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No 845

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

Auctions in which Losers Set the Price

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February 29, 2008

Abstract

We study auctions of a single asset among symmetric bidders with affiliated values. We show that the second-price auction minimizes revenue among all efficient auction mechanisms in which only the winner pays, and the price only depends on the losers' bids. In particular, we show that the k-th price auction generates higher revenue than the second-price auction, for all $k > 2$. If rationing is allowed, with shares of the asset rationed among the t highest bidders, then the $(t+1)$-st price auction yields the lowest revenue among all auctions with rationing in which only the winners pay and the unit price only depends on the losers' bids. Finally, we compute bidding functions and revenue of the k-th price auction, with and without rationing, for an illustrative example much used in the experimental literature to study first-price, second-price and English auctions.

Journal of Economic Literature Classification Numbers: D44, D82.

Keywords: Auctions, Second-Price Auction, English Auction, k-th Price Auction, Affiliated Values, Rationing, Robust Mechanism Design.

1 Introduction

We study auctions of a single asset among symmetric bidders with affiliated values, that satisfy the following three properties: 1. The bidder with the highest signal wins. 2. Only the winner pays. 3. The price only depends on the losers' signals. Auction mechanisms in

[1] We would like to thank the associate editor and referees for their useful comments. Ilia Tsetlin is grateful to the Centre for Decision Making and Risk Analysis at INSEAD for supporting this project.

this class are defined by a price function $p^1(\cdot)$ which maps the signals of the losers into the price paid by the winners. The second-price auction is an example of such a mechanism. Other examples include the k-th price auction, with $k > 2$, in which the highest bidder wins and pays a price equal to the k-th highest bid.

Property 1 says that the auction is efficient. The properties of efficiency and that losers do not pay hold in all standard auctions. The third property, that the price paid by the winner is determined by the losing bids, is a robustness property; it holds in any ex-post incentive compatible mechanism (see Bergemann and Morris, 2005, for a recent discussion of robustness in mechanism design and ex-post incentive compatibility). In an auction that satisfies our third property, a bidder does not need to worry about manipulating the price, because the price does not depend on his bid; his bid only determines whether he wins or loses. This property captures an important feature of an ex-post incentive compatible auction, without going as far as requiring no regret after all possible signal-profile realizations.[1]

We show that the second-price auction minimizes revenue among all p^1-auctions. In particular, for all $k > 2$, the k-th price auction generates higher revenue than the second-price auction.

We also consider rationing. Auctions with rationing have been used to model *initial public offerings* (IPO's) by Parlour and Rajan (2005). As they point out, in a typical IPO there is excess demand at the offer price, and shares are rationed to investors. Rationing schemes are used more widely than just in IPO's, for example to sell tickets to sport and entertainment events. With risk neutral bidders, lottery qualification auctions (see Harstad and Bordley, 1996) are formally equivalent to rationing. In such auctions the highest bidders win lottery tickets for the assignment of an asset.

Parlour and Rajan (2005) studied a sealed-bid, uniform price auction, in which the winners are the t highest bidders and the price is the $(t + 1)$-st highest bid. Each of the t winners receives a share whose value, like in uniform rationing, does not depend on the bids. They showed that rationing may raise the issuer's revenue. (See also Bulow and Klemperer, 2002,

[1] Ex-post incentive compatible mechanisms have the no-regret property that no buyer would want to revise his decision after observing the rivals' behavior (signals).

for a discussion of the potential benefits of rationing in common value auctions.)

We study p^t auctions, a generalization of p^1 auctions, and show that all p^t-auctions yield higher revenue than the auction studied by Parlour and Rajan (2005). For example, revenue can be raised by leaving the number of winners and rationing rule unchanged, but stipulating that the price is some bid lower than the highest losing bid.

Kagel and Levin (1993) were the first to study a special case of the 3-rd price auction with independent private values. They found such an auction useful from an experimental point of view, because its predictions differ in important ways from those of first- and second-price auctions. Wolfstetter (2001) used revenue equivalence to derive the bidding function in the k-th price auction, with $k > 2$, for the general model with independent private values.

Besides shedding theoretical light on the affiliated values model, our results could prove quite useful in the experimental testing of (Bayesian) Nash equilibrium theory. We elaborate on this point in the concluding section.

