Conformity and Bounded Rationality in Games with Many Players

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Abstract

Intepret a set of players all playing the same pure strategy and all with similar attributes as a society. Is it consistent with self interested behaviour for a population to organise itself into a relatively small number of societies? In a companion paper we characterized how large " must be, in terms of parameters describing individual games, for an equilibrium to exhibit conformity in pure strategies. In this paper we provide a wide class of games where such conformity is boundedly rational, that is, where " can be chosen to be small.

1 Conformity, bounded rationality and equilibrium

We suggest that in games with many players common elements of bounded rational behavior are the use of pure strategies and conformity in the sense that a player is inclined to choose the same strategy as players he perceives as similar to himself. If this is the case, and we take Nash equilibrium as an outcome of fully rational behavior, then the consistency of boundedly rational behavior and rationality requires the existence of an approximate Nash equilibrium exhibiting conformity. In this paper we provide a family of games with many players where the desired equilibrium exists.

This paper extends, in important respects, previous results of Wooders, Cartwright and Selten, WCS, (2002) and Cartwright and Wooders, CW, (2003a). In WCS we treat collections of games with complete information and demonstrate existence of an approximate Nash equilibrium in pure strategies and conformity. The class of games considered in WCS are all derived from a common, underlying structure. In contrast to the earlier research in WCS, CW treats individual games. Also, CW introduces a new notion of conformity that allows individuals within the same society to adhere to the same social norms – that is, to play the same strategy – while taking on different roles in that society. For example, according to one social norm, in females cook dinner and males mow the lawn. Given an individual game, CW determines bounds, depending on the parameters describing the game, so that if ε is larger than the bounds, an ε-equilibrium exhibiting conformity exists. Roughly, the parameters describing a game are the number of 'player classes' and a measure of the closeness of players within classes. The novel features of CW are thus the notion of conformity, the notion of player classes in strategic games, and the treatment of individual games rather than games with many players (as in WCS and the prior literature on purification of Nash equilibria).

An important question not addressed in CW is whether, for games with many players - large games - the parameters describing individual games, and thus ε, can be chosen to be small. In this paper we introduce a framework of games with incomplete information and demonstrate conditions under which the numbers of player classes can be chosen to be relatively small while the distance between any two players of the same class is small. Conformity and existence of an approximate equilibrium in pure strategies then follows from our prior results.

To treat a family of games of incomplete information we take as given a set of attributes Ω, a set of player types T and a set of actions A. A player's attribute is assumed to be publicly observable while a player's type, determined by nature, is not. A universal payoff function h details the payoff of a player as a function of his attribute, type and action and the attributes, types and actions of the complementary player set. A universal beliefs function b details the probability distribution over type profiles - players are assumed to have consistent beliefs with respect to this distribution. We refer to the tuple $G = (\Omega; A; T; b; h)$ as a non-cooperative pregame. A player set and an attribute function, assigning an attribute to each player, induce, through the pregame, a game.

We provide conditions on a pregame so that all sufficiently large games induced from that pregame have an approximate Nash equilibrium in pure strategies that is consistent with conformity. To formalize the idea of consistency with conformity we introduce the notion of a society. A society is defined as a collection of players who all play the same strategy and who all have attributes in some convex subset of attribute space. A strategy vector induces a partition of the population into societies and, in interpretation, the fewer are the number of societies then the stronger the conformity. In our main result we provide a bound on the number of societies induced that is independent of the number of players. Thus, in large populations societies must also be large.

Following CW we permit an endogenous assignment of roles within a society. A player is assigned a role according to some probability distribution determined by the society and can make his action choice conditional on his role. This approach allows us to model the case where players could be seen as conforming (or belonging to the same society) even though they may perform di¤erent actions: Given that players can make action choice conditional on role or type, two players can play the same strategy and yet (because they have been allocated di¤erent roles or types) play di¤erent actions.

As well as treating the bounded rationality of conformity in pure strategies we also treat in isolation the bounded rationality of playing pure strategies and the bounded rationality of conformity. In both cases we provide su¢ cient conditions on a pregame for the existence of an approximate Nash equilibrium satisfying the desired properties - either one in pure strategies or one consistent with conformity.

Elaborating further on the prior literature, WCS provide a family of games with many players for which there exists an approximate Nash equilibrium in pure strategies that partitions the player set into a bounded number of societies. Two limitations of the results due to WCS are: …rst, it only treats games of perfect information which, amongst other things, does not allow us to model an assignment of roles within a society. Second, the bound on number of societies is proportional to the number of strategies; given that the framework of WCS can be extended to allow a countable set of strategies (see Cartwright and Wooders 2003b) this appears a signi…cant limitation.

In the companion paper CW we consider the bounded rationality of conformity in pure strategies for arbitrary games of incomplete information.

This is done by introducing the concept of a (±; Q)-class game - any …nite game is (±; Q)-class game for some values of ± and Q. Given a (±; Q)-class game a bound on " permitting existence of a Bayesian Nash "-equilibrium in pure strategies consistent with social conformity, is provided as a function of ± and Q. The approach of CW has the advantage of treating individual games and permitting incomplete information. In addition, CW are able to bound the number of societies independently of the number of strategies. CW, however, only put a bound on the " for which there exists an "-equilibrium satisfying the desired properties - they do not provide conditions under which the " is small.

