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DEPARTEMENT D'ECONOMIE

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Abstract: In this paper, we consider optimal reinsurance from an insurer's point of view. Given a (low) ruin probability target, insurers want to find the optimal risk transfer mechanism, i.e. either a proportional or a nonproportional reinsurance treaty. Since it is usually admitted that reinsurance should lower ruin probabilities, it should be easy to derive an efficient Monte Carlo algorithm to link ruin probability and reinsurance parameter. Unfortunately, if it is possible for proportional reinsurance, this is no longer the case in nonproportional reinsurance. Some examples where reinsurance might increase ruin probabilities are given at the end, when claim arrival and claim size are not independent.

Keywords: Dependence; Reinsurance; Ruin probability; Solvency requirements

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Reinsurance, ruin and solvency issues: some pitfalls

Arthur Charpentier[1]

Abstract

In this paper, we consider optimal reinsurance from an insurer's point of view. Given a (low) ruin probability target, insurers want to find the optimal risk transfer mechanism, i.e. either a proportional or a nonproportional reinsurance treaty. Since it is usually admitted that reinsurance should lower ruin probabilities, it should be easy to derive an efficient Monte Carlo algorithm to link ruin probability and reinsurance parameter. Unfortunately, if it is possible for proportional reinsurance, this is no longer the case in nonproportional reinsurance. Some examples where reinsurance might increase ruin probabilities are given at the end, when claim arrival and claim size are not independent.

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1 Introduction and motivation

Reinsurance can be defined as the transfer of risk from a direct insurer, the cedant, to a second insurance carrier, the reinsurer. Basically, if an individual risk is too big for insurance company, or if the potential loss of the entire portfolio is too heavy, the insurance company can buy a reinsurance protection.

There should be links between reinsurance and ruin probabilities. For instance, (15) "*focuses on the impact that risk transfer instruments, such as reinsurance and catastrophe bonds, have on the performance of insurers. [...] A typical insurance company's goal is to operate under two somewhat conflicting constraints: a safety first constraint and a return on assets constraint. The first relates to both a target ruin probability level and a target insolvency level; the second is to satisfy the firm's shareholders and investors.*" Assume that an insurer sets a target ruin probability (over a given time horizon) - based on its risk appetite - but if the firm cannot meet this level of insolvency risk (with a given strategy), then "*it must take steps to reduce the amount of risk in its portfolio*" and a natural technique considered in (15) is to "*purchase reinsurance.*

Some details can be found in recent literature. As mentioned in (12), "*reinsurance plays an important role in reducing the risk in an insurance portfolio.*" (21) claimed that "*if we choose ruin probability as a risk measure, the goal of reinsurance is to reduce this probability to a certain chosen level*". And finally, (9) is even more specific about risk reduction: "*reinsurance is able to offer additional underwriting capacity for cedants, but also to reduce the probability of a direct insurer's ruin*".

In this paper, we will discuss the later, showing that this statement is a standard pitfall and that reinsurance can actually increase ruin probability, which should lead risk managers or regulators to be more cautious about reinsurance plans of (re)insurance companies.

Hence, we formulate the solvency problem as follows: given a ruin probability target *and* a capital amount u, could it be possible to design a reinsurance treaty (either proportional or nonproportional) with optimal parameter $\theta^\star$? An more precisely, is there a simple relationship between ruin probability, economic capital u and reinsurance ? In Section 2, classical results on that problem will briefly be recalled, mainly based not on ruin probability, but on bounds for ruin probability. Then in Sections 3 and 4, proportional and nonproportional reinsurance treaties will be consided. As we will see, in the proportional case, reinsurance always lower ruin probabilities and a simple Monte Carlo algorithm can be designed. Unfortunately, this will not be the case for nonproportional reinsurance since reinsurance can actually increase ruin probability. Simple examples will hep to understand how this counterintuitive result can be obtained.

2 The standard mathematical framework

A reinsurance treaty, is an '*equitable*' transfer of the risk of a risk, from one entity (the insurance company, also called ceding company) to another (the reinsurance company), in exchange for a fee called premium. The reinsurance company repays a function of the paid loss.

