent h's first period endowments can be decomposed into a marketed part and a non-marketed part, where the latter part lies orthogonal to the marketed subspace under the inner product ρ. We write

$$\tilde{e}^h = \tilde{e}^h_{\mathrm{M}} + \tilde{e}^h_{\perp}$$

and have by definition $\tilde{e}^h_\perp \cdot_\rho z = 0$ for all $z \in \langle A \rangle$. This decomposition is uniquely determined. We define the marketed endowments by $\tilde{e}_{\mathrm{M}} = \sum_{h=1}^{H} \tilde{e}^h_{\mathrm{M}}$. The market portfolio θ_{M} is then defined as the unique portfolio satisfying

$$A\theta_{\mathrm{M}} = \tilde{e}_{\mathrm{M}}.$$

Note that it may happen that $q^* \cdot \theta_{\mathrm{M}} = 0$, even when $\tilde{e} \gg 0$.[1] However, to simplify matters, we assume $q^* \cdot \theta_{\mathrm{M}} \neq 0$.

Given a competitive equilibrium (c^*, θ^*, q^*), we define β_θ for a portfolio $\theta \in \mathbb{R}^J$ by

$$\beta_\theta = \frac{\mathrm{cov}(r_\theta, r_{\theta_{\mathrm{M}}})}{\mathrm{var}(r_{\theta_{\mathrm{M}}})}.$$

Then the following result can be found in the literature (see e.g. Willen (1997) for a derivation in the GEI-framework).

Theorem 1: *Under the assumptions that all agents have quadratic utility, that* $\mathrm{var}(\tilde{e}_{\mathrm{M}}) > 0$, *and that there is a riskless bond, each equilibrium* (c^*, θ^*, q^*) *of* $\mathcal{E}$ *with equilibrium consumption* $(c^{1*}, \ldots, c^{H*}) \gg 0$ *has the following properties.*

[1] For a vector $x \in \mathbb{R}^m$ we use the notation $x \geq 0$ if $x \in \mathbb{R}^m_+$, $x > 0$ if $x \in \mathbb{R}^m_+ \setminus \{0\}$, and $x \gg 0$ if $x \in \mathbb{R}^m_{++}$.

1. *The CAPM-pricing formula holds; when* $q^* \cdot \theta_{\mathrm{M}} \neq 0$, *then for each* $\theta \in \mathbb{R}^J$,

$$E(r_\theta) - R^{\mathrm{f}} = \beta_\theta(E(r_{\theta_{\mathrm{M}}}) - R^{\mathrm{f}}). \tag{3}$$

2. *The pricing vector satisfies* $\pi^*_A = \alpha_1 1_S - \alpha_2 \tilde{e}_{\mathrm{M}}$, *with* $\alpha_1 > \alpha_2 \mathrm{E}(\tilde{e})$ *and* α_2 *strictly positive.*

Note that for the case where the endowments are spanned, i.e. where $e^h_\perp = 0$ for all h, the pricing formula reduces to the standard CAPM-formula (see Magill and Quinzii (1996)).

It might be sensible to define the market portfolio somewhat differently as a portfolio of risky assets only. In this case define $\widehat{\theta}_{\mathrm{M}} = (0, \theta_{\mathrm{M},2}, \ldots, \theta_{\mathrm{M},J})$. If we define $\widehat{\beta}_\theta = \mathrm{cov}(r_\theta, r_{\widehat{\theta}_{\mathrm{M}}})/\mathrm{var}(r_{\widehat{\theta}_{\mathrm{M}}})$ it turns out that the pricing formula still holds. After some substitutions, one obtains

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \widehat{\beta}_\theta(\mathrm{E}(r_{\widehat{\theta}_{\mathrm{M}}}) - R^{\mathrm{f}}).$$

Finally, note that the concept of marketed endowments is not needed to define the pricing vector. Since $\tilde{e}_\perp$ is orthogonal to $\langle A \rangle$, the pricing vector can also be defined by $\tilde{\pi}^*_A = \alpha_1 1_S - \alpha_2 \tilde{e}$. Of course it then no longer holds that $\tilde{\pi}^*_A \in \langle A \rangle$. Therefore income streams not in $\langle A \rangle$ are typically priced differently by $\tilde{\pi}^*_A$ than by π^*_A.

3 CAPM and CRRA: Three Examples

As we have discussed in the introduction, Theorem 1 can only be obtained when one is willing to make very restrictive assumptions. Magill and Quinzii (1996) comment on the CAPM: "As we indicated above these models are interesting since they lead to clearcut results which have strong intuitive appeal. However the restrictive nature of the hypothesis made could cast doubt on the generality of the results." In particular, the assumption that all agents maximize a quadratic utility function is unattractive because it implies increasing absolute risk aversion. A more realistic assumption, and one commonly made in macroeconomics and finance, is that agents' preferences exhibit constant relative risk aversion. The question we want to address in this paper is how much actual equilibrium prices will differ from the predictions of CAPM in this more realistic setting.