In this paper we address some of the issues that arise from WCS and CW. First, we extend the pregame framework of WCS to permit incomplete information. We then demonstrate a connection between (±; Q)-class games and games induced from a pregame. This allows us to apply the results of CW and in doing so provide a family of games where the use of pure strategies and conformity can be consistent with individually rational behavior. Further, in the results of this paper, the number of societies is bounded independently of the number of strategies.

We proceed as follows: Section 2 introduces de…nitions and notation and Section 3 reviews the de…nition of a (±; Q)-class game. In Section 4 we treat conformity, in Section 5 we treat pure strategies and in Section 6 we treat conformity in pure strategies. In Section 7 we conclude.

2 De…nitions and notation

We begin this section by de…ning a Bayesian game. The pregame framework is then introduced and we demonstrate how Bayesian games can be induced through a pregame. Next, we consider the strategies available to players in a Bayesian game and discuss expected payo¤s. We conclude the section with the definition of a Nash equilibrium.

2.1 A Bayesian Game

A Bayesian game Γ is given by a tuple $(N; A; T; g; u)$ where N is a finite player set, A is a set of action profiles, T is a set of type profiles, g is a probability distribution over type profiles and u is a set of utility functions. We define these components in turn.

Let $N = \{1; \ldots; n\}$ be a finite player set, let A denote a finite set of actions and let T denote a finite set of types. 'Nature' assigns each player a type. Informed of his own type but not the types of his opponents, each player chooses an action. We say that a game is a game of perfect information if $|T| = 1$. Let $A \equiv A^N$ be the set of action profiles and let $T \equiv T^N$ be the set of type profiles. Given action profile a and type profile t we let a_i and t_i denote respectively the action and type of player $i \in N$.

A player's payoff depends on the attributes, actions and types of players. Formally, in game Γ, for each player $i \in N$ there exists a utility function $u_i : A \times T \to \mathbb{R}$. In interpretation $u_i(a; t)$ denotes the payoff of player i if the action profile is a and the type profile t. Let u denote the set of utility functions.

A player, once informed of his own type, selects an action without knowing the types of the other players. A player therefore forms beliefs over the types he expects others to be. These beliefs are represented by a function p_i where $p_i(t_{-i}|t_i)$ denotes the probability that player i assigns to type profile $(t_i; t_{-i})$ given that i is of type t_i. Throughout we will assume consistent beliefs. Formally, for some probability distribution g over type profiles, we assume:

$$p_i(t_{-i}|t_i) = \frac{g(t_i; t_{-i})}{\sum_{t'_{-i} \in T_{-i}} g(t_i; t'_{-i})} \quad (1)$$

for all $i \in N$ and $t_i \in T$.[1]

2.2 Pregames

Let Ω be a compact metric space, called an attribute space and let N be a finite player set. A function α mapping from N to Ω is called an attribute function. The pair $(N; \alpha)$ is a population. In interpretation, an attribute function ascribes an attribute to each player in a population. Taking as given a finite set of actions A and types T a population $(N; \alpha)$ induces a Bayesian game $\Gamma(N; \alpha) \equiv (N; A; T; g^{\alpha}; u^{\alpha})$ as we now formalize.

Denote by W the set of all mappings from $\Omega \times A \times T$ into Z_+; the non-negative integers. A member of W is called a weight function. Given population $(N; \alpha)$ we say that a weight function $w_{\alpha;a;t}$ is relative to action profile a and type profile t if and only if:

$$w_{\alpha;a;t}(\omega; a^l; t^z) = \left|\left\{ i \in N : \alpha(i) = \omega; a_i = a^l \text{ and } t_i = t^z \right\}\right|.$$

Thus, $w(\omega; a^l; t^z)$ denotes the number of players with attribute ω and type t^z who take action a^l.

A universal payoff function h maps $\Omega \times A \times T \times W$ into R_+, the non-negative real numbers. The function h will determine payoff functions for every game induced by the pregame. Given a population $(N; \alpha)$, the payoff of a player will depend on his attribute, his action, his type and the weight function induced by the attributes, actions and types of the complementary player set. Formally:

$$u_i^{\alpha}(a; t) \stackrel{\text{def}}{=} h(\alpha(i); a_i; t_i; w_{\alpha;a;t}).$$

Denote by D the set of all mappings from $\Omega \times T$ into Z_+. A member of D is called a type function. Given population $(N; \alpha)$ we say that type

[1] We assume that the denominator of (1) is always positive - i.e. there is positive probability that a player $i \in N$ will be of type t_i for each $t_i \in T$.

function $d_{\alpha;t}$ is relative to type profile t if:

$$d_{\alpha;t}(\omega; t^*) = |\{i \in N : \alpha(i) = \omega \text{ and } t_i = t^*\}|:$$

Thus, $d_{\alpha;t}(\omega; t^*)$ denotes the number of players with attribute ω and type t^*.[2] A universal beliefs function b maps D into $[0;1]$. The value $\mathrm{b}(d_{\alpha;t})$ is interpreted as the probability of type profile t. Formally:

$$g^{\alpha}(t) \stackrel{\text{def}}{=} \mathrm{b}(d_{\alpha;t})$$

where g^{α} is the probability distribution over type profiles induced by b and α. Players are assumed to have consistent beliefs with respect to g^{α}. It is important to realize the differences between functions g^{α} and b. Function g^{α} is defined relative to a population $(N;\alpha)$ and its domain is T^N. Function b, however, is defined independently of any specific game and has domain D.

