Non-convex Aggregate Technology and Optimal Economic Growth*

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Abstract

This paper examines a model of optimal growth where the aggregation of two separate well behaved and concave production technologies exhibits a basic non-convexity. First, we consider the case of strictly concave utility function: when the discount rate is either low enough or high enough, there will be one steady state equilibrium toward which the convergence of the optimal paths is monotone and asymptotic. When the discount rate is in some intermediate range, we find sufficient conditions for having either one equilibrium or multiple equilibria steady state. Depending to whether the initial capital per capita is located with respect to a critical value, the optimal paths converge to one single appropriate equilibrium steady state. This state might be a poverty trap with low per capita capital, which acts as the extinction state encountered in earlier studies focused on S-shapes production functions. Second, we consider the case of linear utility and provide sufficient conditions to have either unique or two steady states when the discount rate is in some intermediate range . In the latter case, we give conditions under which the above critical value might not exist,

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and the economy attains one steady state in finite time, then stays at the other steady state afterward.

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1 Introduction

Problems in the one-sector optimal economic growth model where the production technology exhibits increasing return at first and decreasing return to scale afterward have received earlier attention. Skiba (1978), examined this convex-concave technology in continuous time and provided some results, which were further extended rigorously in Majumdar and Mitra (1982) for a discrete time setting. Further, Majumdar and Nermuth (1982) considered irreversible investment and arbitrary increasing production function which is twice continuous and differentiable.

With a convex-concave technology giving rise to S-shaped production functions, the time discount rate plays an important role: when the future utility is heavily discounted, the optimal program converges monotonically to the "low" steady state - the extinction corresponding to a degenerated state characterized by vanishing long run capital stock- while in the opposite case, it tends in the long run to the optimal steady state, usually referred to as the Modified Golden Rule (MGR). If the rate of interest falls into an intermediate range of future discounting, the convergence now depends upon the initial stock of capital (see e.g. Dechert and Nishimura (1983)). Further, Majumdar and Mitra (1983) examined these questions in optimal growth model with a linear utility function while Mitra and Ray (1984) considered nonsmooth technologies and showed that any optimal path approaches asymptotically the set of steady states. Recently, Kamihigashi and Roy (2006) generalized to production functions which are increasing and upper semi-continuous. They obtained for the case of linear utility that any optimal path, which is strictly monotone, either converges to zero or reaches a positive steady state in finite time and possibly jumps among different steady states. Furthermore, Kamihigashi and Roy (2007) gave general conditions in a Nonsmooth, Nonconvex model of optimal growth to have a steady state, to obtain that an optimal path converges either to zero or to a well determined steady state, or to infinity.

In the present paper, we put emphasis on the existence of many technology-blueprint books, where each technology is well behaved and strictly concave, but the aggregation of theses technologies gives rise to some local non-convex range. To get in touch with a real problem akin to what we are studying, consider an economy with two technology possibilities. The α-technology requires less infrastructure expenditure such as investment in roads and highways and provides at the low range of capital labor ratio a higher production, thus higher consumption possibility than the β-technology. Alternatively, the β-technology requires heavy infrastructure investment, allows less production at the low range of capital labor ratio but a much greater production at high range of capital labor ratio. Intuitively, which technology would be relevant depends upon how the economy is initially endowed with capital and labor, and how their inhabitants evaluate the present consumption relative to future consumption.

For the purpose of expositional simplicity, consider two Cobb-Douglas technologies - mutually exclusive for reason of some setup technological cost - depicted in the Figure below where output per capita is a function of the capital-labor ratio. The intersection of the production graphs is located at point C where $k = 1$.Therefore the α-technology is relatively more efficient than the β-technology when $k \leq 1$, but less efficient when $k \geq 1$. The two production graphs have a common tangent passing through A and B. Thus, the aggregate production which combines both the α-technology and the β-technology exhibits a non-convex range depicted by the contour ACB. Clearly, the type of non-convexity in the aggregate technology we consider is quite different from those studied in the literature. The extinction - a degenerated state (0,0) - that we rule out by the Inada conditions imposed on the technology does no longer play the role of an attractor as in earlier studies mentioned above. Yet, there exist two MGR long run equilibria: $\widehat{k}_\alpha$, the " low" steady state - sometime referred to as the poverty trap- and $\widehat{k}_\beta$, the "high" steady state. The "low" steady state is an attractor in this case, and we must ask which of these two states will effectively be the equilibrium, and how the latter will be attained over time.

For the case of strictly concave utility as the objective function, we shall show that when future discounting is high enough, the equilibrium is the optimal steady state $\widehat{k}_\alpha$ corresponding to the technology that is relatively more efficient at the low capital per head. Conversely, when future discounting is low, the equilibrium is the optimal steady state $\widehat{k}_\beta$ corresponding to the technology that is relatively more efficient at high capital per head. For any