We assume that all agents have constant relative risk aversion utility functions of the form

$$\begin{aligned} v^h(c) &= \frac{c^{1-\gamma^h}}{1-\gamma^h}, \quad \gamma^h \neq 1, \\ v^h(c) &= \log(c), \quad \gamma^h = 1, \end{aligned}$$

where γ^h is the coefficient of relative risk aversion.

In this section we consider three simple examples, two of which can be solved analytically, to show that whether or not equilibrium returns are close to CAPM depends crucially on the joint distribution of all asset payoffs.

3.1 Exact CAPM with a Linear Factor Structure

Suppose that there is single representative agent, whose initial endowments consist of his labor income plus dividends from his asset holdings. Denote the agent's possibly stochastic labor income by l. Suppose there are N identically and independently distributed random variables (or 'factors') ε_i which are all independent to l and have mean zero. Suppose that there is a riskless bond and $J-1$ stocks in unit net supply where for $j = 2, ..., J$,

$$d_j = \mu_j 1_S + \sum_{i=1}^{N} \phi_j^i \varepsilon^i, \quad \sum_{j=2}^{J} \phi_j^i = 1 \text{ for all } i. \tag{4}$$

Note that with this setup arbitrary first and second moments of asset payoffs can be constructed. Higher moments of one individual asset or the market can be matched by the choice of ε, but higher moments of all assets cannot be matched. We now show that with this construction, the CAPM pricing formula (3) holds exactly, independently of the representative agent's utility.

Since there is a representative agent we can price all assets by pricing the factors. Since the factors are all iid and independent of l, by symmetry, they must all have the same price, q_ε. If q_1 denotes the price of the bond, then the price of stock j is given by

$$q_j = \mu_j q_1 + q_\varepsilon \sum_{i=1}^{N} \phi_j^i.$$

The return of stock j is given by

$$\left(\mu_j 1_S + \sum_{i=1}^{N} \phi_j^i \varepsilon^i \right) / q_j$$

and the return of the market portfolio by

$$\left(\sum_{j=2}^{J} \left(\mu_j 1_S + \sum_{i=1}^{N} \phi_j^i \varepsilon^i \right) \right) / \sum_{j=2}^{J} q_j.$$

After some manipulations, it follows that stock j's market $\widehat{\beta}$ is given by

$$\widehat{\beta}_j = \frac{\sum_{k=2}^{J} q_k}{q_j} \frac{\sum_{i=1}^{N} \phi_j^i}{N}.$$

We have that the CAPM formula (3) holds if and only if

$$\frac{\mu_j}{q_j} - \frac{1}{q_1} = \frac{\sum_{k=2}^J q_k}{q_j} \left(\frac{\sum_{k=2}^J \mu_k}{\sum_{k=2}^J q_j} - \frac{1}{q_1} \right) \frac{\sum_{i=1}^N \phi_j^i}{N}.$$

Multiplying by $q_j = \mu_j q_1 + q_\varepsilon \sum_{i=1}^N \phi_j^i$ and substituting $\sum_{k=2}^J q_k = q_1 \sum_{k=2}^J \mu_k + N q_\varepsilon$ reveals that this equation (and therefore CAPM pricing) always holds.

Note that in this setup CAPM pricing follows purely from the absence of arbitrage, holds independently of the representative agent's preferences and has nothing to do with mean-variance analysis.

3.2 No CAPM with Different Factors

We now give a simple example that shows that the CAPM pricing formula might be completely useless for explaining cross sectional returns when asset payoffs are not generated by identically distributed factors.

Consider an economy with a representative agent with CRRA utility function and risk aversion of γ. As before, the agent's initial endowments consist of his labor income plus dividends from his asset holdings. In addition to a bond there are two risky stocks which are independently distributed. There are two independently distributed factors, ε^1 and ε^2. The first factor is 0.8 with probability 2/3 and 1.4 with probability 1/3. The second factor is 1.2 with probability 2/3 and 0.6 with probability 1/3. The two stocks' dividends are

$$d_1 = \varepsilon^1 \text{ and } d_2 = \varepsilon^2 + \mu 1_S$$

for some $\mu \geq 0$. The stocks are in unit net supply. The agent has a non-stochastic labor income of $0.2 - \mu$ in the second period. Suppose the agent also has 2 units available for consumption in the first period and does not discount the future, i.e. $\delta = 1$. Since the assets are in unit net supply, it follows that $\tilde{e} = 0.2 + \varepsilon^1 + \varepsilon^2$.

For $\mu = \frac{E((\varepsilon^1 - \varepsilon^2)/\tilde{e}^\gamma)}{E(1/\tilde{e}^\gamma)}$ the equilibrium prices of the two stocks are identical. Since both stocks have the same covariance with aggregate endowments, CAPM predicts that the excess return of the two stocks must be equal. However, the equilibrium excess returns are quite different and depend on γ as the following table shows.

The key to this example lies in the fact that on the margin, a CRRA agent prefers the dividends of asset 1 to the dividends of asset 2 - therefore for the same expected payoffs, asset 2 must be cheaper than asset 1 and its returns higher. A mean-variance agent with quadratic utility, on the other hand, would be indifferent between ε^1 and ε^2 since they have the same mean and variance.