A pregame is given by a tuple

$$G = (\Omega; A; T; \mathrm{b}; h);$$

consisting of a compact metric space Ω of attributes, finite action and type sets A and T, a universal beliefs function $\mathrm{b} : D \to [0;1]$ and a universal payoff function $h : \Omega \times A \times T \times W \to R_+$. As discussed above we refer to a population $(N;\alpha)$ as inducing, through the pregame, a Bayesian game

$$\Gamma(N;\alpha) \equiv (N; A; T; g^{\alpha}; u^{\alpha}):$$

2.3 Strategies and expected payoffs

Take as given a population $(N;\alpha)$ and induced Bayesian game $(N; A; T; g^{\alpha}; u^{\alpha})$. As discussed above, knowing his own type, but not those of his opponents a player chooses an action. A pure strategy details the action a player will

[2] Note that $d_{\alpha;t}$ is a projection of $w_{\alpha;a;t}$ onto $\Omega \times T$.

take for each type $t^z \in T$ and is given by a function $s^k : T \to A$ where $s^k(t^z)$ is the action taken by the player if he is of type t^z. Denote the set of pure strategies by S and let $K = |A|^{|T|} = |S|$ denote the number of pure strategies.

A (mixed) strategy is given by a probability distribution over the set of pure strategies. The set of strategies is denoted by $\Delta(S)$. Given a strategy x we denote by $x(s^k)$ the probability that a player takes pure strategy $s^k \in S$. We denote by $x(a^l|t^z)$ the probability that a player takes action a^l given that he is of type t^z. Let $\Sigma = \Delta(S)^N$ denote the set of strategy vectors. We refer to a strategy vector m as degenerate if for all $i \in N$ and $t^z \in T$ there exists some action a^l for which $m_i(a^l|t^z) = 1$.

We assume that players are motivated by expected payoffs. Given a strategy vector σ, a type $t^z \in T$ and beliefs about the type profile p_i^α the probability that player i puts on the action profile-type profile pair $a = (a_1; ...; a_n)$ and $t = (t_1; ...; t_{i-1}; t^z; t_{i+i}; ...; t_n)$ is given by:

$$\Pr(a; t_{-i}|t^z) \stackrel{\text{def}}{=} p_i^\alpha(t_{-i}|t^z)\sigma_1(a_1|t_1)...\sigma_i(a_i|t^z)...\sigma_n(a_n|t_n).$$

Thus, given any strategy vector σ, for any type $t^z \in T$ and any player i of type t^z, the expected payoff of player i can be calculated. Let $U_i^\alpha(\cdot|t^z) : \Sigma \to R$ denote the expected utility function of player i conditional on his type being t^z where:

$$U_i^\alpha(\sigma|t^z) \stackrel{\text{def}}{=} \sum_{a \in A} \sum_{t_{-i} \in T_{-i}} \Pr(a; t_{-i}|t^z) u_i^\alpha(a; t_z; t_{-i}).$$

Denote by EW the set of functions mapping $\Omega \times A \times T$ into R_+, the non-negative reals. We refer to $ew; eg \in EW$ as expected weight functions. Given a population $(N; \alpha)$ we say that an expected weight function $ew_{\alpha;\sigma}$ is relative to strategy profile σ if and only if:

$$ew_{\alpha;\sigma}(\omega; a^l; t^z) = \sum_{a \in A} \sum_{t \in T} w_{\alpha;a;t}(\omega; a^l; t^z) \Pr(a; t)$$

for all ω, a^l and t^z. Thus, $ew_{\alpha,\sigma}(\omega, a^l, t^z)$ denotes the expected number of players of attribute ω who will have type t^z and play action a^l. Note that this expectation is taken before any player is aware of his type.

2.4 Nash equilibrium

The standard definition of a Bayesian Nash equilibrium applies. A strategy vector σ is a Bayesian Nash ε-equilibrium (or informally an approximate Bayesian Nash equilibrium) if and only if:

$$U_i^{\alpha}(\sigma_i, \sigma_{-i}|t^z) \geq U_i^{\alpha}(x, \sigma_{-i}|t^z) - \varepsilon$$

for all $x \in \Delta(S)$, all $t^z \in T$ and for all $i \in N$. We say that a Bayesian Nash ε equilibrium m is a Bayesian Nash ε-equilibrium in pure strategies if m is degenerate.

3 (δ, Q)-class games

Informally, a game $\Gamma(N;\alpha) = (N, A, T, g^{\alpha}, u^{\alpha})$ is a (δ, Q)-class game if the population N can be partitioned into Q subsets $N_1, ..., N_Q$, called classes, where (1) any two players in the same class are 'δ-substitutes' for each other and (2) roughly, the payoff to a player depends only on his own strategy choice and the 'aggregate strategy' of the players in each class. The concept of a (δ, Q)-class game was introduced in CW.

To formally define a (δ, Q)-class game we require notions of approximate substitute players. Take as given a game $\Gamma(N;\alpha)$ and a partition of the player set $N = \{N_1, ..., N_Q\}$.

Partition N is a δ_I-interaction substitute partition when: For any two strategy vectors $\sigma^1, \sigma^2 \in \Sigma$ if:

$$\sum_{i \in N_q} \sigma_i^1(s^k) = \sum_{i \in N_q} \sigma_i^2(s^k);$$

for all N_q and all $s^k \in S$, then:

$$\left| U_i^{\alpha}(x; \sigma^1_{-i} | t^z) - U_i^{\alpha}(x; \sigma^2_{-i} | t^z) \right| \leq \delta_I$$

for any player $i \in N$ and any strategy $x \in \Delta(S)$.

