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Abstract

An extensive literature in economics uses a continuum of random variables to model individual random shocks imposed on a large population. Let H denote the Hilbert space of square-integrable random variables. A key concern is to characterize the family of all H-valued functions that satisfy the law of large numbers when a large sample of agents is drawn at random. We use the iterative extension of an infinite product measure introduced in [6] to formulate a "sharp" law of large numbers. We prove that an H-valued function satisfies this law if and only if it is both Pettis-integrable and norm integrably bounded.

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1 Introduction

There is a large and growing literature on an economic system with individual uncertainty and many agents. Formally, a continuum of random variables is used to model the random shocks imposed on the large number of agents. The papers [2] and [3] by Al-Najjar, and [9] by Khan and Sun, are closely related to this literature. They raise a key issue, which is to find a necessary and sufficient condition so that "risk in the continuum economy can be consistently estimated from risks in large, randomly drawn samples of agents." An idea recently developed in [6] enables us to offer an answer.

More precisely, consider a bounded function f with values in the Hilbert space $H = L^2(\Omega)$ of square-integrable random variables. Then, is weak measurability enough for f to satisfy the law of large numbers (LLN) when agents are sampled at random? Despite the claim stated as the Main Theorem of [2], a negative answer emerges in [3] and [9].[1]

The corrigendum [3] also suggests "measurability of the covariance structure" (which in [9] is called "joint measurability of the autocorrelation function") as an alternative to "weak measurability". As pointed out in Section 4 of [9], however, this condition is strictly stronger than any of the four conditions (including "weak measurability") in the Main Theorem of [2].[2]

By contrast, it follows from the main result of this note that all four conditions can be made equivalent provided one restates the LLN for H-valued functions appearing as condition (iv) in the Main Theorem of [2]. Specifically, our Theorem 1 characterizes all the H-valued functions satisfying a "sharp" LLN as those which are both Pettis integrable and norm integrably bounded.[3]

To obtain this result, we use the iteratively complete infinite product probability space constructed in [6]. The resulting concept of an iteratively null set is more general than the usual concept of a null set in the corresponding infinite product probability space. The sharp LLN requires the convergence of sample averages only for all sequences outside an iteratively null set.

In the sequel, we introduce the iteratively complete product in Section 2. The main result is stated in Section 3 and proved in Section 4.

2 Iteratively Complete Products

Let $(T_k, \mathcal{T}_k, \lambda_k)$, $k \in \mathbb{N}$ be any sequence of complete and countably additive probability spaces. Then let

$$\mathbb{P}^n := (\prod_{k=1}^{n} T_k, \otimes_{k=1}^{n} \mathcal{T}_k, \otimes_{k=1}^{n} \lambda_k), \quad \mathbb{P}^\infty := (\prod_{k=1}^{\infty} T_k, \otimes_{k=1}^{\infty} \mathcal{T}_k, \otimes_{k=1}^{\infty} \lambda_k)$$

respectively denote the product of the first n probability spaces, and the infinite product of the entire sequence of probability spaces.

We can always assume that the above product probability spaces are complete in the sense that subsets of measure zero are included as measurable sets with zero measure. Nevertheless, a stronger form of "iterative" completion for the products $\mathbb{P}^n$ and $\mathbb{P}^\infty$ was needed for the main result of [6] — namely, the essential equivalence of pairwise and mutual conditional

[1]Specifically, condition (iv) of this Main Theorem is actually strictly stronger than the three equivalent conditions (i)–(iii). Yet in [2] it was claimed that "The force of the theorem lies in establishing the equivalence of four seemingly unrelated aspects of the process f."

[2]That is, the new condition (i$'$) in [3] is even strictly stronger than condition (iv).

[3]Unlike [2], our LLN allows functions that may not be bounded. We note that when an H-valued function is bounded, it is trivially norm integrably bounded, and also Pettis integrable — see Remark 1 below.

independence. This stronger completion includes those "iteratively null" sets which are not product measurable, even though their indicator functions have value zero for iterated integrals of all orders.