γ	Excess return 1	Excess return 2	CAPM prediction
2	7.7	10.1	8.9
4	13.7	20.7	17.2
6	15.2	25.7	20.4

TABLE 1: Equilibrium excess returns

3.3 Heterogeneous Agents and Identical Factors

While the argument in Section 3.1 shows that given a linear factor structure one might expect CAPM to provide a good approximation to prices if there is only a single agent, one has to compute equilibria in order to assess how well CAPM predicts equilibrium prices in economies with heterogeneous agents and incomplete markets. From now on we examine economies with three heterogeneous agents, representing classes of agents with low, medium and high incomes.

Each agent is endowed with an initial portfolio $(0, \theta_-^h)$ of the riskless bond and the available stocks with current income, representing current labor income plus dividends from θ_-^h, $e_0^1 = 2/3$, $e_0^2 = 1$, and $e_0^3 = 4/3$, and with stochastic future labor income given by some $l^h \in \mathbb{R}_{++}^S$. For each household h, the labor incomes l_s^h are generated by S independent draws from some given distribution. The first agent has no capital income, $\theta_-^1 = 0$. For the other agents we have $\theta_-^2 = 1/3 \cdot 1_{J-1}$ and $\theta_-^3 = 2/3 \cdot 1_{J-1}$. Agents have heterogeneous von Neumann-Morgenstern utility functions with CRRA, identical uniform probabilities over states and identical discount factors $\delta^h = 0.95$.

The assets available are given by a riskless bond and 7 stocks. In most examples the dividends of asset j depend on a single common factor which we now denote by $f \in \mathbb{R}^S$ as well as on an idiosyncratic factor $\varepsilon^j \in \mathbb{R}^S$. We denote asset j's load in the common factor by ϕ_j, varying from 0.25 to 1.75 in steps of 0.25. Aggregate consumption and capital share in the examples are calibrated to yearly US data. The expected growth rate of aggregate consumption equals two percent and the standard deviation of both the factor and the idiosyncratic shock determining the dividends are about 0.13 - giving an overall standard deviation of the stock market of about 0.17. The standard deviation of labor income is chosen to be around 0.10 and labor income constitutes around 2/3 of total income. The eleven random variables in the model are therefore $((l^h)_{h=1,\dots,H}, f, (\varepsilon^j)_{j=2,\dots,J})$.

Note that the construction of asset dividends now differs from (4) in that factor weights no longer add up to one.

As a first example we analyze the case where the realization of each random variable is either high or low with equal probabilities, and all random variables are independent.

The minimal state space to achieve this consists of $2^{11} = 2,048$ states. More specifically we have that

$$\begin{aligned} l_s^h &\in \{2/3 \cdot (1.02 - 0.1), 2/3 \cdot (1.02 + 0.1)\}, \\ f_s &\in \{-0.13, 0.13\}, \\ \varepsilon_s^j &\in \{-0.13, 0.13\}. \end{aligned}$$

Dividends of asset j are then determined by

$$d_s^j = 1/3 \cdot 1/7 \cdot (1.02 + \sqrt{\phi_j} f_s + \varepsilon_s^j).$$

We have taken the square of the factor load ϕ_j in the specification of the dividends in order to give it an interpretation as the (approximate) stock's market β.

We choose heterogeneous coefficients of risk aversion: $\gamma^1 = 6$, $\gamma^2 = 4$ and $\gamma^3 = 2$.

With these specifications we compute the equilibrium prices and portfolio-holdings and compare them to the predictions of the CAPM in Figure 1. In Herings and Kubler (2002) we develop an algorithm that is tailored to the finance GEI-model with one good per state, and that is independent of the number of states. We use this algorithm to approximate equilibria numerically.

The solid line in the figure is the security market line, i.e. the CAPM relationship between a portfolio's β and its risk premium. The actual equilibrium expected returns of the seven securities are depicted by + and lie all almost exactly on the security market line. CAPM turns out to be an extraordinarily good predictor for the actual equilibrium returns of assets in this example. This is surprising for two reasons. First, the asset payoffs and dividends do not satisfy the assumptions in Section 3.1. Secondly, the introduction of agents' heterogeneity does not alter the cross section of equilibrium returns significantly.

Although the graph of Figure 1 looks very convincing, it is clear that we need more objective measures to quantify the deviation of equilibrium prices and portfolio-holdings from the CAPM predictions.

The most straightforward approach is to measure the accuracy of CAPM-pricing is to take the Mean Squared Error (MSE), which is defined by

$$\text{MSE} = \sqrt{\frac{1}{J-1} \sum_{j=2}^{J} (r_j^* - \widehat{r}_j)^2},$$

where r_j^* denotes the equilibrium expected return of asset j and $\widehat{r}_j$ the prediction by CAPM, using (3) and the true equilibrium interest rates and market return.