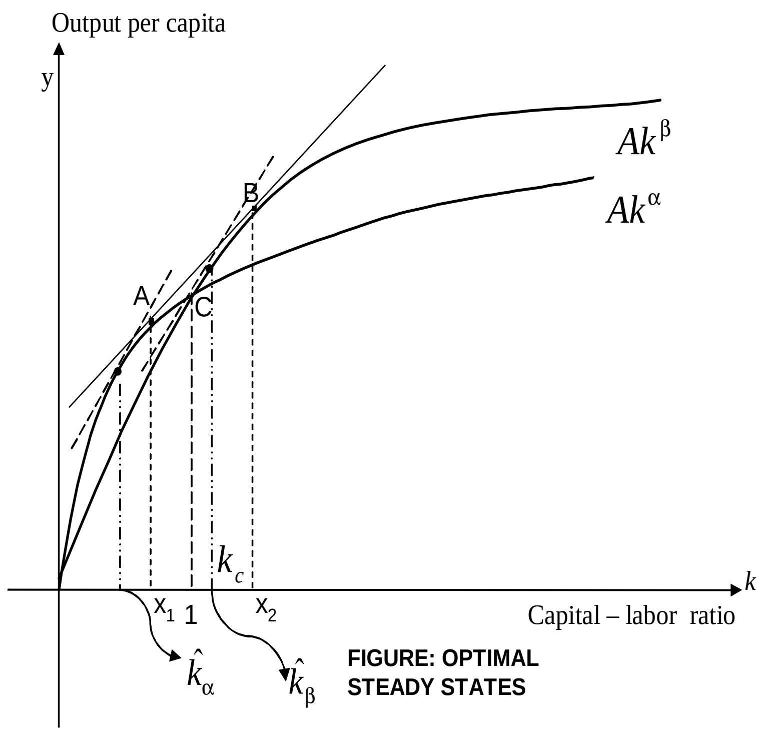

FIGURE: OPTIMAL STEADY STATES

value of the initial capital stock in these cases, the convergence to the optimal steady state equilibrium is assured. In contrast, when future discounting is in some intermediate range, there might exist two optimal steady states and the dynamic convergence now depends on the initial stock of capital k_0. We show that there exists a critical value k_c such that every optimal path from $k_0 < k_c$ will converge to $\widehat{k}_\alpha$, and every optimal path from $k_0 > k_c$ will converge to $\widehat{k}_\beta$. We also provides sufficient conditions to have two optimal steady states or to have just one optimal steady state. The convergence to the steady state is monotonic and asymptotic.

When the utility function is linear (that means that the social planner is the firm who maximizes the intertemporal profit), the optimal paths from a positive initial capital labor ratio will reach in finite time one of the two positive optimal steady states. Necessary and sufficient condition either to have uniqueness of steady state $\widehat{k}_\beta$ or to have positive poverty trap $\widehat{k}_\alpha$are obtained. Furthermore, we also provide the condition for having a critical value k_c as in the case with strictly concave utility, and the condition under which such critical value does not exist. In the latter case, the economy attains one steady state in finite time, then jumps to the other steady state afterward.

The paper is organized as follows. In Section 2, we specify our model. In Section 3, we provide a complete analysis of the optimal growth paths with strictly concave utility. In Section 4, we consider the case of linear utility as in Kamihigashi and Roy (2006), and in Section 5, we summarize our findings and provide some concluding comments.

2 The Model

The economy produces a homogeneous good according two possible Cobb-Douglas technologies, the $\alpha-$technology $f_\alpha(k) = Ak^\alpha$, and the β-technology $f_\beta(k) = Ak^\beta$ where k denotes the capital per head and $0 < \alpha < \beta < 1$. The efficient technology will be $y = \max\left\{Ak^\alpha, Ak^\beta\right\} = f(k)$.

The convexified economy is defined by $cof(k)$ where co stands for convex-hull. It is the smallest concave function minorized by f. Its epigraph, i.e. the set $\{(k, \lambda) \in R_+ \times R_+ : cof(k) \geq \lambda\}$ is the convex hull of the epigraph of

$f, \{(k,\lambda) \in R_+ \times R_+ : f(k) \geq \lambda\}$ (see figure 1). One can check that $cof = f_\alpha$ for $k \in [0, x_1]$, $cof = f_\beta$ for $k \in [x_2, +\infty[$, and affine between x_1 and x_2. More explicitly, we have

$$\alpha A x_1^{\alpha-1} = \beta A x_2^{\beta-1} = \frac{Ax_1^\alpha - Ax_2^\beta}{x_1 - x_2}$$

which implies

$$x_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1-\beta}{\beta-\alpha}}$$

and

$$x_2 = \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1-\alpha}{\beta-\alpha}}.$$

In our economy , the social utility is represented by $\sum_{t=0}^{t=+\infty} \gamma^t u(c_t)$ where γ is the discount factor and c_t the consumption. In period t, this consumption is constrained by the net output $f(k_t) - k_{t+1}$, where k_t denotes the per head capital stock available at date t.

The optimal growth model can be written as

$$\max \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

under the constraints

$$\forall t \geq 0, c_t \geq 0, k_t \geq 0, c_t \leq f(k_t) - k_{t+1}, \text{ and } k_0 > 0 \text{ is given.}$$

We assume that the utility function u is strictly concave, increasing, continuously differentiable, $u(0) = 0$ and (Inada Condition) $u'(0) = +\infty$. The discount factor γ satisfies $0 < \gamma < 1$.

Let V denote the value-function, i.e.

$$V(k_0) = \max \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

under the constraints

$$\forall t \geq 0, c_t \geq 0, k_t \geq 0, c_t \leq f(k_t) - k_{t+1}, \text{ and } k_0 \geq 0 \text{ is given.}$$

Remark 1: Before proceeding the analysis, we wish to say that our technology specification used for aggregation purpose in this paper is not restrictive. Indeed, consider the following production function $f(k) = \max\{Ak^\alpha, Bk^\beta\}$, with $A \neq B$. Define $\widetilde{k} = \frac{k}{\lambda}$, $\widetilde{c} = \frac{c}{\lambda}$, $v(c) = u(\frac{c}{\lambda})$, where λ satisfies $A\lambda^\alpha = B\lambda^\beta$. Let $A' = A\lambda^\alpha = B\lambda^\beta$. It is easy to check that the original optimal growth model behind becomes

$$\max \sum_{t=0}^{+\infty} \gamma^t v(\widetilde{c}_t)$$

under the constraints

$$\forall t \geq 0, \widetilde{c}_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \leq \widetilde{f}(\widetilde{k}_t) - \widetilde{k}_{t+1}, \text{ and } \widetilde{k}_0 > 0 \text{ is given.};$$

where $\widetilde{f}(x) = \max\{A'x^\alpha, A'x^\beta\}$.

