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On the vertices of the k-additive core

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Abstract

The core of a game v on N, which is the set of additive games ϕ dominating v such that $\phi(N) = v(N)$, is a central notion in cooperative game theory, decision making and in combinatorics, where it is related to submodular functions, matroids and the greedy algorithm. In many cases however, the core is empty, and alternative solutions have to be found. We define the k-additive core by replacing additive games by k-additive games in the definition of the core, where k-additive games are those games whose Möbius transform vanishes for subsets of more than k elements. For a sufficiently high value of k, the k-additive core is nonempty, and is a convex closed polyhedron. Our aim is to establish results similar to the classical results of Shapley and Ichiishi on the core of convex games (corresponds to Edmonds' theorem for the greedy algorithm), which characterize the vertices of the core.

Key words: cooperative games; core; k-additive games; vertices

1 Introduction

Given a finite set N of n elements, and a set function $v : 2^N \to \mathbb{R}$ vanishing on the empty set (called hereafter a *game*), its *core* $\mathcal{C}(v)$ is the set of additive set

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functions ϕ on N such that $\phi(S) \geq v(S)$ for every $S \subseteq N$, and $\phi(N) = v(N)$. Whenever nonempty, the core is a convex closed bounded polyhedron.

In many fields, the core is a central notion which has deserved a lot of studies. In cooperative game theory, it is the set of imputations for players so that no subcoalition has interest to form [18]. In decision making under uncertainty, where games are replaced by *capacities* (monotonic games), it is the set of probability measures which are coherent with the given representation of uncertainty [19]. More on a combinatorial point of view, cores of convex games are also known as base polytopes associated to supermodular functions [13,9], for which the greedy algorithm is known to be a fundamental optimization technique. Many studies have been done along this line, e.g., by Faigle and Kern for the matching games [8], and cost games [7]. In game theory, which will be our main framework here, related notions are the selectope [3], and the Shapley value with many of its variations on combinatorial structures (see, e.g., [1]).

It is a well known fact that the core is nonempty if and only if the game is balanced [4]. In the case of emptiness, an alternative solution has to be found. One possibility is to search for games more general than additive ones, for example k-additive games and capacities proposed by Grabisch [10]. In short, k-additive games have their Möbius transform vanishing for subsets of more than k elements, so that 1-additive games are just usual additive games. Since any game is a k-additive game for some k (possibly $k = n$), the k-additive core, i.e., the set of dominating k-additive games, is never empty provided k is high enough. The authors have justified this definition in the framework of cooperative game theory [15]. Briefly speaking, an element of the k-additive core implicitely defines by its Möbius transform an imputation (possibly negative), which is now defined on groups of at most k players, and no more on individuals. By definition of the k-additive core, the total worth assigned to a coalition will be always greater or equal to the worth the coalition can achieve by itself; however, the precise sharing among players has still to be decided (e.g., by some bargaining process) among each group of at most k players.

In game theory, elements of the core are imputations for players, and thus it is natural that they fulfill monotonicity. We call monotonic core the core restricted to monotonic games (capacities). We will see in the sequel that the core is usually unbounded, while the monotonic core is not.

The properties of the (classical) core are well known, most remarkable being the result characterizing the core of convex games, where the set of vertices is exactly the set of additive games induced by maximal chains (or equivalently by permutations on N) in the Boolean lattice $(2^N, \subseteq)$. This has been shown by Shapley [17], and later Ichiishi proved the converse implication [12]. This result is also known in the field of matroids, since vertices of the base poytope can be found by a greedy algorithm.

A natural question arises: is it possible to generalize the Shapley-Ichiishi theorem for k-additive (monotonic) cores? More precisely, can we find the set of vertices for some special classes of games? Are they induced by some generalization of maximal chains? The paper shows that the answer is more complex than expected. It is possible to define notions similar to permutations and maximal chains, so as to generate vertices of the k-additive core of $(k+1)$-monotone games, a result which is a true generalization of the Shapley-Ichiishi theorem, but this does not permit to find all vertices of the core. A full analytical description of vertices seems to be difficult to find, but we completely explicit the case $k = n - 1$.

After a preliminary section introducing necessary concepts, Section 3 presents our basic ingredients, that is, orders on subsets of at most k elements, and achievable families, which play the role of maximal chains in the classical case. Then Section 4 presents the main result on the characterization of vertices for

$(k+1)$-monotone games induced by achievable families.

2 Preliminaries

Throughout the paper, $N := \{1, \ldots, n\}$ denotes a set of n elements (players in a game, nodes of a graph, etc.). We use indifferently 2^N or $\mathcal{P}(N)$ for denoting the set of subsets of N, and the set of subsets of N containing at most k elements is denoted by $\mathcal{P}^k(N)$, while $\mathcal{P}^k_*(N) := \mathcal{P}^k(N) \setminus \{\emptyset\}$. For convenience, subsets like $\{i\}, \{i, j\}, \{2\}, \{2, 3\}, \ldots$ are written in the compact form $i, ij, 2, 23$ and so on.

A *game* on N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The set of games on N is denoted by $\mathcal{G}(N)$. For any $A \in 2^N \setminus \{\emptyset\}$, the *unanimity game centered on* A is defined by $u_A(B) := 1$ iff $B \supseteq A$, and 0 otherwise.

A game v on N is said to be:

(i) *additive* if $v(A \cup B) = v(A) + v(B)$ whenever $A \cap B = \emptyset$;

(ii) *convex* if $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, for all $A, B \subseteq N$;

(iii) *monotone* if $v(A) \leq v(B)$ whenever $A \subseteq B$;

(iv) *k-monotone* for $k \geq 2$ if for any family of k subsets $A_1, \ldots A_k$, it holds

$$v(\bigcup_{i=1}^k A_i) \geq \sum_{\substack{K \subseteq \{1,\ldots,k\} \\ K \neq \emptyset}} (-1)^{|K|+1} v(\bigcap_{j \in K} A_j)$$

(v) *infinitely monotone* if it is k-monotone for all $k \geq 2$.

Convexity corresponds to 2-monotonicity. Note that k-monotonicity implies k'-monotonicity for all $2 \leq k' \leq k$. Also, $(n-2)$-monotone games on N are infinitely monotone [2]. The set of monotone games on N is denoted by $\mathcal{MG}(N)$, while the set of infinitely monotone games is denoted by $\mathcal{G}_\infty(N)$.

Let v be a game on N. The *Möbius transform* of v [16] (also called *dividends* of v, see Harsanyi [11]) is a function $m : 2^N \to \mathbb{R}$ defined by:

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} v(B), \quad \forall A \subseteq N.$$

The Möbius transform is invertible since one can recover v from m by:

$$v(A) = \sum_{B \subseteq A} m(B), \quad \forall A \subseteq N.$$

If v is an additive game, then m is non null only for singletons, and $m(\{i\}) = v(\{i\})$. The Möbius transform of u_A is given by $m(A) = 1$ and m is 0 otherwise.

A game v is said to be k-*additive* [10] for some integer $k \in \{1, \ldots, n\}$ if $m(A) = 0$ whenever $|A| > k$, and there exists some A such that $|A| = k$, and $m(A) \neq 0$.

Clearly, 1-additive games are additive. The set of games on N being at most k-additive (resp. infinitely monotone games at most k-additive) is denoted by $\mathcal{G}^k(N)$ (resp. $\mathcal{G}^k_\infty(N)$). As above, replace $\mathcal{G}$ by $\mathcal{MG}$ if monotone games are considered instead.