Informally, N is a δ_I-interaction substitute partition if a player's payoff changes by at most δ_I when other players of the same class 'exchange' strategies, with his own strategy choice held constant.

Partition N is a δ_P-individual substitute partition when: For any N_q, for any two players $i, j \in N_q$ and for any strategy vector $\sigma \in \Sigma$ such that $\sigma_i = \sigma_j$:

$$\left| U_i^{\alpha}(x; \sigma_{-i} | t^z) - U_j^{\alpha}(x; \sigma_{-j} | t^z) \right| \leq \delta_P$$

for any strategy $x \in \Delta(S)$.

Informally, N is a δ_P-individual substitute partition if the payoffs of any two players in the same class, when they both play the same strategy and the strategies of other players are held constant, are within δ_P.

Partition N is a δ_C-strategy switching partition when: For any two strategy vectors $\sigma^1, \sigma^2 \in \Sigma$ if:

$$\sum_{i \in N_q} \left| \sigma_i^1(s^k) - \sigma_i^2(s^k) \right| \leq 1, \qquad (2)$$

for all N_q and all $s^k \in S$ then:

$$\left| U_i^{\alpha}(x; \sigma^1_{-i} | t^z) - U_i^{\alpha}(x; \sigma^2_{-i} | t^z) \right| \leq \delta_C$$

for any player $i \in N$ and any strategy $x \in \Delta(S)$.

Thus, given a small proportional change in the 'aggregate strategy' of a class, if N is a δ_C-strategy switching partition then payoffs will change by at most δ_C.

Game $\Gamma(N;\alpha)$ is said to be a $(\delta_I;\delta_P;\delta_C;Q)$-class game if there exists a partition N of the player set into Q classes such that N is a δ_I-interaction substitute partition, a δ_P-individual substitute and a δ_C-strategy switching partition. If $\delta_I;\delta_P;\delta_C \leq \delta$ then we refer to $\Gamma(N;\alpha)$ as a $(\delta;Q)$-class game. Given a $(\delta;Q)$-class game $\Gamma(N;\alpha)$ we refer to a partition N as a proper partition of the player set into classes if it is a δ-substitute partition and δ-strategy switching partition.

4 Games With Many Players

We will assume throughout a relatively mild continuity property with respect to attributes. This assumption, introduced in WCS, dictates that a player's payoff is relatively invariant to a small perturbation of the attributes of players (including himself). Such an assumption would be satisfied, for example, in a private goods economy where individual preferences depend only on own consumption of commodities. Formally:

Continuity in attributes: Pregame $G = (\Omega;A;T;\Theta;h)$ is said to satisfy continuity in attributes when: for any $\varepsilon > 0$ and any two induced games $\Gamma(N;\alpha)$ and $\Gamma(N;\alpha')$, if, for all $i \in N$,

$$\mathrm{dist}(\alpha(i);\alpha'(i)) < \varepsilon$$

then, for all $i \in N$, all $t^z \in T$ and any strategy vector $\sigma \in \Sigma$:

$$\left| U_i^{\alpha}(\sigma|t^z) - U_i^{\alpha'}(\sigma|t^z) \right| < \varepsilon.$$

We note that the assumption of continuity in attributes considers a change in attributes while the strategies are held constant. The assumption comprises essentially two distinct elements: (1) A player should be relatively indifferent to small changes in the attributes of others - this would suggest, amongst other things, that the probability distribution over types is largely unaffected by a small change in the attribute function. (2) Players with similar attributes receive similar payoffs. This later point will clearly have a role to play in demonstrating the existence of δ_P-individual substitute partitions for small values of δ.

4.1 Societies

We define a society. Given a game $(N;\alpha)$ and a strategy vector σ we interpret a set of players D as a society if (i) there exists some strategy $x \in \Delta(S)$ such that $\sigma_i = x$ for all $i \in D$, and (ii) for any player $i \in N$, if $\alpha(i) \in con(\alpha(D))$ then $i \in D$.[3] Thus, any two players belonging to a society D must play the same strategy. Furthermore, to any society D we can associate a convex subset Ω_D of attribute space Ω with the properties that any player belonging to D has an attribute in Ω_D while there exists no player who has an attribute in Ω_D that does not belong to D. [4]

We say that a strategy vector σ induces a partition of the player set into Q societies if there exists a Q member partition of the player set $N = \{N_1; ...; N_Q\}$ such that each subset N_q is a society.

Given a population $(N;\alpha)$ we say that a partition $N = \{N_1; ...; N_Q\}$ is a partition of $(N;\alpha)$ into convex subsets if there exists a partition $\{\Omega_1; ...; \Omega_Q\}$ of Ω into convex subsets with the property that if $i \in N_q$ then $\alpha(i) \in \Omega_q$ for

[3] Where $con(\alpha(D))$ denotes the convex hull of $\alpha(D)$.

[4] This is a stronger notion of conformity than used by WCS. In WCS condition (ii) becomes: for any player $i \in N$ if $\alpha(i) \in int(con(\alpha(D)))$ then $i \in D$ where $int(A)$ denotes the interior of set A.

all $i \in N$.[5]

4.2 Conformity

In this section we demonstrate that for a large family of games we can put a bound Q on the number of societies, where Q is independent of population size, such that any game within this family has an approximate Nash equilibrium that partitions the population into at most Q societies. Note that in this section we make no assumption that players use pure strategies.