3 Analysis of the optimal growth paths

The preliminary results are summarized in the following proposition.

Proposition 1 *(i) For any $k_0 \geq 0$, there exists an optimal growth path $(c_t^*, k_t^*)_{t=0,\dots,+\infty}$ which satisfies:*

$$\forall t, 0 \leq k_t^* \leq M = \max\left[k_0, \widetilde{k}\right], 0 \leq c_t^* \leq f(M),$$

where $\widetilde{k} = f(\widetilde{k})$.

(ii) If $k_0 > 0$, then $\forall t, c_t^* > 0, k_t^* > 0, k_t^* \neq 1$, and we have Euler equation

$$u'(c_t^*) = \gamma u'(c_{t+1}^*) f'(k_{t+1}^*).$$

(iii) Let $k_0' > k_0$ and $(k_t'^*)$ be an optimal path associated with k_0'. Then we have: $\forall t, k_t'^* > k_t^*$.

(iv) The optimal capital stocks path is monotonic and converges to an optimal steady state. Here, this steady state will be either $\widehat{k}_\alpha = (\gamma A\alpha)^{\frac{1}{1-\alpha}}$ or $\widehat{k}_\beta = (\gamma A\beta)^{\frac{1}{1-\beta}}$.

Proof: (i) The proof of this statement is standard and may be found in Le Van and Dana (2003), chapter 2. (ii) From Askri and Le Van (1998), the value-function V is differentiable at any $k_t^*, t \geq 1$. Moreover, $V'(k_t^*) =$

$u'(f(k_t^*)-k_{t+1}^*)f'(k_t^*)$ and this excludes that $k_t^*=1$ since 1 is the only point where f is not differentiable. From Inada Condition, we have $c_t^*>0, k_t^*>0, \forall t$. Hence, Euler Equation holds for every t.

(iii) It follows from Amir (1996) that $k_0'>k_0$ implies $\forall t, k_t'^*>k_t^*$. From Euler Equation we have

$$u'(f(k_0)-k_1^*)=\gamma V'(k_1^*)$$

and

$$u'(f(k_0')-k_1^*)=\gamma V'(k_1'^*).$$

If $k_1^*=k_1'^*$ then $k_0=k_0'$: a contradiction. Hence, $k_1^*<k_1'^*$. By induction, $\forall t>1, k_t'^*>k_t^*$.

(iv) First assume $k_1^*>k_0$. Then the sequence $(k_t^*)_{t\geq 2}$ is optimal from k_1^*. From (iii), we have $k_2^*>k_1^*$. By induction, $k_{t+1}^*>k_t^*, \forall t$. If $k_1^*<k_0$, using the same argument yields $k_{t+1}^*<k_t^*, \forall t$. Now if $k_1^*=k_0$, then the stationary sequence $(k_0,k_0,...,k_0,...)$ is optimal.

We have proved that any optimal path (k_t^*) is monotonic. Since, from (1), it is bounded, it must converge to an optimal steady state k^s. If this one is different from zero, then the associated optimal steady state consumption c^s must be strictly positive from Inada Condition. Hence, from Euler Equation, either $k^s=\widehat{k}_a$ or $k^s=\widehat{k}_b$ since it could not equal 1.

It remains to prove that (k_t^*) cannot converge to zero. On the contrary, for t large enough, say greater than some T, we have $u'(c_t^*)>u'(c_{t+1}^*)$ since $f'(0)=+\infty$. Hence, $c_{t+1}^*>c_t^*$ for every $t\geq T$. In particular, $c_{t+1}^*>c_T^*>0$, $\forall t>T$. But $k_t^*\to 0$ implies $c_t^*\to 0$: a contradiction. ■

We obtain the following corollary:

Corollary 1 *If $\gamma A\alpha>1$, then any optimal path from $k_0>0$ converges to $\widehat{k}_\beta$. If $\gamma A\beta<1$, then any optimal path from $k_0>0$ converges to $\widehat{k}_\alpha$.*

Proof: In Proposition 1, we have shown that any optimal path (k_t^*) converges either to $\widehat{k}_\alpha$ or to $\widehat{k}_\beta$. But when $\gamma A\alpha>1$, we have $\widehat{k}_\alpha>1$, $f(\widehat{k}_\alpha)=A(\widehat{k}_\alpha)^\beta$ and $f'(\widehat{k}_\alpha)=\beta A(\widehat{k}_\alpha)^{\beta-1}\neq\frac{1}{\gamma}$. Consequently, $\widehat{k}_\alpha$ could not be an optimal steady state. Therefore, (k_t^*) cannot converge to $\widehat{k}_\alpha$. From the statement (iv) in Proposition 1, it converges to $\widehat{k}_\beta$.

Similarly, when $\gamma A\beta < 1$, any optimal path from $k_0 > 0$ converges to $\widehat{k}_\alpha$. ■

In our Figure, when $\widehat{k}_\alpha \geq 1, \alpha-$ technology is clearly less efficient than $\beta-$technology, thus $\widehat{k}_\alpha$ is not the optimal steady state. Similarly for $\widehat{k}_\beta \leq 1$. In these cases, there will be an unique optimal steady state. But when the discount factor is in an intermediate range defined by $\gamma A\alpha \leq 1 \leq \gamma A\beta$, there exists more than one such state. We now give an example where $\widehat{k}_\alpha$ and $\widehat{k}_\beta$ are both optimal. Since x_1 and x_2 are independent of A and γ, we can choose A and γ such that

$$\alpha A x_1^{\alpha-1} = \beta A x_2^{\beta-1} = \frac{1}{\gamma}, \text{ with } 0 < \gamma < 1.$$

It is easy to check that x_1 and x_2 are optimal steady states for the convexified technology and hence for our technology. Since $x_1 = \widehat{k}_\alpha$, $x_2 = \widehat{k}_\beta$, we have found two positive optimal steady states. It will be shown in the following proposition.