- Proportional reinsurance: Quota-Share

The quota-share treaty is a treaty between the ceding company and the reinsurer to share premiums (paid by insured to the insurance company) and losses (claimed by the insured) with the same proportion, i.e. for some (fixed) $\alpha \in (0, 1]$

$$\begin{cases} \text{claim loss } X: \alpha X \text{ paid by the cedant, } (1-\alpha)X \text{ paid by the reinsurer,} \\ \text{premium } P: \alpha P \text{ kept by the cedant, } (1-\alpha)P \text{ transfered to the reinsurer.} \end{cases}$$

- Nonproportional reinsurance: Excess-fo-Loss

The excess-of-loss treaty is a treaty which only responds if the loss suffered by the insurer exceeds a certain amount, 'retention' (upper limit), 'priority' or 'deductible' (denoted d).

$$\begin{cases} \text{claim loss } X: \min\{X, d\} \text{ paid by the cedant, } \max\{0, X - d\} \text{ paid by the reinsurer,} \\ \text{premium } P: P_d \text{ kept by the cedant (the net premium), } P - P_d \text{ transfered to the reinsurer.} \end{cases}$$

We consider here the classical Cramér-Lundberg framework (see e.g. (4), (11) or (2)), i.e. assume that

A1 claims arrival is driven by an homogeneous Poisson process, $N_t \sim \mathcal{P}(\lambda t)$, so that durations between consecutive arrivals $T_{i+1} - T_i$ are independent $\mathcal{E}(\lambda)$ random variables,

A2 claim size $X_1, \cdots, X_n, \cdots$ are i.i.d. non-negative random variables,

A3 claim size is independent of the claim arrival process.

If the X_i's have finite exponential moments, claims are said to have *light tails*, while if the X_i's do not have finite exponential moments, claims have *heavy tails*. And finally let $Y_t = \sum_{i=1}^{N_t} X_i$ denote the aggregate amount of claims during period $[0, t]$.

The "*pure premium*" for a given period of time (usually one year) is the expected value of the aggregate loss. Hence, the pure premium required over period $[0, t]$ is

$$\pi_t = \mathbb{E}(Y_t) = \mathbb{E}(N_t)\mathbb{E}(X) = \underbrace{\lambda\mathbb{E}(X)}_{\pi} t.$$

Note that more general premiums can be considered, e.g.

- safety loading proportional to the pure premium, $\pi_t = [1 + \lambda] \cdot \mathbb{E}(Y_t)$,

- safety loading proportional to the variance, $\pi_t = \mathbb{E}(Y_t) + \lambda \cdot Var(Y_t)$,

- safety loading proportional to the standard deviation, $\pi_t = \mathbb{E}(Y_t) + \lambda \cdot \sqrt{Var(Y_t)}$,

- entropic premium (exponential expected utility) $\pi_t = \dfrac{1}{\alpha} \log\left(\mathbb{E}(e^{\alpha Y_t})\right)$,

- Esscher premium $\pi_t = \dfrac{\mathbb{E}(X \cdot e^{\alpha Y_t})}{\mathbb{E}(e^{\alpha Y_t})}$,

- Wang distorted premium $\pi_t = \displaystyle\int_0^\infty \Phi\left(\Phi^{-1}\left(\mathbb{P}(Y_t > x)\right) + \lambda\right) dx$,

In this paper, as in the classical Cramér-Lundberg model, assume that

A4 premium π_t is a linear function, i.e. $\pi_t = \pi t$ where π is the premium per unit of time (say one year).

The general expression is that $\pi_t = \varphi(Y_t)$. Among standard assumptions on the premium principle, recall the following,

A5 the premium is homogeneous (of order 1), i.e. $\varphi(\lambda X) = \lambda\varphi(X)$, for all $\lambda > 0$.

A6 there is a *safety loading*, i.e. $\varphi(X) \geq \mathbb{E}(X)$.

