Algorithms for graphs embeddable with few crossings per edge[*]

Hans L. Bodlaender[†]   Alexander Grigoriev[‡]

Abstract. We consider graphs that can be embedded on a surface of bounded genus such that each edge has a bounded number of crossings. We prove that many optimization problems, including maximum independent set, minimum vertex cover, minimum dominating set and many others, admit polynomial time approximation schemes when restricted to such graphs. This extends previous results by Baker [1] and Eppstein [3] to a much broader class of graphs.

1   Introduction

Already more than two decades ago, Baker [1] showed that the *maximum independent set* and many other NP-hard optimization problems on graphs admit polynomial time approximation schemes (PTAS) when restricted to planar graphs. The basic idea of Baker's algorithm was to remove the vertices in every kth level of a breadth first search tree (BFS) and to solve the problem on the remaining components by a dynamic programming algorithm. Baker proved that from k ways of choosing which set of levels to remove there is at least one which only decreases the size of the maximum independent set by a factor of at most $(k-1)/k$. Moreover, remaining components after levels deletion are k-outerplanar graphs, and dynamic programming can solve the problem on these components efficiently.

Recently, Eppstein in [3] observed that the results by Baker [1] can be extended to any minor-closed family of graphs satisfying so-called diameter-treewidth property. This implies that the problem admits a PTAS if restricted to bounded-genus graphs. This result has been generalized to other minor-closed classes; in particular, Grohe gave PTAS's for several problems, for any minor-closed family that does not contain all graphs [2].

In this paper we continue the line of investigations — in which way can Baker's technique be further extended? Revisiting Eppstein [3] result, we observe that the restriction that the class of graphs must be minor-closed can be relaxed. By moving from the input graph to an auxiliary graph obtained by

replacing each crossing by a vertex and back, we can obtain Baker-type PTAS's for several problems on graph that are embeddable on a surface of bounded genus (e.g., the plane) with a bounded number of crossings per edge.

2 Problem and Definitions

We illustrate the basic ideas of the PTAS on the maximum independent set problem. Given a graph $G = (V, E)$, we look for a maximum cardinality independent set in G, i.e. a vertex subset $V' \subseteq V$ such that no two vertices from V' are adjacent by an edge from E. Let $n = |V|$. This problem is known to be NP-hard even for planar graphs. The problem admits a PTAS if restricted to planar graphs [1] and even to bounded-genus graphs [3].

Definition 2.1 (Good embedding). *We call an embedding of graph G on a surface S of genus g a* good embedding *if it satisfies the following conditions: (i) all vertices of the graph are given as distinct points in S; (ii) no two edge crossings happen in the same point in S; (iii) for any edge no vertex of the graph, except the endpoints of the edge, is situated on the edge.*

Definition 2.2 (Crossing parameter). *Let the* crossing parameter *φ of a graph (on surface S) be the minimum over all good embeddings on S of the maximum over all edges e of the number of edge crossings of e.*

Through this paper we assume that a good embedding of G is given and both the crossing parameter φ and the genus g of S are bounded by some constants. Clearly, the graph is planar if $g = 0$ and $\varphi = 0$.

3 The Polynomial Time Approximation Scheme

We now describe our polynomial time approximation scheme for the maximum independent set problem on graphs with bounded crossing parameter on bounded genus. Consider the following algorithm $\mathcal{A}$ which is a revised version of the algorithms by Baker [1] and Eppstein [3].

Construct the graph $G' = (V', E')$ obtained from G by replacing each edge crossing by a vertex. W.l.o.g. let $\varphi < k$. Build an arbitrary breadth first search tree T of G, with root v_0, and consider the levels of the tree (i.e., vertex sets with equal distance to v_0). Suppose we remove every kth level in T from G' together with its φ successive levels. This decomposes G' into a collection of subgraphs $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ where each subgraph $H_t = (V_t, E_t)$ is induced by $k - \varphi - 1$ consecutive levels in T of G'.