Proposition 2 *Assume $\gamma A\alpha \leq 1 \leq \gamma A\beta$. If $\gamma A\alpha$ is close to 1, then any optimal path (k_t^*) from $k_0 > 0$ converges to $\widehat{k}_\beta$. If $\gamma A\beta$ is close to 1, then (k_t^*) converges to $\widehat{k}_\alpha$.*

Proof: First, observe that when $\gamma A\alpha \leq 1$ then $f(\widehat{k}_\alpha) = A\left(\widehat{k}_\alpha\right)^\alpha$ and when $1 \leq \gamma A\alpha$, $f(\widehat{k}_\beta) = A\left(\widehat{k}_\beta\right)^\beta$. Now consider the case $\gamma A\alpha = 1 < \gamma A\beta$. We have $\widehat{k}_\alpha = 1$ and $A > 1$.

It is well-known that given $k_0 > 0$, there exists a unique optimal path from k_0 for the β-technology. Moreover, this optimal path converges to $\widehat{k}_\beta$. Observe that the stationary sequence $\left(\widehat{k}_\alpha, \widehat{k}_\alpha, ..., \widehat{k}_\alpha, ...\right)$ is feasible from $\widehat{k}_\alpha$, for the β-technology, since it satisfies $0 \leq \widehat{k}_\alpha = 1 < A\left(\widehat{k}_\alpha\right)^\beta = A$. Hence, if $\left(\widetilde{k}_t\right)$ is an optimal path for β-technology starting from $\widehat{k}_\alpha$ and if (k_t) is an optimal path of our model starting also from $\widehat{k}_\alpha$, we will have

$$\sum_{t=0}^{\infty}\gamma^t u(f(\widehat{k}_\alpha)-\widehat{k}_\alpha) < \sum_{t=0}^{\infty}\gamma^t u(f(\widetilde{k}_t)-\widetilde{k}_{t+1}) \leq \sum_{t=0}^{\infty}\gamma^t u(f(k_t)-k_{t+1}) = V(\widehat{k}_\alpha).$$

This amounts to be certain that $\widehat{k}_\alpha$ can not be an optimal steady state. Hence, any optimal path from $k_0 > 0$ must converge to $\widehat{k}_\beta$.

Since $\widehat{k}_\alpha$ is continuous in γ, V continuous and since $\sum_{t=0}^{\infty} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) < V(\widehat{k}_\alpha)$ when $\gamma A\alpha = 1$, this inequality still holds when $\gamma A\alpha$ is close to 1 and less than 1. In other words, $\widehat{k}_\alpha$ is not an optimal steady state when $\gamma A\alpha$ is close to 1 and less than 1. Consequently, any optimal path with positive initial value will converge to $\widehat{k}_\beta$.

Similar argument applies when $\gamma A\beta$ is near one but greater than one. ■

What then happens when the discount factor is within an intermediate range? We now would like to show :

Proposition 3 *Assume $\gamma A\alpha < 1 < \gamma A\beta$. If both $\widehat{k}_\alpha$ and $\widehat{k}_\beta$ are optimal steady states then there exists a critical value k_c such that every optimal path from $k_0 < k_c$ will converge to $\widehat{k}_\alpha$, and every optimal path from $k_0 > k_c$ will converge to $\widehat{k}_\beta$.*

Proof: Consider at first $k_0 < \widehat{k}_\alpha$. Since $\widehat{k}_\alpha$ is optimal steady state, we have $k_t^* < \widehat{k}_\alpha, \forall t > 0$. Since the sequence (k_t^*) is increasing, bounded from above by $\widehat{k}_\alpha$, it will converge to $\widehat{k}_\alpha$. Similarly, when $k_0 > \widehat{k}_\beta$, any optimal path converges to $\widehat{k}_\beta$.

Let $\overline{k} = \sup \left\{ k_0 : k_0 \geq \widehat{k}_\alpha \right\}$ such that any optimal path from k_0 converges to $\widehat{k}_\alpha$. Obviously, $\overline{k} \leq \widehat{k}_\beta$, since $\widehat{k}_\beta$ is optimal steady state.

Let $\underline{k} = \inf \left\{ k_0 : k_0 \leq \widehat{k}_\beta \right\}$ such that any optimal path from k_0 converges to $\widehat{k}_\beta$. Obviously, $\underline{k} \geq \widehat{k}_\alpha$, since $\widehat{k}_\alpha$ is optimal steady state.

We now claim that $\overline{k} = \underline{k}$.

It is obvious that $\overline{k} \leq \underline{k}$. Now, if $\overline{k} < \underline{k}$, then take k_0, k_0' which satisfy $\overline{k} < k_0 < k_0' < \underline{k}$. From the definitions of $\overline{k}$ and $\underline{k}$, there exist an optimal path from $k_0, (k_t^*)$, which converges to $\widehat{k}_\beta$ and an optimal path from $k_0', (k_t'^*)$, which converges to $\widehat{k}_\alpha$. For t large enough, $k_t'^* < k_t^*$, which is impossible since $k_0 < k_0'$ (see Proposition 1, statement (iii)).

