OPTIMAL NONLINEAR POLICIES FOR NON-UTILITARIAN MOTIVES

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1. Introduction

This paper studies optimization by a principal, e.g. a government or a firm, whose choice among social alternatives is constrained to those it can decentralize, as in the nonlinear taxation literature, but whose ranking of these alternative equilibria is not constrained to be of any special form, such as utilitarian or profit-maximising. Accordingly, we shall define welfare directly on the various quantities arising in the economy, on the consumption vectors of consumers in full detail, without necessarily processing this information in any particular way.

The exercise is of interest, we think, partly for purely 'theoretical' (or 'mathematical') reasons, namely to provide a framework of analysis for the incentives problem as such and to explore features of solutions due to its general structure and not to any special maximands. In this sense, these are notes on the problem of 'optimization subject to optimization', where the maximands at the two levels are not necessarily related in any given form. To this view of the exercise, some might answer that only a few special cases really matter at all, such as utilitarian taxation or pricing by governments or benevolent firms, or nonlinear pricing by profit maximizers. But even then there is a case for gaining perspective on these models, to aid our understanding of each special problem or indeed of their relation, by placing them in a suitable wider context that contains them.

Alongside the above remarks on formalism or interpretation, I offer the following more practical motivations for the exercise. I start with the usual utilitarian objective in mind, and introduce various reasons why this may be too restrictive in applications.

(i) Egalitarianism.

It has been argued strongly, for example by Sen (1973), that utilitarianism is not the natural vehicle to capture a concern with equality, despite the principal's freedom to 'concavify' utility functions before adding them. For one thing, the distribution of utilities may matter, e.g. some notion of distance between top and bottom utilities. This would call for a general individualistic approach (welfare defined on utilities), not necessarily Paretian (i.e. increasing in its arguments), let alone utilitarian. Furthermore, there is no special reason why 'egalitarianism' should have to be defined on utilities at all, and maybe it is the distribution of income itself that worries a particular 'egalitarian' government, or of cross-section consumption at a point in time rather than in a life-cycle sense. This would require welfare to be defined on consumption vectors directly.

(ii) Paternalism.

A second type of reason for relaxing the utilitarian, or even individualistic framework is given by paternalistic considerations: that consumers' preferences are socially 'wrong' in some respect, or that equivalently they act on the basis of the wrong information. For example, it is an old view to regard utility time-discounts ('impatience') as myopia (see, e.g. Pigou, 1929, Ch. II of Part I), which the government should not abide by in comparing social states. Similarly, various forms of social security schemes we observe in practice can be argued, as Diamond (1977) does, to be an important instance of paternalistic behaviour by the government, as they impose floors on individual consumption of certain goods and services as well as on total consumption per period (as the ability to borrow against future security payments is usually very limited). Otherwise, if redistribution and insurance were the only purposes of the scheme, a simpler poll transfer ought often to have been observed in place of it.

(iii) Other objectives.

One would often wish to move further away from the usual utilitarian set-up than we may have suggested above. For example, the utilitarian nonlinear-tax model we have discussed elsewhere applies directly to the problem of utilitarian pricing by public firms, whenever charges can be made nonlinear in quantities.[1/] But more often than not are public utilities required to include profits in their maximands in some form alongside their consumers' welfare proper (Goldman _et al._, 1977, allow for this), with the non-negativity condition on profits, imposed in the usual utilitarian model, being only an extreme form of this interdependence.

Similarly, governments often wish to maximize national income, or tax revenue, or employment, or to bring balance of payments considerations to bear on the decision to tax certain goods more than others. Clearly, all these variables should ideally not be given any weight in themselves but only insofar as they affect welfare indirectly. But governments do behave in this Dutch-school-like 'flexible targets' form, presumably because the short-run and gross economic indicators are all too important for them - just as private-firm managers may well care for non-profit variables. Whatever the objective function, the incentives problem remains essentially the same.

The outline of the paper is as follows. Section 2 presents the model and discusses aspects of it: individual behaviour and participation, the welfare function and the interpretation of the model for some special cases. Section 3 derives and discusses necessary conditions for an optimum, including the solution for optima with corners, which had not been considered in detail or generality in the literature; and section 4 discusses features of optima: end-points taxation, which takes a readily intuitable form for the general case; some further implications of the end-points result for special cases, in particular revenue or profit maximization; relations between tax rates; and some remarks on the signs of distortions in the optimum - i.e. on tax rates. A brief stock-taking is offered in section 5. Throughout, I emphasize interpretation rather than points of rigour, paying attention to only some of the latter which arise or take a different form given the present context, or which had not been discussed before.

2. The Model

2.1. Preferences, skills and income

Consider an economy whose consumers' utility functions are

$$u = u(a,b,h) \tag{1}$$

where a is a numeraire commodity, b possibly a vector of other goods which for expediency we shall mostly refer to as a scalar, and h a parameter that captures individual ability, income, tastes or whichever central difference amongst consumers a given model is to concentrate on. We assume h to follow density f(h), and denote the support of the latter (the smallest interval that contains all h's actually observed in the population) by $[\underline{h}, \bar{h}]$, although we shall later feel free to give a different interpretation to these 'extreme' values of h.

We impose the convention that, unlike preferences, consumers' opportunity sets are all identical. This essentially amounts to defining a and b as they occur in production (efficiency hours of work rather than time worked, for example), and measuring them as net trades with the market. Thus, if endowments do differ with h, these differences simply affect the utility different consumers derive from a given pair of trades (a,b), if final consumption is what matters. This is already allowed for by (1)[2/].