We introduce a second assumption:

Risk Neutrality property: We say that a pregame G satisfies the risk neutrality property when: for any population $(N;\alpha)$ and any two strategy profiles $\sigma;\tilde{\sigma} \in \Sigma^N$ with expected weight functions $ew_{\alpha;\sigma}$ and $ew_{\alpha;\tilde{\sigma}}$ respectively, where:

$$ew_{\alpha;\sigma}(\omega; a^l; t^z) = ew_{\alpha;\tilde{\sigma}}(\omega; a^l; t^z)$$

for all $\omega; a^l$ and t^z, if $\sigma_i = \tilde{\sigma}_i$ then:

$$U_i^{\alpha}(\sigma|t^z) = U_i^{\alpha}(\tilde{\sigma}|t^z)$$

for any $t^z \in T$.

The risk neutrality property requires players to be risk neutral with respect to the strategies of others. For example, consider two players i and j who both have attribute ω and consider some other player l. The risk neutrality property dictates that player l should be indifferent as to whether (i) player i plays strategy s^1 and player j plays strategy s^2, (ii) player j plays strategy s^1 and player i plays strategy s^2, and (iii) both players choose strategy

[5] Of course the sets Ω_q are required to be only relatively convex since Ω may not be convex.

s^1 and s^2 with probability one half. There are many instances where this assumption would appear mild - we consider one case later.

Before stating our …rst Theorem, we recall that it follows from Theorem 2 of CW that any (0; 0; ±; Q(¿))-class Bayesian game has a Bayesian Nash equilibrium (a 0-equilibrium) with the property that any two players in the same class play the same strategy. Our …rst Theorem demonstrates that for any game induced by a pregame there is a 'near-by' (0; 0; ±; Q) game for some ± and Q. The existence of such games allows us to infer properties of games induced by pregames.

Theorem 1: Consider a pregame G = (- ; A; T ; b; h) that satis…es the risk neutrality property. Given real number ¿ > 0 there is a real number Q(¿) such that for any population (N; ®) there is another population (N; ®) satisfying dist(®(i); ®(i)) < ¿ for all i 2 N and, for some ±, the induced game ¡(N; ®) is a (0; 0; ±; Q(¿))-class Bayesian game. Furthermore, there exists a partition N of N that is both a proper partition into classes for game ¡(N; ®) and a partition of (N; ®) into convex subsets.

Proof: Partition - into convex subsets $-_1; :::; -_Q$ each of diameter less than ¿ > 0. For each subset $-_q$ choose and …x an attribute $!_q$ 2 $-_q$. Consider an arbitrary game ¡(N; ®). De…ne attribute function ® as follows for all i 2 N:

$$®(i) = !_q \text{ if and only if } ®(i) \ 2 \ -_q.$$

Clearly dist(®(i); ®(i)) < ¿ for all i 2 N. For each q, de…ne N_q = fi 2 N : ®(i) = $!_q$g. We conjecture that the partition N ´ fN_1; :::; N_Qg satis…es the desired properties. We note that all players of the same class have the same attribute. It is, thus, immediate from the de…nition of individual substitute partitions that N is a 0-individual substitute partition for ¡(N; ®). Also, by the risk neutrality property N is a 0-interaction substitute partition for

$\Gamma(N;\bar{\alpha})$. Thus, game $\Gamma(N;\bar{\alpha})$ is a $(0;0;\delta;Q)$-class Bayesian game and $\mathcal{N}$ is a proper partition of N. It is immediate that $\mathcal{N}$ partitions $(N;\alpha)$ into convex subsets.$\square$

Theorem 1 leads to Proposition 1, a consequence of the above result and Theorem 2 of CW:

Proposition 1: Consider a pregame $G = (\Omega; A; T; b; h)$ that satisfies the risk neutrality property and continuity in attributes. Given real number $\varepsilon > 0$ there exists real number $Q_1(\varepsilon) > 0$ such that for any population $(N;\alpha)$ the induced game $\Gamma(N;\alpha)$ has a Bayesian Nash ε-equilibrium that induces a partition of the player set into $Q_1(\varepsilon)$ societies.

Proof: Given an $\varepsilon > 0$, define $\tau \equiv \frac{1}{2}\varepsilon$. By Theorem 1 there exists real number $Q(\tau)$ such that for any population $(N;\alpha)$ there exists a population $(N;\bar{\alpha})$ such that $\max_{i \in N}\{dist(\alpha(i);\bar{\alpha}(i))\} < \tau$ and the induced game $\Gamma(N;\bar{\alpha})$ is a $(0;0;\delta;Q)$-substitute Bayesian game for some δ. Further there exists a proper partition of N into classes $\mathcal{N}$ for game $\Gamma(N;\bar{\alpha})$ that is a partition of $(N;\alpha)$ into convex subsets. Theorem 2 of CW states that any $(0;0;\delta;Q)$-class Bayesian game has a Bayesian Nash 0-equilibrium m with the property that any two players of the same class play the same strategy. Thus, game $\Gamma(N;\bar{\alpha})$ has a Bayesian Nash 0-equilibrium m with the property that any two players of the same class play the same strategy. By continuity in attributes, for all $i \in N$:

$$\left| U_i^{\alpha}(x; m_{-i}|t^z) - U_i^{\bar{\alpha}}(x; m_{-i}|t^z) \right| < \frac{\varepsilon}{2}.$$

for all $x \in \Delta(S)$ and $t^z \in T$. Thus:

$$U_i^{\alpha}(m_i; m_{-i}|t^z) > U_i^{\bar{\alpha}}(x; m_{-i}|t^z) - \varepsilon$$

for all $x \in \Delta(S)$. This completes the proof.$\square$

If the number of attributes is finite then we can go further. Of course with a finite number of attributes conformity is less interesting. The following Theorem, also a consequence of Theorem 2 of CW, simply states that there exists a possibly mixed strategy where all players who are identical play the same strategy.