Posit $k_c = \overline{k} = \underline{k}$ and conclude. ■

Remark 2: Let now $\widehat{k}_\alpha$ and $\widehat{k}_\beta$, depicted in our Figure, be two potential optimal steady states and ask the question which of them will be the long run equilibrium in the optimal growth model or both of them are the long run equilibria. In the following proposition, we give sufficient conditions for $\widehat{k}_\alpha$ (respectively $\widehat{k}_\beta$) to be an optimal steady state or both $\widehat{k}_\alpha$, $\widehat{k}_\beta$ be optimal steady states. Before stating this proposition, we mention two lemmas. Their proofs may be found in Kamihigashi and Roy (2007, page 442).

Let $F(x) = \gamma f(x) - x, \forall x \geq 0$.

Lemma 1 *Let (k_t) be an optimal capital path which is nonstationary. Then there exists t such that $F(k_0) < F(k_t)$.*

Lemma 2 *Suppose there exists $\widehat{k}$ which satisfies $F(\widehat{k}) = \sup_{x\geq 0} F(x)$. Then $\widehat{k}$ is an optimal steady state.*

We now set up the following conditions:

Condition 1: $\frac{(A\gamma\alpha)^{\frac{\alpha}{1-\alpha}}}{(A\gamma\beta)^{\frac{\beta}{1-\beta}}} > \frac{1-\beta}{1-\alpha} \Leftrightarrow F(\widehat{k}_\alpha) > F(\widehat{k}_\beta)$

Condition 2: $\frac{(A\gamma\alpha)^{\frac{\alpha}{1-\alpha}}}{(A\gamma\beta)^{\frac{\beta}{1-\beta}}} < \frac{1-\beta}{1-\alpha} \Leftrightarrow F(\widehat{k}_\alpha) < F(\widehat{k}_\beta)$

Condition 3: $\frac{(A\gamma\alpha)^{\frac{\alpha}{1-\alpha}}}{(A\gamma\beta)^{\frac{\beta}{1-\beta}}} = \frac{1-\beta}{1-\alpha} \Leftrightarrow F(\widehat{k}_\alpha) = F(\widehat{k}_\beta)$

Proposition 4 *Assume $A\gamma\alpha \leq 1 \leq A\gamma\beta$.*

(i) Under condition 1, $\widehat{k}_\alpha$ is an optimal steady state. The stationary sequence $(\widehat{k}_\alpha)$ is the unique optimal sequence from $\widehat{k}_\alpha$.

(ii) If $\frac{(A\gamma\alpha)^{\frac{\alpha}{1-\alpha}}}{(A\gamma\beta)^{\frac{\beta}{1-\beta}}} > \frac{1-\zeta_\alpha\beta}{1-\zeta_\alpha\alpha} \Leftrightarrow \gamma f(\widehat{k}_\alpha) - \zeta_\alpha \widehat{k}_\alpha > \gamma f(\widehat{k}_\beta) - \zeta_\alpha \widehat{k}_\beta$ where $\zeta_\alpha = \frac{u'(f(\widehat{k}_\beta)-\widehat{k}_\alpha)}{u'(f(\widehat{k}_\alpha)-\widehat{k}_\alpha)} < 1$, then $\widehat{k}_\alpha$ is the unique optimal steady state. Any optimal path from $k_0 > 0$ will converge to $\widehat{k}_\alpha$.

(iii)Under the condition 2, $\widehat{k}_\beta$ is an optimal steady state. The stationary sequence $(\widehat{k}_\beta)$ is the unique optimal sequence from $\widehat{k}_\beta$.

(iv) If

$$\widehat{k}_\beta < A\widehat{k}_\alpha^\alpha,$$

and if

$$\frac{(A\gamma\alpha)^{\frac{\alpha}{1-\alpha}}}{(A\gamma\beta)^{\frac{\beta}{1-\beta}}} < \frac{1-\zeta_\beta\beta}{1-\zeta_\beta\alpha} \Leftrightarrow \gamma f(\widehat{k}_\alpha) - \zeta_\beta \widehat{k}_\alpha < \gamma f(\widehat{k}_\beta) - \zeta_\beta \widehat{k}_\beta$$

where $\zeta_\beta = \frac{u'(f(\widehat{k}_\alpha)-\widehat{k}_\beta)}{u'(f(\widehat{k}_\beta)-\widehat{k}_\beta)} < \frac{1}{\beta}$ *, then* $\widehat{k}_\beta$ *is the unique optimal steady state. Any optimal path from* $k_0 > 0$ *will converge to* $\widehat{k}_\beta$

(v) Under condition 3, both $\widehat{k}_\alpha$ *and* $\widehat{k}_\beta$ *are optimal steady states. In this case, we have a critical value* k_c *stated in the preceding proposition.*

Proof: *Proof of (i):* It is easy to check that $\widehat{k}_\alpha$ is the unique maximizer of F between 0 and 1 while $\widehat{k}_\beta$ is the unique maximizer for $x \geq 1$. Moreover, F increases from 0 to $\widehat{k}_\alpha$ and decreases from $\widehat{k}_\alpha$ to 1. It then increases from 1 to $\widehat{k}_\beta$ and decreases after. Under the condition $F(\widehat{k}_\alpha) > F(\widehat{k}_\beta)$, we have $F(\widehat{k}_\alpha) = \max_{x\geq 0} F(x)$. From Lemma 2, $\widehat{k}_\alpha$ is an optimal steady state. If there exists another optimal path (k_t) from $\widehat{k}_\alpha$, this one must be nonstationary and satisfies $F(k_t) \leq F(\widehat{k}_\alpha), \forall t$. That contradicts Lemma 1. There exists therefore a unique optimal path from $\widehat{k}_\alpha$.