It may be useful to relate (1) to special structures that have been studied in the literature. To study labour supplies and the generation of income, it is natural to focus on the way the wage rate varies across consumers, so that we get h = wage, with a and b representing, by our previous convention, consumption and gross income. This gives (1) the form

$$u = U(a, b/h), \tag{2}$$

which is Mirrlees' (1971) well-known income-tax model. Alternatively, still concentrating on income differences across consumers, one could disregard labour-supply decisions and set h = income, directly, so as to focus attention on the expenditure side of behaviour. Then, writing x for consumption of numeraire, b for the commodity subject to taxation or discriminatory pricing and $a = a(b)$ for the total cost of the latter, individual budget constraints reduce to $x + a = h$. Thus, if preferences are otherwise identical among consumers, a unique underlying $U(x,b)$ becomes

$$u = U(h-a, b), \tag{3}$$

again as in (1). This is the utility-structure used by Goldman et. al. (1977) as well as (implicitly) by Roberts (1978) and Willig (1978).

I assume u to be strictly concave, twice differentiable and, for convenience, increasing in a and b. We also need to make sure that h is an unequivocal ordinal index for consumers' economic behaviour. For this, writing s for the marginal rate of substitution of a for b,

$$s(a,b,h) \equiv u_b/u_a, \tag{4}$$

I assume

$$s_h \equiv \partial s/\partial h > 0 \ . \qquad (5)$$

That is, indifference curves always turn in the same direction, at each point in (a,b)-space, as h increases. Thus, a and b are weakly monotonic in h whatever the budget line consumers face – in fact (5) implies more, that a is non-increasing and b non-decreasing in h, but this is purely by our convention on the actual signs of s_h and of u_a, u_b. Single-signedness is what matters. It is easy to check what (5) amounts to in the examples considered above: (2) satisfies it if (not only if) consumption a is non-inferior (Seade, 1978), while (3) does iff the taxed good b is non-inferior.

2.2. Social Welfare

Following the motivation given in the introduction, we make welfare dependent directly on each individual's consumption of a and b. With a continuum of consumers, these "lists" of quantities actually are entire functions of h. For lack of standard notation, I shall denote by $a|$ the whole arc of the a-allocation:

$$a| \ \equiv \ a\Big|_{\underline{h}}^{\bar{h}} \equiv \{a(h) \quad \text{for} \quad \underline{h} \leqslant h \leqslant \bar{h}\} \qquad (6)$$

(cf. the usual under-bar for vectors), and similarly write $b|$ for the arc of b(h) and $u|$ for the arc of utilities across values of h. Welfare will thus be a general[3/]

$$W \ = \ W(a| \ , b| \ ). \qquad (7)$$

Particular examples of interest are easily written down to put (7) to work on applications such as those suggested in the introduction. But as soon as one does that, one loses sight of the common structure of alternative examples, and generality. I guess that the usual resort to special forms of (7) is largely due to lack of notation for functionals which is easy to write and grasp.[4/] By way of contrast, we notice that for a problem closely related to the present exercise, but where the representation of consumers by a continuum is unnecessary, Diamond and Mirrlees (1971) do use the discrete counterpart to (7) for as long as the analysis permits, treating the individualistic form as a special case worthy of further study. Nevertheless, it will prove convenient not to insist on 'too' much generality, and adopt a form of (7) which is both general enough for our purposes, and easier to handle. Thus, I impose additive separability of welfare across consumers,

$$W = \int_{\underline{h}}^{\bar{h}} \hat{u}(a,b,h)\, f(h)\, dh \,, \tag{8}$$

where the function $\hat{u}(a,b,h)$ has a natural interpretation: it is the <u>social</u> (or principal's) <u>utility</u> from h's consumption. This still allows W to depend in any linear form on (private or paternalistic) utilities, on the variance of utilities or of specific goods such as income, on aggregates such as profits or revenue, or variables depending on aggregates (linearly, but only local changes are considered), such as various macro-objectives.

One can notionally relax additive separability, replacing the derivatives of the functional (8) at any given point, $\hat{u}_a$, $\hat{u}_b$ by the corresponding derivatives of the functional (7) with respect to local arc-changes,[5/] $W_a(h)$, $W_b(h)$. One would then just modify accordingly all the equations we shall obtain below, and perhaps draw a qualitative feel as to how non-separability of W would affect the optima. But the actual computation of solutions would be complicated enormously, as the usual set of differential equations one obtains for the description of the optimum would be replaced by an integro-differential or a difference-differential system with forward and backward 'memory' (derivatives da/dh and db/dh depending, at each point, on the state of the system at other points in both directions), which do not appear to have been studied at all in the mathematical literature.

2.3. Constraints

The principal's problem is to make an optimal selection, relative to its objective (8), of two allocation-functions $a(h)$, $b(h)$ which should arise through decentralisation and meet some 'isoperimetrical' constraint on total demands, which can variously be interpreted, for example, as a production constraint, government's revenue requirement, or minimum permissible profits for a public firm.[6/] Linearizing, this constraint is

$$\int_{\underline{h}}^{\bar{h}} \{ a(h) + p\, b(h) \} \; f(h) \; dh \leq A \, , \qquad (9)$$

where p is the relative shadow price of b at the optimal equilibrium.

Let us turn now to the decentralisation condition on the allocations. All the government can do, we assume, is to offer consumers a budget set, a set of points in (a,b)-space from which to choose their consumption. If the (north-east) frontier of this set is smooth, interior individual maximization for people taking part in the scheme imposes

$$u_a a' + u_b b' = 0, \tag{10}$$

where $a' \equiv da(h)/dh$, etc; this is derived and discussed in Seade (1977). The ratio a'/b' is the trade-off between the two goods as faced by the consumer at the margin, so that (10) is an <u>envelope condition</u> for the economy: tangency of preferences with the budget constraint for each consumer. On the other hand, this tangency may come 'from the wrong side', i.e. give a minimum. It is a simple exercise in the use of indifference-curve diagrams to check, given (10) and assumption (5) on preferences ($s_h > 0$), that maximization occurs iff $b' \geq 0$ for all h. This is rigorously proved by Mirrlees (1976, appendix). We must, therefore, ensure that[7/]