Definition 1 *A set $E \subseteq \prod_{k=1}^n T_k$ is said to be* ***iteratively null*** *if for every permutation π on $\{1, \ldots, n\}$, the iterated integral*

$$\int_{t_{\pi(1)} \in T_{\pi(1)}} \cdots \int_{t_{\pi(n)} \in T_{\pi(n)}} 1_E(t_1, t_2, \ldots, t_n)\, d\lambda_{\pi(n)}(t_{\pi(n)}) \ldots d\lambda_{\pi(1)}(t_{\pi(1)}) \tag{1}$$

is well-defined with value zero, where 1_E is the indicator function of the set E in $\prod_{k=1}^n T_k$; in other words, for $\lambda_{\pi(1)}$-a.e. $t_{\pi(1)} \in T_{\pi(1)}$, $\lambda_{\pi(2)}$-a.e. $t_{\pi(2)} \in T_{\pi(2)}$, $\ldots$, $\lambda_{\pi(n)}$-a.e. $t_{\pi(n)} \in T_{\pi(n)}$, one has $(t_1, t_2, \ldots, t_n) \notin E$.

The following two propositions taken from [6] show that one can extend the product probability spaces $\mathbb{P}^n$ and $\mathbb{P}^\infty$ by including all the iteratively null sets, and then forming the iterated completion.

Proposition 1 *Given any $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the family of all iteratively null sets in $\prod_{k=1}^n T_k$. Then there exists a uniquely defined complete and countably additive probability space $\bar{\mathbb{P}}^n = (\prod_{k=1}^n T_k, \bar{\otimes}_{k=1}^n \mathcal{T}_k, \bar{\otimes}_{k=1}^n \lambda_k)$ that satisfies the Fubini property, with:*

1. *$\bar{\otimes}_{k=1}^n \mathcal{T}_k$ as the σ-algebra $\sigma([\otimes_{k=1}^n \mathcal{T}_k] \cup \mathcal{E}_n)$, which is equal to the collection $[\otimes_{k=1}^n \mathcal{T}_k]\, \Delta \mathcal{E}_n := \{\, B \Delta E : B \in \otimes_{k=1}^n \mathcal{T}_k,\ E \in \mathcal{E}_n \,\}$;*
2. *$[\bar{\otimes}_{k=1}^n \lambda_k]\,(B \Delta E) = [\otimes_{k=1}^n \lambda_k]\,(B)$ whenever $B \in \otimes_{k=1}^n \mathcal{T}_k$ and $E \in \mathcal{E}_n$.*

Proposition 2 *There exists a countably additive probability space, denoted by $\bar{\mathbb{P}}^\infty = (\prod_{k=1}^\infty T_k, \bar{\otimes}_{k=1}^\infty \mathcal{T}_k, \bar{\otimes}_{k=1}^\infty \lambda_k)$, in which $\bar{\otimes}_{k=1}^\infty \mathcal{T}_k$ is the σ-algebra generated by the union $\mathcal{G} := \cup_{n=1}^\infty \mathcal{G}_n$ of the families $\mathcal{G}_n$ of cylinder sets taking the form $A \times \prod_{k=n+1}^\infty T_k$ for some $A \in \bar{\otimes}_{k=1}^n \mathcal{T}_k$, whereas $\bar{\otimes}_{k=1}^\infty \lambda_k$ is the unique countably additive extension to this σ-algebra of the set function $\mu : \mathcal{G} \to [0,1]$ defined so that $\mu(A \times \prod_{k=n+1}^\infty T_k) := \bar{\otimes}_{k=1}^n \lambda_k(A)$ for all $A \in \bar{\otimes}_{k=1}^n \mathcal{T}_k$. Moreover, for any $D \in \bar{\otimes}_{k=1}^\infty \mathcal{T}_k$, there exist $B \in \otimes_{k=1}^\infty \mathcal{T}_k$ and $E \in \bar{\otimes}_{k=1}^\infty \mathcal{T}_k$ such that $D = B \Delta E$ and $[\bar{\otimes}_{k=1}^\infty \lambda_k]\,(E) = 0$.*

The countably additive probability space $\bar{\mathbb{P}}^\infty$ will be called the *iterated completion* of $\mathbb{P}^\infty$. It will also be called the *iteratively complete product space.* When all the probability spaces $(T_k, \mathcal{T}_k, \lambda_k)$ $(k \in \mathbb{N})$ are copies of $(T, \mathcal{T}, \lambda)$, let $(T^\infty, \bar{\mathcal{T}}^\infty, \bar{\lambda}^\infty)$ denote the iterated completion of the infinite product probability space, and let t^∞ denote a general element $\{t_n\}_{n=1}^\infty$ of T^∞.

3 The Main Result

Let $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ both be countably additive complete probability spaces. Given any $S \in \mathcal{T}$, let $1_S : T \to \{0, 1\}$ denote the indicator function of the set S. Let H denote any Hilbert space, not only the space $L^2(\Omega)$ of random variables with a finite second moment. Given any $b, b' \in H$, let $\langle b, b' \rangle$ denote their inner product.