The classical solvency problem is the following: given a ruin probability target, e.g. 0.1%, find capital u such that,

$$\psi(T, u) = 1 - \mathbb{P}(u + \pi t \geq Y_t, \forall t \in [0, T]) = 1 - \mathbb{P}(S_t \geq 0, \forall t \in [0, T]) = \mathbb{P}(\inf\{S_t, t \in [0, T]\} < 0) = 0.1\%,$$

where $S_t = u + \pi t - Y_t$ denotes the insurance company surplus. After reinsurance, the net surplus is then

$$S_t^{(\theta)} = u + \pi^{(\theta)} t - \sum_{i=1}^{N_t} X_i^{(\theta)},$$

where $\pi^{(\theta)}$ is the net premium, and

$$\begin{cases} X_i^{(\theta)} = \theta X_i, & \theta \in [0, 1], \text{ for quota share treaties, denoted } \alpha \text{ afterward} \\ X_i^{(\theta)} = \min\{\theta, X_i\}, & \theta > 0, \text{ for excess-of-loss treaties, denoted } d \text{ afterward.} \end{cases}$$

In practice, ruin probability is usually a target imposed by the shareholders, regulatory administrations, or the market (and competitors). In obtain to obtain a *AAA* ranking from ranking agencies (namely Moodys, Standard & Poor's or Fitch IBCA), or a *AA* ranking (quality borrowers, a bit higher risk than *AAA*), the default (or ruin) probability should be lower than 0.03% over one year (or from 0.02% to 0.06% over 5 years). Hence, we are dealing with extremely rare events. Instead of targeting a ruin probability level, (6) and Chapter 9 in (8) suggest to target an upper bound of the ruin probability. In the case of light tailed claims, it is possible to obtain an upper bound for ruin probability: let γ denote the '*adjustment coefficient*', defined as the unique positive root of

$$\lambda + \pi\gamma = \lambda M_X(\gamma), \text{ where } M_X(t) = \mathbb{E}(\exp(tX)).$$

The Lundberg inequality (see Section 7.6 in (8)) states that

$$0 \leq \psi(T, u) \leq \psi(\infty, u) \leq \exp[-\gamma u].$$

(10) proposed an improvement in the case of finite horizon ($T < \infty$). (6) studied the impact of reinsurance treaties on those two upper bounds in the case of exponential claims.

Instead of upper bounds, it is also possible to consider approximations of ruin probability. (3) and (22) proposed some approximations in the case where $\mathbb{E}(|X|^3) < \infty$. But those approximation are valid only when $u \to \infty$. Finally, using some Gaussian approximation (14) proved that the cedant's non-ruin probability decreases with the deductible. But in this paper the goal is to focus on the exact ruin probability level (instead of an upper bound).

3 Proportional reinsurance

With proportional reinsurance, if $1 - \alpha$ is the ceding ratio, the surplus process for the company is

$$S_t^{(\alpha)} = u + \pi_\alpha t - \sum_{i=1}^{N_t} \alpha X_i \tag{1}$$

with $\alpha \in [0, 1]$. Without further assumptions (such as A1, A2 or A3), we can prove that reinsurance can always lower ruin probability.

Proposition 3.1. *In the most general model with an homogeneous premium principle (A4), consider a proportional reinsurance treaty with quota share α, then ruin probability is decreasing with α.*

Proof. From equation 1, if the premium principle is homogeneous, then $\pi_\alpha = \alpha\pi$, and therefore

$$\begin{aligned} \psi(u, T, \alpha) &= \mathbb{P}(\exists t \in [0, T], S_t^{(\alpha)} < 0) = \mathbb{P}(\exists t \in [0, T], (1 - \alpha)u + \alpha S_t < 0) \\ &\leq \mathbb{P}(\exists t \in [0, T], \alpha S_t < 0) = \psi(u, T). \end{aligned}$$