We recall the fundamental following result.

Proposition 1 *[5] Let v be a game on N. For any $A, B \subseteq N$, with $A \subseteq B$, we denote $[A, B] := \{L \subseteq N \mid A \subseteq L \subseteq B\}$.*

(i) Monotonicity is equivalent to

$$\sum_{L \in [i, B]} m(L) \geq 0, \quad \forall B \subseteq N, \quad \forall i \in B.$$

(ii) For $2 \leq k \leq n$, k-monotonicity is equivalent to

$$\sum_{L \in [A, B]} m(L) \geq 0, \quad \forall A, B \subseteq N, A \subseteq B, \quad 2 \leq |A| \leq k.$$

Clearly, a monotone and infinitely monotone game has a nonnegative Möbius transform.

The *core* of a game v is defined by:

$$\mathcal{C}(v) := \{\phi \in \mathcal{G}^1(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \text{ and } \phi(N) = v(N)\}.$$

A *maximal chain* in 2^N is a sequence of subsets $A_0 := \emptyset, A_1, \ldots, A_{n-1}, A_n := N$ such that $A_i \subset A_{i+1}$, $i = 0, \ldots, n-1$. The set of maximal chains of 2^N is denoted by $\mathcal{M}(2^N)$.

To each maximal chain $C := \{\emptyset, A_1, \ldots, A_n = N\}$ in $\mathcal{M}(2^N)$ corresponds a unique permutation σ on N such that $A_1 = \sigma(1)$, $A_2 \setminus A_1 = \sigma(2)$, $\ldots$, $A_n \setminus A_{n-1} = \sigma(n)$. The set of all permutations over N is denoted by $\mathfrak{S}(N)$. Let v be a game. Each permutation σ (or maximal chain C) induces an additive game ϕ^σ (or ϕ^C) on N defined by:

$$\phi^\sigma(\sigma(i)) := v(\{\sigma(1), \ldots, \sigma(i)\}) - v(\{\sigma(1), \ldots, \sigma(i-1)\})$$

or

$$\phi^C(\sigma(i)) := v(A_i) - v(A_{i-1}), \quad \forall i \in N.$$

with the above notation. The following is immediate.

Proposition 2 *Let v be a game on N, and C a maximal chain of 2^N. Then*

$$\phi^C(A) = v(A), \quad \forall A \in C.$$

Theorem 1 *The following propositions are equivalent.*

(i) *v is a convex game.*

(ii) *All additive games ϕ^σ, $\sigma \in \mathfrak{S}(N)$, belong to the core of v.*

(iii) $\mathcal{C}(v) = \mathrm{co}(\{\phi^\sigma\}_{\sigma \in \mathfrak{S}(N)})$.

(iv) $\mathrm{ext}(\mathcal{C}(v)) = \{\phi^\sigma\}_{\sigma \in \mathfrak{S}(N)}$,

where $\mathrm{co}(\cdot)$ *and* $\mathrm{ext}(\cdot)$ *denote respectively the convex hull of some set, and the extreme points of some convex set.*

(i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv) are due to Shapley [17], while (ii) $\Rightarrow$ (i) was proved by Ichiishi [12].

A natural extension of the definition of the core is the following. For some integer $1 \leq k \leq n$, the *k-additive core* of a game v is defined by:

$$\mathcal{C}^k(v) := \{\phi \in \mathcal{G}^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \phi(N) = v(N)\}.$$

In a context of game theory where elements of the core are imputations, it is natural to consider that monotonicity must hold, i.e., the imputation allocated to some coalition $A \in \mathcal{P}_*^k(N)$ is larger than for any subset of A. We call it the *monotone k-additive core*:

$$\mathcal{MC}^k(v) := \{\phi \in \mathcal{MG}^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \phi(N) = v(N)\}.$$

We introduce also the *core of k-additive infinitely monotone games*:

$$\mathcal{C}_\infty^k(v) := \{\phi \in \mathcal{G}_\infty^k(N) \mid \phi(A) \geq v(A), \quad \forall A \subseteq N, \text{ and } \phi(N) = v(N)\}.$$

The latter is introduced just for mathematical convenience, and has no clear application. Note that $\mathcal{C}(v) = \mathcal{C}^1(v) = \mathcal{C}_\infty^1(v)$.

3 Orders on $\mathcal{P}_*^k(N)$ and achievable families

As our aim is to give a generalization of the Shapley-Ichiishi results, we need counterparts of permutations and maximal chains. These are given in this section. Exact connections between our material and permutations and maximal chains will be explicited at the end of this section. First, we introduce total orders on subsets of at most k elements as a generalization of permutations.

We denote by $\prec$ a total (strict) order on $\mathcal{P}_*^k(N)$, $\preceq$ denoting the corresponding weak order.

(i) $\prec$ is said to be *compatible* if for all $A, B \in \mathcal{P}^k_*(N)$, $A \prec B$ if and only if $A \cup C \prec B \cup C$ for all $C \subseteq N$ such that $A \cup C, B \cup C \in \mathcal{P}^k_*(N)$, $A \cap C = B \cap C = \emptyset$.

(ii) $\prec$ is said to be $\subseteq$-*compatible* if $A \subset B$ implies $A \prec B$.

(iii) $\prec$ is said to be *strongly compatible* if it is compatible and $\subseteq$-compatible.

We introduce the *binary order* $\prec^2$ on 2^N as follows. To any subset $A \subseteq N$ we associate an integer $\eta(A)$, whose binary code is the indicator function of A, i.e., the ith bit of $\eta(A)$ is 1 if $i \in A$, and 0 otherwise. For example, with $n = 5$, $\{1, 3\}$ and $\{4\}$ have binary codes 00101 and 01000 respectively, hence $\eta(\{1, 3\}) = 5$ and $\eta(\{4\}) = 8$. Then $A \prec^2 B$ if $\eta(A) < \eta(B)$. This gives

$$1 \prec^2 2 \prec^2 12 \prec^2 3 \prec^2 13 \prec^2 23 \prec^2 123 \prec^2 4 \prec^2 14 \prec^2 24 \prec^2 \\ 124 \prec^2 34 \prec^2 134 \prec^2 234 \prec^2 1234 \prec^2 5 \prec^2 \dots \quad (1)$$

Note the recursive nature of $\prec^2$. Obviously, $\prec^2$ is a strongly compatible order, as well as all its restrictions to $\mathcal{P}^k_*(N)$, $k = 1, \dots, n-1$.

We introduce now a generalization of maximal chains associated to permutations. Let $\prec$ be a total order on $\mathcal{P}^k_*(N)$. For any $B \in \mathcal{P}^k_*(N)$, we define

$$\mathcal{A}(B) := \{A \subseteq N \mid [A \supseteq B] \text{ and } [\forall K \subseteq A \text{ s.t. } K \in \mathcal{P}^k_*(N), \text{ it holds } K \preceq B]\}$$

the *achievable family of B*.