$$b' \geq 0. \tag{11}$$

Notice that (10) and (11) also hold for most h's consuming on a <u>corner of the budget set</u>: there, a' and b' are simply zero over an h-range. It is only <u>corners of the allocation functions</u> $(a(h), b(h))$, which are not unrelated to the previous ones, that pose difficulties for these equations. But corners of the former kind are important and will be studied for optima, while those of the latter kind are not, in that they will only arise at a few values of h - at any rate if the variational analysis is to be applicable at all. One can allow for such non-differentiabilities of a(h) and b(h) by using the following condition due to Mirrlees (1976), which generalises (10) and has a more familiar envelope-form: namely, that $du/dh = \partial u/\partial h$. In order to use quantities directly as controls for the optimization, as I will do below, it turns out to be more convenient to write this condition in an integrated form,

$$u(a,b,h) = \int_{\underline{h}}^{h} u_j(a,b,j)\, dj + u^o , \tag{10'}$$

where u^o is the utility an $\underline{h}$-man receives in the given allocation. It simplifies the exposition and underlying derivations, however, if one uses (10) as a surrogate for (10'); details using (10') can be left to interested readers. The optimality conditions that emerge are essentially identical in the two cases, with the multiplier for (10) being simply the integral of that for (10'), and corners require separate attention anyhow.

The above conditions, (10) and (11), are necessary and sufficient for an individual optimum from amongst the set of possibilities offered by the government. This applies only to consumers actually maximizing on that set, which in certain contexts (the usual closed-economy optimal-tax set-up) can be taken to be <u>all</u> consumers in the population. More generally, however, consumers have the option of leaving the market altogether, becoming tax-exiles for example. A <u>participation constraint</u> is required: that each consumer who does stay in, derives not less utility than a certain minimum $\bar{u}$, presumably his best alternative net of costs involved, which would normally differ across consumers:

$$u(a(h),\ b(h),h) \geq \bar{u}\ (h). \tag{12}$$

It would be incorrect to impose (12) for all h in the population, however, for it must only hold for values of h 'captured' by the scheme. A full treatment of the problem must incorporate a choice of the captured ranges of h as one of the controls, and only apply (12) to the relevant ranges of h. One can clearly not say, for the general case, what the partition of the population into participants and leavers will be like in the optimal equilibrium, as this will depend among other things on the nature of the exogenous schedule $\bar{u}(h)$ in (12). But it is clear that interior necessary conditions for the case when the marginal values of h between participation regimes are chosen optimally, must be the same as those one would obtain by treating those optimal marginal h's as fixed - only the relevant end-point conditions will be sensitive to this added dimension of choice. We can therefore think of the interval $[\underline{h},\bar{h}]$ as denoting a given participation arc rather than the whole, exogenously given population; if these consumers are to be induced to stay in in the optimum, clearly (12) must hold for each of them. We want to study the nature of the tax or price schedules these (and other !) consumers will face in the optimum.

One last point one should mention in this connection, is that it is not now clear whether the domain of welfare should be the set of consumers who stay in, or all the population; it can be either. But by additive separability, this does not upset the optimization within each participation range - it only affects the choice of the extent of participation.

3. The optimum : characterization

3.1. Necessary conditions

So as to facilitate reference of multipliers to associated constraints, let us write in full the Lagrangean for the problem:

$$\max_{a,b} \int_{\underline{h}}^{\bar{h}} F \, dh \,, \qquad \text{where} \qquad (13)$$

$$F \equiv \{ [\hat{u} - \lambda(a + pb)]f + \mu (u_a a' + u_b b') + \nu b' + \pi(u - \bar{u})\},$$

where the arguments of all functions have been omitted. I neglect non-negativity constraints on (a,b), which can be brought in in given cases as the need arises.

First order conditions for (13) are constraints (9), (10), (11) and (12), the first one and last two of these having complementary slack with $\lambda \geq o$, $\nu \geq o$, $\pi \geq o$, respectively, plus the following:

$$(\hat{u}_a - \lambda) f + (\pi - \mu') \; u_a = \mu \, u_{ah}, \qquad (14)$$

$$(\hat{u}_b - \lambda p) f + (\pi - \mu') \; u_b = \mu \, u_{bh} + \nu' \,, \qquad (15)$$

with transversality conditions $\mu u_a = \mu u_b + \nu = o$ (see Seade, 1977, pp. 224-5) at either end-point whose value of h is <u>fixed</u>, i.e. one which is not a frontier between participation-regimes, as discussed at the end of the previous section. For such fixed-h end-points[8/], hence,

$$\mu(\underline{h}) = \mu(\bar{h}) = \nu(\bar{h}) = \nu(\bar{h}) = o. \qquad (16)$$

On the other hand, at arc end-points interior to the population the relevant part of (16) does not hold, but a continuity condition on end-point utilities replaces it in closing the system, as we shall see below.

To put conditions (14)-(15) in a more useful form, it is most convenient to treat values of h where consumption is changing separately from those where it is not.

It is clear from assumption (5) on preferences $(s_h > o)$, that for demands actually observed in the population, consumption is constant at $(\bar{a},\bar{b})$ over an h-range if and only if $(\bar{a},\bar{b})$ lie on a corner of the opportunity set (tax function) consumers face. That is, discussing constancy of a(h), b(h) is tantamount to discussing possible corners of the optimal tax function.

3.2. Taxes on smooth arcs

Over an h-range where consumption is changing, the value of ν is and remains zero, so that ν' in (15) vanishes. It will be easier to interpret the first-order conditions if we define

$$\hat{s} \equiv \hat{u}_b/\hat{u}_a. \tag{17}$$

This is the social (or principal's) marginal rate of substitution, or constant-welfare trade-off, on an h-man's consumption of a(h), b(h), derived from the social evaluation of his consumption, $\hat{u}(\cdot,\cdot;h)$. Eliminating terms in $(\pi - \mu')$ from (14) and (15) and re-arranging, one obtains

$$\lambda(s - p) = \hat{u}_a (s - \hat{s}) + u_a s_h \mu/f, \tag{18}$$

where s is h's private marginal rate of substitution of a for b, defined in (4). The left-hand side of (18) is the distortion on the (relative) price of b an h-man should face in the optimum: the excess of marginal price he pays (= s) over producers' price p.