$\square$

Assuming that there was ruin (without reinsurance) before time T, if the insurance had ceded a proportion $1 - \alpha^*$ of its business, where

$$\alpha^* = \frac{u}{u - \inf\{S_t, t \in [0, T]\}},$$

there would have been no ruin (at least on the period $[0, T]$). If $\psi(T, u, \alpha)$ denote the ruin probability associated to reinsurance cover α, then

$$\psi(T, u, \alpha) = \psi(T, u) \cdot \mathbb{P}(\alpha^* \leq \alpha | \text{ruin}).$$

From a numerical point of view, one need to estimate ruin probability without reinsurance, and the distribution of α^* given that there was a ruin at time $\tau \in [0, T]$. More generally, if

$$\alpha^* = \frac{u}{u - \min\{S_t, t \in [0, T]\}} \mathbf{1}(\min\{S_t, t \in [0, T]\} < 0) + \mathbf{1}(\min\{S_t, t \in [0, T]\} \geq 0),$$

$$\psi(T, u, \alpha) = \psi(T, u) \cdot \mathbb{P}(\alpha^* \leq \alpha).$$

The optimal (ex-post) reinsurance program can be visualized on Figure 1.

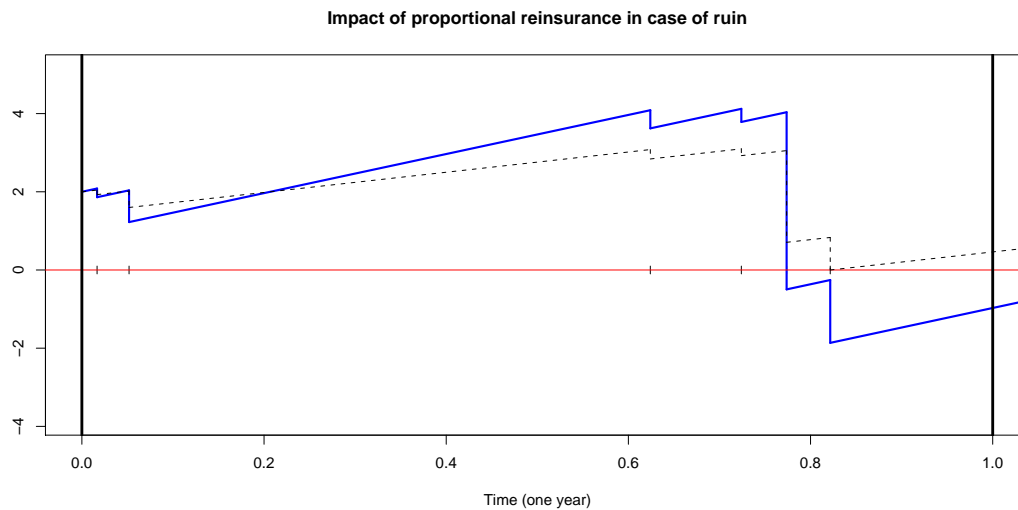**Impact of proportional reinsurance in case of ruin**

Figure 1: Proportional reinsurance helps to decrease ruin probability, the plain line is the gross surplus, and the dotted line the cedant surplus with a reinsurance treaty.

In that case, the algorithm to plot the ruin probability as a function of the reinsurance share is simply the following

```
RUIN <- 0; ALPHA <- NA
for(i in 1:Nb.Simul){
  T <- rexp(N,lambda); T <- T[cumsum(T)<1]; n <- length(T)
  X <- r.claims(n); S <- u+premium*cumsum(T)-cumsum(X)
  if(min(S)<0) { RUIN <- RUIN +1
                 ALPHA <- c(ALPHA,u/(u-min(S))) }
  }
  rate <- seq(0,1,by=.01); proportion <- rep(NA,length(rate))
for(i in 1:length(rate)){
proportion[i]=sum(ALPHA<rate[i])/length(ALPHA)
}
plot(rate,proportion*RUIN/Nb.Simul)
```