A different approach consists of the following. It is well-known that $\pi_A^* \in \langle 1_S, \widetilde{e}_{\mathrm{M}} \rangle$ is sufficient for CAPM-pricing. That this is necessary as well follows from the observation that otherwise π_A^* is equal to the sum of its projection on $\langle 1_S, \widetilde{e}_{\mathrm{M}} \rangle$ plus a non-zero orthogonal

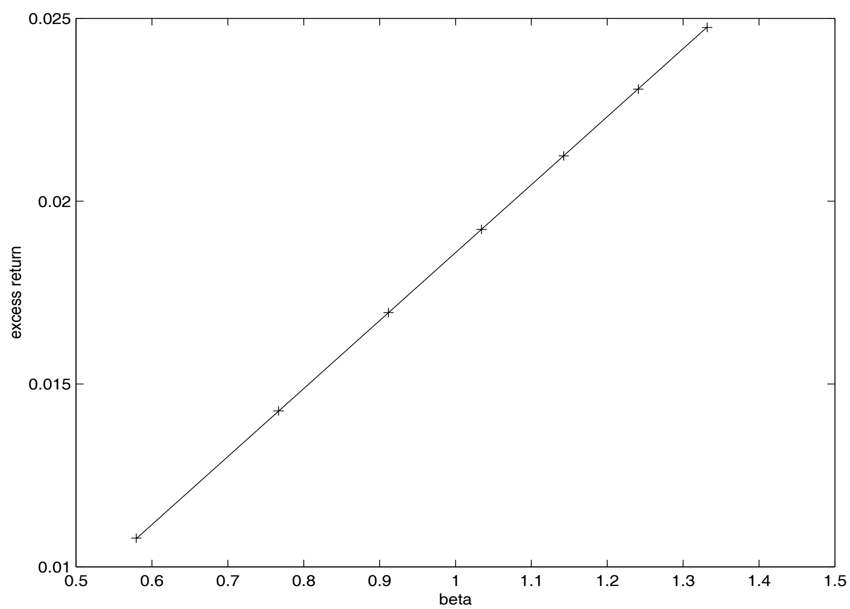

FIGURE 1: Security market line with high-low returns.

part in $\langle A \rangle$ under the inner product ρ. When CAPM-pricing is valid, the orthogonal part should have zero price, which is obviously not the case when priced by π_A^*. Therefore, an interesting alternative to MSE is to take the OLS R^2 of the regression with

$$\text{proj}_{\langle A \rangle} \left(\frac{\partial_{c_1^h} u^h(c^{h*})/\rho_1}{\partial_{c_0^h} u^h(c^{h*})}, \dots, \frac{\partial_{c_S^h} u^h(c^{h*})/\rho_S}{\partial_{c_0^h} u^h(c^{h*})} \right)$$

as regressand and 1_S and $\tilde{e}_{\text{M}}$ as regressors. Notice that this measure is independent of h. We call it Pricing R^2.

The following table confirms that CAPM provides an outstanding prediction for the economy under consideration.

R^{f}	1.0633
Equity Premium	0.0185
MSE	0.0000530
Pricing R^2	0.99999998

Table 2: CAPM for CRRA preferences and two-point distributions.

4 Robustness of CAPM

In order to show that the predictions of CAPM are a good approximation for equilibria in a wide variety of economic settings we compute several hundred examples. We assume that there are $S = 32,768$ states of nature. Using a large number of states guarantees that our final samples are good approximations of continuous distributions. By taking a large number of states we rule out finite sample effects on the prices of assets. When we replicate the experiment and generate economies out of a newly drawn sample, the equilibrium will be almost the same if the number of states is sufficiently large.

We test the robustness of our results to variations in the distributions of endowments and assets. We consider three different families of return processes and compute 100 randomly generated examples within each class. We show the histograms of MSE and Pricing R^2. In all histograms the scaling is taken identically, so that results for different models can be compared easily.

4.1 Uniform Returns

In order to verify whether our results depend crucially on all factors having the same distribution we now assume uniformly distributed shocks, which all have different support.

We also allow for some variation in the ratio of labor income to total income, in the variance of the factor and in the variance of the idiosyncratic shocks.

More specifically, we start each example by randomly generating parameters a_1, a_2, a_3 and a_4, where

$$\begin{aligned} a_1 &\sim \mathrm{U}(1.02 \cdot 0.5, 1.02 \cdot 0.9), \\ a_2 &\sim \mathrm{U}(1.02 \cdot 1.1, 1.02 \cdot 1.5), \\ a_3 &\sim \mathrm{U}(-0.5, -0.1), \\ a_4 &\sim \mathrm{U}(0.1, 0.5). \end{aligned}$$

Given a realization for $a_1, \ldots, a_4$, we continue the construction of the economy by taking independent drawings for l_s^h, f_s and ε_s^j, where

$$\begin{aligned} l_s^h &\sim \mathrm{U}(2/3 \cdot 0.8, 2/3 \cdot 1.24), \\ f_s &\sim \mathrm{U}((a_1 - a_2)/2, (a_2 - a_1)/2), \\ \varepsilon_s^j &\sim \mathrm{U}(a_3, a_4). \end{aligned}$$

Finally, dividends are determined by

$$d_s^j = 1/3 \cdot 1/7 \cdot (\frac{a_1 + a_2}{2} + \sqrt{\phi_j} f_s + \varepsilon_s^j).$$

Given the realizations for the parameters a_1 and a_2, $1/3 \cdot 1/7 \cdot (a_1 + a_2)/2$ equals expected dividends from asset j. The realization of the factor belongs to the interval $[(a_1 - a_2)/2, (a_2 - a_1)/2]$ and the realizations of the idiosyncratic shocks to the interval $[a_3, a_4]$. The expected labor income and the variance of labor income are taken as before.