It is of interest to note that, for a given value of $\mu(h)$, the last term in (18) is exactly the usual one describing the optimal utilitarian distortion - indeed this is obvious from (18) itself, as $s = \hat{s}$ under utilitarianism (or individualism, more generally). To this component of the tax, the term $\hat{u}_a(s - \hat{s})$ is now added, which has a natural interpretation: it measures the social value of the divergence between private and social preferences. More precisely, it is the compensating change in the quantity of numeraire the individual reckons he needs for constant utility as b changes, over society's computed compensation, $\hat{s}$, both valued at the social value of numeraire in h's hands, $\hat{u}_a(\cdot,\cdot;h)$. Thus, the first term in (18) arises directly from differences between social and private preferences, and might be called the 'paternalistic' motive for taxation. It might also be viewed as the 'first-best' component of the tax, as it describes the line on which the first-best set of allocations would fall, were it achievable. In contrast, the second term in (18) is the 'individualistic', or 'second-best' motive for the tax, which arises purely from the incentives nature of the problem, from the need to resort to decentralization of equilibrium allocations.

The above remarks are only suggestive of interpretation, and not meant to be operational, for the actual value of $\mu(h)$ will depend critically on the choice of the function $\hat{u}(\cdot,\cdot;h)$, as well as on the remaining data for the problem. This $\mu(h)$ we obtain by direct integration of (14):

$$\mu(h) = \mu(\underline{h}) + \int_{\underline{h}}^{h} \{\frac{\hat{u}_a - \lambda}{u_a} f + \pi\} \exp\left(-\int_{h'}^{h} \frac{u_{ah}}{u_a} dj\right) f dh' . \quad (19)$$

If $\underline{h}$ is fixed (no exclusion at the bottom), $\mu(\underline{h}) = o$. If $\bar{h}$ is fixed, $\mu(\bar{h}) = o$.

The characterization of the optimum, leaving apart possible constancy-ranges of consumption, is now complete. At each point, either $\pi = o$ or utility is given by (12). Using this, equations (10), (18) and (19) can be transformed into a system of three differential equations in $a(h)$, $b(h)$ and $\mu(h)$, or in $\pi(h)$, $b(h)$ and $\mu(h)$. The particular solution of interest will be the one that satisfies condition (9) on total demands plus, at the bottom, $\mu(\underline{h}) = o$ if $\underline{h}$ is fixed, or else $u(a(\underline{h}), b(\underline{h}), \underline{h}) = \bar{u}(\underline{h})$ 9/; similarly for $\bar{h}$.

3.3. Corners

We now turn to intervals of h where consumption remains constant at a corner. To simplify things, I assume away the participation constraint, which would seem unlikely to start biting within a corner, but which could easily be accounted for if necessary.

Over an interval of constancy of the allocation functions, say for $j \in [h_o, h_1]$, the term ν' in (15) does not generally vanish. However, at both end-points of that interval the multiplier ν is 'just' biting, $\nu(h_o) = \nu(h_1) = o$, and everywhere in between it is non-negative. Moreover, it is easy to show that ν (as well as μ) is continuous throughout the schedule[10/]. It follows, integrating (14) and (15), that

$$\int_{h_o}^{j} \{(\hat{u}_a - \lambda) f - \mu' u_a - \mu u_{ah}\} \, dh \equiv o, \tag{20}$$

and $$\nu(j) = \int_{h_o}^{j} \{(\hat{u}_b - \lambda p) f - \mu' u_b - \mu u_{bh}\} \, dh \geq o, \tag{21}$$

both $\forall \; j \in [h_o, h_1]$, and that

$$\int_{h_o}^{h_1} \{(\hat{u}_b - \lambda p) f - \mu' u_b - \mu u_{bh}\} \, dh = o. \tag{22}$$

All one needs to know about a range of constancy is how long it should be, i.e. the value of h_1 if we think of the solution as being worked out from the bottom. Equation (22) can be used to find h_1 (with (21) providing a continuous check as the integration is performed), for all the other information (22) requires is known to us on reaching the point h_o[11/]. In particular, notice that (14), and hence the solution for $\mu(h)$, (19), are independent of ν and hold at corners too. This characterization, however, is indirect and not at all operational, involving rather complicated expressions for μ and μ'. To simplify things, notice that when $a(h)$ and $b(h)$ are constant, say at $(\bar{a}, \bar{b})$, $du_a/dh = u_{ah}$. We can therefore integrate by parts the middle terms of these equations:

$$\int_{h_o}^{j} \mu' u_a \, dh = \mu u_a \Big|_{h_o}^{j} - \int_{h_o}^{j} \mu u_{ah} \, dh , \tag{23}$$

similarly for u_b in (21)-(22). The last terms of (23) and (20) cancel each other, as do those in the corresponding equations for u_b. Hence, (20)-(21) becomes

$$\int_{h_o}^{j} (\hat{u}_a - \lambda)\, f\, dh = u_a^j \mu^j - u_a^o \mu^o ,$$

$$\int_{h_o}^{j} (\hat{u}_b - \lambda p)\, f\, dh \geq u_b^j \mu^j - u_b^o \mu^o ,$$

(with equalities at $j = h_1$), where u_a^o denotes $u_a\,(\bar{a},\bar{b},h_o)$, $\mu^j \equiv \mu(j)$, and so on. Eliminating μ^j from here, we finally obtain

$$\int_{h_o}^{j} \{(\hat{u}_b - \lambda p) - s^j(\hat{u}_a - \lambda)\}\; f\, dh \geq u_a^o \mu^o (s^j - s^o), \quad (24)$$