Proof of (ii): Observe that for any $\zeta \in (0,1)$, we have $\frac{1-\zeta\beta}{1-\zeta\alpha} > \frac{1-\beta}{1-\alpha}$. Thus under the condition $\gamma f(\widehat{k}_\alpha) - \zeta\widehat{k}_\alpha > \gamma f(\widehat{k}_\beta) - \zeta\widehat{k}_\beta$, $\widehat{k}_\alpha$ is an optimal steady state. To prove that $\widehat{k}_\beta$ is not an optimal steady, observe that the stationary sequence $(\widehat{k}_\alpha)$ is feasible from $\widehat{k}_\beta$. Let

$$\delta_T = u(f(\widehat{k}_\beta) - \widehat{k}_\alpha) + \sum_{t=1}^{T} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) - \sum_{t=0}^{T} \gamma^t u(f(\widehat{k}_\beta) - \widehat{k}_\beta).$$

One has:

$$\begin{aligned}
\delta_T \quad &\geq \quad u'(f(\widehat{k}_\beta) - \widehat{k}_\alpha)(\widehat{k}_\beta - \widehat{k}_\alpha) + \sum_{t=1}^{T} \gamma^t u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha)(f(\widehat{k}_\alpha) - \widehat{k}_\alpha - f(\widehat{k}_\beta) + \widehat{k}_\beta) \\
&\geq \quad \gamma u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) f(\widehat{k}_\alpha) - u'(f(\widehat{k}_\beta) - \widehat{k}_\alpha)\widehat{k}_\alpha \\
&- \quad [\gamma u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) f(\widehat{k}_\beta) - u'(f(\widehat{k}_\beta) - \widehat{k}_\alpha)\widehat{k}_\beta] \\
&+ \quad \gamma u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha)[F(\widehat{k}_\alpha) - F(\widehat{k}_\beta)] + ... + \gamma^{T-1} u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha)[F(\widehat{k}_\alpha) - F(\widehat{k}_\beta)] \\
&+ \quad \gamma^T u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha)(\widehat{k}_\beta - \widehat{k}_\alpha).
\end{aligned}$$

Letting T converge to infinity, one gets

$$\lim_{T\to+\infty} \delta_T \geq u'(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) \left[\gamma f(\widehat{k}_\alpha) - \zeta\widehat{k}_\alpha - (\gamma f(\widehat{k}_\beta) - \zeta\widehat{k}_\beta)\right] > 0.$$

That proves that $\widehat{k}_\beta$ is not an optimal steady state. Any optimal path from $k_0 > 0$ will converge to $\widehat{k}_\alpha$.

Proof of (iii): It is similar to the proof of (i).

Proof of (iv): It suffices to prove that $\widehat{k}_\alpha$ is not an optimal steady state. One can check that conditions in (iv) imply that $\widehat{k}_\beta$ is feasible from $\widehat{k}_\alpha$, and $F(\widehat{k}_\beta) > F(\widehat{k}_\alpha)$. As in the proof of (ii), we find

$$u(f(\widehat{k}_\alpha) - \widehat{k}_\beta) + \sum_{t\geq 1} \gamma^t u(f(\widehat{k}_\beta) - \widehat{k}_\beta) > \sum_{t\geq 1} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha).$$

Proof of (v): We have $F(\widehat{k}_\alpha) = F(\widehat{k}_\beta) = \max_{x\geq 0} F(x)$. It follows from Lemma 2 that both $\widehat{k}_\alpha$ and $\widehat{k}_\beta$ are optimal steady states. Existence of critical value follows from Proposition 3 ■

Remark 3: The condition given in (v) is satisfied by the points x_1, x_2 (see our Figure) in Remark 2.

Remark 4: The existence of critical value is recognized since the paper by Dechert and Nishimura (1983). See also, for the continuous time setting, Askenazy and Le Van (1999). But in these models, the technology is convex-concave. The low steady state is unstable while the high is stable. An optimal path converges either to zero (extinction) or to the high steady state. In our model, with a technology which is concave-concave, any optimal path converges either to the high steady state or to the low steady state which is strictly positive. This latter acts as an attractor unless the discount rate is very low.

Remark 5: In the proposition 4, (i) and (iii) own much to Kamihigashi and Roy (2007). Uniqueness of the steady state in (ii) and (iv) , and the critical value k_c in (v) are peculiar to our model of concave-concave technology.

4 The case of linear utility function

In this section, which is much inspired from Kamihigashi and Roy (2006), we consider the case of linear utility function. Earlier consideration of this case has been given in Majumdar and Mitra (1983) with a convex-concave technology. Here, with the concave-concave technology, we provide necessary and sufficient conditions either to have uniqueness of steady state or, in case of two steady states $\widehat{k}_\alpha$ and $\widehat{k}_\beta$, to have the poverty trap $\widehat{k}_\alpha$as an equilibrium. We also show that the economy attains a steady state in finite time, then jumps to the other steady state and stays there forever.

Assume in this section that the rate of discount falls in the intermediate range, i.e. $A\gamma\alpha < 1 < A\gamma\beta$. There will be two potential optimal steady states $\widehat{k}_\alpha$, $\widehat{k}_\beta$, with $\widehat{k}_\alpha < \widehat{k}_\beta$. We will use the following lemmas, the proofs of which are given in Kamihigashi and Roy (2006, respectively p.335 and p. 333)

Lemma 3 *If the utility function is linear and the production function f is strictly increasing, upper semicontinuous, satisfies $f(0) = 0$ and $\exists\bar{x}, \forall x \in (0, \bar{x}), \gamma f'(x) > 1$, then every optimal path (k_t) from $k_0 > 0$ satisfies $k_t \geq \bar{x}$ for every t large enough.*

Lemma 4 *If the utility function is linear and the production function f is strictly increasing, upper semicontinuous and satisfies $f(0) = 0$ then for any optimal path (k_t), one has two alternatives:*

$$\begin{aligned} \lim_{t\to+\infty} k_t &= 0 \\ \exists T, \forall t \geq T, k_t &\in S, \end{aligned}$$

where S is the set of positive steady states.