Definition 2 *Let f be any function from $(T, \mathcal{T}, \lambda)$ to H.*

1. *The function f is said to be* ***scalarly equivalent to*** *another function g from $(T, \mathcal{T}, \lambda)$ to H if, for any $b' \in H$, the real-valued functions $\langle f(t), b'\rangle$ and $\langle g(t), b'\rangle$ are equal for λ-a.e. $t \in T$.*

2. *The function f is said to satisfy the* ***classical (resp., sharp) law of large numbers*** *if there exists $a \in H$ such that for λ^∞-a.e. (resp., $\bar{\lambda}^\infty$-a.e.) $t^\infty \in T^\infty$, one has $\lim_{n\to\infty} \|a - \frac{1}{n}\sum_{k=1}^{n} f(t_k)\| = 0$. Let $\mathrm{LLN}(H)$ (resp., $\mathrm{SLLN}(H)$ denote the (linear) space of all functions from T to H that satisfy the classical (resp., sharp) law of large numbers.*

3. *The function f is said to be* ***Pettis integrable*** *if there exists $b \in H$ such that, for all $b' \in H$, the real-valued function $\langle f(\cdot), b'\rangle$ on T is λ-integrable, with $\int_T \langle f(t), b'\rangle d\lambda = \langle b, b'\rangle$. Then the vector b is said to be the* ***Pettis integral****.*

4. *The function f is said to be* ***norm integrably bounded*** *if there exists a* **dominant** *λ-integrable function $f^* : T \to \mathbb{R}_+$ for $\|f\|$, in the sense that $\|f(t)\| \leq f^*(t)$ for λ-a.e. $t \in T$.*

From now on, let $\mathcal{L}(H)$ denote the (linear) space of all functions f from $(T, \mathcal{T}, \lambda)$ to H that are both Pettis integrable and norm integrably bounded. It is known in the literature that $\mathrm{LLN}(\lambda^\infty, H) \subseteq \mathcal{L}(H)$ but the equality does not hold in general; see [9] and its references.

The following theorem, however, shows that when the measure λ^∞ is replaced by its iterated completion $\bar{\lambda}^\infty$, not only does the strengthened inclusion $\mathrm{SLLN}(H) \subseteq \mathcal{L}(H)$ hold, but it becomes an equality. This equality provides a very general characterization for a continuum of random variables to satisfy the classical law of large numbers in the iterated completion of the product probability space. Moreover, an obvious corollary of our results is that $\mathrm{LLN}(H)$ is in general a proper subset of $\mathrm{SLLN}(H)$.

Theorem 1 *Let f be a function from T to H. A necessary and sufficient condition for f to satisfy the sharp law of large numbers is that f is Pettis integrable and norm integrably bounded. That is,* $\mathrm{SLLN}(H) = \mathcal{L}(H)$.

Remark 1 *Let f be a norm bounded function from T to H as in [2]. Suppose f is weakly measurable, in the sense that for all $b' \in H$, the real-valued function $t \mapsto \langle f(t), b'\rangle$ on T is $\mathcal{T}$-measurable. Then f must be Pettis integrable, hence $f \in \mathrm{SLLN}(H)$ by Theorem 1. Conversely, if $f \in \mathrm{SLLN}(H)$, then Theorem 1 implies that f is Pettis integrable and hence weakly measurable. It follows that conditions (i)–(iii) of the Main Theorem of [2] are each equivalent to the weaker condition (iv′) $f \in \mathrm{SLLN}(H)$ rather than to the original condition (iv) (which can be stated as $f \in \mathrm{LLN}(H)$).*[4]

4 The Proof

The following two lemmas are taken from [7].

Lemma 1 *For each $n \in \mathbb{N}$, let S_n be a subset of T whose λ-outer measure is one. Then the $\bar{\lambda}^\infty$-outer measure of $\prod_{n=1}^{\infty} S_n$ is also one.*

[4]See also Section 6 of [9] for a detailed discussion of the four conditions.

Lemma 2 *Let g be a real-valued function on T. Suppose there is a real constant c such that*

$$\lim_{n\to\infty} \frac{g(t_1)+\ldots+g(t_n)}{n} = c, \tag{2}$$

for $\bar{\lambda}^\infty$-a.e. $t^\infty \in T^\infty$. Then g is λ-integrable, with $\int_T g(t)d\lambda = c$.

The proof of following lemma adapts some ideas in [8] to the setting of iteratively complete products.