Proposition 2: Consider a pregame G that satisfies the risk neutrality property and where the number of attributes is a finite integer Q. For any population $(N;\alpha)$ the induced game $\Gamma(N;\alpha)$ has a Bayesian Nash 0-equilibrium that induces a partition of the player set into Q societies.

Proof: Take as given game $\Gamma(N;\alpha)$. Let $\Omega = \{\omega_1;\ldots;\omega_Q\}$ be the space of attributes and let $\mathcal{N} = \{N_1;\ldots;N_Q\}$ denote the partition of the player set where $i \in N_q$ if and only if $\alpha(i) = \omega_q$. Partition $\mathcal{N}$ is a 0-interaction substitute partition and a 0-substitute partition. Thus, $\Gamma(N;\alpha)$ is a $(0;0;\delta;Q)$-class Bayesian game. Theorem 2 of CW states that any $(0;0;\delta;Q)$-class Bayesian game has a Bayesian Nash 0-equilibrium m with the property that any two players of the same class play the same strategy. This completes the proof. $\blacksquare$

5 Pure Strategy Equilibrium

The risk neutrality property proves insufficient for the existence of an approximate Nash equilibrium in pure strategies. We therefore introduce a stronger large game property. First, taking a population $(N;\alpha)$ as given, we define a metric on the space EW_α of expected weight functions:

$$\text{dist2}(ew;eg) = \frac{1}{|N|}\sum_{\omega\in\alpha(N)}\sum_{a^l\in A}\sum_{t^z\in T}\left|ew(\omega;a^l;t^z) - eg(\omega;a^l;t^z)\right|$$

for any $ew;eg \in EW_\alpha$. Thus, two expected weight functions are 'close' if the expected proportion of players with each attribute and each type that are playing each action are close.

Large game property: We say that a pregame G satisfies the large game property when: for any $\varepsilon > 0$, any population $(N;\alpha)$ and any two strategy profiles $\sigma;\sigma' \in \Sigma^N$ where:

$$dist2(\tilde{w}_{\alpha;\sigma};\tilde{w}_{\alpha;\sigma'}) < \varepsilon$$

if $\sigma_i = \sigma'_i$ then:

$$|U_i^{\alpha}(\sigma|t^{\zeta}) - U_i^{\alpha}(\sigma'|t^{\zeta})| < \varepsilon:$$

for any $t^{\zeta} \in T$.

If a pregame satisfies the large game property then we can think of payoff functions as satisfying one principal condition - a player is nearly indifferent to a change in the proportion of players of each attribute and of each type playing each action (provided his own strategy is unchanged); thus, the behavior of no one individual or small group of individuals can have large effects on a player's payoff. This contrasts with the risk neutrality property where one individual can have a large influence.[6] Risk neutrality is also required to hold under the large game property but an assumption of risk neutrality is mild in this context; with large player sets, finite sets of pure strategies and finite types the law of large numbers dictates that the actual proportion of players playing each action will, with high probability, be close to the expected proportion (Kalai 2002).

We state our second theorem:

Theorem 2: Consider a pregame G that satisfies the large game property. Given real numbers $\delta > 0$ and $\tau > 0$ there are integers $\eta(\delta;\tau)$ and $Q(\delta;\tau)$ such that for any population $(N;\alpha)$, where $|N| > \eta(\delta;\tau)$, there exists a

[6] This could be the case if there is a unique player with a certain attribute.

similar population (N;®) with dist(®(i);®(i)) < ¿ for all i 2 N and the induced game ¡(N;®) is a (±;Q(±;¿))-substitute Bayesian game. Further there exists a proper partition into classes N for game ¡(N;®) that is a partition of (N;®) into convex subsets.

Proof: Suppose that the statement of the lemma is false. Then there is some $\bar{\pm}$ > 0 and $\bar{¿}$ > 0, such that for any real number $\overline{Q}$ and for each integer º there is a population (N°;®°) where jN°j > º and such that for no population (N°;®°) where max$_{i2N}$ fdist(®(i);®(i))g < $\bar{¿}$ is the induced game ¡(N°;®°) a (±;$\overline{Q}$)-substitute Bayesian game.

Partition - into convex subsets -$_1$;:::;-$_Q$ each of diameter less than $\bar{¿}$. To each subset -$_q$ choose and …x an attribute !$_q$. For each (N°;®°) de…ne the attribute function ®° as follows: for all i 2 N°:

®°(i) = !$_q$ if and only if ®(i) 2 -$_q$.