The relationship between ruin probability and the cedant's share can be visualized on Figure 2. Since proportional reinsurance can only lower ruin probability, it is natural to observe an increasing function. Further, ruin probability is null when the cedant share is 0 (all the risk is transfered, and the initial

capital is still u). On that numerical example, if the cedant company want to have a ruin probability half lower than the one without any reinsurance treaty, the insurance company should keep 1/3 of the risk when claims are Pareto distributed with tail index close to 1 (extremely heavy tails) and 4/5 of the risk when claims are Pareto distributed with tail index equal to 3 (lighter tails). Again, those proportions are coherent with intuition: with heavy tailed claims, proportional reinsurance does not reduced ruin probability very efficiently.
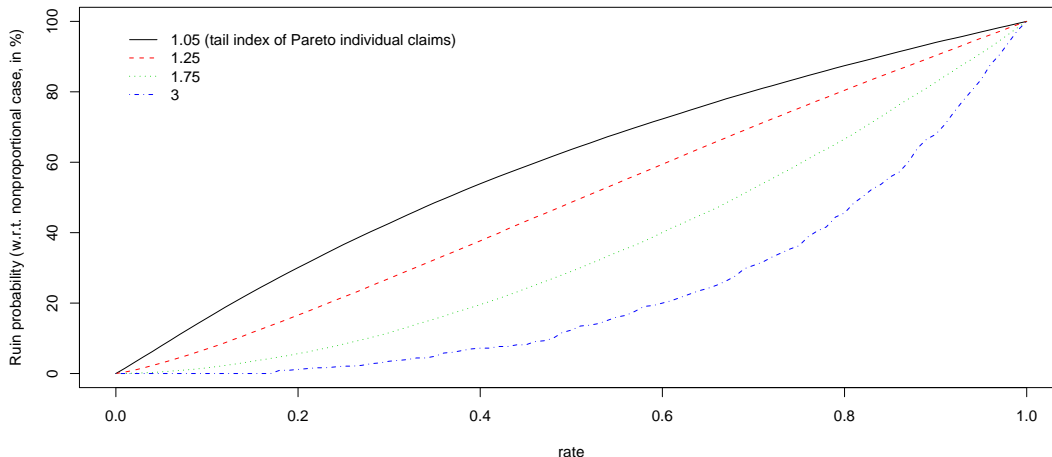

Figure 2: Ruin probability as a function of the cedant's share, for different tails for claim size.

4 Nonproportional reinsurance

With nonproportional reinsurance, if $d \geq 0$ is the priority of the reinsurance contract, the surplus process for the company is

$$S_t^{(d)} = u + \pi^{(d)}t - \sum_{i=1}^{N_t} \min\{X_i, d\} \text{ where } \pi^{(d)} = \mathbb{E}(S_1^{(d)}) = \mathbb{E}(N_1) \cdot \mathbb{E}(\min\{X_i, d\}).$$

Here the problem is that it is possible to have a lot of small claims (smaller than d), and to have ruin with the reinsurance cover (since $p^{(d)} < p$ and $\min\{X_i, d\} = X_i$ for all i if claims are no very large), while there was no ruin without the reinsurance cover (see Figure 3).

In the two following section, we will show that it is possible to increase ruin probability with reinsurance, in the case of a nonhomogneous Poisson process, and dependent claim size.

4.1 Nonhomogeneous Poisson process and dependent claim size

Nonhomogeneous Poisson processes have been considered in the context of ruin probability calculation e.g. in (18) or (13). Hence, a first possible extension is a two assume different periods, with different λ_t.

For instance, we consider a two periods, and a Poisson process with intensity λ_1 on the first period (say $[0, T/2)$) and λ_2 on the second period (say $[T/2, T]$). It is also possible to assume that claim size have distribution F_1 on period 1 and F_2 on period 2.

Remark 4.1. *This model can be interesting e.g. when dealing with hurricanes. The 'hurricane season' is from May till October, and during this period, the intensity is much higher (see (20)). This can be used also for car insurance, since there are usually more claims in winter than in summer.*

Dependence between claim sizes and claim arrivals has been motivated in (1) for instance. It can be introduced easily based on *conditional independence*: given the period of arrival, the claim size distribution is either F_1 or F_2, or more precisely here F_{θ_1} and F_{θ_2} (where the two distributions are in the same parametric family, where we assume, for convenience that θ denotes the expected value).