Lemma 3 *If a function f from T to H satisfies the sharp law of large numbers, then it is norm integrably bounded.*

Proof: Let $f \in \mathrm{SLLN}(H)$, with $\|a - \frac{1}{n}\sum_{k=1}^n f(t_k)\| \to 0$ for $\bar{\lambda}^\infty$-a.e. $t^\infty \in T^\infty$. Let D be the set of all $t^\infty \in T^\infty$ such that $\lim_{n\to\infty} \|\frac{1}{n} f(t_n)\| = 0$. Because

$$\frac{1}{n} f(t_n) = -\left[a - \frac{1}{n}\sum_{k=1}^{n} f(t_k)\right] + \frac{n-1}{n}\left[a - \frac{1}{n-1}\sum_{k=1}^{n-1} f(t_k)\right] + \frac{1}{n}a,$$

it is easy to see that $\bar{\lambda}^\infty(D) = 1$.

Let $g(\cdot)$ be the **upper λ-envelope** of $\|f(\cdot)\|$, in the sense that g is a $\mathcal{T}$-measurable function from T to $\mathbb{R}_+ \cup \{\infty\}$ satisfying: (i) $g(t) \geq \|f(t)\|$ for all $t \in T$; (ii) for any $\mathcal{T}$-measurable function h from T to $\mathbb{R}_+ \cup \{\infty\}$, the λ-inner measure of the set $\{t \in T : \|f(t)\| \leq h(t) < g(t)\}$ is zero (see [8, p. 302]). For each $n \in \mathbb{N}$, define $S_n := \{t \in T : g(t) \leq 2\|f(t)\| \text{ or } \|f(t)\| \geq n\}$. Let $h_n(t) := \min\{n,\ g(t)/2\}$ for each $t \in T$. Then the function h_n is $\mathcal{T}$-measurable. Also, it is clear that $\|f(t)\| < h_n(t) < g(t)$ for all $t \in T \setminus S_n$ (even when $g(t) = \infty$). By definition of the upper λ-envelope, therefore, the set $T \setminus S_n$ must have λ-inner measure zero, implying that the λ-outer measure of S_n is one. Lemma 1 says that then the set $\prod_{n=1}^\infty S_n$ also has $\bar{\lambda}^\infty$-outer measure one, and so therefore does $D \cap \prod_{n=1}^\infty S_n$.

Fix any $t^\infty \in D \cap \prod_{n=1}^\infty S_n$. Since $\lim_{n\to\infty} \|\frac{1}{n} f(t_n)\| = 0$, one has $\|f(t_n)\| < n$ for sufficiently large n, and then $t_n \in S_n$ implies that $0 \leq g(t_n) \leq 2\|f(t_n)\|$. Hence, $\lim_{n\to\infty} \frac{1}{n} g(t_n) = 0$. But g is $\mathcal{T}$-measurable by definition, so $\lim_{n\to\infty} \frac{1}{n} g(t_n) = 0$ for all t^∞ in some $\mathcal{T}^\infty$-measurable superset E of $D \cap \prod_{n=1}^\infty S_n$. Since the $\bar{\lambda}^\infty$-outer measure of $D \cap \prod_{n=1}^\infty S_n$ is one, it follows that $\bar{\lambda}^\infty(E) = \lambda^\infty(E) = 1$.

Given any $t^\infty \in T^\infty$, let $\phi(t^\infty) := \sup_{n\in\mathbb{N}} \frac{1}{n} g(t_n)$. Then $\phi(t^\infty)$ is finite for all $t^\infty \in E$. Because g is $\mathcal{T}$-measurable, the function $\phi : T \to \mathbb{R}_+ \cup \{\infty\}$ must be $\mathcal{T}^\infty$-measurable. So there exists a positive integer K such that

$$\lambda^\infty\left(\{\, t^\infty \in T^\infty : \phi\left(t^\infty\right) < K \,\}\right) > 1/2. \tag{3}$$

For each $n \in \mathbb{N}$, let $\alpha_n := \lambda\left(\{t \in T : g(t) \geq nK\}\right)$. Because λ^∞ is a product measure, it is evident that

$$\lambda^\infty\left(\{\, t^\infty \in T^\infty : \phi\left(t^\infty\right) < K \,\}\right) = \prod_{n=1}^{\infty}(1-\alpha_n). \tag{4}$$

Clearly (3) and (4) imply $\prod_{n=1}^\infty (1-\alpha_n) > 1/2$. But $\ln(1-\alpha_n) \leq -\alpha_n$, so

$$\sum_{n=1}^{\infty} \alpha_n \leq -\sum_{n=1}^{\infty} \ln(1-\alpha_n) < -\ln(1/2) = \ln 2 < \infty. \tag{5}$$

This implies that $\lim_{n\to\infty} \alpha_n = 0$, and so $\lambda\left(\{t \in T : g(t) = \infty\}\right) = 0$.