$\forall\; j \in [h^o, h^1]$, and

$$\int_{h_o}^{h_1} \{(\hat{u}_b - \lambda p) - s^1 (\hat{u}_a - \lambda)\}\, f\, dh = u_a^o \mu^o (s^1 - s^o). \quad (25)$$

Equation (25) determines, in a rather direct form, the solution-value for h_1, or equivalently the 'exit' right-slope of the tax function at the corner, s^1. We need not worry about multiplicities of solutions, which in all probability will be a feature of (25): the relevant solution is the <u>smallest</u> $h_1 \geq h_o$ that solves it, because at that point ν in (21) has fallen from having a strictly positive value to being zero, and moving beyond that point will generically render ν negative, which we detect in a simple form as a violation of (24). (This is not inconsistent with a possible degenerate case of two corners merging into a single one at h_1 - the value of ν 'bouncing back' into the positive after the point h_1. The computation would simply re-start there).

Equation (25) has a simple enough structure as to invite interpretation, but I have not been able to find an intuitive explanation for it. It, and (24), can be further simplified, in search of an explanation, if we consider the utilitarian case and treat h_1 as being 'close enough' to h_o, so that $s^j - s^o \simeq s^o_h (j - h_o)$. Then, (24) requires that, within a corner,

$$\int_{h_o}^{j} \{\lambda(s^j - s^o) - u_a (s^j - s)\} fdh \geq o, \tag{26}$$

with equality being the signal for the next smooth regime to start.

4. Features of optima

Optimal tax schedules can have very diverse properties, depending critically on the principal's objective and the way h is assumed to enter individual preferences (given by the structure of the problem at hand), and even on the specific functions being used. This is true for the familiar utilitarian taxation (or pricing) problem, and consequently all the more so in the present more general model. Only a few properties of taxes are known to hold fairly generally under utilitarianism: the non-negativity of optimal income tax rates (Mirrlees, 1971); the no-distortion (e.g. zero marginal tax) requirement at end-points where bounching does not occur (Seade, 1977); Atkinson-Stiglitz's (1976) condition on the undesirability of differential taxation across commodities entering utility separately from h; these are three such features that come to mind.[12/] Our purpose in this section is to see what becomes of these results when more general optimal schedules are considered. I omit various details of proofs or of precise conditions under which results hold, whenever these follow closely their counterparts in the above-mentioned articles.

4.1. Tax treatment at the end-points

Let us ignore for the moment the participation constraint. The values $\underline{h}$ and $\bar{h}$ are then fixed and correspond with the actual extreme values of h in the population, rather than with policy-determined extreme points of a given taxation interval, where it borders with possible h-ranges on either side where people choose to withdraw from the scheme.

With fixed $\underline{h}$ or $\bar{h}$, the transversality condition (16) applies, so that $\mu(\underline{h}) = o$, or $\mu(\bar{h}) = o$ respectively. It follows, under weak regularity conditions on the way preferences vary with h (boundedness of u_{ih}), that $s_h\mu/f$ also vanishes at the points $\underline{h}$ and $\bar{h}$, regardless of whether density f is zero or not there (Seade, 1977, p. 228). One must be cautious, however, as to what implications to draw from this for tax rates: the bottom value of the parameter for the population, $\underline{h}$, can be identified with the bottom of the tax schedule only if there is no bunching of consumers with $h > \underline{h}$ at the consumption point observed for $\underline{h}$. Otherwise, if a non-zero range of consumers $[\underline{h}, h^*]$ all maximize utility at the same point, (a corner solution) on the tax schedule, e.g. choosing not to work at all, then it is still true that $\mu(\underline{h}) = o$, but the bottom tax rate now relates, via equation (18), to the lowest h with an interior solution, namely h^* ; and $\mu(h^*)$ will not normally be zero. If there is no bunching, however, the bottom of the h-distribution and of the (interior) tax schedule can be identified with each other, and the optimal tax rate for that point is indeed given by (18) with its last term set equal to zero. That is, without exclusion or bunching,

$$\lambda(s - p) = \hat{u}_a (s - \hat{s}) , \quad \text{at } h = \underline{h}, \bar{h} . \qquad (27)$$

A corollary to this is the result (Seade, 1977) that optimal tax or price schedules for utilitarian objectives, whatever the structure of the specific problem at hand may be, display no distortion at the end-points in the absence of bunching[13/]. (The same is true for <u>each</u> tax schedule, if b and accordingly s and p, are interpreted as vectors). The generalization to more general individualistic social preferences, which is the general case for which $s \equiv \hat{s}$, is immediate from (27).

Before turning to the interpretation of the more general form of the end-points condition (27), it is of interest to briefly discuss the crucial role of the no-bunching and no-exclusion assumptions in securing this result. For ease of exposition, let us refer to the utilitarian closed-economy case. As suggested in Seade (1977, pp. 231 f.), the no-distortion requirement follows intuitively from the fact that, without bunching, a marginal tax at an end-point has <u>all</u> the population on one side of it, so that the motive for distorting prices, essentially to raise revenue from some to pump it back directly or indirectly to others, is not met. The end-point distortion's yield is, under those circumstances, a pure deadweight waste. But if there is bunching, say at $\underline{h}$, there will be a certain fraction of the population not paying any tax on the bottom rate, so that some distortion will (normally) remain desirable on distributional grounds. Similar remarks apply when there is 'exclusion' - if, say, $\underline{h}$ is only the lowest h-value among those who choose to stay in in the optimum. In that case, a change in the marginal tax at the bottom changes the position of the bottom point itself, the value of $\underline{h}$, affecting revenue, distribution and welfare in other forms than those suggested by the simple argument above.