Figures 2a-b show that the ability of CAPM to predict portfolio-holdings and excess returns is robust to variations in the distribution of shocks.

Figure 2 shows that CAPM is an excellent predictor for the class of CRRA utility functions and uniform factors. While the factors are no longer iid, in most cases the MSE is around $1 \cdot 10^{-4}$. The worst Pricing R^2 found is 0.9999.

The high values of the Pricing R^2 provides very useful information for the pricing of assets. Recall that the price of asset j is given by $\pi_A^* \cdot d^j$. Any vector that is highly correlated with π_A^* should lead to a similar price for asset j. In particular, when the Pricing R^2 is close to one, CAPM is bound to give almost exact equilibrium prices and the use of CAPM leads to a low MSE.

4.2 Log-Normal Returns and Different Risk Aversions

We now vary the factor structure (4) further my assuming that all asset payoffs are the product of iid factors with log-normal distribution. Furthermore we vary agents' degree of risk aversion.

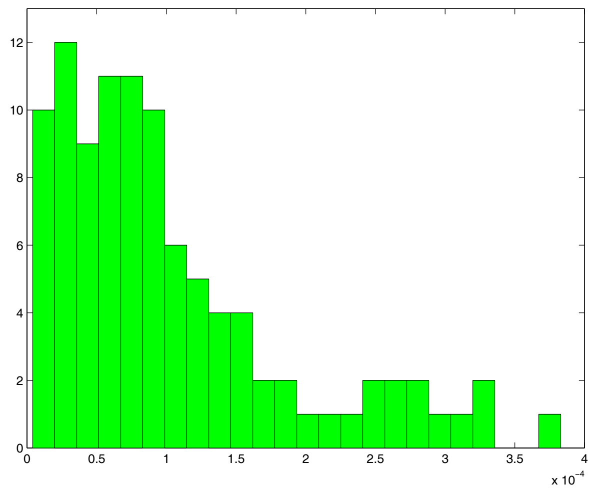

FIGURE 2A: Uniform: MSE.

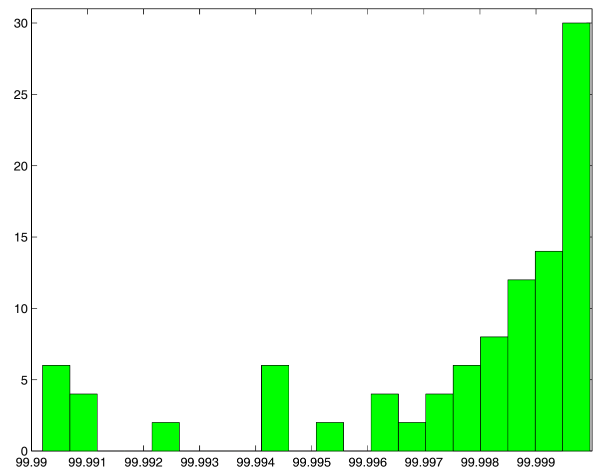

FIGURE 2B: Uniform: 100· Pricing R^2.

Throughout this section we assume that all random variables are log-normally distributed, so l_s^h, f_s, and ε_s^j are drawn independently from a log-normal distribution. The log-normal distribution with mean μ and variance σ^2 is denoted by $\mathrm{LN}(\mu, \sigma^2)$. As before asset 1 is the riskless bond. For $j \geq 2$, we define asset j's dividend to be

$$d_s^j = 1/3 \cdot 1/7 \cdot 1.02 \cdot f_s^j \cdot \varepsilon_s^j$$

and we choose

$$\begin{aligned} l_s^h &\sim \mathrm{LN}(2/3 \cdot 1.02, (2/3)^2 \cdot 0.01), \\ f_s^j &\sim \mathrm{LN}(1, \phi_j \cdot 0.0161), \\ \varepsilon_s^j &\sim \mathrm{LN}(1, 0.0161). \end{aligned}$$

The actual $(f_s^j)_{j=2}^J$ are all based on a single realization of a normal random variable. For each asset j, we linearly transform the realization of this random variable in such a way that after taking the exponent a log-normally distributed random variable with mean 1 and variance $c_j \cdot 0.0161$ results. The construction of the random variables implies that all dividends themselves are log-normally distributed. To get a similar variance of the entire stock market as before the variance of the factors and the idiosyncratic shock have to be chosen to be 0.0161 instead of 0.0169. Notice that the factor realization does not enter linearly in the formula for the asset's dividends, an assumption that is made in most models describing factor economies.