The paper is organized as follows. The next section introduces the model. Section 3 introduces p^t auctions, with $t \geq 1$, and derives the main results of the paper. In Section 4 we use an illustrative example to examine k-th price auctions with and without rationing and the English auction. Section 5 concludes.

2 The Model

A single object is auctioned to N risk-neutral bidders. Bidder i, $i = 1, 2, \ldots, N$, observes the realization x_i of a signal X_i. Denote with $s = (x_1, ..., x_N)$ the vector of signal realizations. Let $s \vee s'$ be the component-wise maximum and $s \wedge s'$ be the component-wise minimum of s and s'. As in Milgrom and Weber (1982), signals are drawn from a distribution with a joint pdf $f(s)$, which is symmetric in $x_1, ..., x_N$ and satisfies the affiliation property:

$$f(s \vee s')f(s \wedge s') \geq f(s)f(s') \qquad \text{for all } s, s'. \tag{1}$$

If the inequality holds strictly, we say that the signals are strictly affiliated. The support of f is $[\underline{x}, \overline{x}]^N$, with $-\infty < \underline{x} < \overline{x} < +\infty$. We also assume that f is differentiable.

The value V_i of the object to bidder i is a function of all signals: $V_i = u(X_i, \{X_j\}_{j \neq i})$. The function $u(\cdot)$ is non-negative, bounded, differentiable, increasing in each variable, and symmetric in the other bidders' signal realizations x_j, $j \neq i$. The model with affiliated private values corresponds to valuation function $u(X_i, \{X_j\}_{j \neq i}) = X_i$; that is, bidder i's valuation depends only on his own signal.

In studying the equilibrium of a given auction, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with Y^n the n-th highest signal of bidders $2, 3, ..., N$ (i.e., all bidders except bidder 1).

Define

$$v_t(x, y) = E\left[V_1 | X_1 = x, Y^t = y\right].$$

Affiliation implies that $v_t(x, y)$ is increasing in both arguments, and hence differentiable almost everywhere (see Milgrom and Weber, 1982, Theorem 5).

3 p^t-Auctions

Parlour and Rajan (2005) model bookbuilding and rationing in initial public offerings as a sealed-bid, uniform-price auction in which the winners are the t highest bidders and the unit price is the $(t+1)$-st highest bid. Each of the t winners receives a share of the asset whose value does not depend on the bids (uniform sharing, where each winner receives a share $1/t$, is a special case), and pays his share of the unit price. The bidding function in such an auction is

$$\beta_{t+1}(x) = E\left[V_1 | X_1 = x, Y^t = x\right]. \quad (2)$$

This is the same as the bidding function in a uniform auction for t objects with bidders having unit demand and the price being the $(t+1)$-st bid (e.g., see Milgrom 1981). When

$t = 1$, the Parlour-Rajan auction coincides with the second-price auction. More generally, after rescaling payoff functions by $1/t$, uniform rationing of a single object to the t highest of N bidders is strategically equivalent to selling t objects to N bidders with unit demand.

By the revelation principle (see Myerson, 1981), given any auction, or mechanism, there is an equivalent direct mechanism where bidders directly report their signals to a designer, and it is an equilibrium for all bidders to report truthfully. A direct mechanism can be thought of as a proxy auction in which each bidder reports a signal to a proxy bidder who then bids on his behalf in the true auction.

Let $r_1, ..., r_N$ be the bidders' reported signal values in decreasing order ($r_1 \geq r_2 \geq ... \geq r_N$). We are interested in the class of (direct) auction mechanisms, called p^t-auctions, which satisfy the following three properties: 1. The bidders with the t highest signals win ($t \geq 1$), and the share that each winner gets does not depend on the bids. 2. Only the winners pay; they pay their share of the unit price p^t. 3. The uniform unit price does not depend on the winners' signals and it is a weakly increasing function of the losers' signals, $p^t = p^t(r_{t+1}, r_{t+2}, ..., r_N)$.

Properties 1 and 2 are satisfied by all standard auctions. If $t = 1$, so that there is no rationing, Property 1 implies that the auction is efficient. Property 3 captures an important feature of an ex-post incentive compatible auction, without going as far as requiring no regret after all possible signal-profile realizations. In an auction that satisfies it, bidders cannot directly manipulate the price.