Let us now seek an interpretation for the end-points optimality condition (27). A possible tack would be simply to adapt the 'efficiency vs. equality' argument used for the utilitarian version of (27): that is, for the general case, that at the end-points the second-best motive for the tax disappears, and all the weight should be given to the 'efficiency' element, correcting only for differences between individual and social preferences, as in (27), and not for the government's inability to allocate consumption directly. But this view of (27) is not very telling, nor does it add anything to our understanding of the simple result for the utilitarian case. Instead, let us rewrite this expression as

$$s = \frac{\hat{u}_b - \lambda p}{\hat{u}_a - \lambda} \equiv \hat{\sigma} \qquad \text{at} \quad h = \underline{h}, \bar{h} \tag{28}$$

This condition has an interesting interpretation, which in particular puts the utilitarian result in a different light, as an instance of a more general principle. Notice that $\hat{u}_b$ represents the social gain from giving an extra unit of b to an h-man, while λp is the cost of doing that, the unit production cost of b. Hence $\hat{u}_b - \lambda p$ is the social profit of the given man's consumption of b at the margin. Similarly, $\hat{u}_a - \lambda$ is the social marginal profit from adding to his consumption of a. The right-hand side of (28), which we have denoted by $\hat{\sigma}$, thus emerges as (minus) the slope of the social 'net' indifference curve between consumption of a and b by an h-man. Hence, in contrast with the straightforward gross counterpart to this measure defined on (social) utilities alone ($\hat{s} \equiv \hat{u}_b/\hat{u}_a$), $\hat{\sigma}$ computes social-utility trade-offs taking due account of the cost-side of consumption. Therefore (28) is a condition for generalized Pareto efficiency (in a non-Paretian world) at the end-points, i.e. tangency of individual and social net indifference curves. The reason for this is of course not that preferences embodied in s matter *per se*, but that without it it is possible to increase $\hat{u}$ at (and near) the end-points at no extra resource cost, as exemplified in Figure 1.

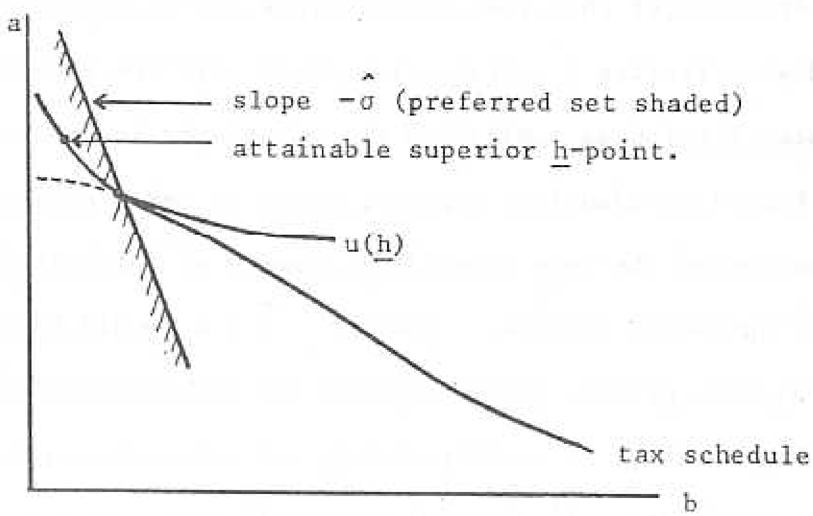

Figure 1: Inefficiency at an end-point

4.2. Implications of the end-points condition : examples

With a utilitarian welfare function, (28) reduces to the simple $s = p$ at the end-points, as mentioned above. One can, however, still write this condition as $u_b/u_a = (u_b - \lambda p)/(u_a - \lambda)$, and interpret it in terms of private vs. social net preferences as above.

It would be useful to explore the type of implications (28) may have for taxes in non-utilitarian set-ups. Consider first Mirrlees' (1971) income tax model, retaining its given structure on consumers' preferences and behaviour, but let welfare be non-utilitarian. For example, very high incomes may be unwanted *per se*, out of direct regard for the distribution of income that emerges. If so, the constant-utility consumption-cost of extra income for the top-man is higher in $\hat{u}$ than in u, i.e. $\hat{s} > s$. From (27), this then implies $p > s$ at the top, where p is in this case the real wage rate paid by producers. Therefore, the marginal tax rate at the top turns out to be positive in the optimum, despite all purely Paretian considerations to the contrary. Preferences that give rise to this may be what many of us have in mind when feeling a certain discomfort with the end-point no-distortion prescription as applied to the top of the income scale. A similar, perhaps less plausible argument could be made for the bottom of the income tax on the very same lines, namely an interest in narrowing the spread of incomes. There, $\hat{s} < s$ would reflect a greater social than private preference for the working poor to earn more by working harder. I would probably not subscribe to this view, but it sounds familiar. If adopted, one would have $s > p$ at the bottom (a higher net than gross marginal wage), which means a negative marginal income tax, again upsetting a rather general property of utilitarian income taxes: the non-negativity of marginal rates to which we return below.

A second non-utilitarian example of interest - both intrinsic interest and as an illustration of the issues involved - is that of a revenue-maximizing government or firm. For small movements around an optimum, and a correspondingly fixed value of aggregate transfer income (profits), maximizing revenue is equivalent to minimizing private consumption (of leisure and goods), so that the welfare function (9) reduces to

$$W = - \int_{\underline{h}}^{\bar{h}} (a + pb)\, f\, dh. \tag{29}$$

That is, $\hat{u}(a,b,h) = -(a + pb)$ here. The same notation applies to the profit-maximizing firm (see footnote 6). The special, striking feature of this example, is that in the optimum the constraint (10) on total demands will not be tight: one is maximizing W subject to a floor on W. Hence the multiplier dual to (10) vanishes: $\lambda = o$. Also, from (29), $\hat{u}_a = -1$, $\hat{u}_b = -p$. Substituting these values into (28), we get the same no-distortion condition as under utilitarianism, $s = p$, so that optimal tax schedules for revenue (or profit) maximization have a zero distortion at either end-point where bunching and exclusion do not arise.