We now assume that all agents have constant relative risk aversion and we draw γ^h, $h = 1, 2, 3$, from a uniform distribution on the interval $[0.5, 10]$. As before we compute 100 examples - Figures 3a-b report the analogues of Figures 2a-b for the CRRA case.

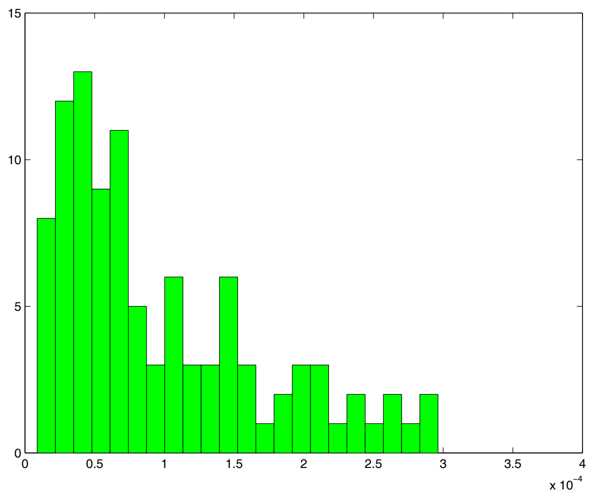

FIGURE 3A: CRRA: MSE.

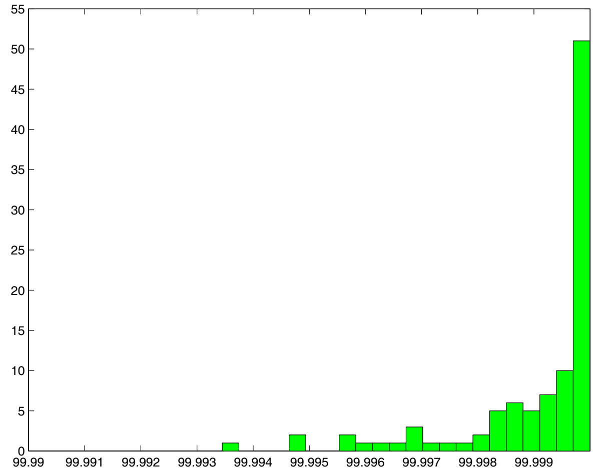

FIGURE 3B: CRRA: 100· Pricing R^2.

As before CAPM is a very good prediction for actual equilibrium returns.

4.3 Two Common Factors

In this subsection we generate a number of economies where risky assets depend on two common factors, f and $\widehat{f}$, and factor loads for each one of the assets are randomly drawn. On top of this, also the importance of the idiosyncratic shock is randomly determined.

We start each example by randomly generating, for each asset $j = 2, \ldots, J$, parameters ϕ_j, $\widehat{\phi}_j$, and i_j. These parameters represent the load in factor 1, the load in factor 2 and the importance of the idiosyncratic shock. More specifically it holds that

$$\begin{aligned} \phi_j &\sim \mathrm{U}(0,2), \\ \widehat{\phi}_j &\sim \mathrm{U}(0,2), \\ i_j &\sim \mathrm{U}(0,4). \end{aligned}$$

Labor income, the two factors and assets' idiosyncratic shocks are independently log-normally distributed, so l_s^h, f_s, $\widehat{f}_s$, and ε_s^j are drawn from a log-normal distribution,

$$\begin{aligned} l_s^h &\sim \mathrm{LN}(2/3 \cdot 1.02, (2/3)^2 \cdot 0.01), \\ f_s^j &\sim \mathrm{LN}(1, \phi_j \cdot 0.0161), \\ \widehat{f}_s^j &\sim \mathrm{LN}(1, \widehat{\phi}_j \cdot 0.0161), \\ \varepsilon_s^j &\sim \mathrm{LN}(1, i_j \cdot 0.0161). \end{aligned}$$

Finally, dividends are determined by

$$d_s^j = 1/3 \cdot 1/7 \cdot 1.02 \cdot f_s^j \cdot \widehat{f}_s^j \cdot \varepsilon_s^j.$$

The way to generate f_s^j, $j = 2, \ldots, J$, from a single realization of a normally distributed random variable is the same as in Section 4.2. The same applies to the other factor.

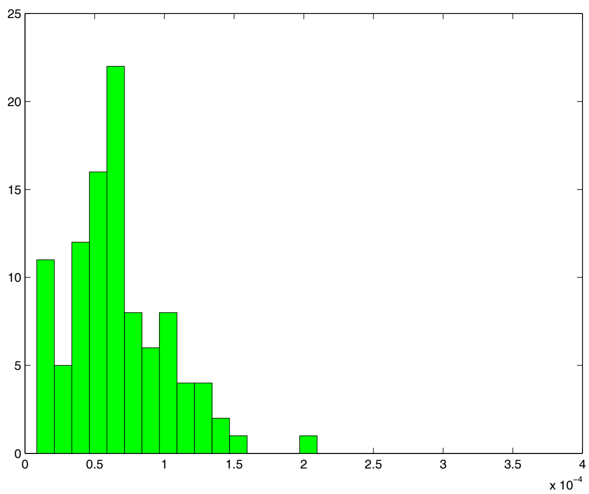

FIGURE 4A: Two-factor: MSE.