Example 1: Consider $n = 3$, $k = 2$, and the following order: $1 \prec 2 \prec 12 \prec 13 \prec 23 \prec 3$. Then

$$\mathcal{A}(1) = \{1\}, \quad \mathcal{A}(2) = \{2\}, \quad \mathcal{A}(12) = \{12\}, \quad \mathcal{A}(13) = \mathcal{A}(23) = \emptyset, \\ \mathcal{A}(3) = \{3, 13, 23, 123\}.$$

Proposition 3 *$\{\mathcal{A}(B)\}_{B \in \mathcal{P}^k_*(N)}$ is a partition of $\mathcal{P}(N) \setminus \{\emptyset\}$.*

Proof: Let $\emptyset \neq C \in \mathcal{P}(N)$. It suffices to show that there is a unique $B \in \mathcal{P}^k_*(N)$ such that $C \in \mathcal{A}(B)$. Let $K_1, K_2, \dots, K_p$ be the nonempty collection of subsets of C in $\mathcal{P}^k(N)$, assuming $K_1 \prec K_2 \prec \cdots \prec K_p$. Then $C \in \mathcal{A}(K_p)$ is the unique possibility, since any B outside the collection will fail to satisfy the condition $B \subseteq C$, and any $B \neq K_p$ inside the collection will fail to satisfy the condition $K_p \preceq B$. ■

Proposition 4 *For any $B \in \mathcal{P}^k_*(N)$ such that $\mathcal{A}(B) \neq \emptyset$, $(\mathcal{A}(B), \subseteq)$ is an inf-semilattice, with bottom element B.*

Proof: If $\mathcal{A}(B) \neq \emptyset$, any $C \in \mathcal{A}(B)$ contains B, hence every $K \subseteq B \subseteq C$, $K \in \mathcal{P}^k_*(N)$, is such that $K \preceq B$. Hence $B \in \mathcal{A}(B)$, and it is the smallest element.

Let $K, K' \in \mathcal{A}(B)$, assuming $\mathcal{A}(B)$ contains at least 2 elements (otherwise, we are done). $K \in \mathcal{A}(B)$ is equivalent to $K \supseteq B$ and any $L \subseteq K$, $L \in \mathcal{P}^k_*(N)$ is such that $L \preceq B$. The same holds for K'. Therefore, $K \cap K' \supseteq B$, and if $L \subseteq K \cap K'$, $L \in \mathcal{P}^k_*(N)$, then $L \subseteq K$ and $L \subseteq K'$, which entails $L \preceq B$. Hence $K \cap K' \in \mathcal{A}(B)$. ■

From the above proposition we deduce:

Corollary 1 *Let $B \in \mathcal{P}^k_*(N)$ and $\prec$ be some total order on $\mathcal{P}^k_*(N)$. Then $\mathcal{A}(B) \neq \emptyset$ if and only if for all $C \in \mathcal{P}^k_*(N)$, $C \subseteq B$ implies $C \preceq B$. Consequently, if $|B| = 1$ then $\mathcal{A}(B) \neq \emptyset$.*

Corollary 2 *$\mathcal{A}(B) \neq \emptyset$ for all $B \in \mathcal{P}^k_*(N)$ if and only if $\prec$ is $\subseteq$-compatible.*

It is easy to build examples where achievable families are not lattices.

EXAMPLE 2: Consider $n = 4, k = 2$ and the following order: 2, 3, 24, 12, 4, 13, 34, 1, 23, 14. Then $\mathcal{A}(23) = \{23, 123, 234\}$, and $1234 \notin \mathcal{A}(23)$ since $14 \succ 23$.

Assuming $\mathcal{A}(B)$ is a lattice, we denote by $\check{B}$ its top element.

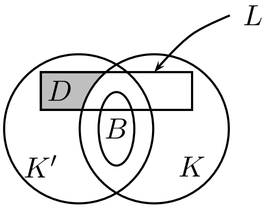
Proposition 5 *Let $\prec$ be a total order on $\mathcal{P}_*^k(N)$. Consider $B \in \mathcal{P}_*^k(N)$ such that $\mathcal{A}(B)$ is a lattice. Then it is a Boolean lattice isomorphic to $(\mathcal{P}(\check{B} \setminus B), \subseteq)$.*

Proof: It suffices to show that $\mathcal{A}(B) = \{B \cup K \mid K \subseteq \check{B} \setminus B\}$. Taking $\check{K} := \check{B} \setminus B$, we have $B \cup \check{K} \in \mathcal{A}(B)$. Hence, any $L \subseteq B \cup \check{K}$, $L \in \mathcal{P}_*^k(N)$, is such that $L \preceq B$. This is a fortiori true for $L \subseteq B \cup K$, $L \in \mathcal{P}_*^k(N)$, $\forall K \subseteq \check{K}$. Hence $B \cup K$ belongs to $\mathcal{A}(B)$, for all $K \subseteq \check{K}$. ■

Proposition 6 *Assume $\prec$ is compatible. For any $B \in \mathcal{P}_*^k(N)$ such that $\mathcal{A}(B) \neq \emptyset$, $\mathcal{A}(B)$ is the Boolean lattice $[B, \check{B}]$.*

Proof: If $\mathcal{A}(B)$ is a lattice, we know by Prop. 5 that it is a Boolean lattice with bottom element B. Since we know that $\mathcal{A}(B)$ is an inf-semilattice by Prop. 4, it remains to show that $K, K' \in \mathcal{A}(B)$ implies $K \cup K' \in \mathcal{A}(B)$. Assume $K \cup K' \notin \mathcal{A}(B)$. Then there exists $L \subseteq K \cup K'$, $L \in \mathcal{P}_*^k(N)$ such that $L \succ B$. Necessarily, $L \setminus K \neq \emptyset$, otherwise $L \subseteq K$ and $K \in \mathcal{A}(B)$ imply $L \prec B$, a contradiction. Similarly, $L \setminus K' \neq \emptyset$. Moreover, $L \not\subseteq B$ since $\mathcal{A}(B) \neq \emptyset$ (see Cor. 1).

We consider $D := L \setminus K$, not empty by definition of L. Since $L \setminus D \subseteq K$ and $L \setminus D \in \mathcal{P}_*^k(N)$, we have $L \setminus D \preceq B$, with strict inequality since $L \setminus D$ has elements outside $K \cap K'$, hence outside B (see Figure below).



Suppose first that $|B| < k$, and let $D := \{i, j, \ldots\}$. We have $B \cup l \in \mathcal{P}_*^k(N)$ and $B \cup l \subseteq K'$ for any $l \in D$, which implies $B \cup l \prec B$. Taking $l = i$, by compatibility, $L \setminus D \prec B$ implies $(L \setminus D) \cup i \prec B \cup i \prec B$. By compatibility again, $(L \setminus D) \cup i \prec B$ implies $(L \setminus D) \cup i \cup j \prec B \cup j \prec B$. Continuing the process till all elements of D have been taken, we finally end with $L \prec B$, a contradiction.

Secondly, assume that $|B| = k$. Take $K'' \subset B$ such that $K'' \supseteq L \cap B$ and $|K'' \cup D| = k$, which is always possible by assumption. Since $K'' \subset B \subseteq K$ and $K'' \in \mathcal{P}_*^k(N)$, we have $K'' \prec B$. Then

(i) Either $L \setminus D \prec K'' \prec B$. By compatibility, $L \setminus D \prec K''$ implies $L \prec K'' \cup D$. Since $K'' \cup D \in \mathcal{P}_*^k(N)$ and $K'' \cup D \subseteq K'$, we deduce that $K'' \cup D \prec B$, hence $L \prec B$, a contradiction.