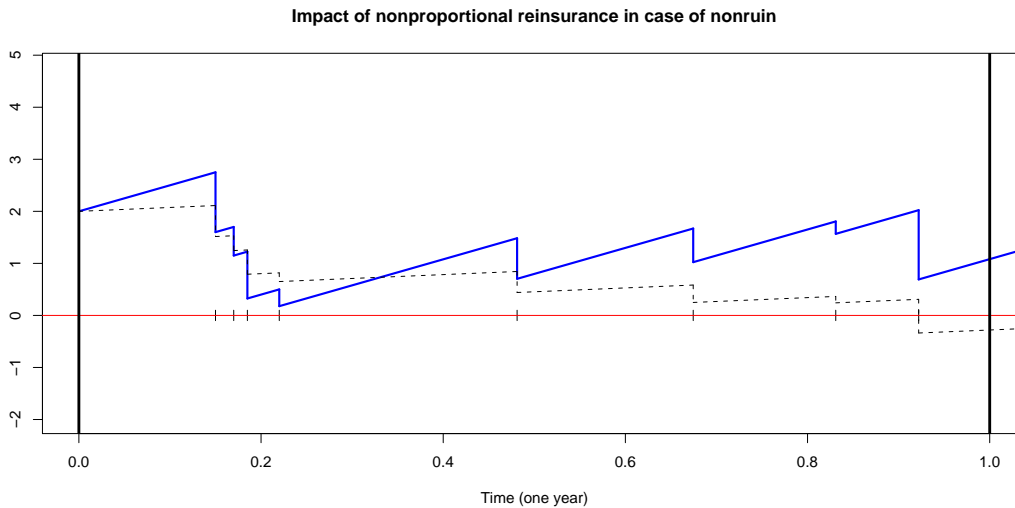**Impact of nonproportional reinsurance in case of nonruin**

Figure 3: Case where nonproportional reinsurance can cause ruin, the plain line is the gross surplus, and the dotted line the cedant surplus with a reinsurance treaty.

The case $(\lambda_1 > \lambda_2)$ and $(\theta_1 > \theta_2)$ corresponds to

$$\begin{cases} \text{period 1: a lot of "large" claims} \\ \text{period 2: a small number of "small" claims} \end{cases}$$

while the case $(\lambda_1 > \lambda_2)$ and $(\theta_1 < \theta_2)$ corresponds to

$$\begin{cases} \text{period 1: a lot of "small" claims} \\ \text{period 2: a small number of "large" claims} \end{cases}$$

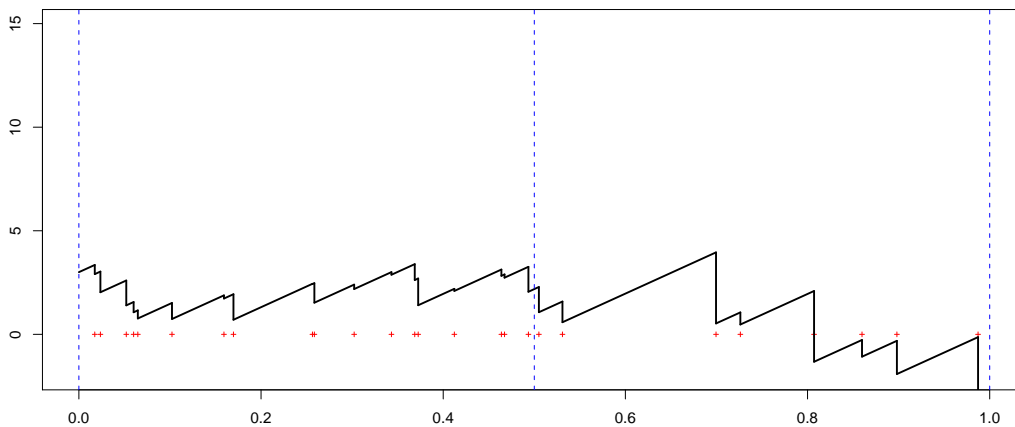A path generations of the later case can be visualized on Figure 4.1.