This result, as stated, holds with the same generality as that of its utilitarian counterpart, i.e. regardless of the specific structure of preferences imposed by the problem at hand. But a crucial difference with the utilitarian result arises through violations of the no-bunching, no-exclusion provisos made in the statement of the result. Under utilitarianism, these conditions may or may not be met, depending on the alternatives to participation consumers of different kinds have, $\bar{u}(h)$ in (12), as well as on the specific form of their preferences, density and so on. It is therefore natural, short of performing a taxonomy of possibilities or specific discussion of special cases, simply to rule out by assumption both bunching and non-participation in the statement of the end-points result. For the revenue maximizer the situation is rather different: no optimum can exist before and unless the participation constraint bites. Intuitively, the government's exploitation of consumers can only be stopped by its own wish (the choice of W), electoral or other form of defeat (not considered), or migration. Formally, putting $\lambda = o$ and $\hat{u}_a = -1$ in (19), the latter becomes

$$\mu(h) = \mu(\underline{h}) + \int_{\underline{h}}^{h} (\pi - \frac{f}{u_a}) \exp(\cdot) \, dh. \qquad (30)$$

Now suppose that all consumers participate in the scheme. This implies that $\pi(h) \equiv o \; \forall h$, and that $\underline{h}$ and $\bar{h}$ are fixed as given in the population, so that (16) applies at both ends, $\mu(\underline{h}) = \mu(\bar{h}) = o$. But from (30), with $\pi = o$, $\mu' < o \; \forall h$, which implies a contradiction. Thus it is a general property of optimal taxes for revenue-maximization that some consumers are induced to opt out of the system [14/] - including the possibility of an endogenous mortality rate for a slave economy. Depending on which ranges of h first choose to leave, which can clearly not be ascertained without assumptions on the fine structure of the problem, the no-distortion result may hold at $\underline{h}$, $\bar{h}$, none or both.

4.3. Other properties of optimal taxes

While leaving routine details out, let us see what the counterparts are, for the present formulation, of other well-known features of utilitarian taxes.

(i) Commodity taxes

A result by Atkinson and Stiglitz (1976) for utilitarianism states that no distortion should be imposed within a vector of commodities x^1 which enter utility in a weakly separable form from h: $u = U(\phi(x^1), x^2, h)$, where the vector (x^1, x^2) is the same as (a,b), but split in some different form, with b being interpreted as a vector here. Under the present more general social preferences, this rule is modified in much the same form as the utilitarian no-distortion condition for the end-points was. That is, given separability of x^1, only an 'efficient' distortion amongst the components of x^1 should be imposed, given precisely by expression (28) again, where the rates of substitution and relative prices s, $\hat{s}$, p, apply to pairs of commodities within the vector x^1. An immediate implication of this is that a counterpart to Atkinson-Stiglitz's result applies to revenue or profit maximization too: relative mark-ups over marginal costs should be the same for all commodities in each consumer's basket, if those commodities enter utilities in a separable form from consumer's 'income' h. This follows from (28), noting again that, under profit maximization, $\lambda = o$, $u_a - 1$, $u_b = - p$.

(ii) Rates of taxation

Mirrlees(1971) shows that, for the income tax case (u as in (2)),

$d\,u_a/dh < o$ implies positivity of the optimal marginal income tax at all levels of income where it is defined. His argument, and this result, extend directly to the general utilitarian problem: u_a decreasing in h implies a positive marginal tax of a relative to b throughout the schedule. That is, rather naturally, taxation will be used to transfer numeraire in the direction where it is more 'useful' in consumption.

The problem, alas, lies in that the final schedule of u_a's is an endogenous component of the whole optimization exercise. One would expect single-signedness of du_a/dh in a wide range of unspecified cases, but exceptions can easily arise; the condition is not a valid 'primitive' assumption and needs to be checked for particular cases. For the income tax problem, for example, it can be shown to hold if leisure and consumption are non-inferior and non-complementary in the Edgeworth (cardinal) sense (Seade, 1978). For utility given by (3), as used in the pricing literature, $d\,u_a/dh = - u_{11} > o$, so that mere concavity of u ensures that the distortion is unambiguously of one sign: a positive mark-up above marginal cost, as a means to exact more surplus from high-income consumers, whose u_a is lower.

With non-utilitarian preferences the above result still holds, now stated in terms of single-signedness of $d\,\hat{u}_a/dh$, but more importantly now applying only to the 'second-best' element of the tax in (18). The total sign of the distortion will depend on the interaction between this effect and the 'efficiency' element of the tax, the first term in (18), $\hat{u}_a(s - \hat{s})$. For example, referring again to the income-tax problem, one could think of a government interested in redistribution (in an ex-post sense), with $d\hat{u}_a/dh < o$, but whose views on leisure are more puritanical than consumers', so that it would like to see people, generally, work harder. The first of these motives for taxation, on the limit to redistribution due to decentralization, will call for a positive marginal tax throughout, tailing-off towards zero near the end points of the schedule; the latter motive will be of the opposite sign: a marginal-subsidy incentive to effort. The latter could not possibly dominate throughout in most cases of interest, but any such effect would render marginal rates negative in some ranges of income, approaching the end-point values as given by (28).

5. Concluding remarks

The purpose of this paper was to provide a formulation of the optimization-cum-incentives problem which would, on the one hand, unify the two main special cases that had been studied before, utilitarianism and profit-maximization, bringing out their common structure and results; while at the same time would put the problem in a more general form. This generality is not only meant to cater for further examples than those mentioned above, as some we suggested in the introduction, but more importantly it is intended to put special cases in perspective, hopefully enhancing our understanding of the working of the incentives problem, or of specific results.