The k-th price auction with rationing, with $k \geq t + 1$, in which the t highest bidders win and pay a unit price equal to the k-th highest bid, corresponds to a p^t-auction with a price function $p^t(r_k)$ that only depends on r_k. The first-price auction, clearly, is not equivalent to any p^t-auction. The English auction, on the other hand, corresponds to a p^1-auction (an auction with no rationing) with a price function $p^1(r_2, ..., r_N)$ that depends on the reports of all losers.

We now derive a (necessary) equilibrium condition that must be satisfied by a p^t-auction.

Theorem 1 (*The Indifference Condition*) *A p^t-auction must satisfy the following condition*

$$E\left[V_1|X_1=x,Y^t=x\right]=E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1=x,Y^t=x\right], \tag{3}$$

together with the boundary condition

$$p^t(\underline{x},...,\underline{x})=E\left[V_1|X_1=\underline{x},Y^t=\underline{x}\right]. \tag{4}$$

Proof. Let $f_{t:N-1}\left(y_t,...,y_{N-1}|X_1=x\right)$ denote the marginal density of $Y^t,...,Y^{N-1}$ conditional on $X_1=x$, and $f_t(y_t|X_1=x)$ denote the marginal density of Y^t conditional on $X_1=x$. If all bidders different from bidder 1 truthfully bid their signals, then the payoff of bidder 1 when his type is x and he reports r is proportional to[2]

$$U(x,r)=\int_{\underline{x}}^{r}E\left[\left(v_t(x,Y^t)-p^t\left(Y^t,...,Y^{N-1}\right)\right)|X_1=x,Y^t=y_t\right]f_t(y_t|X_1=x)dy_t. \tag{5}$$

The first-order condition for maximization with respect to r can be written as:

$$v_t(x,r)=E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1=x,Y^t=r\right].$$

In equilibrium, bidder 1 must bid $r=x$; hence (3) holds. ∎

In a p^t-auction a bidder's payoff is only affected by his own bid when he is tied for a win. In such a case, the marginal benefit of winning the object is $E\left[V_1|X_1=x,Y^t=x\right]$, while the marginal cost is $E\left[P|X_1=x,Y^t=x\right]$. Optimality, condition (3), requires the two to be equal.

The indifference condition (3) and the boundary condition (4) are first order conditions. Lemma 1, proven in the Appendix, shows that they are sufficient for a truthful equilibrium of a p^t-auction if either there are affiliated private values, or an additional assumption is satisfied.

[2] The constant of proportionality equals the expected share of the asset that bidder 1 would get, were he to win.

Assumption 1 *One of the following two conditions holds.*

(1) For all values of x and r, it is

$$\frac{\partial E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1=x,Y^t=r\right]}{\partial x}\leq\frac{\partial v_t(x,r)}{\partial x}.$$

(2) Let $f_{t+1:N-1}(y_{t+1},...,y_{N-1}|X_1=x,Y^t=r)$ be the density of $Y^{t+1},...,Y^{N-1}$ conditional on $X_1=x$ and $Y^t=r$. The function $\varsigma(\cdot)$ defined by

$$\varsigma(x,r)=\frac{v_t(x,r)}{f_{t+1:N-1}\left(r,r,...,r|X_1=x,Y^t=r\right)}$$

is increasing in x for all values of r.

Part (1) of Assumption 1 requires that an increase in bidder 1's type x has a larger impact on the expected value of bidder 1 than on the expected unit price at auction, conditional on bidder 1 winning the auction and bidding as a type r, the highest losing type. This is a natural assumption, which is always satisfied if signals are independent, because in such a case the expected value of p^t does not depend on x. The appealing feature of part (2) of Assumption 1 is that it imposes no restriction on the p^t function. It is also always satisfied if signals are independent, because in such a case the denominator of $\varsigma(x,r)$ does not depend on x, while $v_t(x,r)$ increases with x.

Lemma 1 *Suppose that either there are private values, or Assumption 1 holds. Then conditions (3) and (4) are sufficient for a p^t-auction to be well defined.*

We are now ready to show that the auction with rationing studied by Parlour and Rajan (2005) minimizes revenue among all p^t-auctions.