(ii) Or $K'' \prec L \setminus D \prec B$. Since $(L \setminus D) \cap (B \setminus K'') = \emptyset$, from compatibility $K'' \prec L \setminus D$ implies $B = K'' \cup (B \setminus K'') \prec (L \setminus D) \cup (B \setminus K'')$. We have by assumption $|(L \setminus D) \cup (B \setminus K'')| = |L| \leq k$ and $(L \setminus D) \cup (B \setminus K'') \subseteq K$, from which we deduce $(L \setminus D) \cup (B \setminus K'') \prec B$. Hence we get $B \prec B$, a contradiction.

$\blacksquare$

The following example shows that compatibility is not a necessary condition.

Example 3: Consider $n = 4$, $k = 2$, and the following order: 1, 3, 2, 12, 23, 13, 4, 14, 24, 34. This order is not compatible since $3 \prec 2$ and $12 \prec 13$.

We obtain:

$$\mathcal{A}(1) = 1, \quad \mathcal{A}(3) = 3, \quad \mathcal{A}(2) = 2, \quad \mathcal{A}(12) = 12, \quad \mathcal{A}(23) = 23, \quad \mathcal{A}(13) = \{13, 123\},$$
$$\mathcal{A}(4) = 4, \quad \mathcal{A}(14) = 14, \quad \mathcal{A}(24) = \{24, 124\}, \quad \mathcal{A}(34) = \{34, 134, 234, 1234\}.$$

All families are lattices.

In the above example, $\prec$ was $\subseteq$-compatible. However, this is not enough to ensure that achievable families are lattices, as shown by the following example.

Example 4: Let us consider the following $\subseteq$-compatible order with $n = 4$ and $k = 2$:

$$3 \prec 4 \prec 34 \prec 2 \prec 24 \prec 1 \prec 13 \prec 12 \prec 23 \prec 14.$$

Then $\mathcal{A}(23) = \{23, 123, 234\}$.

We give some fundamental properties of achievable families when they are lattices, in particular of their top elements.

Proposition 7 *Assume $\prec$ is compatible, and consider a nonempty achievable family $\mathcal{A}(B)$, with top element $\check{B}$. Then $\{\mathcal{A}(B_i) \mid B_i \in \mathcal{P}^k_*(N), B_i \subseteq \check{B}, \mathcal{A}(B_i) \neq \emptyset\}$ is a partition of $\mathcal{P}(\check{B}) \setminus \{\emptyset\}$.*

Proof: We know by Prop. 3 that all $\mathcal{A}(B_i)$'s are disjoint. It remains to show that (1) any $K \subseteq \check{B}$ is in some $\mathcal{A}(B_i)$, $B_i \subseteq \check{B}$, and (2) conversely that any K in such $\mathcal{A}(B_i)$ is a subset of $\check{B}$.

(1) Assume $K \in \mathcal{A}(B_i)$, $B_i \not\subseteq \check{B}$. Then $B_i \subseteq K \subseteq \check{B}$, a contradiction.

(2) Assume $K \in \mathcal{A}(B_i)$, $B_i \subseteq \check{B}$, and $K \not\subseteq \check{B}$. Then there exists $l \in K$ such that $l \notin \check{B}$ (and hence not in B_i). Note that this implies $B_i \cup l \prec B_i$, provided $|B_i| < k$. First we show that $l \prec j$ for any $j \in B_i$. Since $K \supseteq B_i \cup \{l\}$, we deduce that for any $j \in B_i$, $\{j, l\} \prec B_i$ and $l \prec B_i$. If $B_i = \{j\}$, we can further deduce that $l \prec j$. Otherwise, if $B_i = \{j, j'\}$, from $\{j, l\} \prec \{j, j'\}$ and

$\{j', l\} \prec \{j, j'\}$, by compatibility $l \prec j$ and $l \prec j'$. Generalizing the above, we conclude that $l \prec j$ for all $j \in B_i$.

Next, if $l \notin \check{B}$, it means that for some $B' \subseteq \check{B}$ such that $B' \cup l \in \mathcal{P}_*^k(N)$, we have $B' \cup l \succ B$ (otherwise l should belong to $\check{B}$). We prove that $B' \not\supseteq B_i$. The case $|B_i| = k$ is obvious, let us consider $|B_i| < k$. Suppose on the contrary that $B' = B_i \cup L$, $L \subseteq N \setminus B_i$. Then $B_i \cup l \prec B_i$ implies that $B' \cup l = B_i \cup l \cup L \prec B_i \cup L \prec B$, the last inequality coming from $B_i \cup L \subseteq \check{B}$, $B_i \cup L \in \mathcal{P}_*^k(N)$. But $B' \cup l \succ B$, a contradiction.

Choose any $j \in B_i \setminus B'$. Since $j \succ l$, we deduce $B' \cup j \succ B' \cup l \succ B$, but since $B' \cup j \subseteq \check{B}$ and $B' \cup j \in \mathcal{P}_*^k(N)$, it follows that $B' \cup j \prec B$, a contradiction. ■

Proposition 8 *Let $\prec$ be a compatible order on $\mathcal{P}_*^k(N)$. For any $B \in \mathcal{P}_*^k(N)$ such that $\mathcal{A}(B)$ is nonempty, putting $\check{B} := \{i_1, \ldots, i_l\}$ with $i_1 \prec \cdots \prec i_l$, then necessarily there exists $j \in \{1, \ldots, l\}$ such that $B = \{i_j, \ldots, i_l\}$.*

Proof: Assume $\check{B} \neq B$, otherwise we have simply $j = 1$. Consider i_j the element in B with the lowest index in the list $\{1, \ldots, l\}$. Let us prove that all successors $i_{j+1}, \ldots, i_l$ are also in B. Assume $j < l$ (otherwise we are done), and suppose that $i_{j'} \notin B$ for some $j < j' \leq l$. Then by compatibility, $B = (B \setminus i_j) \cup i_j \prec (B \setminus i_j) \cup i_{j'}$. Since $(B \setminus i_j) \cup i_{j'} \subseteq \check{B}$ and $(B \setminus i_j) \cup i_{j'} \in \mathcal{P}_*^k(N)$, the converse inequality should hold. ■

Proposition 9 *Assume that $\prec$ is strongly compatible. Then for all $B \subseteq N$, $|B| < k$, $\check{B} = B$.*

Proof: By Prop. 6 and Cor. 2, we know that $\mathcal{A}(B)$ is a Boolean lattice with top element denoted by $\check{B}$. Suppose that $\check{B} \neq B$. Then there exists $i \in \check{B} \setminus B$, and $B \cup i \in \mathcal{A}(B)$. Remark that $|B \cup i| \leq k$ and $\mathcal{A}(B \cup i) \ni B \cup i$ by Prop. 6 and Cor. 2 again. This contradicts the fact that the achievable families form a partition of $\mathcal{P}_*^k(N)$ (Prop. 3). ■

Proposition 10 *Let $\prec$ be a strongly compatible order on $\mathcal{P}_*^k(N)$, and assume w.l.o.g. that $1 \prec 2 \prec \cdots \prec n$. Then the collection $\check{\mathcal{B}}$ of $\check{B}$'s is given by:*

$$\check{\mathcal{B}} = \Big\{\{1, 2, \ldots, l\} \cup \{j_1, \ldots, j_{k-1}\} \mid l = 1, \ldots, n-k+1 \\ \text{and } \{j_1, \ldots, j_{k-1}\} \subseteq \{l+1, \ldots, n\}\Big\} \bigcup \Big\{A \subseteq N \mid |A| < k\Big\}.$$

If $\prec$ is compatible, then $\check{\mathcal{B}}$ is a subcollection of the above, where some subsets of at most $k-1$ elements may be absent.