Given any fixed $t \in T$ with $g(t) < \infty$, let m be the smallest integer such that $g(t) < mK$. Then $g(t) \in [nK, \infty)$ for $n \in \{1, \ldots, m-1\}$, and so $\sum_{n=1}^{\infty} 1_{[nK,\infty)}(g(t)) = m - 1$. It follows that

$$g(t) \leq K + K \sum_{n=1}^{\infty} 1_{[nK,\infty)}(g(t)) \tag{6}$$

for all $t \in T$ with $g(t) < \infty$. Because $\lambda(\{t \in T : g(t) = \infty\}) = 0$, the definition of α_n implies that $\int_T 1_{[nK,\infty)}(g(t))d\lambda = \alpha_n$. It follows from (5) and (6) that $\int_T g d\lambda \leq K + K \sum_{n=1}^{\infty} \alpha_n < K(1 + \ln 2) < \infty$.

Finally, let f^* be the function from T to $\mathbb{R}_+$ such that $f^*(t) = g(t)$ when $g(t) < \infty$ and $f^*(t) = 0$ when $g(t) = \infty$. Clearly f^* is a dominant λ-integrable function for $\|f\|$, so f is norm integrably bounded. ∎

Lemma 4 *Let $f \in \mathcal{L}(H)$ be scalarly equivalent to the zero function. Then $f \in \mathrm{SLLN}(H)$.*

Proof: Let g be a dominant λ-integrable function for $\|f\|$. For each $k \in \mathbb{N}$, let X_k be the random variable defined on $(T^\infty, \mathcal{T}^\infty, \lambda^\infty)$ by $X_k(t^\infty) := [g(t_k)]^2$. Since $\mathbb{E}X_1^{1/2} < \infty$ and the variables X_k are i.i.d., the Marcinkiewicz–Zygmund Theorem for the case $p = 1/2$ and $c = 0$ (see [4], p. 125) implies that $n^{-2} \sum_{k=1}^{n} X_k(t^\infty) \to 0$ for λ^∞-a.e. $t^\infty \in T^\infty$. Because $\|f(\cdot)\| \leq g(\cdot)$, we have $n^{-2} \sum_{k=1}^{n} \|f(t_k)\|^2 \to 0$ for λ^∞-a.e. $t^\infty \in T^\infty$.

For any $t^\infty \in T^\infty$, we have

$$\left\| \frac{1}{n} \sum_{k=1}^{n} f(t_k) \right\|^2 \leq \frac{1}{n^2} \sum_{k=1}^{n} \|f(t_k)\|^2 + \frac{2}{n^2} \sum_{1 \leq j < k \leq n} |\langle f(t_j), f(t_k) \rangle| . \tag{7}$$

Because f is scalarly equivalent to zero, for any $h \in H$ one has $\langle f(t), h \rangle = 0$ for λ-a.e. $t \in T$. In particular, for any $t' \in T$, one has $\langle f(t), f(t') \rangle = 0$ for λ-a.e. $t \in T$. Hence there exists a $\mathcal{T} \bar{\otimes} \mathcal{T}$-measurable set $D \subseteq T \times T$ such that $(\bar{\lambda} \times \bar{\lambda})(D) = 1$ and $\langle f(t), f(t') \rangle = 0$ for all $(t, t') \in D$. For each pair $j, k \in \mathbb{N}$, let D_{jk} denote the set of all sequences $t^\infty \in T^\infty$ such that $(t_j, t_k) \in D$, and define $D^* := \cap_{j=1}^{\infty} \cap_{k=j+1}^{\infty} D_{jk}$. Then for all $t^\infty \in D^*$ one has $\langle f(t_j), f(t_k) \rangle = 0$ for all $j, k \in \mathbb{N}$ with $j < k$. Obviously $\bar{\lambda}^\infty(D_{jk}) = 1$ for each $j, k \in \mathbb{N}$, so $\bar{\lambda}^\infty(D^*) = 1$ also.