Theorem 2 *The p^t-auction in which the unit price is the $(t+1)$-st bid, generates the lowest expected revenue among all p^t-auctions.*

Proof: Let R^t be the revenue in a p^t-auction with price function $p^t(\cdot)$, and let R^t_{t+1} be the revenue in the Parlour-Rajan auction. It follows from (3) and (2) that, conditional on

$X_1 \geq x = Y^t$,

$$\begin{aligned} E\left[R_{t+1}^t | X_1 \geq x = Y^t\right] &= \beta_{t+1}^t(x) \\ &= E\left[p^t\left(Y^t, ..., Y^{N-1}\right) | X_1 = x, Y^t = x\right] \\ &\leq E\left[p^t\left(Y^t, ..., Y^{N-1}\right) | X_1 \geq x, Y^t = x\right] \\ &= E\left[R^t | X_1 \geq x = Y^t\right], \end{aligned}$$

where the inequality follows from affiliation. Taking expectations of both sides yields $E\left[R_{t+1}^t\right] \leq E\left[R^t\right]$. Under strict affiliation the inequality is strict if p^t is strictly increasing in at least one r_i, $i > t+1$. ∎

Theorem 2 does not contradict the main message of Parlour and Rajan (2005). They showed that with common values rationing may raise the issuer's revenue. Theorem 2 shows that there are many auctions with rationing that yield even higher revenue than the auction they proposed. For example, revenue would be raised by leaving the number of winners and the rationing rule unchanged, but stipulating that the price is some bid lower than the highest losing bid.

If values are private, then the Parlour and Rajan auction with rationing always yields less revenue than the second-price auction. In such a case, revenue in the second-price auction is the expected value of the second order statistic out of the N bidders' signals, while in the Parlour and Rajan auction revenue is the expected value of the $(t+1)$-st order statistic, with $t > 1$.

The main result for the important special case of no rationing, p^1-auctions, follows as a corollary of Theorem 2.

Corollary 1 *The second-price auction generates the lowest expected revenue among all p^1-auctions.*

Under strict affiliation, the second-price auction yields strictly less revenue than any p^1-auction in which the price strictly increases with at least one losing bid different from the second highest bid.

The bidder with the second highest signal, say bidder 2, is the price setter in a second-price auction. It follows directly from the indifference condition that bidder 2's bid in a second-price auction is equal to the expected price in a p^1-auction, conditional on bidder 2's signal being tied with the winner's signal. However, because signals are affiliated and bidder 2 has the second highest signal, the expected price in a p^1-auction conditional on bidder 2 being tied with the highest bidder is an underestimate of the true expected price. It follows that in the class of p^1-auctions, expected revenue is minimized by the second-price auction. Thus, in particular, a k-th price auction generates higher revenue than the second-price auction, for all $k > 2$.

In the special case of affiliated private values, the English and the second-price auction are equivalent and yield the same revenue. It follows that in such a case any p^1-auction not identical to the second-price auction (for example, the k-th price auction) yields higher revenue than the English auction. In general, the English auction does not necessarily maximize revenue in the class of p^1-auctions.

4 An Illustrative Example

In this section, we discuss the best known analytically solvable example of auctions with affiliated values. We will derive equilibrium bidding functions and revenue results for the k-th price ($k \geq 2$) and the English auctions with rationing.

Example 1 *There is a single object and N bidders. Conditional on $V = v$, each bidder's signal is drawn independently from a uniform distribution on $[v - \frac{1}{2}, v + \frac{1}{2}]$, where the random variable (or signal) V corresponds to the object's common value component. Bidder i's payoff consists of a private-value and a common-value component, with weights λ and $(1 - \lambda)$ respectively, $0 \leq \lambda \leq 1$. It is $u(\cdot) = \lambda X_i + (1 - \lambda)V$. The random variable V has a diffuse prior; that is, it is uniformly distributed on $[-M, M]$ with $M \to \infty$.*

This example has been extensively used in the experimental literature to study first-price, second-price, and English auctions in the two polar cases of pure private ($\lambda = 1$) and pure common values ($\lambda = 0$); see Kagel, Harstad and Levin (1987), Kagel and Levin (2002), and Parlour et al. (2007). Klemperer (2004, pp. 55-57) presents the equilibria and revenue comparisons of first-price, second-price and English auctions for the pure common-value case in which $\lambda = 0$. Parlour and Rajan (2005) study a few variations of this example with $\lambda = 0$, including some in which the signal distribution is not uniform and the random variable V has finite support, rather than being diffuse over the real line. These variations have the advantage of making the model more realistic (e.g., V is bounded above and below), but come at the cost of having to resort to numerical methods in order to calculate bidding functions near the boundary of the signal support and expected revenue. In the interior of the signal support, on the other hand, the bidding functions correspond to the analytically solvable version of the example we study.[3]