Proof: From Prop. 9, we know that $\check{\mathcal{B}}$ contains all subsets having less than k elements. This proves the right part of "$\bigcup$" in $\check{\mathcal{B}}$. By Prop. 9 again, the left part uniquely comes from those B's of exactly k elements. Take such a B. From Prop. 8, we know that $\check{B}$ cannot contain elements ranked after the last one of B in the sequence $1, 2, \ldots, n$. In other words, letting $B := \{l, j_1, \ldots, j_{k-1}\}$, with l the lowest ranked element, we know that $\check{B} = B' \cup \{l, j_1, \ldots, j_{k-1}\}$, with all elements of B' ranked before l. It remains to show that necessarily B' contains all elements from 1 to l excluded. Assume $j \notin B'$, $1 \leq j < l$. Then it should exist $K \in \mathcal{P}_*^k(N)$, $j \in K \subseteq \check{B} \cup j$, such that $K \succ B$. Since $|B| = k$, it cannot be that $K \supseteq B$, so that say $j' \in B$ is not in K. Hence we have $j \prec j'$, and by compatibility, $K = (K \setminus j) \cup j \prec (K \setminus j) \cup j'$. Now, $(K \setminus j) \cup j' \in \mathcal{P}_*^k(N)$ and $(K \setminus j) \cup j' \subseteq \check{B}$, which entails $(K \setminus j) \cup j' \prec B$, so that $K \prec B$, a contradiction.

Finally, consider that $\prec$ is only compatible. Then by Cor. 2, there exists $B \in \mathcal{P}^k_*(N)$ such that $\mathcal{A}(B) = \emptyset$. This implies that there exist some proper subsets of B in $\mathcal{P}^k_*(N)$ ranked after B, let us call K the last ranked such subset. Then $|K| < k$, and $\mathcal{A}(K) \neq \{K\}$ since it contains at least B, because all subsets of B are ranked before K by definition of K. Hence K does not belong to $\check{\mathcal{B}}$. ■

We finish this section by explaining why achievable families induced by orders on $\mathcal{P}^k_*(N)$ are generalizations of maximal chains induced by permutations. Taking $k = 1$, $\mathcal{P}^1_*(N) = N$, and total orders on singletons coincide with permutations on N. Trivially, any order on N is strongly compatible, so that all achievable families are nonempty lattices. Denoting by σ the permutation corresponding to $\prec$, i.e., $\sigma(1) \prec \sigma(2) \prec \cdots \prec \sigma(n)$, then

$$\mathcal{A}(\{\sigma(j)\}) = [\{\sigma(j)\}, \{\sigma(1), \ldots, \sigma(j)\}],$$

i.e., the top element $\{\check{\sigma(j)}\}$ is $\{\sigma(1), \ldots, \sigma(j)\}$. Then the collection of all top elements $\{\check{\sigma(j)}\}$ is exactly the maximal chain associated to σ.

4 Vertices of $\mathcal{C}^k(v)$ induced by achievable families

Let us consider a game v and its k-additive core $\mathcal{C}^k(v)$. We suppose hereafter that $\mathcal{C}^k(v) \neq \emptyset$, which is always true for a sufficiently high k. Indeed, taking at worst $k = n$, $v \in \mathcal{C}^n(v)$ always holds.

4.1 General facts

A k-additive game v^* with Möbius transform m^* belongs to $\mathcal{C}^k(v)$ if and only if it satisfies the system

$$\sum_{\substack{K\subseteq A\\|K|\leq k}} m^*(K) \geq \sum_{K\subseteq A} m(K), \quad A \in 2^N \setminus \{\emptyset, N\} \tag{2}$$

$$\sum_{\substack{K\subseteq N\\|K|\leq k}} m^*(K) = v(N). \tag{3}$$

The number of variables is $N(k) := \binom{n}{1} + \cdots + \binom{n}{k}$, but due to (3), this gives rise to a $(N(k)-1)$-dim closed polyhedron. (2) is a system of $2^n - 2$ inequalities. The polyhedron is convex since the convex combination of any two elements of the core is still in the core, but it is not bounded in general. To see this, consider the simple following example.

Example 5: Consider $n = 3$, $k = 2$, and a game v defined by its Möbius transform m with $m(i) = 0.1$, $m(ij) = 0.2$ for all $i, j \in N$, and $m(N) = 0.1$. Then the system of inequalities defining the 2-additive core is:

$$\begin{aligned}
m^*(1) &\geq 0.1\\
m^*(2) &\geq 0.1\\
m^*(3) &\geq 0.1\\
m^*(1) + m^*(2) + m^*(12) &\geq 0.4\\
m^*(1) + m^*(3) + m^*(13) &\geq 0.4\\
m^*(2) + m^*(3) + m^*(23) &\geq 0.4\\
m^*(1) + m^*(2) + m^*(3) + m^*(12) + m^*(13) + m^*(23) &= 1.
\end{aligned}$$

Let us write for convenience $m^* := (m^*(1), m^*(2), m^*(3), m^*(12), m^*(13), m^*(23))$. Clearly $m^*_0 := (0.2, \quad 0.1, \quad 0.1, \quad 0.2, \quad 0.2, \quad 0.2)$ is a solution, as well as

$m_0^* + t(1,\ 0,\ 0,\ -1,\ 0,\ 0)$ for any $t \geq 0$. Hence $(1,\ 0,\ 0,\ -1,\ 0,\ 0)$ is a ray, and the core is unbounded.

For the monotone core, from Prop. 1 (i) there is an additional system of $n2^{n-1}$ inequalities

$$\sum_{\substack{K \in [i,L] \\ |K| \leq k}} m^*(K) \geq 0, \quad \forall i \in N, \forall L \ni i. \tag{4}$$

For monotone games, Miranda and Grabisch [14] have proved that the Möbius transform is bounded as follows:

$$-\binom{|A|-1}{l'_{|A|}} v(N) \leq m(A) \leq \binom{|A|-1}{l_{|A|}} v(N), \quad \forall A \subseteq N,$$

where $l_{|A|}, l'_{|A|}$ are given by:

(i) $l_{|A|} = \dfrac{|A|}{2}$, and $l'_{|A|} = \dfrac{|A|}{2} - 1$ if $|A| \equiv 0 (\mathrm{mod} \quad 4)$

(ii) $l_{|A|} = \dfrac{|A|-1}{2}$, and $l'_{|A|} = \dfrac{|A|-3}{2}$ or $l'_{|A|} = \dfrac{|A|+1}{2}$ if $|A| \equiv 1 (\mathrm{mod} \quad 4)$

(iii) $l_{|A|} = \dfrac{|A|}{2} - 1$, and $l'_{|A|} = \dfrac{|A|}{2}$ if $|A| \equiv 2 (\mathrm{mod} \quad 4)$

(iv) $l_{|A|} = \dfrac{|A|-3}{2}$ or $l_{|A|} = \dfrac{|A|+1}{2}$, and $l'_{|A|} = \dfrac{|A|-1}{2}$ if $|A| \equiv 3 (\mathrm{mod} \quad 4)$.

Since $v(N)$ is fixed and bounded, the monotone k-additive core is always bounded.