Given game (N°;®°) let N° = fN$^°_1$;:::;N$^°_Q$g denote the partition of the player set such that i 2 N$^°_q$ if and only if ®°(i) = !$_q$. We note that the value Q is …xed independently of the game (N°;®°). The partition N° is a 0-individual substitute partition for all º and, given the large game property, a $\bar{\pm}$-interaction substitute partition. Also, for su¢ciently large º, by the large game property, N° is a $\bar{\pm}$-strategy switching partition. Thus, game ¡(N°;®°) is a (±;Q)-substitute Bayesian game.¥

Our third proposition demonstrates the existence of an approximate Nash equilibrium in pure strategies and obtains a puri…cation result as a consequence of the puri…cation result in CW for (±;Q) class-games.

Proposition 3: Consider a pregame G that satis…es the large game property and continuity in attributes. Given real number " > 0 there exists real number $\eta_2(\varepsilon) > 0$ such that any induced game $\Gamma(N;\alpha)$ where $|N| > \eta_2(\varepsilon)$ has a Bayesian Nash ε-equilibrium in pure strategies.

Proof: Let $\delta \equiv \frac{1}{6}\varepsilon$. By Theorem 2 there are real numbers η and Q such that for any population $(N;\alpha)$, where $|N| > \eta$, there exists a population $(N;\alpha')$ such that $\max_{i \in N}\{dist(\alpha(i);\alpha'(i))\} < \delta$ and the induced game $\Gamma(N;\alpha')$ is a $(\delta;Q)$-substitute Bayesian game. Theorem 1 of CW states that any $(\delta;Q)$-class game has a Nash 4δ-equilibrium. Let m be a Nash 4δ-equilibrium of game $\Gamma(N;\alpha')$. By continuity in attributes, for all $i \in N$:

$$\left|U_i^{\alpha}(x;m_{-i}|t^z) - U_i^{\alpha'}(x;m_{-i}|t^z)\right| < \delta:$$

for all $x \in \Delta(S)$ and $t^z \in T$. Thus:

$$U_i^{\alpha}(m|t^z) > U_i^{\alpha'}(x;m_{-i}|t^z) - 6\delta$$

for all $x \in \Delta(S)$. This completes the proof. ∎

6 Conformity in Pure Strategies with Roles

Following the approach of CW we consider the possibility that players may conform in their choice of strategy yet play different actions. The existence of imperfect information permits this as a player's action is conditional on his type. We assume that players can endogenously create imperfect information through an allocation of roles within a society. To simplify the analysis we assume that play 'begins' with a game of perfect information.

Take as given a pregame $G = (\Omega; A; T; b; h)$ where $|T| = 1$. Games induced through this pregame are games of perfect information. Assume that there exists a set of roles $R = \{r^1; \ldots; r^K\}$. Consider game $\Gamma(N;\alpha)$. Let $\mathcal{R} \subseteq R^N$ be the set of role profiles. Take as given a probability distribution f over the set of role profiles $\mathcal{R}$ where $f(r)$ denotes the probability of role profile r. We consider a Bayesian game with endogenous roles $\Gamma(f)(N;\alpha)$. In game $\Gamma(f)(N;\alpha)$ roles are (Harsanyi) types. Thus, roles are randomly allocated to players, a player can make his action choice conditional on his role and makes his choice of action knowing his role but not those of players in the complementary player set. A player's payoff, however, does not depend directly on the role profile. We assume that players have consistent beliefs with respect to the distribution over roles f. Formally, we can define game $\Gamma(f)(N;\alpha) = (N; A; T(f); g^{\alpha}(f); u^{\alpha}(f))$ to satisfy:

1. $T(f) \equiv R$,
2. for all $r \in R$,

$$g^{\alpha}(f)(r) \equiv f(r)$$

3. $u_i^{\alpha}(f)(a;r) \equiv u_i^{\alpha}(a)$ for all $a \in A$, $r \in R$ and all $i \in N$.

Condition 1 states that roles are equivalent to types. Condition 2 states that players have consistent beliefs with respect to the distribution of roles. Condition 3 states that payoffs are not directly effected by the role profile.

We highlight that roles and a probability distribution over role profiles are defined relative to a specific game $\Gamma(N;\alpha)$ rather than a pregame. This reflects the idea that roles are endogenously created within a population. Thus, it is more natural to think of a probability distribution over roles taking as given a specific game $\Gamma(N;\alpha)$. Note that this observation is also reflected in the statement of Proposition 4, to follow.

As discussed further by CW, to retain a notion of society in which players can truly be seen as conforming, assumptions are required on the probability distribution over roles f. For instance, it seems desirable, if players are conforming, that every player should have an equal chance of being each role; if this were not the case then it might be argued that players who are playing the same strategy are not exhibiting the same behaviour. This motivates our first condition.

Within class anonymity: A probability distribution over roles f satisfies within class anonymity if the probability that a player from a class N_q will have role r^k is (a priori) identical for all players belonging to that class. Formally, if $i, j \in N_q$ for some q then:

$$\sum_{r \in R: r_i = r^k} f(r) = \sum_{r \in R: r_j = r^k} f(r)$$

for all $r^k \in R$.

To motivate the next requirement, consider the example of a male-female household following the roles of 'he goes out to work, she stays home'. For this norm to be successful, it is necessary that no player, knowing the structure of society – the number of players with each role in his or her class – after roles are assigned, wishes to change role assignment. This motivates a second condition.

Within class determination: Given a role profile r let $z(r; k; q)$ be the number of players in class N_q who have role r^k. A probability distribution over roles f is within class determined if for any class N_q and for any two role profiles r and $\bar{r}$, if $f(r), f(\bar{r}) > 0$ then $z(r; k; q) = z(\bar{r}; k; q)$ for all classes q and for all $r^k \in R$.