Proposition 3 *In Example 1, the bidding function in a k-th price auction with rationing is given by*

$$\beta_k^t(x) = x + \frac{k-1}{N} - \frac{1}{2} + \lambda \left[\frac{1}{2} - \frac{t}{N}\right].$$

The expected revenue in a k-th price auction with rationing, conditional on $V = v$, is

$$E[R_k^t | V = v] = v + \lambda \left[\frac{1}{2} - \frac{t}{N}\right] - \frac{N+1-k}{N(N+1)}.$$

The proof is in the appendix. The bidding function and revenue in a k-th price auction with rationing satisfy the following properties. (1) The bid and revenue are increasing functions of k. (2) The bid decreases (and revenue need not increase) with the number of bidders N. (3) The bid and revenue increase with the weight λ attached to the private-value component if and only if $t < N/2$. (4) For fixed k and $\lambda > 0$, the bid and revenue decrease with the rationing parameter t.

[3] We should stress that only for the case of λ "sufficiently close" to 1, we have been able to establish existence of equilibrium (i.e., that the second order conditions hold). For other values of λ the bidding functions we present in Proposition 3 are the only increasing symmetric equilibrium candidates.

In the third-price auction with independent private values and uniform distribution of types, Kagel and Levin (1993) showed that the bid function satisfy property (2). Wolfstetter (2001) demonstrated that in the general model with independent private values, the bid function satisfies properties (1) and (2). As is well known, with independent signals revenue equivalence holds, and hence for fixed t revenue does not depend on k.

Property (4) shows that, at least in this example, rationing is not beneficial in a k-th price auction: any auction with rationing in which the price is the k-th highest bid yields less revenue than the very same auction without rationing ($t = 1$). This does not contradict Parlour and Rajan (2005), who claimed that rationing raises bids. They assumed $k = t + 1$, and if one makes such an assumption, then indeed the bid increases with rationing (i.e., with t), provided $\lambda < 1$; that is, provided values are not purely private.

In an English auction with rationing, bidding stops when there are only t bidders left. Each of them is allocated a share of the asset and pays a share of the unit price, the bid of the last bidder to drop out of the auction.

Proposition 4 *In Example 1, suppose bidder* 1 *with signal* x *is left with* t *opponents in an English auction with rationing, and hence all signals* $Y^{t+1}, ..., Y^{N-1}$ *have been revealed during the bidding. Then bidder 1 bids*

$$\beta_E^t(x) = x + (1-\lambda)\left[(y_{N-1} + 1 - x)\frac{t}{t+1} - \frac{1}{2}\right].$$

The expected revenue in an English auction with rationing, conditional on $V = v$, *is*

$$E[R_E^t|V = v] = v + \frac{1}{2} - \lambda\frac{t+1}{N+1} - (1-\lambda)\left(\frac{1}{(N+1)(t+1)} + \frac{1}{2}\right).$$

The proof is in the appendix. The bidding function and revenue in an English auction with rationing satisfy the following properties. (1) Revenue increases with the number of bidders N. (2) The bid and revenue may increase or decrease with the weight λ attached to the private-value component. (3) For $\lambda < 1$, the bid increases with the rationing parameter t.

Revenue increases with the rationing parameter t if and only if $t < \sqrt{\frac{1-\lambda}{\lambda}} - 1$. In particular, with common values ($\lambda = 0$), an increase in the rationing parameter increases both bids and revenue, while with private values an increase in rationing reduces revenue. Note here the contrast with the k-th price auction, where rationing is never beneficial.

According to the standard interpretation of the "linkage principle" (see Milgrom and Weber, 1982, Milgrom, 1987, Krishna and Morgan, 1997, Krishna, 2002, and Klemperer, 2004), if the price the winner pays in an efficient auction with affiliated signals and common values is more statistically linked to the other bidders' signals, then expected revenue is higher. Since in a k-th price auction the price only depends on "one other bidder's information," this would seem to imply that the expected revenue is higher in an ascending than in any k-th price auction. It is thus interesting to observe that in the case of common values (i.e., $\lambda = 0$) and without rationing (i.e., $t = 1$) the revenue in an English auction is higher than in a k-th price auction if and only if $k < \frac{N+2}{2}$. The English auction does not maximize revenue in the class of p^1-auctions.[4]

This result and the result that with private values the k-th price auction always generates higher revenue than the English auction are related to Lopomo (2000). He showed, using a two-bidder example, that there are auctions yielding greater revenue than the English auction, in which losers do not pay. However, the mechanism in Lopomo's example does not satisfy the property that the price only depends on the losers' bids; it is substantially more complex than p^1-auctions (especially k-th price auctions), and it is not easy to generalize beyond the two-bidder case.