For $\mathcal{C}_\infty^k(v)$, using Prop. 1 (ii) system (4) is replaced by a system of $N(k) - n$ inequalities:

$$m^*(K) \geq 0, \quad K \in \mathcal{P}_*^k(N), |K| > 1. \tag{5}$$

Since in addition we have $m^*(\{i\}) \geq m(\{i\})$, $i \in N$ coming from (2), m^* is bounded from below. Then (3) forces m^* to be bounded from above, so that $\mathcal{C}_\infty^k(v)$ is bounded.

In summary, we have the following.

Proposition 11 *For any game v, $\mathcal{C}^k(v)$, $\mathcal{MC}^k(v)$ and $\mathcal{C}_\infty^k(v)$ are closed convex $(N(k)-1)$-dimensional polyhedra. Only $\mathcal{MC}^k(v)$ and $\mathcal{C}_\infty^k(v)$ are always bounded.*

The following result about rays of $\mathcal{C}^k(v)$ is worthwile to be noted.

Proposition 12 *The components of rays of $\mathcal{C}^k(v)$ do not depend on v, but only on k and n.*

Proof: For any polyhedron defined by a system of m inequalities and n variables (including slack variables) $\mathbf{Ax} = \mathbf{b}$, it is well known that its conical part is given by $\mathbf{Ax} = \mathbf{0}$, and that rays (also called basic feasible directions) are particular solutions of the latter system with $n-m$ non basic components all equal to zero but one (see, e.g., [6]). Hence, components of rays do not depend on $\mathbf{b}$.

Applied to our case, this means that components of rays do not depend on v, but only on k and n. ■

4.2 A Shapley-Ichiishi-like result

We turn now to the characterization of vertices induced by achievable families. Let v be a game on N, m its Möbius transform, and $\prec$ be a total order on $\mathcal{P}^k_*(N)$. We define a k-additive game $v_\prec$ by its Möbius transform as follows:

$$m_\prec(B) := \begin{cases} \sum_{A\in\mathcal{A}(B)} m(A), & \text{if } \mathcal{A}(B) \neq \emptyset \\ 0, & \text{else} \end{cases} \tag{6}$$

for all $B \in \mathcal{P}^k_*(N)$, and $m_\prec(B) := 0$ if $B \notin \mathcal{P}^k_*(N)$.

Due to Prop. 3, $m_\prec$ satisfies $\sum_{B\subseteq N} m_\prec(B) = \sum_{B\subseteq N} m(B) = v(N)$, hence $v_\prec(N) = v(N)$.

This definition is a generalization of the definition of ϕ^σ or ϕ^C (see Sec. 2). Indeed, denoting by σ the permutation on N corresponding to $\prec$, we get:

$$\begin{aligned} m_{\prec}(\{\sigma(i)\}) &= \sum_{A \subseteq \{\sigma(1),\dots,\sigma(i-1)\}} m(A \cup \sigma(i)) \\ &= \sum_{A \subseteq \{\sigma(1),\dots,\sigma(i)\}} m(A) - \sum_{A \subseteq \{\sigma(1),\dots,\sigma(i-1)\}} m(A) \\ &= v(\{\sigma(1),\dots,\sigma(i)\}) - v(\{\sigma(1),\dots,\sigma(i-1)\}) = \phi^{\sigma}(\{\sigma(i)\}) = m^{\sigma}(\{\sigma(i)\}), \end{aligned}$$

where m^σ is the Möbius transform of ϕ^σ (see Sec. 2).

Proposition 13 *Assume that $\mathcal{A}(B)$ is a nonempty lattice. Then $v_{\prec}(\check{B}) = v(\check{B})$ if and only if $\{\mathcal{A}(C) \mid C \in \mathcal{P}^k_*(N), C \subseteq \check{B}, \mathcal{A}(C) \neq \emptyset\}$ is a partition of $\mathcal{P}(\check{B}) \setminus \{\emptyset\}$.*

Proof: We have by Eq. (6)

$$v_{\prec}(\check{B}) = \sum_{\substack{C \subseteq \check{B} \\ C \in \mathcal{P}^k_*(N) \\ \mathcal{A}(C) \neq \emptyset}} m_{\prec}(C) = \sum_{\substack{C \subseteq \check{B} \\ C \in \mathcal{P}^k_*(N) \\ \mathcal{A}(C) \neq \emptyset}} \sum_{K \in \mathcal{A}(C)} m(K). \tag{7}$$

On the other hand, $v(\check{B}) = \sum_{K \subseteq \check{B}} m(K)$. To ensure $v_{\prec}(\check{B}) = v(\check{B})$ for any v, every $K \subseteq \check{B}$ must appear exactly once in the last sum of (7), which is equivalent to the desired condition. ■

The following is immediate from Prop. 13 and 7.

Corollary 3 *Assume $\prec$ is compatible, and consider a nonempty achievable family $\mathcal{A}(B)$. Then $v_{\prec}(\check{B}) = v(\check{B})$.*

Proposition 14 *Let us suppose that all nonempty achievable families are lattices. Then v k-monotone implies that $v_{\prec}$ is infinitely monotone.*

Proof: It remains to show that $m_{\prec}(B) \geq 0$ for any B such that $1 < |B| \leq k$. For all such B satisfying $\mathcal{A}(B) \neq \emptyset$,

$$m_{\prec}(B) = \sum_{A \in \mathcal{A}(B)} m(A) = \sum_{A \in [B, \check{B}]} m(A).$$

Since $1 < |B| \leq k$, by Prop. 1, it follows from k-monotonicity that $m_{\prec}(B) \geq 0$ for all $B \in \mathcal{P}_*^k(N)$. ■

The next corollary follows from Prop. 6.

Corollary 4 *Let us suppose that $\prec$ is compatible. Then v k-monotone implies that $v_{\prec}$ is infinitely monotone.*

Theorem 2 *v is $(k+1)$-monotone if and only if for all compatible orders $\prec$, $v_{\prec}(A) \geq v(A)$, $\forall A \subseteq N$.*

Proof: For any compatible order $\prec$, and any $A \subseteq N$, $A \neq \emptyset$, by compatibility and Prop. 6, we can write

$$v_{\prec}(A) = \sum_{\substack{B \subseteq A \\ B \in \mathcal{P}_*^k(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{C \in [B, \check{B}]} m(C). \tag{8}$$

Let $C \subseteq A$. Then by Prop. 3, $C \in \mathcal{A}(B)$ for some $B \subseteq A$. Indeed $B \subseteq C \subseteq A$. Hence (8) writes

$$v_{\prec}(A) = v(A) + \sum_{\substack{B \subseteq A \\ B \in \mathcal{P}_*^k(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C). \tag{9}$$

($\Rightarrow$) Let us take any compatible order $\prec$. By (9), it suffices to show that

$$\sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C) \geq 0, \quad \forall B \subseteq A, B \in \mathcal{P}_*^k(N), \mathcal{A}(B) \neq \emptyset. \tag{10}$$

For simplicity define $\mathcal{C}$ as the set of subsets C satisfying the condition in the summation in (10). If $\check{B} \subseteq A$, then $\mathcal{C} = \emptyset$, and so (10) holds for such B's. Assume then that $\check{B} \setminus A \neq \emptyset$. Let us take $i \in \check{B} \setminus A$. Then $C_0 := B \cup i$ is a minimal element of $\mathcal{C}$, of cardinality $1 < |B| + 1 \leq k + 1$. Observe that

$[C_0, \check{B}] \subseteq \mathcal{C}$, and that it is a Boolean sublattice of $[B, \check{B}]$. Hence, $(k+1)$-monotonicity implies that $\sum_{C \in [C_0, \check{B}]} m(C) \geq 0$ (see Prop. 1).