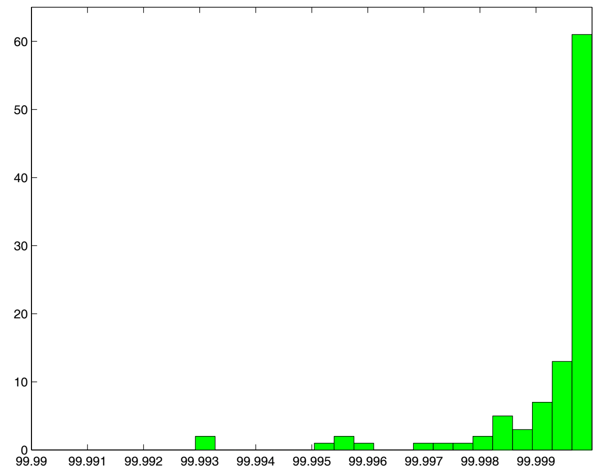

FIGURE 4B: Two-factor: 100· Pricing R^2.

Surprisingly, CAPM pricing is still a very good prediction for equilibrium returns. Although the asset payoffs now differ substantially from a linear factor structure, pricing errors are very small.

4.4 Options

Since markets are incomplete and utilities are not quadratic, the introduction of an option on one of the assets will generally change all equilibrium prices. Therefore one might expect that the introduction of an option worsens CAPM-pricing considerably. Furthermore, given the robustness of CAPM in the earlier examples, it is interesting to see if it is possible to give an equilibrium pricing formula for options in incomplete markets via CAPM.

Another reason to introduce an option is that this is an asset with the capacity to seriously alter the higher order moments of an asset portfolio. One possible explanation for our results obtained so far is that asset markets are very incomplete, which makes it difficult for households to change the higher order moments of the returns of their portfolios. Although households care for higher order moments, the mix of marketed assets makes it difficult to affect the higher order moments. With the introduction of an option this clearly changes. Agents have then a possibility to limit downwards risk, which is exactly the kind of risk agents with CRRA utility functions are concerned about, but mean-variance optimizers are not.

In order to investigate this issue more closely we introduce a call option on the most risky asset. Specifically we have a 9-th security which pays $\max(d_s^j - X, 0)$ in state s, with X the strike price of the call option.

Suppose we consider the uniquely determined equilibrium pricing vector π_A^* of the economy without the option, and we use this pricing vector to price the option. Given the reasoning of the previous paragraph, at those prices one would expect the call option (in combination with the bond) to be more attractive to the agents than the stock, exactly because of the higher order moments. So the equilibrium price of the call option should be higher than the one computed by CAPM-pricing, in order to make that asset less appealing. As a consequence, the expected equilibrium return of the call option should be less than the one predicted by CAPM.

To examine different options, we draw X out of the uniform distribution for each example. To avoid options that are either too far in or too far out of the money we determine in each example the minimal dividend paid out by asset 8, $\underline{d}^8 = \min_{s=1,\ldots,S} d_s^8$, and the maximal dividend paid out, $\overline{d}^8 = \max_{s=1,\ldots,S} d_s^8$. We then draw X out of a uniform distribution on $[0.5 \cdot (1.02 + \underline{d}^8), 0.5 \cdot (1.02 + \overline{d}^8)]$. Note that 1.02 is the expected dividend of asset 8. The strike price is always between the average of the minimal dividend and the expected dividend, and the average of the expected dividend and the maximal dividend. The results are given in Figures 5a-b.

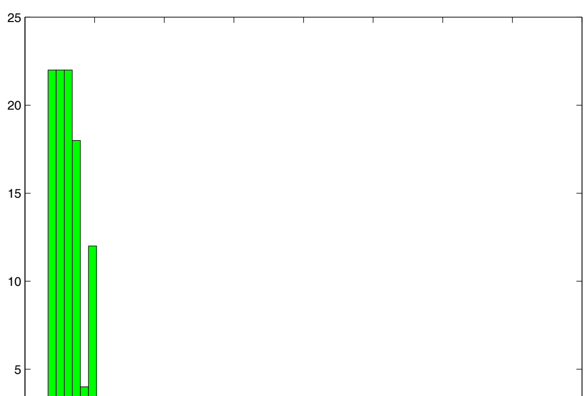

FIGURE 5A: Option: MSE.

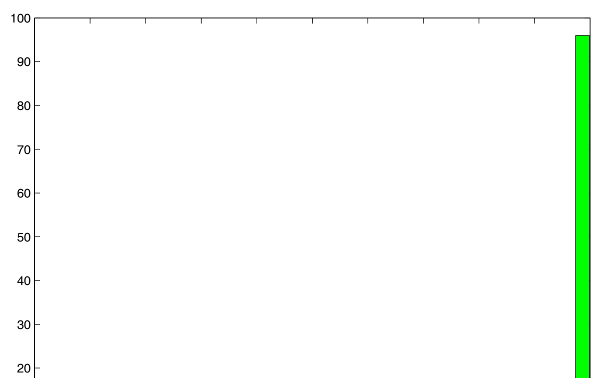

FIGURE 5B: Option: 100· Pricing R^2.