5 Conclusions

We have shown that the second-price auction minimizes revenue in the class of efficient auctions in which the price paid by the winners depends only on the losing bids, and losers

[4] By Proposition 3, in the example studied in this section the N-th price auction maximizes revenue among all k-th price auctions. We have been unable to establish that this is the case in the general model, or to find a counterexample.

do not pay (p^1-auctions). When an asset is rationed to t bidders, setting the unit price to be the $(t+1)$-st bid minimizes revenue among the class of p^t-auctions, a generalization of p^1-auctions.

We do not advocate for the use in practice of p^t-auctions as a way to increase revenue. The criticism raised about the practical use of the second-price auction (e.g., see Rothkopf, 2007) also applies to p^t-auctions. In our view, a potentially important application of our results is experimental testing of (Bayesian) Nash equilibrium theory. In studying auctions with affiliated values, experimentalists have typically used a pure private-value and a pure common-value version of a simple example of the general model. We have provided closed form solutions of the bid function and revenue of the k-th price auction for a generalization of this example, in which values have a private and a common value component and rationing is allowed. We have derived several additional predictions that could prove useful in experimental studies (e.g., in a k-th price auction with rationing, the bid and revenue increase with k and decrease with the rationing parameter t, while in an English auction the bid always increases with t unless values are purely private, and revenue increases with t if there are common values and decreases with t if there are private values).

In auctions with common or affiliated values, experimental subjects (especially inexperienced ones) do not behave fully in accordance with the predictions of equilibrium theory. Instead, they fall prey of the *winner's curse*; they do not entirely take into account that winning conveys the bad news that all other bidders have lower value estimates (e.g., see Kagel and Levin, 2002, Kagel, Harstad and Levin, 1987, and Parlour et al., 2007). In the equilibrium of a k-th price auction (with or without rationing) a bidder must bid above his value estimate conditional on being tied with the winner. It seems then reasonable to conjecture that in such auctions with affiliated values there might be less overbidding relative to the equilibrium prediction; an underestimate of the strategic need to bid above one's own value estimate may counteract the winning curse. Testing experimentally this conjecture and the other theoretical results concerning k-th price auctions could lead to interesting new insights about the predictive power of Bayesian Nash equilibrium theory.

Appendix

Proof of Lemma 1. We need to show that when all other bidders bid truthfully in the p^t-auction, it is optimal for bidder 1 also to bid truthfully. If all other bidders bid truthfully, the payoff of type x of bidder 1 bidding as type r is $U(x;r)$, defined in (5). Differentiating with respect to r gives that $\frac{\partial U(x;r)}{\partial r}$ is proportional to

$$\left\{v_t(x,r) - E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 = x, Y^t = r\right]\right\} f_t(r|X_1 = x). \tag{7}$$

$\frac{\partial U(x;r)}{\partial r} = 0$ for $r = x$ and it has the same sign as $x-r$ if there are private values (because in that case $v_t(x,r) = x$ and $E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 = x, Y^t = r\right]$ is increasing in r by affiliation), or if part (1) of Assumption 1 holds. It follows that $r = x$ is a global maximizer of $U(x;r)$ if there are private values, or part (1) of Assumption 1 holds.

We now prove that part (2) of Assumption 1 is also a sufficient condition. The expression in (7) is proportional to

$$\left\{v_t(x,r) - E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 = x, Y^t = r\right]\right\} f_t(r|X_1 = x),$$

which has the same sign as

$$\begin{aligned}\Delta(x,r) &= \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1 = x, Y^t = r\right)} - \frac{E\left[p^t\left(Y^t,...,Y^{N-1}\right)|X_1 = x, Y^t = r\right]}{f_{t+1:N-1}\left(r,..,r|X_1 = x, Y^t = r\right)} \\ &= \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1 = x, Y^t = r\right)} \\ &\quad - \int_{\underline{x}}^{r} .. \int_{\underline{x}}^{y_{N-2}} p^t(r, y_{t+1}, .., y_{N-1}) \frac{f_{t+1:N-1}\left(y_{t+1},..|X_1 = x, Y^t = r\right)}{f_{t+1:N-1}\left(r,...,r|X_1 = x, Y^t = r\right)} dy_{N-1}..dy_{t+1}.\end{aligned}$$