From the results in the last two paragraphs, (7) implies that, as $n \to \infty$, so $\|\frac{1}{n} \sum_{k=1}^{n} f(t_k)\| \to 0$ for $\bar{\lambda}^\infty$-a.e. $t^\infty \in T^\infty$. Hence $f \in \mathrm{SLLN}(H)$. ∎

Proof of Theorem 1:
First, we prove necessity. Let f be a function from T to H such that, for some $a \in H$, one has $\lim_{n\to\infty} \|a - \frac{1}{n} \sum_{k=1}^{n} f(t_k)\| = 0$ for $\bar{\lambda}^\infty$-a.e. $t^\infty \in T^\infty$. Take any fixed $b' \in H$, and let $g(\cdot)$ be the real-valued function $\langle f(\cdot), b' \rangle$ on T. A routine calculation shows that $\lim_{n\to\infty} \frac{1}{n}[g(t_1) + \ldots + g(t_n)] = \langle a, b' \rangle$ for $\bar{\lambda}^\infty$-a.e. $t^\infty \in T^\infty$. Then Lemma 2 implies that g is λ-integrable, with $\int_T g(t)d\lambda = \langle a, b' \rangle$. Hence, f is Pettis integrable and has a as its Pettis integral. Lemma 3 implies that f is also norm integrably bounded.

Second, we prove sufficiency. Let f be any function in $\mathcal{L}(H)$. Since f is Pettis integrable, so is the function $1_S(\cdot)f(\cdot)$ for each $S \in \mathcal{T}$ — see [1, Theorem 11.51] or [5, p. 52]. Let $\nu(S)$ denote the H-valued Pettis integral of $1_S(\cdot)f(\cdot)$. It follows that $\|\nu(S)\|^2 = \langle \nu(S), \nu(S) \rangle = \int_T \langle (1_S f)(t), \nu(S) \rangle d\lambda$. By hypothesis, there exists a λ-integrable function $f^* : T \to \mathbb{R}_+$ such that $\|f(t)\| \leq f^*(t)$ for λ-a.e. $t \in T$, and so $\langle (1_S f)(t), \nu(S) \rangle \leq (1_S f^*)(t)\|\nu(S)\|$. Hence $\|\nu(S)\|^2 \leq \int_T (1_S f^*)(t)\|\nu(S)\| d\lambda$. So even when $\|\nu(S)\| = 0$, one has

$$\|\nu(S)\| \leq \int_S f^*(t) d\lambda. \tag{8}$$

Let $T_1, T_2, \ldots, T_n$ be any partition of T into n pairwise disjoint $\mathcal{T}$-measurable subsets. Then (8) implies that

$$\sum_{k=1}^{n} \|\nu(T_k)\| \leq \sum_{k=1}^{n} \int_{T_k} f^*(t) d\lambda = \int_T f^*(t) d\lambda < +\infty.$$

It follows that ν is an H-valued σ-additive measure of bounded variation. Moreover, (8) also implies that the vector measure ν is absolutely continuous w.r.t. λ.

Next, we shall show that f is scalarly equivalent to a Bochner integrable function ϕ from $(T, \mathcal{T}, \lambda)$ to H. Because H, as a reflexive Banach space, has the Radon–Nikodym property (see [5, p. 82]), there exists a Bochner integrable function ϕ from $(T, \mathcal{T}, \lambda)$ to H such that $\nu(S)$ is the Bochner integral $\int_S \phi(t) d\lambda$ for each $S \in \mathcal{T}$. Now the Bochner integral, when it exists, must equal the Pettis integral (see, for example, [1, p. 423]). So given any $h \in H$, it follows that

$$\langle \nu(S), h \rangle = \int_S \langle \phi(t), h \rangle d\lambda = \int_S \langle f(t), h \rangle d\lambda.$$

Because the choice of $S \in \mathcal{T}$ was arbitrary, one has $\langle f(t), h \rangle = \langle \phi(t), h \rangle$ for λ-a.e. $t \in T$. That is, f is scalarly equivalent to ϕ.[5]

Define $\psi := f - \phi$. Because ϕ is Bochner integrable, it follows from [5, p. 45] that $\|\phi\|$ is integrable. Clearly, then, ψ is norm integrably bounded, Pettis integrable, and scalarly equivalent to zero. So Lemma 4 implies that ψ is in SLLN(H). The classical law of large numbers for Bochner integrable functions (see [8]) says that ϕ is in LLN(H), and thus in SLLN(H) as well. Therefore $f = \phi + \psi \in$ SLLN(H). ■

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[5] The argument used in this paragraph is essentially the same as the simple argument on [5, p. 89], where the case of norm bounded functions is considered.