We will apply these lemma to our model and get

Proposition 5 *Assume the utility linear and $A\gamma\alpha < 1 < A\gamma\beta$. Let (k_t) be an optimal path from $k_0 > 0$. Then it will reach in finite time either $\widehat{k}_\alpha$ or $\widehat{k}_\beta$.*

Proof: Since $f'(0) = +\infty$ in our model, from Lemma 3, an optimal path from $k_0 >$ cannot converge to zero. The result follows from Lemma 4. ■

We also use the following lemma (Kamihigashi and Roy, 2006, p.330)

Lemma 5 *Assume the utility function is linear and the production function f is strictly increasing, upper semicontinuous and satisfies $f(0) = 0$. Let k^1, k^2 be optimal steady states such that k^1 is feasible from k^2 and k^2 is feasible from k^1. Then any capital path (k_t) with $k_t \in \{k^1, k^2\}$ is optimal from k^1 and from k^2.*

We now give sufficient and necessary conditions either for $\widehat{k}_\alpha$ (respectively $\widehat{k}_\beta$) to be optimal steady state, or both of them be optimal steady states.

Proposition 6 *Assume $A\gamma\alpha < 1 < A\gamma\beta$.*

(i) Under condition 1, $\widehat{k}_\alpha$ is the unique optimal steady state. Any optimal path from $k_0 > 0$ will reach $\widehat{k}_\alpha$ in finite time.

(ii) Under condition 2 and if $\widehat{k}_\beta < A\widehat{k}_\alpha^\alpha$,then $\widehat{k}_\beta$ is the unique optimal steady state. Any optimal path from $k_0 > 0$ will reach $\widehat{k}_\beta$ in finite time.

(iii) Under condition 3 and if $\widehat{k}_\beta \leq A\widehat{k}_\alpha^\alpha$, then both $\widehat{k}_\beta$ and $\widehat{k}_\alpha$ are optimal steady states In this case there is no critical value k_c.[1]*Moreover, an optimal path will reach either $\widehat{k}_\beta$ or $\widehat{k}_\alpha$ at some date T.*

(iv) Under condition 3 and if Assume $\widehat{k}_\beta > A\widehat{k}_\alpha^\alpha$, then both $\widehat{k}_\beta$ and $\widehat{k}_\alpha$ are optimal steady states. There will be in this case a critical value k_c as in proposition 3.

Proof: *Proof of (i)*: It is obvious that $\widehat{k}_\alpha$ is optimal steady state since it maximizes the function F. We now claim that it is the unique optimal steady state. For that, let

$$\delta_T = u(f(\widehat{k}_\beta) - \widehat{k}_\alpha) + \sum_{t=1}^{T} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) - \sum_{t=0}^{T} \gamma^t u(f(\widehat{k}_\beta) - \widehat{k}_\beta).$$

We have

$$\delta_T = \sum_{t=0}^{T-1} \gamma^t \left(F(\widehat{k}_\alpha) - F(\widehat{k}_\beta) \right) + \gamma^T(\widehat{k}_\beta - \widehat{k}_\alpha),$$

and $\lim_{T\to+\infty} \delta_T = \frac{F(\widehat{k}_\alpha)-F(\widehat{k}_\beta)}{1-\gamma} > 0$. Our claim is true. From Proposition 5, any optimal path from $k_0 > 0$ will reach $\widehat{k}_\alpha$ in finite time.

Proof of (ii): Similarly

$$\lim_{T\to+\infty} \left[u(f(\widehat{k}_\alpha) - \widehat{k}_\beta) + \sum_{t=1}^{T} \gamma^t u(f(\widehat{k}_\beta) - \widehat{k}_\beta) - \sum_{t=0}^{T} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha \right] > 0.$$

As just above, any optimal path from $k_0 > 0$ will reach $\widehat{k}_\beta$ in finite time.

[1] This result is due to Takashi Kamihigashi

Proof of (iii): We prove that there is no critical value k_c. Let $k_0 > 0$. From Lemma 4, an optimal path will reach either $\widehat{k}_\beta$ or $\widehat{k}_\alpha$ at some date T. If it reaches $\widehat{k}_\beta$, the claim is true. If it reaches $\widehat{k}_\alpha$, then from Lemma 5, define $k_t = \widehat{k}_\beta$ for $t \geq T+1$.
Proof of (iv): Apply Proposition 3. ■

In the following proposition, we show that we may have an optimal path which reaches $\widehat{k}_\beta$ (respectively $\widehat{k}_\alpha$) and stays at $\widehat{k}_\alpha$(respectively $\widehat{k}_\beta$) afterward.

We need the following lemma, the proof of which is given in Kamihigashi and Roy (2006, p.333).

Lemma 6 *Let (k_t) be an optimal path from $k_0 > 0$. If for any $t \leq T$, k_t is not a steady state, then $(k_t)_{t=0,..,T+1}$ is strictly monotone.*

Proposition 7 *Assume $A\gamma\alpha < 1 < A\gamma\beta$. Under condition 3 and if $\widehat{k}_\beta \leq A\widehat{k}_\alpha^\alpha$, then for any $k_0 > 0$,we can find an optimal path (k_t^*) which reaches $\widehat{k}_\beta$ (alternatively $\widehat{k}_\alpha$) at some finite time T, then jumps to $\widehat{k}_\alpha$ (alternatively $\widehat{k}_\beta$) and stays there forever afterward .*