In sum, within class anonymity requires that each player in a class has an equal probability of being allocated each role. Within class determination implies that the number of players who have each role can be known with certainty ex ante - the only uncertainty is who will have each role. These are strong requirements on f: We propose they capture the notion that players in the same class who play the same strategy are conforming to some norm of behavior.

Before stating our final result we introduce one further definition. We use the concept of ex-post Nash equilibrium as introduced by Kalai (2002). Ex-post Nash implies that, knowing the action profile and the type profile, no player has a strong incentive to change her own action. Formally, given population $\Gamma(N;\alpha)$ an action profile, type profile pair $a;t$ is said to be ε ex-Post Nash if for all $i \in N$:

$$u_i^{\alpha}(a;t) \geq u_i^{\alpha}(a^{l};a_{-i};t) - \varepsilon$$

for all $a^{l} \in A$. A strategy profile σ is said to be a Bayesian ε ex-Post Nash equilibrium if it yields an ε ex-Post Nash action profile, type profile pair with probability one. If a strategy vector is a Bayesian ex-Post Nash equilibrium then, as discussed further by Kalai (2002), no player would wish to change his action after knowing the types (or roles) and the actions of the other players. The proof is based on that of CW (Theorem 3).

Proposition 4: Consider a pregame G that satisfies the large game property and continuity in attributes and where $|T| = 1$. Given a real number $\varepsilon > 0$ there are real numbers $\eta_4(\varepsilon) > 0$ and $Q_4(\varepsilon)$ such that for any population $(N;\alpha)$ where $|N| > \eta_4(\varepsilon)$ there exists a probability distribution over role profiles f (that is within class anonymous and determined) with the property that game $\Gamma(f)(N;\alpha)$ has a Bayesian ε ex-Post Nash equilibrium in pure strategies that induces a partition of the player set into at most $Q_4(\varepsilon)$ societies.

Proof: Let $\delta \equiv \frac{1}{12}\varepsilon$. By Theorem 2 there are real numbers η and Q such that for any population $(N;\alpha)$, where $|N| > \eta$, there exists a population $(N;\bar{\alpha})$ such that $dist(\alpha(i);\bar{\alpha}(i)) < \delta$ for all $i \in N$ and the induced game

$\Gamma(N;\bar{\alpha})$ is a $(\delta;Q)$-substitute Bayesian game. Further there exists a proper partition into classes $\mathcal{N}$ for game $\Gamma(N;\bar{\alpha})$ that is a partition of $(N;\alpha)$ into convex subsets.

Theorem 3 of CW states that any given any $(\delta;Q)$-class game $\Gamma(N;\bar{\alpha})$ and proper partition $\mathcal{N}$ there exists a probability distribution over role profiles f (that is within class anonymous and determined) such that $\Gamma(f)(N;\bar{\alpha})$ has a Bayesian 10δ ex–Post Nash equilibrium with the property that every player of the same class plays the same pure strategy. Let m be such an equilibrium of game $\Gamma(N;\bar{\alpha})$. By continuity in attributes, for all $i \in N$:

$$\left| U_i^{\alpha}(x; m_{-i}|t^z) - U_i^{\bar{\alpha}}(x; m_{-i}|t^z) \right| < \delta.$$

for all $x \in \Delta(S)$ and $t^z \in T$. Thus:

$$U_i^{\alpha}(m_i; m_{-i}|t^z) > U_i^{\bar{\alpha}}(x; m_{-i}|t^z) - 12\delta$$

for all $x \in \Delta(S)$. This completes the proof. $\square$

7 Conclusion

In this paper we provide a family of games with many players for which there exists an approximate Nash equilibrium in pure strategies exhibiting conformity. A strategy vector exhibits conformity when the population could be partitioned into a relatively small number of societies - players in the same society play the same strategy and have similar attributes. The existence of roles within a society was permitted, thus allowing the possibility that players play the same strategy and yet perform different actions.

Our results complement and extend those due to WCS and CW. In WCS we also provide a family of games for which there exists an approximate Nash equilibrium in pure strategies exhibiting conformity. The current paper, however, extends that of WCS in considering games of imperfect information. This allows a different interpretation of conformity and of a society. As a consequence we are able to bound the number of societies independently of the number of strategies (in contrast to WCS). In CW we treat individual games and provide a bound on the ε, depending on the parameters describing the game, allowing existence of a Nash ε-equilibrium in pure strategies exhibiting conformity. CW do not, however, demonstrate that in large games this bound, and thus ε, can be taken to be small. This paper applies the results of CW in focussing on large games.

References

[1] Cartwright, E.J. and M.H. Wooders (2003a) "Bounded rationality in arbitrary games with many players," University of Warwick Department of Economics Working Paper no 672. On-line at http://www2.warwick.ac.uk/fac/soc/economics/research/papers/twerp672-revised.pdf

[2] Cartwright, E.J. and M.H. Wooders (2003b) "On approximate equilibrium in pure strategies in games with many players," University of Warwick Working Paper (forthcoming).

[3] Kalai, E. (2002) "Ex-post stability in large games," Northwestern University Discussion Paper 1351.

[4] M. Wooders, E. Cartwright and R. Selten (2002) "Social conformity in games with many players," University of Warwick Department of Economics Working Paper no 589 (revised), on-line at http://www2.warwick.ac.uk/fac/soc/economics/research/papers/2001/.