Consider $j \in \check{B} \setminus A$, $j \neq i$. If no such j exists, then $[C_0, \check{B}] = \mathcal{C}$, and we have shown (10) for such B's. Otherwise, define $C_1 := B \cup j$ and the interval $[C_1, \check{B} \setminus i]$, which is disjoint from $[C_0, \check{B}]$. Applying again $(k+1)$-monotonicity we deduce that $\sum_{D \in [C_1, \check{B}]} m(D) \geq 0$. Continuing this process until all elements of $\check{B} \setminus A$ have been taken, the set $\mathcal{C}$ has been partitioned into intervals $[B \cup i, \check{B}], [B \cup j, \check{B} \setminus i], [B \cup k, \check{B} \setminus \{i, j\}], \ldots, [B \cup l, A \cup l]$ where the sum of $m(C)$ over these intervals is non negative by $(k+1)$-monotonicity. Hence (10) holds in any case and the sufficiency is proved.

($\Leftarrow$) Consider $K, L \subseteq N$ such that $1 < |K| \leq k+1$ and $L \supseteq K$. We have to prove that $\sum_{C \in [K, L]} m(C) \geq 0$. Without loss of generality, let us assume for simplicity that $K := \{i, i+1, \ldots, l\}$ and $L := \{1, \ldots, l\}$, with $l - k \leq i < l \leq n$. Define $B := K \setminus i = \{i+1, \ldots, l\}$ and $A := L \setminus i$. Take a total order on $\mathcal{P}_*^k(N)$ as follows:

(i) put first all subsets in $\mathcal{P}_*^k(L)$, with increasing cardinality, except B which is put the last

(ii) then put remaining subsets in $\mathcal{P}_*^k(N)$ such that they form a compatible order (for example: consider the above fixed sequence in $\mathcal{P}_*^k(L)$ augmented with the empty set as first element of the sequence, then take any subset D in $N \setminus L$ belonging to $\mathcal{P}_*^k(N)$, and add it to any subset of the sequence, discarding subsets not in $\mathcal{P}_*^k(N)$. Do this for any subset D of $N \setminus L$).

(iii) subsets in $\mathcal{P}_*^k(L)$ with same cardinality are ordered according to the lexicographic order, which means in particular $1 \prec 2 \prec \cdots \prec l$.

[1] For example, with $n = 5, l = 4, i = 3, k = 3$:

$$1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 14 \prec 23 \prec 24 \prec 34 \prec 123 \prec 124 \prec 134 \prec 234 \prec 4 \prec 5 \prec 51 \prec 52 \cdots$$

One can check that such an order is compatible[1]. By construction, we have $\mathcal{A}(B) = [B, L]$. Indeed, for any $C \in \mathcal{A}(B)$, any subset of C in $\mathcal{P}^k_*(N)$ is ranked before B. Moreover, $[K, L] = [B \cup i, L] = \{C \in \mathcal{A}(B) \mid C \not\subseteq A\}$. Now, take any $B' \neq B$ in $\mathcal{P}^k_*(L)$ such that $B' \subseteq A$. Let us prove that any $C \in \mathcal{A}(B')$ is such that $i \notin C$, or equivalently $C \subseteq A$. Indeed, up to the fact that B is ranked last, the sequence $\mathcal{P}^k_*(L)$ forms a strongly compatible order. Adapting slightly Prop. 9, it is easy to see that if $|B'| < k$, then either $\check{B}' = B'$ or $\mathcal{A}(B') = \emptyset$, the latter arising if $B' \supset B$. Then trivially any $C \in \mathcal{A}(B')$ satisfies $C \subseteq A$. Assume now $|B'| = k$. If B' contains some $j \prec i$, then $B' \cup i$ cannot belong to $\mathcal{A}(B')$ since by lexicographic ordering $B' \cup i \setminus j$ is ranked after B', which implies that for any $C \in \mathcal{A}(B')$, $i \notin C$. Hence, the condition $i \in C$ can be true for some $C \in \mathcal{A}(B')$ only if all elements of B' are ranked after i. But since $B = \{i + 1, \dots, l\}$, this implies that either $B' = B$, a contradiction, or B' does not exist (if $|B| < k$).

Let us apply the dominance condition for $v_\prec(A)$. Using (9), dominance is equivalent to write:

$$\sum_{\substack{B \subseteq A \\ B \in \mathcal{P}^k_*(N) \\ \mathcal{A}(B) \neq \emptyset}} \sum_{\substack{C \in [B, \check{B}] \\ C \not\subseteq A}} m(C) \geq 0.$$

Using the above, this sum reduces to $\sum_{C \in [K,L]} m(C) \geq 0$. This finishes the proof. ■

The following is an interesting property of the system $\{(2), (3)\}$.

Proposition 15 *Let $\prec$ be a compatible order. Then the linear system of equalities $v_\prec(\check{B}) = v(\check{B})$, for all $\check{B}$'s induced by $\prec$, is triangular with no zero on the diagonal, and hence has a unique solution.*

Proof: We consider w.l.o.g. that $1 \prec 2 \prec \cdots \prec n$ and consider the binary order $\prec^2$ for ordering variables $m^*(B)$.

Delete all variables such that $\mathcal{A}(B) = \emptyset$, and consider the list of subsets in $\mathcal{P}^k_*(N)$ corresponding to non deleted variables. Take all B's in the list, and their corresponding $\check{B}$'s (always exist since by compatibility, $\mathcal{A}(B)$ is a lattice). They are all different by Prop. 3, so we get a linear system of the same number of equations (namely $v_\prec(\check{B}) = v(\check{B})$) and variables. Take one particular equation corresponding to B. Then variables used in this equation are necessarily $m^*(B)$ itself (because $\check{B} \supseteq B$), and some variables ranked before B in the binary order. Indeed, if $\check{B} = B$, then all variables used in the equation are ranked before B by $\prec^2$. If $\check{B} \neq B$, supersets B' of B in $\mathcal{P}^k_*(N)$ are ranked after B by $\prec^2$ (because $\prec^2$ is $\subseteq$-compatible), and ranked before B by $\prec$ (otherwise $\mathcal{A}(B)$ would not contain $\check{B}$), but since they contain B, necessarily $\mathcal{A}(B') = \emptyset$, so corresponding variables are deleted.

Hence the system is triangular. ■

Note that the proof holds under the condition that all achievable families are lattices, so compatibility is even not necessary.

Theorem 3 *Let v be a $(k+1)$-monotone game. Then*

(i) If $\prec$ is strongly compatible, then $v_\prec$ is a vertex of $\mathcal{C}^k(v)$.

(ii) If $\prec$ is compatible, then $v_\prec$ is a vertex of $\mathcal{C}^k_\infty(v)$.