Figure 4: Nonhonomgenous Poisson process, and claim size dependent on claim arrival process.

Consider the following case

A1' claims arrival is driven by an nonhomogeneous Poisson process, with two periods, 1 and 2, and parameters $\lambda_1 = 1.5\lambda$ and $\lambda_2 = 0.5\lambda$ respectively, with $\lambda = 20$,

A2' claim size are bounded distributions, either uniform or Beta, with distributions F_1 during period 1 and F_2 during period 2, where F_1 is the uniform distribution over $[0, 4/3]$ and F_2 is the uniform distribution over $[0, 4]$.

A3' claim size is dependent of the claim arrival process (dependence by mixture)

For convenience, assume that the premium is here the pure premium[2]. For the numerical application on Figure 4.1, we assume that u represent 15% of the yearly (gross) premium.
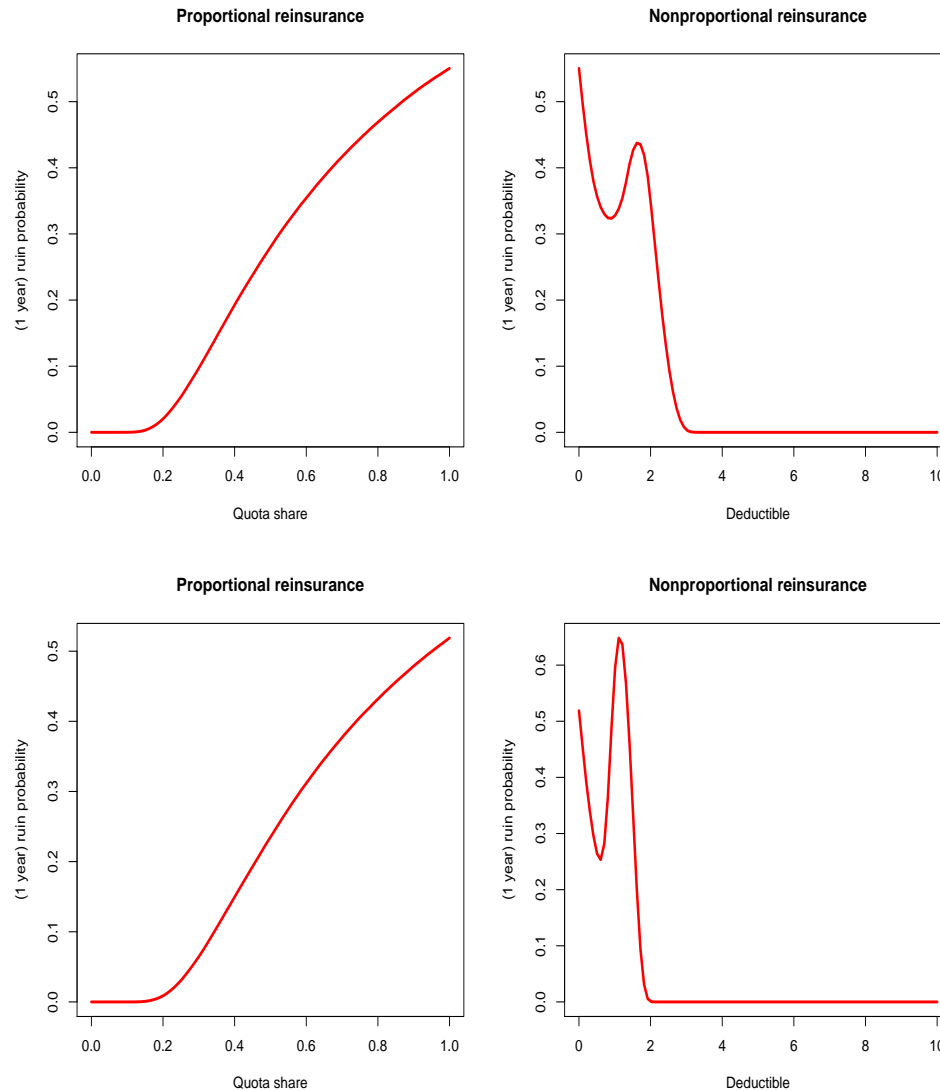

Figure 5: Nonhonomgenous Poisson process, and bounded claim size (uniform on top and beta below), dependent on claim arrival process.