By affiliation, for all $y_j < r$, $j = 2,...,N-1$, the expression

$$\frac{f_{t+1:N-1}\left(y_{t+1},..,y_{N-1}|X_1 = r, Y^t = r\right)}{f_{t+1:N-1}\left(r,..,r|X_1 = r, Y^t = r\right)} - \frac{f_{t+1:N-1}\left(y_{t+1},..,y_{N-1}|X_1 = x, Y^t = r\right)}{f_{t+1:N-1}\left(r,..,r|X_1 = x, Y^t = r\right)}$$

has the same sign of $x-r$. Thus, for $x>r$

$$\begin{aligned}\Delta(x,r) &\geq \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)} \\ &- \int_{\underline{x}}^{r}..\int_{\underline{x}}^{y_{N-2}} p^t(r,y_{t+1},..,y_{N-1})\frac{f_{t+1:N-1}\left(y_{t+1},..|X_1=r,Y^t=r\right)}{f_{t+1:N-1}\left(r,...,r|X_1=r,Y^t=r\right)}dy_{N-1}..dy_{t+1} \\ &= \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)}-\frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)},\end{aligned}$$

which is positive by part (2) of Assumption 1. It follows that when $x>r$, it is $\frac{\partial U(x;r)}{\partial r}\geq 0$, and hence it is profitable for bidder 1 to increase his bid. Similarly, for $x<r$

$$\begin{aligned}\Delta(x,r) &\leq \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)} \\ &- \int_{\underline{x}}^{r}..\int_{\underline{x}}^{y_{N-2}} p^t(r,y_{t+1},..,y_{N-1})\frac{f_{t+1:N-1}\left(y_{t+1},..|X_1=r,Y^t=r\right)}{f_{t+1:N-1}\left(r,...,r|X_1=r,Y^t=r\right)}dy_{N-1}..dy_{t+1} \\ &= \frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)}-\frac{v_t(x,r)}{f_{t+1:N-1}\left(r,..,r|X_1=x,Y^t=r\right)},\end{aligned}$$

which is negative by part (2) of Assumption 1. When $x<r$ it is profitable for bidder 1 to decrease his bid. This completes the proof. ∎

Proof of Proposition 3. One can show (e.g., see Klemperer, 2004) that:

$$E\left[V|X_1=x,Y^t=x\right]=x-\frac{1}{2}+\frac{t}{N}.$$

Furthermore, since $E\left[Y^{k-1}|V\right]$ is equal to the $(k-1)$-st highest value out of $N-1$ draws from a uniform on $\left[V-\frac{1}{2},V+\frac{1}{2}\right]$, it is

$$E\left[Y^{k-1}|V\right]=V+\frac{1}{2}-\frac{k-1}{N},$$

and hence it follows that

$$\begin{aligned} E\left[Y^{k-1}|X_1=x,Y^t=x\right] &= E\left[E\left[Y^{k-1}|V\right]|X_1=x,Y^t=x\right] \\ &= E\left[V+\frac{1}{2}-\frac{k-1}{N}|X_1=x,Y^t=x\right] \\ &= x-\frac{k-(1+t)}{N}. \end{aligned}$$

Looking for a linear equilibrium $\beta_k^t(x)=a+bx$ of the k-th price auction with rationing, we can write equation (3) as

$$\lambda x+(1-\lambda)E\left[V|X_1=x,Y^t=x\right]=a+bE\left[Y^{k-1}|X_1=x,Y^t=x\right],$$

or,

$$\lambda x+(1-\lambda)\left[x-\frac{1}{2}+\frac{t}{N}\right]=a+b\left[x-\frac{k-(1+t)}{N}\right].$$

Hence it is $b=1$ and $a=\frac{k-1}{N}-\frac{1}{2}+\lambda\left[\frac{1}{2}-\frac{t}{N}\right]$. This gives the bidding function.