Consider a subgraph G_t of G induced by vertices $V_t \cap V$. Since number of crossings per edge is at most φ and we removed $\varphi + 1$ consecutive levels from G', we have that after deletion of levels there is no an edge $e \in E$ such that its two endpoints belong to two different subgraphs $G_{t'}$ and $G_{t''}$. Therefore, for each choice of k we have a subgraph of G formed by a collection of subgraphs $G_1, G_2, \ldots, G_r$. By arguments similar to Baker's [1] there is a choice of the deleting set of levels which leads to a subgraph approximating the optimum within a factor $(k - \varphi - 1)/k$.

Now, let us show that the treewidth of G_t is bounded by $O(k)$ for all $t = 1, \ldots, r$. Consider a subgraph H_t induced by levels $r+1, r+2, \ldots, r+s$ in T of G' where $s = k - \varphi - 1 = O(k)$. Consider a minor of G' obtained by contraction of the first r levels in T to a single vertex and deletion of all levels above $r+s$. Clearly, this minor is a graph of genus g. Moreover, it has a diameter of at most $2(k - \varphi - 1) = O(k)$. By Eppstein [3] the treewidth of such a minor is $O(gk)$. Therefore, H_t as a subgraph of such a minor has the treewidth of at most $O(gk)$ as well.

Now, let us estimate how much the treewidth of G_t and H_t can differ. Construct a graph H_t' from H_t by replacing each vertex v in H_t that represents an edge crossing, say e_1 and e_2, by two adjacent vertices v_1 and v_2 representing e_1 and e_2 respectively. Let v_1 be adjacent to all vertices corresponding to the neighborhood of v representing e_1, and let v_2 be adjacent to all vertices corresponding to the neighborhood of v representing e_2. It is not hard to see that a tree decomposition of H_t of treewidth d can be turned into a tree decomposition of H_t' of treewidth at most $2d + 1$. One can also observe that we can select for each edge in G_t a path in H_t' between its endpoints, such that these paths do not have internal vertices in common. Thus, G_t is a minor of H_t' and hence the treewidth of G_t is at most twice the treewidth of H_t plus one, and thus $O(gk) = O(k)$ as required. Hence, the maximum independent set for G_t can be found in time $O(n2^{O(k)})$ by a dynamic programming algorithm, using standard treewidth techniques. As a result we have the following theorem.

Theorem 3.1. *Algorithm $\mathcal{A}$ outputs an independent set of graph G of size at least $1 - O(1/k)$ times the optimum in time $O(kn2^{O(k)})$, and thus, there is a PTAS for maximum independent set for graphs given with an embedding on a surface of bounded genus and with bounded crossing parameter.* $\square$

For each of the following problems (and many others), we can also obtain in a similar way, using techniques similar to those of Baker [1], a PTAS for graphs embeddable on a surface of bounded genus with bounded crossing parameter: minimum vertex cover, minimum dominating set, minimum edge dominating set, minimum triangle matching, maximum H-matching, maximum tile salvage.

4   More on the Crossing Parameter

In this section we give some results on the computational complexity of the crossing parameter and some properties of the class of graphs with bounded crossing parameter.

Theorem 4.1. *The problem to determine if a given graph G can be embedded on the plane with crossing parameter 1 is NP-complete.* $\square$

Corollary 4.1.1. *When P$\neq$NP, there does not exist a polynomial time 2-approximation algorithm for finding the crossing parameter of a graph on the plane.* $\square$

Notice, however, that several natural classes of graphs have a bounded crossing parameter on the plane. For instance, graphs of intersections of objects in

the plane with bounded objects density (disk graphs with bounded density are special case of these); graphs with bounded degree and bounded tree width; planar graphs.

Observation 4.1. *The class of graphs with an embedding on the plane with crossing parameter 1 is not closed under taking minors. In fact, every graph is a minor of a graph with crossing parameter 1: take any good embedding, and then add a new vertex of degree two between every two successive crossings.*

From work on the crossing *number* of graphs (the minimum total number of crossings in a planar embedding), we can also obtain bounds on the crossing parameter (on the plane). E.g., the crossing number of a complete graph with n vertices is $\Theta(n^4)$ [4], hence its crossing parameter is $\Theta(n^2)$.

References

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