As far as results are concerned I was expecting fewer to hold for the general case, or to be less amenable to intuition, than under utilitarianism; but this turned out not to be the case. In particular, the optimal taxation of (or pricing to) the 'richest' and 'poorest' members of the community takes a simple and intuitable form when the principal's objective is not spelled out. Moreover, the same 'efficiency' condition that must hold at the end-points, describes the way in which taxes on commodities which are separable from individuals' features in utility should be related, generalizing the well-known no-excise-taxes result by Atkinson and Stiglitz (1976). When applied to the profit maximization case, this condition reduces to exactly the same form found by these authors for utilitarianism, which simplifies the kind of price-discrimination a monopolist selling many goods should or would use.

Unlike the above, most other results we discussed held only for special cases we considered, such as the general requirement for revenue or profit maximizers to exclude part of the population, or the conditions under which the distortions on relative prices can be signed. Considerable generalizations should be possible in many of these cases.

Finally, results apart, it was also of interest to note the simplicity taken by the conditions describing the optimum in the general case: the decomposition of taxes into a 'first-best' or 'paternalistic', and a 'decentralization' motives, and the simple treatment which can be given to corners of optima, which had not been treated in detail in one case, or not at all in most others, in the contexts of previous analyses.

Footnotes

A preliminary version of this paper was read at the 1978 Winter Symposium of the Econometric Society (Sindelfingen, Germany; January) and at seminars in Hull, Nuffield College and Warwick (S.S.R.C. Economic Theory Study Group). I am grateful to the members of these seminars, and in particular to James Mirrleeś and David Starrett for useful remarks.

1/ The interpretation of the taxation model for the utilitarian pricing of consumption goods whose retrading can be prevented was noted in Seade (1977, p. 229). An implication was, as also noted there, that no distortion should be imposed at the top and perhaps bottom (if all consumers are buyers) of optimal pricing schedules. Recent papers from the pricing literature have studied essentially the same model, under a special assumption on preferences (equation (3), below) which arises naturally in that context, and obtained again the no-distortion result, among others. See Goldman et al. (GLS, 1977; but see below), Roberts (1978) and Willig (1978). GLS allow for a somewhat more general maximand (giving a weight to profits, alongside utilities) and GLS and Roberts derive other features of the model they consider, some of which we refer to below.

2/ Finally, we assume away differences in consumption sets across consumers, of an essential kind or induced by the redefinition and measurement of (a,b) as described. Roughly, these differences can be dealt with extending all consumption sets to the union of all of them, and setting $u = -\infty$ for points outside h's consumption set (or, rather, setting marginal utilities of coming 'into' the set equal to infinity). Clearly, the same allocations will be achievable under the two descriptions.

3/ I avoid calling (7) Bergsonian because this term is very often (but incorrectly) applied to the individualistic welfare function - apart from the fact that Bergson's definition was in a finite-dimensional context.

4/ A functional analyst would of course simply write (7) as $W(a,b)$, where the arcs a and b are mere points in a suitable space, but this is no less abstract than (7) and can be inconvenient if notation is to allow for particular realisations of the functions a and b too.

5/ The Volterra derivative of the functional W w.r.t. a, at the point h', is the limit solution to

$$\delta W = \int_{t_2}^{t_1} W_a(h)\ \delta a\ dh, \quad \text{where } h' \in (t_o,t_1),\ t_1 - t_o \to o,\ \delta a \to o.$$

See Ryder and Heal (1973), or Volterra (1959).

6/ The first interpretation of (9) is obvious. For the second, write $t(h)$ for the amount of tax an h-man ends up paying in a given equilibrium and y for his gross transfer income. The individual budget constraint is then $a + pb + t = y$ which, imposing $\int t \geq R$ (revenue), yields (9) with $A \equiv \int y - R$. For (9) as profits $\geq - A$, think of pb as production cost and $-a$ as the price-function for consumers, their required outlay on buying quantity b.

7/ We have or can not place an upper limit on b', i.e. on the 'speed' at which demands vary across different people. The limit case $b'(h_d) = \infty$ amounts essentially to a discontinuity or multi-valuedness of $b(h)$ at h_d, i.e., the tax function running along h_d's relevant indifference curve over an arc. Goldman et al. (1977) give an example of this possibility and Mirrlees (1971) gives conditions under which discontinuities are ruled out for the income-tax case (Theorem 2-v.) I assume away discontinuities of allocations. Were they to arise, our analysis would apply all the same to each continuous arc, with the point of discontinuity h_d playing the role of $\bar{h}$ and $\underline{h}$ for two successive continuous arcs - only transversality conditions would be affected at these points : equilibrium utility must be continuous in h through the point h_d (see note 9, below).

8/ Notice that, with a fixed, 'captive' population as in Seade (1977), one always gets $\mu(\underline{h}) = \mu(\bar{h}) = o$, despite the fact that this may or may not translate into zero distortions at the endpoints (cf. (18), below), depending on whether bunching (of, still, participants !) occurs there. This point is often overlooked in interpreting (16) directly in terms of taxes.

9/ That is, equality in (12) must always hold at the boundary points, between regimes where (12) is met and those where it is violated. This follows from continuity of equilibrium allocations of utility, itself imposed by continuity of $u(\cdot,\cdot;h)$ on h.

10/ Multipliers are continuous inside smooth arcs of the allocation-functions $a(h)$, $b(h)$; at corners of these (where a constancy range starts and ends), standard variational analysis requires continuity of $F_{a'}$ and $F_{b'}$ (see Hadley and Kemp, 1971, p. 37), i.e., in this case, of μu_a and $\mu u_b + \nu$, which imply the result. This continuity property of the multipliers is essential for the present argument and at other points below.

11/ The only previous solution for corners I am aware of is in Mirrlees (1971, Theorem 2-iv), who uses essentially forms of (21) and (22) for the income-tax case, with μ and μ' from (19).

12/ Another general property of optimal taxes is Mirrlees' (1976) very general but obscure Pareto-efficiency condition, on a relation that must hold between optimal nonlinear taxes on different commodities.

13/ Versions of this result for the utilitarian problem were noted in footnote 1.

14/ Goldman et al. (1977) and Roberts (1978) derive versions of this result for profit maximization with consumers described by (3).

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