Letting Y_N^k be the k-th highest value out of N draws from a uniform on $\left[V-\frac{1}{2},V+\frac{1}{2}\right]$, the expected revenue in a k-th price auction with rationing, conditional on $V=v$, is

$$\begin{aligned} E[R_k^t|V=v] &= E[\beta_k^t\left(Y_N^k\right)|V=v] \\ &= \left[v+\frac{1}{2}-\frac{k}{N+1}\right]+\frac{k-1}{N}-\frac{1}{2}+\lambda\left[\frac{1}{2}-\frac{t}{N}\right] \\ &= v+\lambda\left[\frac{1}{2}-\frac{t}{N}\right]-\frac{N+1-k}{N(N+1)}. \end{aligned}$$

This completes the proof. ∎

Proof of Proposition 4. Let $f(x|v)$ be the density of x conditional on v; it is equal to 1 for $v\in\left[x-\frac{1}{2},x+\frac{1}{2}\right]$ and zero otherwise. Its associated distribution in the interior of the support is $F(x|v)=x-v+\frac{1}{2}$. Suppose bidder 1 with signal x is left with t opponents, and hence all signals $Y^{t+1},...,Y^{N-1}$ have been revealed during the bidding. Then bidder 1 knows that $v \in \left[x - \frac{1}{2}, y_{N-1} + \frac{1}{2}\right]$. The bidding function is:

$$
\begin{aligned}
\beta_E(x, y_t, ..., y_{N-1}) &= \lambda x + (1-\lambda) E[V | X_1 = x, Y^t = x, Y^{t+1} = y_{t+1}, ..., Y^{N-1} = y_{N-1}] \\
&= \frac{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}} [\lambda x + (1-\lambda) v] f^2(x|v)[1 - F(x|v)]^{t-1} f(y_{t+1}|v)...f(y_{N-1}|v) dv}{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}} f^2(x|v)[1 - F(x|v)]^{t-1} f(y_{t+1}|v)...f(y_{N-1}|v) dv} \\
&= \lambda x + (1-\lambda) \frac{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}} v \left(\frac{1}{2} - x + v\right)^{t-1} dv}{\int_{x-\frac{1}{2}}^{y_{N-1}+\frac{1}{2}} \left(\frac{1}{2} - x + v\right)^{t-1} dv} \\
&= \lambda x + (1-\lambda) \frac{\int_0^{y_{N-1}-x+1} \left(z + x - \frac{1}{2}\right) z^{t-1} dz}{\int_0^{y_{N-1}-x+1} z^{t-1} dz} \\
&= \lambda x + (1-\lambda) \left(x - \frac{1}{2} + \frac{\int_0^{y_{N-1}-x+1} z^t dz}{\int_0^{y_{N-1}-x+1} z^{t-1} dz} \right) \\
&= \lambda x + (1-\lambda) \left(x - 1/2 + \frac{t}{t+1} \frac{(y_{N-1} - x + 1)^{t+1}}{(y_{N-1} - x + 1)^t} \right) \\
&= \lambda x + (1-\lambda) \left[x - 1/2 + (y_{N-1} - x + 1) \frac{t}{t+1} \right].
\end{aligned}
$$

If Y_N^m is the m-th highest value out of N draws from a uniform on $\left[V - \frac{1}{2}, V + \frac{1}{2}\right]$, then $E[Y_N^m | V = v] = v + \frac{1}{2} - \frac{m}{N+1}$, and revenue in the English auction conditional on $V = v$ is

$$
\begin{aligned}
E[R_E^t | V = v] &= E\left[Y_N^{t+1} \left(1 - \frac{(1-\lambda)t}{t+1}\right) + \frac{(1-\lambda)t}{t+1} Y_N^N + (1-\lambda) \left(\frac{t}{t+1} - \frac{1}{2}\right) | V = v \right] \\
&= \left(v + \frac{1}{2} - \frac{t+1}{N+1}\right) \left(1 - \frac{(1-\lambda)t}{t+1}\right) + \frac{(1-\lambda)t}{t+1} \left(v + \frac{1}{2} - \frac{N}{N+1}\right) + (1-\lambda) \left(\frac{t}{t+1} - \frac{1}{2}\right) \\
&= v + \frac{1}{2} - \frac{1+\lambda t}{N+1} - \frac{N}{N+1} \frac{(1-\lambda)t}{t+1} + (1-\lambda) \left(\frac{t}{t+1} - \frac{1}{2}\right) \\
&= v + \frac{1}{2} - \lambda \frac{t+1}{N+1} - (1-\lambda) \left(\frac{1}{(N+1)(t+1)} + \frac{1}{2}\right).
\end{aligned}
$$

This completes the proof. ▮

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