Proof: We will prove that there exists an optimal path which reaches $\widehat{k}_\beta$ and stays at $\widehat{k}_\alpha$ forever afterward. The proof of the other alternative is similar.
(a) If $k_0 \geq \widehat{k}_\beta$, since $\widehat{k}_\beta \leq A\widehat{k}_\beta^\beta \leq Ak_0^\beta$, the path $(k_t^*) = (k_0, \widehat{k}_\beta, \widehat{k}_\alpha, \widehat{k}_\alpha, ..., \widehat{k}_\alpha, ...)$ is optimal. Indeed,

$$\begin{aligned}
\sum_{t=0}^{+\infty} \gamma^t (f(k_t^*) - k_{t+1}^*) &= f(k_0) + \sum_{t=0}^{+\infty} \gamma^t F(k_{t+1}^*) \\
&= F(k_0) + F(\widehat{k}_\beta) + \sum_{t=1}^{+\infty} \gamma^t F(\widehat{k}_\alpha) \\
&= F(k_0) + \sum_{t=0}^{+\infty} \gamma^t F(\widehat{k}_\alpha) \\
&\geq F(k_0) + \sum_{t=0}^{+\infty} \gamma^t F(k_t) = \sum_{t=0}^{+\infty} \gamma^t (f(k_t) - k_{t+1})
\end{aligned}$$

for any feasible sequence (k_t) from k_0.

(b) If $k_0 \in (\widehat{k}_\alpha, \widehat{k}_\beta)$,since $\widehat{k}_\beta \leq A\widehat{k}_\alpha^\alpha$, we repeat the preceding arguments to show that $(k_0, \widehat{k}_\beta, \widehat{k}_\alpha, \widehat{k}_\alpha, ..., \widehat{k}_\alpha, ...)$ is optimal from k_0.

(b) If $k_0 \in (\widehat{k}_\alpha, \widehat{k}_\beta)$,since $\widehat{k}_\beta \leq A\widehat{k}_\alpha^\alpha$, we repeat the preceding arguments to show that $(k_0, \widehat{k}_\beta, \widehat{k}_\alpha, \widehat{k}_\alpha, ..., \widehat{k}_\alpha, ...)$ is optimal from k_0.

(c). Let $k_0 \in (0, \widehat{k}_\alpha)$. Any optimal path, before reaching $\widehat{k}_\alpha$ or $\widehat{k}_\beta$, since it cannot converge to zero, will be strictly increasing (from Lemma 6). There exists T' such that $k^*_{T'} \geq \widehat{k}_\alpha$. The path $(k_0, k_1^*, ..., k^*_{T'}, \widehat{k}_\beta, \widehat{k}_\alpha, \ldots, \widehat{k}_\alpha \ldots)$ is optimal. ■

Remark 6: In Proposition 6, the uniqueness of the steady state and the possibility of a critical value k_c are proper to our model and have hopefully some theoretical value-added. The condition for uniqueness here is weaker than that arising from the case of strictly concave utility function.

5 Concluding comments

In this paper, the type of technological non-convexity under consideration assigns to the poverty trap - a non degenerated state - the role of an attractor exactly as the state of extinction in the case of S-shaped production functions encountered in earlier studies on optimal growth. When future discounting is high enough, precisely when $\gamma A\beta < 1$, it is shown that the resulting long run equilibrium is in fact $\widehat{k}_\alpha > 0$. For any value of the initial capital stock, the convergence to this equilibrium is monotonic. On the other hand, when future discounting is relatively low, precisely when $\gamma A\alpha > 1$, the same result will be obtained but with the equilibrium optimal steady state $\widehat{k}_\beta$. When future discounting is in some middle range, i.e. when $\gamma A\alpha < 1 < \gamma A\beta$, there might exist two optimal steady states and the dynamic convergence will depend on the initial stock of capital. We show that there is a critical value of per capita capital stock k_c such that every optimal paths from $k_0 < k_c$ will converge to $\widehat{k}_\alpha$, and every optimal paths from $k_0 > k_c$ will converge to $\widehat{k}_\beta$. Also, we provide sufficient conditions for $\widehat{k}_\alpha$ (respectively $\widehat{k}_\beta$) to be optimal steady states or both $\widehat{k}_\alpha$, $\widehat{k}_\beta$ be optimal steady states.

The case of linear utility is interesting to consider. When future discounting is in some middle range, we give sufficient conditions either to have uniqueness of steady state or, in case of several steady states, to have a poverty trap defined as the "low" steady state. We also show in the latter case that there is not necessarily a critical value k_c, and the convergence to the "low" steady state might be carried out in finite time, with a jump to the other "high" steady state afterward. But we may also have the converse, that is the optimal paths reach at first the "high" steady state and fall down to the "low" steady state afterward. We also provide the condition under which such a critical value k_c still plays its role in the convergence to equilibria as in the case of strictly concave utility studied earlier.

Several useful remarks can be further made. First, it is conceivable that the results obtained in this paper are unaffected when either one or both production technologies entails some fixed costs, i.e. positive output is made possible only if the capital per capita exceeds a threshold level, but their aggregation exhibits the kind of non-convexity depicted in our Figure. Second, for the economist-statisticians, this paper hopefully highlights the importance of informations other than those contained in the technology-blueprint book. Under either high or low future discounting, only one technology is relevant in the sense that it is the chosen technology in long run equilibrium. This certainly helps identifying the production function for data aggregation task. If the future discount rate falls in a range defined by $\gamma A\alpha < 1 < \gamma A\beta$, then the computation of the critical capital stock k_c is essential in view of the determination of the relevant production technology at stake. Third, when there are several production technologies, it is possible to proceed with pair-wise aggregation in order to determine the relevant technology for long run equilibrium. Assume that we have a third technology, say the ε- technology, to take into account. Pair-wise aggregation of α and β-technology allows us to eliminate the α- technology, say. Therefore, we now have to perform the same analysis with β - technology and ϵ-technology, and so on so forth when several technologies are at stake. Pair-wise consideration in this way would help determining the relevant technology corresponding to the optimal steady state.

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