The MSE in Figure 5 refers to the MSE of the pricing of the stocks only. The option is analyzed in detail in Figure 6. It turns out that the MSE are comparable to the ones given before. The Pricing R^2 is somewhat less good than before, but is still excellent. Surprisingly, we have found no systematic effect of the introduction of the option on the price of asset 8. In some examples the introduction of an option raised the price above the CAPM-prediction, in others it has been lower.

Figure 6 analyzes the pricing of the option by CAPM. According to CAPM, a call option is a very risky asset. It has zero pay-offs in bad states of nature, and very high in good states of nature. The covariance of a call option with the market portfolio is very high, which is also clear from Figure 6, where it is shown that the option's β varied from 5 to 35 in the economies generated. Notice that, as we expected, there is indeed an over-prediction of the expected return of an option by CAPM. In all economies generated, CAPM underpriced the call option. The misprediction was relatively small when the option's β is low, say below 10, but may get quite severe for call options with a very high strike price, which are the ones with a high β. Notice, however, that a higher β of an option also corresponds to a higher excess return, which makes the relative misprediction less bad. Still, the over-prediction of call option returns is more than linearly increasing in an option's β, whereas the excess return itself is still roughly linear.

It is surprising that the Pricing R^2 and the MSEs of stocks remained so good in all economies, even when the option was sometimes seriously under-priced by CAPM. In fact, it may even be perceived as an inconsistency that the Pricing R^2 is virtually exactly correct, and the option is seriously mispriced. Indeed, when CAPM-pricing is highly correlated with π^*_A, almost all assets are priced very well. The only exceptions are those like options with a very high strike price. Such an asset pays off in a few (less than 10) states of the 32,768 only. A high correlation with π^*_A is not inconsistent with a fairly different state price in a negligible fraction of states only.

5 Conclusion

In order to show that the CAPM-pricing formula holds in a general equilibrium model with heterogeneous agents , one needs strong assumptions either on preferences or on dividends and endowments (see Berk (1996)). It is possible to construct simple examples with agents who have CRRA utility in which an asset's β implies little about its equilibrium excess return.

However, examining the robustness of CAPM by computing equilibria, we find that CAPM provides an excellent approximation to equilibrium excess returns and portfolio-holdings for a wide variety of dividends and endowments. One possible explanation is that the dividend structures we consider are 'close' to a linear factor structure which guarantees that CAPM holds exactly if there is a single agent in the economy and that labor incomes are chosen to be independent of all asset payoffs. Nevertheless, the computational results are very surprising: Both the introduction of heterogeneous agents and substantial deviations from the linear factor structure seem to have very small quantitative effects on

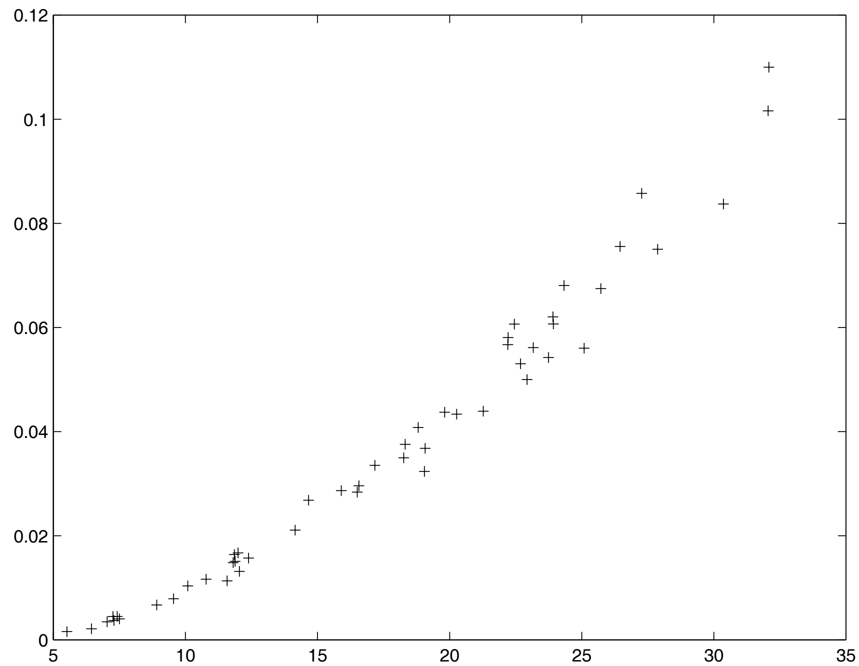

FIGURE 6: Option: over-prediction of return against option's β.

the cross section of equilibrium excess returns.

It has already been noted (see e.g. Heaton and Lucas (1996)) that agents' heterogeneity and independent labor background risk has only small quantitative effects on the equity premium. One contribution of this paper is to show that the effects on cross sectional returns are very small as well. The main contribution of this paper, however, is to show that CAPM provides a good approximation of returns for a wide variety of dividend specifications.

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