Proof: By standard results on polyhedra, it suffices to show that $v_\prec$ is an element of $\mathcal{C}^k(v)$ (resp. $\mathcal{C}^k_\infty(v)$) satisfying at least $N(k)-1$ linearly independent equalities among (2) (resp. among (2) and (5)). Assume $\prec$ is compatible. Then by Cor. 4, $v_\prec$ is infinitely monotone, and it dominates v by $(k+1)$-monotonicity (Th. 2). Moreover, for any $B \in \mathcal{P}^k_*(N)$, $\mathcal{A}(B)$ is either empty or a lattice, hence either $m_\prec(B) = 0$ or $v_\prec(\check{B}) = v(\check{B})$ by Cor. 3. Since if $|B| = 1$, $\mathcal{A}(B) \neq \emptyset$, this gives $N(k)$ equalities in the system defining $\mathcal{C}^k_\infty(v)$, including (3), hence we have the exact number of equalities required, which form a nonsingular system by Prop 15, and (ii) is proved. If the order is strongly compatible, then all achievable families are lattices, which proves the result for $\mathcal{C}^k(v)$, since again by Prop. 15, the system is nonsingular. ■

Remark 1: Vertices induced by (strongly) compatible orders are also vertices of the monotone k-additive core. They are induced only by dominance constraints, not by monotonicity constraints.

Remark 2: Cor. 3 generalizes Prop. 2, while Theorems 2 and 3 generalize the Shapley-Ichiishi results summarized in Th. 1. Indeed, recall that convexity is 2-monotonicity. Then clearly Th. 2 is a generalization of (i) $\Rightarrow$ (ii) of Th. 1, and Th. 3 (i) is a part of (iv) in Th. 1. But as it will become clear below, all vertices are not recovered by achievable families, mainly because they can induce only infinitely monotone games. In particular, $\mathcal{MC}^k(v)$ contains many more vertices.

Let us examine more precisely the number of vertices induced by strongly compatible orders. In fact, there are much fewer than expected, since many strongly compatible orders lead to the same $v_{\prec}$. The following is a consequence of Prop. 10.

Corollary 5 *The number of vertices of $\mathcal{C}^k(v)$ given by strongly compatible orders is at most $\frac{n!}{k!}$.*

Proof: Given the order $1 \prec 2 \prec \cdots \prec n$, a permutation over the last k singletons would not change the collection $\check{\mathcal{B}}$. ■

Note that when $k = 1$, we recover the fact that vertices are induced by all permutations, and that with $k = n$, we find only one vertex (which is in fact the only vertex of $\mathcal{C}^n(v)$), which is v itself (use Prop. 10 and the definition of $m_{\prec}$).

4.3 Other vertices

In this last section we give some insights about other vertices. Even for the (non monotonic) k-additive core, in general for $k \neq 1, n$, not all vertices are induced by strongly compatible orders. However, for the case $k = n-1$, it is possible to find all vertices of $\mathcal{C}^k(v)$. For $1 < k < n-1$ and also for the monotonic core, the problem becomes highly combinatorial.

Theorem 4 *Let v be any game in $\mathcal{G}(N)$, with Möbius transform m.*

(i) If $m(N) > 0$, $\mathcal{C}^{n-1}(v)$ contains exactly 2^{n-1} (if n is even) or $2^{n-1}-1$ (if n is odd) vertices, among which n vertices come from strongly compatible orders. They are given by their Möbius transform:

$$m^*_{B_0}(K) = \begin{cases} m(K), & \text{if } K \not\supseteq B_0 \\ m(K) + (-1)^{|K \setminus B_0|} m(N), & \text{else} \end{cases}$$

for all $B_0 \subset N$ such that $|N \setminus B_0|$ is odd.

(ii) If $m(N) = 0$, then there is only one vertex, which is v itself.

(iii) If $m(N) < 0$, $\mathcal{C}^{n-1}(v)$ contains exactly $2^{n-1}-1$ (if n is odd) or $2^{n-1}-2$ (if n is even) vertices, of which none comes from a strongly compatible order. They are given by their Möbius transform:

$$m^*_{B_0}(K) = \begin{cases} m(K), & \text{if } K \not\supseteq B_0 \\ m(K) - (-1)^{|K \setminus B_0|} m(N), & \text{else} \end{cases}$$

for all $B_0 \subset N$ such that $|N \setminus B_0|$ is even.

Proof: We assume $m(N) \geq 0$ (the proof is much the same for the case $m(N) \leq 0$). We consider the system of $2^n - 1$ inequalities {(2),(3)}, which has

$N(n-1) = 2^n - 2$ variables. We have to fix $2^n - 2$ equalities, among which (3), so we have to choose only one inequality in (2) to remain strict, say for $B_0 \subset N$, $B_0 \neq \emptyset$:

$$\sum_{K \subseteq B_0} m^*(K) > \sum_{K \subseteq B_0} m(K). \tag{11}$$

From the definition of the Möbius transform, we have

$$0 = m^*(N) = \sum_{K \subseteq N} (-1)^{|N \setminus K|} v^*(K).$$

Note that for any $\emptyset \neq K \subseteq N$, $v^*(K)$ is the left member of some inequality or equality of the system. Hence, by turning all inequalities into equalities, we get, by doing the above summation on the system, $0 = m(N)$. Hence, if $m(N) = 0$, there is only one vertex, which is v itself, otherwise taking equality everywhere gives a system with no solution. Since strict inequality holds only for $B_0 \subset N$, we get instead $0 > m(N)$ if $|N \setminus B_0|$ is even, and $0 < m(N)$ if $|N \setminus B_0|$ is odd. The first case is impossible by assumption on $m(N)$, so only the case where $|N \setminus B_0|$ odd can produce a vertex. Note that if $|B_0| = n-1$, we recover all n vertices induced by strongly compatible orders. In total we get $\binom{n}{n-1} + \binom{n}{n-3} + \cdots + \binom{n}{1} = 2^{n-1}$ potential different vertices when n is even, and $2^{n-1} - 1$ when n is odd. Clearly, there is no other possibility.

It remains to show that the corresponding system of equalities is non singular, and eventually to solve it. Assume $B_0 \subset N$ in (11) is chosen. From the linear system of equalities we easily deduce $m^*(K) = m(K)$ for all $K \not\supseteq B_0$. Substituting into all equations, the system reduces to

$$\begin{aligned} \sum_{K \subseteq B \setminus B_0} m^*(B_0 \cup K) &= \sum_{K \subseteq B \setminus B_0} m(B_0 \cup K), \quad \forall B \supset B_0, B \neq N \\ \sum_{K \subset N \setminus B_0} m^*(B_0 \cup K) &= \sum_{K \subset N \setminus B_0} m(B_0 \cup K) + m(N). \end{aligned}$$

B_0 being present everywhere, we may rename all variables after deleting B_0, i.e., we set $N' = N \setminus B_0$, $m'(A) := m(A \cup B_0)$ and $m'^*(A) = m^*(A \cup B_0)$, for all $A \subseteq N'$. The system becomes

$$\sum_{K \subseteq B} m'^*(K) = \sum_{K \subseteq B} m'(K), \quad \forall B \subset N'$$
$$\sum_{K \subset N'} m'^*(K) = \sum_{K \subset N'} m'(K) + m'(N').$$

Summing equations of the system as above, i.e., computing $\sum_{B \subseteq N'} (-1)^{|N' \setminus B|} \sum_{K \subseteq B} m'^*(K)$, we get $m'^*(\emptyset) = m'(\emptyset) + m'(N')$, or equivalently $m^*(B_0) = m(B_0) + m(N)$. Substituting in the above system, we get a system which is triangular (use, e.g., Prop. 15 with $k = n = n'$). We get easily $m^*(K) = m(K) + (-1)^{|K \setminus B_0|} m(N)$, for all $K \supseteq B_0$. ■

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