On this example, we see clearly that increasing the deductible from 1 to 2 is very risky, since the net premium decreases more than the net losses, which increases ruin probability. Note that this result have been obtained with small changes in the values of the Poisson process or the distribution of the claim size (but still being bounded). Hence, when losses and arrivals are correlated, reinsurance can increase ruin probability and lower the solvency of the insurance company.

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[2]Since claim arrival and size are not independent, we cannot use the standard formula stating that the pure premium is the product of frequency and the average cost. Here the pure premium is λ.

4.2 Heterogeneous Poisson process and dependent claim size

A second extension is obtained when the claims arrival process is an heterogeneous Poisson process, and a mixture for claim size. Consider an heterogeneity variable Θ. Given Θ, we consider a classical Carmér-Lundberg model, with intensity λ_θ for the Poisson process, and distribution size F_θ.

Remark 4.2. *Still in the context of hurricanes or large storms, (17) mention that 'Doubly periodic non-homogeneous Poisson models' can be considered: the short term periodicity is the seasonal effect mentioned in the previous section, but we can also consider a long term periodicity, leading to years with a lot of hurricanes, and years with no hurricanes.*

Consider the following case

A1" claims arrival is driven by an heterogeneous Poisson process, i.e. $\lambda_1 = 1.5\lambda$ and $\lambda_2 = 0.875\lambda$, with $\lambda = 20$, with probabilities 20% and 80% respectively, when Θ is either equal to 1 or 2,

A2" claim size are bounded distributions, either uniform or Beta, with distributions F_1 when Θ equals 1 and F_2 when Θ equals 2, where F_1 is the uniform distribution over $[0, 7/2]$ and F_2 is the uniform distribution over $[0, 13/8]$.

A3" claim size is dependent of the claim arrival process (dependence by mixture)

For convenience, assume that the premium is here the pure premium. For the numerical application on Figure 4.2, we assume that u represent 15% of the yearly (gross) premium.
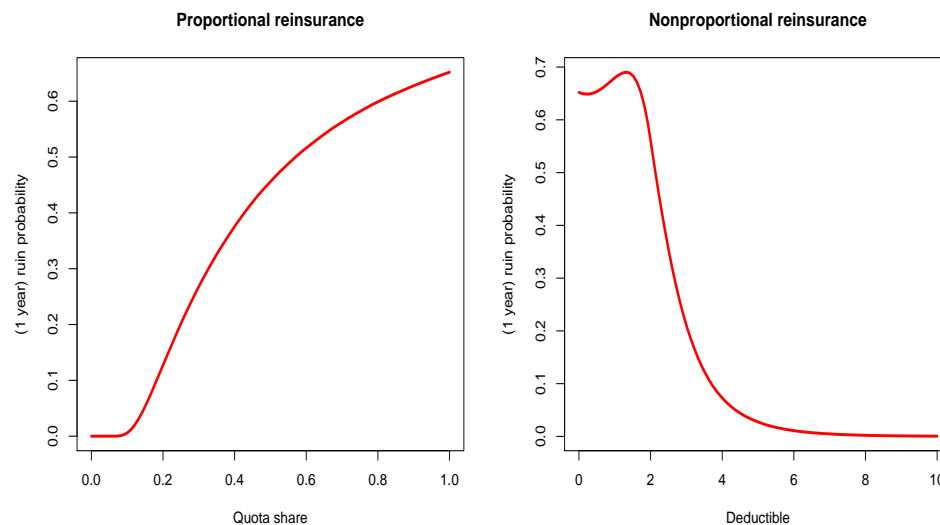

Figure 6: Heterogeneous Poisson process, and bounded claim size (uniform), dependent on claim arrival process.

Similarly here, heterogeneity and dependence can lead to the nonintuitive case where reinsurance leads to a more risky portfolio.

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